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THE NUMBER OF P-VERTICES OF SINGULAR ACYCLIC MATRICES: AN INVERSE PROBLEM

Zhibin Du

School of Software
South China Normal University
Foshan Guangdong 528225, China

School of Mathematics and Statistics
Zhaoqing University
Zhaoqing 526061, China

e-mail: zhibindu@126.com

and

Carlos M. da Fonseca

Kuwait College of Science and Technology
Doha District, Block 4, P.O. Box 27235, Safat 13133, Kuwait

University of Primorska
FAMNIT, Glagoljsaška 8, 6000 Koper, Slovenia

e-mail: c.dafonseca@kcst.edu.kw
carlos.dafonseca@famnit.upr.si

Dedicated to the memory of Prof. Slobodan K. Simić. In honour of Prof. Slobodan K. Simić for his decisive contributions to spectral graph theory.

Abstract

Let A be a real symmetric matrix. If after we delete a row and a column of the same index, the nullity increases by one, we call that index a P-vertex of A. When A is an $n \times n$ singular acyclic matrix, it is known that the maximum number of P-vertices is $n-2$. If T is the underlying tree of A, we will show that for any integer number $k \in \{0, 1, \ldots, n-2\}$, there is a (singular) matrix whose graph is T and with k P-vertices. We will provide illustrative examples.

Keywords: trees, acyclic matrices, singular, multiplicity of eigenvalues, P-set, P-vertices.

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1. Introduction

In 1960, Parter published a groundbreaking research [16] which led into fertile problems in both pure and applied mathematics, on the relation between the multiplicities of eigenvalues of real symmetric matrices and the structure of the corresponding underlying graph. One of the main problems was the relation between the multiplicities of an eigenvalue of a matrix and the submatrix obtained from the deletion of a row and a column of the same index, based on a given underlying graph. A special emphasis was given to trees.

For an undirected graph G, let $\mathcal{S}(G)$ denote the set of all real symmetric matrices sharing G. Each entry of main diagonal of these matrices can assume any real number value.

When studying the multiplicity of 0 as an eigenvalue of a matrix A in $\mathcal{S}(G)$, or in another word, the nullity, we may wonder what are the vertices whose deletion will increase such multiplicity by one. Such vertices are known in the literature as *P-vertices* of A, and the number of such vertices is denoted by $P_\nu(A)$.

In 2005, Kim and Shader proposed in [15] several problems on P-vertices and some related results for matrices whose graphs were paths, i.e., for tridiagonal matrices. These questions were mainly answered in [1, 2, 12] and fully extensions to trees were considered in [4–11].

One of the problems we can find in [15] is to prove that

$$\begin{aligned}&\{P_\nu(A) \mid A \text{ is an } n\times n \text{ nonsingular tridiagonal matrix}\}\\ &= \begin{cases} \{0,1,\dots,n\}, & \text{for even } n,\\ \{0,1,\dots,n-1\}, & \text{for odd } n.\end{cases}\end{aligned}$$

This question was fully answered in [2] and, for general trees, in [10]. Basically we have an inverse problem asking to find a tridiagonal matrix whose number of P-vertices is a given nonnegative integer of a certain interval.

For trees, we were able to develop different techniques in order to characterize the type of trees where the maximum number of P-vertices was attained.

Theorem 1 [10, Theorem 3.1]**.** *Let T be a tree on $n \geqslant 2$ vertices. Then the following statements are equivalent.*

(a) *There exists a nonsingular matrix $A \in \mathcal{S}(T)$ such that $P_\nu(A) = n$.*

(b) *T is a tree obtained from P_2 by a sequence of adding pendant P_2 operations, where adding pendant P_2 operation is referred to the addition of the path P_2 to a vertex of a tree.*

Moreover, we were able to solve a related problem.

Theorem 2 [10, Theorem 7.2]**.** *Let T be a tree on $n \geqslant 2$ vertices. If there exists a nonsingular matrix $B \in \mathcal{S}(T)$ such that $P_\nu(B) = n$, then*

$$\{P_\nu(A) \mid A \textit{ is a nonsingular matrix in } \mathcal{S}(T)\} = \{0,1,\dots,n\}.$$

Consequently, as in the answer to the problem to tridiagonal matrices, combining Theorems 1 and 2, we have the following.

Theorem 3. *Let T be a tree on $n \geqslant 2$ vertices. If T can be obtained from P_2 by a sequence of adding pendant P_2 operations, then*

$$\{P_\nu(A) \mid A \text{ is a nonsingular matrix in } \mathcal{S}(T)\} = \{0, 1, \ldots, n\}.$$

The problems involving singular matrices are in general more difficult to tackle. One of the main issues that one faces is regarding the exact multiplicity of zero, while for nonsingular matrices the multiplicity is exact zero. Nevertheless, we were able in [11] to provide a full characterization of the trees for which there are singular matrices with maximum number of P-vertices.

Theorem 4 [11, Theorem 3.6]**.** *Let T be a tree on $n \geqslant 3$ vertices. The following two conditions are equivalent.*

(a) *There exists a singular matrix $A \in \mathcal{S}(T)$ such that $P_\nu(A) = n - 2$.*

(b) *T is a tree obtained from the star S_3 by a sequence of adding pendant P_2 operations, and each terminal vertex of S_3 is still a terminal vertex of T.*

In the spirit of [15], in this note we aim to consider the following inverse problem. Let A be an $n \times n$ singular matrix whose graph is a tree T, and with maximum number of P-vertices, which is $n-2$, show that for any integer number $k \in \{0, 1, \ldots, n-2\}$, there is a singular matrix whose graph is T and with k P-vertices. We will provide examples of our results. First we will recall and establish new auxiliary results.

2. PRELIMINARIES

We start this section with a result which relates P-vertices with the zero-nonzero pattern of any eigenvector of a singular symmetric matrix associated with 0.

Lemma 5 [14, Theorem 2.1]**.** *Let A be a singular symmetric matrix. If i is a P-vertex of A, then the ith entry of x is 0, for every eigenvector x of A associated with eigenvalue 0.*

The next result is also of great importance. Here $m_A(0)$ denotes the nullity of A and $A(i,j)$ the principal submatrix of A obtained by deleting the i and j rows and columns of A. Similarly, we will use $A(i)$ to represent the principal submatrix of A obtained by deleting the i row and column of A later.

Proposition 6 [14, Proposition 4.7]**.** *Let T be a tree. For the matrix $A \in \mathcal{S}(T)$, if i and j are both P-vertices of A, then $m_{A(i,j)}(0) \neq m_A(0) + 1$.*

Let $A[v]$ represent precisely the diagonal entry of A corresponding to v.

Another useful observation states that if v is a terminal vertex of a tree adjacent to u, and the diagonal entry corresponding to v, of a matrix associated to a given tree, is equal to 0, then u is a P-vertex of such matrix.

Theorem 7 [13, Theorem 8]**.** *Suppose that T is a tree, and v is a terminal vertex of T with unique neighbor u. For $A \in \mathcal{S}(T)$, if $A[v] = 0$, then u is a P-vertex of A.*

Now, let T be the tree as shown in Figure 1, where T_i is a tree obtained from P_2 by a sequence of adding pendant P_2 operations, for each $1 \leqslant i \leqslant k$. Set $|V(T_i)| = n_i$, for $1 \leqslant i \leqslant k$. For each $1 \leqslant i \leqslant k$ and $0 \leqslant t_i \leqslant n_i$, from Theorem 3, there exists a nonsingular matrix $A_i \in \mathcal{S}(T_i)$ such that $P_\nu(A_i) = t_i$.

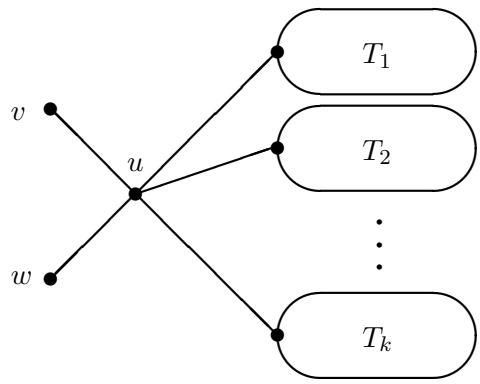

Figure 1. A tree T with extremal number of P-vertices.

Let $\oplus$ denote the direct sum of matrices.

Lemma 8. *Let T be the tree as shown in Figure 1, and $A_1, A_2, \ldots, A_k$ be the nonsingular matrices defined as above. Set A to be the matrix in $\mathcal{S}(T)$ satisfying the following two requirements:*

(a) $A[v] = A[w] = \begin{pmatrix} 0 \end{pmatrix}$;

(b) $A(u) = \begin{pmatrix} 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \end{pmatrix} \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_k$.

Then A is a singular matrix and

$$P_\nu(A) = 1 + t_1 + t_2 + \cdots + t_k.$$

Proof. We will prove this result based on three claims.

Claim 1. *u is a* P*-vertex of A.*

Proof. Since $A[v] = \begin{pmatrix} 0 \end{pmatrix}$, from Theorem 7, u is a P-vertex of A, i.e., $m_{A(u)}(0) = m_A(0) + 1$. Moreover, notice that $m_{A(u)}(0) = 2$, from the construction of A. So $m_A(0) = 1$, and the singularity of A follows. □

Claim 2. *Neither v nor w is a* P*-vertex of A.*

Proof. The proofs about v and w are similar. So we only prove that v is not a P-vertex of A. Since $A[w] = \begin{pmatrix} 0 \end{pmatrix}$, from Theorem 7, u is a P-vertex of $A(v)$, i.e., $m_{A(u,v)}(0) = m_{A(v)}(0) + 1$. Note that $m_{A(u,v)}(0) = 1$, thus $m_{A(v)}(0) = 0$, which implies that v is not a P-vertex of A. Similarly, we could show that w is not a P-vertex of A, either. □

Claim 3. *For any $j \in V(T_i)$, j is a* P*-vertex of A_i if and only if j is a* P*-vertex of A, where $1 \leqslant i \leqslant k$.*

Proof. First suppose that j is a P-vertex of A_i. Recall that u is a P-vertex of A, then $\{u, j\}$ would form a P-set of A. In particular, j is a P-vertex of A.

Next suppose that j is not a P-vertex of A_i. Recall that A_i is nonsingular, which means that $A_i(j)$ is still nonsingular and, equivalently, $m_{A(u,j)}(0) = 2 = m_A(0)+1$. Now it follows that j is not a P-vertex of A either, otherwise, it would lead to a contradiction to Proposition 6. □

Finally, combining Claims 1, 2, and 3, we may deduce that

$$P_\nu(A) = 1 + t_1 + t_2 + \cdots + t_k,$$

as desired. ■

3. MAIN RESULT

From the previous discussion, we may establish now our main result.

Theorem 9. *Let T be a tree on $n \geqslant 2$ vertices. If there exists a singular matrix $B \in \mathcal{S}(T)$ such that $P_\nu(B) = n - 2$, then*

$$\{P_\nu(A) \mid A \text{ is a singular matrix in } \mathcal{S}(T)\} = \{0, 1, \ldots, n-2\}.$$

Proof. For each $0 \leqslant t \leqslant n - 2$, we will present a method for constructing a singular matrix $A \in \mathcal{S}(T)$ such that $P_\nu(A) = t$.

If $t = 0$, then we may set A to be the Laplacian matrix of T. This implies $P_\nu(A) = 0$ from Lemma 5, since it is well-known that $(1, 1, \ldots, 1)^T$ is an eigenvector associated with eigenvalue 0 [3, p. 185].

Suppose in the following that $1 \leqslant t \leqslant n-2$. By Theorem 4, we know that T is a tree obtained from the star S_3 by a sequence of adding pendant P_2 operations, and each terminal vertex of S_3 is still a terminal vertex of T, i.e., T is a tree of the

form depicted in Figure 1, where each T_i is a tree obtained from P_2 by a sequence of adding pendant P_2 operations, for $1 \leqslant i \leqslant k$. In particular, set $n_i = |V(T_i)|$, for $1 \leqslant i \leqslant k$. So

$$n_1 + n_2 + \cdots + n_k = n - 3.$$

Let $t_1, t_2, \ldots, t_k$ be the integers satisfying $0 \leqslant t_i \leqslant n_i$ for each $1 \leqslant i \leqslant k$, and

$$t_1 + t_2 + \cdots + t_k = t - 1.$$

Now by Lemma 8, we may construct a singular matrix $A \in \mathcal{S}(T)$ such that

$$P_\nu(A) = 1 + t_1 + t_2 + \cdots + t_k = t,$$

as desired. ■

4. Examples

Let us consider the following tree

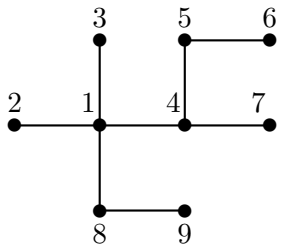

Figure 2

According to Figure 1, T_1 is the tree with vertices $4, 5, 6, 7$, T_2 with vertices $8, 9$, and the vertices u, v, w correspond, respectively, to $1, 2, 3$. Suppose that we want a matrix whose graph is depicted in Figure 2 with 5 P-vertices. Then we may consider the submatrices

$$A_1 = \begin{pmatrix} -2/3 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In case we want a singular matrix with 6 P-vertices, we have for example

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, the matrices

$$\left(\begin{array}{ccc|cccc|cc}
\mathbf{1} & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & \mathbf{-2/3} & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mathbf{0}
\end{array}\right)$$

and

$$\left(\begin{array}{ccc|cccc|cc}
\mathbf{1} & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & \mathbf{0} & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & \mathbf{0} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \mathbf{0} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \mathbf{0} & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)$$

will have, respectively, the required number of P-vertices. In fact the P-vertices of the first matrix are $1, 4, 7, 8, 9$, while of the second $1, 4, 5, 6, 7, 8$, all in the main diagonals in bold.

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