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BALANCEDNESS AND THE LEAST LAPLACIAN EIGENVALUE OF SOME COMPLEX UNIT GAIN GRAPHS

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Abstract

Let $\mathbb{T}_4 = \{\pm 1, \pm i\}$ be the subgroup of 4-th roots of unity inside $\mathbb{T}$, the multiplicative group of complex units. A complex unit gain graph Φ is a simple graph $\Gamma = (V(\Gamma) = \{v_1, \ldots, v_n\}, E(\Gamma))$ equipped with a map $\varphi : \vec{E}(\Gamma) \to \mathbb{T}$ defined on the set of oriented edges such that $\varphi(v_i v_j) = \varphi(v_j v_i)^{-1}$. The gain graph Φ is said to be balanced if for every cycle $C = v_{i_1} v_{i_2} \cdots v_{i_k} v_{i_1}$ we have $\varphi(v_{i_1} v_{i_2})\varphi(v_{i_2} v_{i_3}) \cdots \varphi(v_{i_k} v_{i_1}) = 1$.

It is known that Φ is balanced if and only if the least Laplacian eigenvalue $\lambda_n(\Phi)$ is 0. Here we show that, if Φ is unbalanced and $\varphi(\Phi) \subseteq \mathbb{T}_4$, the eigenvalue $\lambda_n(\Phi)$ measures how far is Φ from being balanced. More precisely, let $\nu(\Phi)$ (respectively, $\epsilon(\Phi)$) be the number of vertices (respectively, edges) to cancel in order to get a balanced gain subgraph. We show that

$$\lambda_n(\Phi) \leq \nu(\Phi) \leq \epsilon(\Phi).$$

We also analyze the case when $\lambda_n(\Phi) = \nu(\Phi)$. In fact, we identify the structural conditions on Φ that lead to such equality.

Keywords: gain graph, Laplacian eigenvalues, balanced graph, algebraic frustration.

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1. Introduction

Let Γ be a simple graph whose set of vertices is $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$. We define $\vec{E}(\Gamma)$ to be the set of oriented edges; in such set we find two copies of each edge of Γ with opposite directions. We write e_{ij} for the oriented edge from v_i to v_j. Given any group $\mathfrak{G}$, a ($\mathfrak{G}$-)*gain graph* is a triple $\Phi = (\Gamma, \mathfrak{G}, \varphi)$ consisting of an *underlying graph* Γ, the *gain group* $\mathfrak{G}$ and a map $\varphi : \vec{E}(\Gamma) \to \mathfrak{G}$ such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1}$ called the *gain function*.

Gain graphs are not only studied in pure mathematics, but also in physics, operations research, psychology and several other research areas (see the annotated bibliography [12] for an updated survey). Most of the basic notions on graphs directly extends to gain graphs. For example, the order $|\Phi|$ of a gain graph $\Phi = (\Gamma, \mathfrak{G}, \varphi)$ is simply $|\Gamma|$, the number of vertices of its underlying graph.

Let $\mathbb{T}$ denote the *circle group*, i.e., the multiplicative group of all complex numbers with norm 1. In other words, $\mathbb{T} = \{z \in \mathbb{C} : z\bar{z} = 1\}$ thought as a subgroup of the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers. For every fixed $k \in \mathbb{N}$, the group $\mathbb{T}$ contains just one subgroups of order k, namely the group of k-th roots of unity $\mathbb{T}_k = \{z \in \mathbb{C} : z^k = 1\}$. For every pair of positive integers (k, d), we have natural inclusions

$$\iota_k : z \in \mathbb{T}_k \mapsto z \in \mathbb{T}, \qquad \text{and} \qquad \iota_k^{dk} : z \in \mathbb{T}_k \mapsto z \in \mathbb{T}_{dk}.$$

$\mathbb{T}$-*gain graphs* are known in literature as *complex unit gain graphs* [7, 8, 10] or *weighted directed graphs* [1, 6].

In [7], the third author extended some fundamental concepts from spectral graph theory like the adjacency, incidence and Laplacian matrices to $\mathbb{T}$-gain graph. Moreover, he established in [8] a suitable setting for studying line graphs of $\mathbb{T}$-gain graphs.

Let $M_{m,n}(\mathbb{C})$ be the set of m-by-n-complex matrices. The *adjacency matrix* $A(\Phi) = (a_{ij}) \in M_{n,n}(\mathbb{C})$ is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If v_i is adjacent to v_j, then $a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})} = \bar{a}_{ji}$. Therefore, $A(\Phi)$ is Hermitian and its eigenvalues are real. The *Laplacian matrix* $L(\Phi)$ is defined as $D(\Gamma) - A(\Phi)$, where $D(\Gamma)$ is the diagonal matrix of vertex degrees. Therefore, $L(\Phi)$ is also Hermitian. As a consequence of [7, Lemma 3.1], the matrix $L(\Phi)$ is positive semidefinite, and therefore, its eigenvalues are nonnegative. We shall always assume they are labeled and ordered according to the following convention

$$0 \le \lambda_n(\Phi) \le \lambda_{n-1}(\Phi) \le \cdots \le \lambda_2(\Phi) \le \lambda_1(\Phi).$$

Let (k, d) be a fixed pair of positive integers. The three gain graphs

$$(1) \qquad \Phi = (\Gamma, \mathbb{T}_k, \varphi), \qquad \Phi' = (\Gamma, \mathbb{T}, \iota \circ \varphi) \qquad \text{and} \qquad \Phi'' = \left(\Gamma, \mathbb{T}_{dk}, \iota_k^{dk} \circ \varphi\right)$$

give rise to the same adjacency and Laplacian matrices. This not only means that every gain function $\varphi : \vec{E}(\Gamma) \to \mathbb{T}_k$ indirectly equips a simple graph Γ with a $\mathbb{T}_{dk}$-gain structure and a complex unit one, but also that the adjacency and Laplacian spectral properties for the three gain graphs (1) are identical.

In this paper we are particularly interested in $\mathbb{T}_4$-gain graphs. Keeping in mind the remark above, a $\mathbb{T}_4$-gain graph can be seen as a complex unit gain graphs with gains in the set $\{\pm 1, \pm i\}$; moreover every spectral result concerning $\mathbb{T}_4$-gain graphs applies as well to $\mathbb{T}_2$-gain graphs, which many readers probably know under the name of *signed graphs*. We point out that the spectral theory of $\mathbb{T}_4$-gain graphs also includes the one of digraphs and mixed graphs as developed, for instance, in [4]. In other words, digraphs and mixed graphs can be re-interpreted as $\mathbb{T}_4$-gain graphs $\Phi = (\Gamma, \varphi)$ such that $\varphi(\vec{E}(\Gamma)) \subseteq \{1, \pm i\}$.

As for the more general theory of biased graphs, balancedness plays a pivotal role for gain graphs (see [11]). Recall that an oriented edge $e_{i_h i_k} \in \vec{E}(\Gamma)$ is said to be *neutral* for $\Phi = (\Gamma, \mathfrak{G}, \varphi)$ if $\varphi(e_{i_h i_k}) = 1$. Similarly, the walk $W = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{l-1} i_l}$ is said to be *neutral* if its *gain*

$$\varphi(W) = \varphi(e_{i_1 i_2})\varphi(e_{i_2 i_3}) \cdots \varphi(e_{i_{l-1} i_l})$$

is equal to 1. An edge set $S \subseteq E$ is *balanced* if every cycle $C \subseteq S$ is neutral. A subgraph is *balanced* if its edge set is balanced.

By [7, Lemma 2.1(2)] or [10, Theorem 2.8], it follows that a connected $\mathbb{T}_4$-gain graph Φ is balanced if and only if the least Laplacian eigenvalue $\lambda_n(\Phi)$ is 0.

Since any unbalanced $\mathbb{T}_4$-gain graph surely contains a balanced gain subgraph, as in the signed case (see [2]) it makes sense to consider the *frustration number* $\nu(\Phi)$ (respectively, the *frustration index* $\epsilon(\Phi)$), i.e., the smallest number of vertices (respectively, edges) whose deletion leads to a balanced gain graph.

Here is the remainder of the paper. In Section 2 we give some basic results in order to keep the paper self-contained. In Section 3 we shall prove that $\lambda_n(\Phi) \leq \nu(\Phi) \leq \epsilon(\Phi)$ for every $\mathbb{T}_4$-gain graph Φ. In the same section we identify conditions on Φ ensuring that $\lambda_n(\Phi) = \nu(\Phi)$, and study how such conditions somehow simplify under some more restrictive structural constraints.

For the reasons explained above, it is not surprising to discover that conditions of our Theorems 3.3 and 3.9 are consistent with those required in [2, Theorems 3.3 and 3.7], where the correspondent problems for $\mathbb{T}_2$-gain graphs are solved.

2. PRELIMINARIES

From now on, a $\mathbb{T}_4$-gain graph will be simply denoted by $\Phi = (\Gamma, \varphi)$. We write $(\Gamma, 1)$ for the $\mathbb{T}_4$-gain graph will all neutral edges.

A *switching function* is any map $\zeta : V(\Gamma) \to \mathbb{T}_4$. Switching the $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$ means replacing φ by φ^ζ, the map defined by $\varphi^\zeta(e_{ij}) = \zeta(v_i)^{-1}\varphi(e_{ij})\zeta(v_j)$ producing the $\mathbb{T}_4$-gain graph $\Phi^\zeta = (\Gamma, \varphi^\zeta)$. We say that $\Phi_1 = (\Gamma, \varphi_1)$ and $\Phi_2 = (\Gamma, \varphi_2)$ (and their gain functions as well) are *switching equivalent* when there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. By writing $\Phi_1 \sim \Phi_2$ or $\varphi_1 \sim \varphi_2$ we mean that Φ_1 and Φ_2 are switching equivalent. A *potential function* for φ is a function $\theta : V(\Gamma) \to \mathbb{T}_4$, such that for every $e_{ij} \in \vec{E}(\Gamma)$, $\theta(v_i)^{-1}\theta(v_j) = \varphi(e_{ij})$.

The following lemma is known to the scholars and holds in the more general context of complex unit gain graphs.

Proposition 2.1. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph. Then the following are equivalent.*

(1) *Φ is balanced.*

(2) $\Phi \sim (\Gamma, 1)$.

(3) *φ has a potential function.*

Proof. The equivalence (1)⇔(2) is guaranteed by [11, Lemma 5.3]. Let now ζ be any switching function. The double implication

$$(\Gamma, \varphi) = (\Gamma, 1)^\zeta \iff \text{the map } \zeta \text{ is a potential function for } \varphi$$

shows that (2)⇔(3). ■

As usual, we denote by A^* the conjugate transpose of $A \in M_{m,n}(\mathbb{C})$. Like [5], we follow the convention that elements of $\mathbb{C}^n$ are always presented as column vectors.

Proposition 2.2. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph.*

$$\lambda_n(\Phi) = \min_{\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^* L(\Phi)\mathbf{x}}{\mathbf{x}^*\mathbf{x}}, \tag{2}$$

and

$$\mathbf{x}^* L(\Phi)\mathbf{x} = \sum_{\substack{e_{ij} \in \vec{E}(\Gamma) \\ i<j}} |x_i - \varphi(e_{ij})x_j|^2. \tag{3}$$

Proof. Since $L(\Phi)$ is Hermitian, Equation (2) essentially comes from [5, Theorem 4.2.2]. Equation (3) specializes [7, Lemma 5.3] to $\mathbb{T}_4$-gain graphs. ■

We end this section by introducing some further notation. Given two graphs Γ_1 and Γ_2, the graphs $\Gamma_1+\Gamma_2$ and $\Gamma_1\vee\Gamma_2$, respectively, denote the disjoint union and the complete join of Γ_1 and Γ_2. We shall often omit the subscript to the identity matrix I_n, to the all-1 matrix J_n, and to the null matrix O_n in $M_{n,n}(\mathbb{C})$ when the size of the matrices is clear from the context. Let A and B be two Hermitian matrices; $A\oplus B$ denotes a diagonal block matrix whose diagonal blocks are A and B. If A and B both belong to $M_{n,n}(\mathbb{C})$, we say that $A\succeq B$ if $A-B$ is positive semidefinite. Finally $A^\dagger$ denotes the Moore-Penrose pseudo-inverse. Recall that $A^\dagger$ is the unique Hermitian matrix such that

$$\text{(i) } A^\dagger A\, A^\dagger = A^\dagger, \qquad \text{(ii) } A\, A^\dagger A = A, \qquad \text{(iii) } A^\dagger A = A\, A^\dagger.$$

Obviously, if A is nonsingular then $A^\dagger = A^{-1}$.

3. ALGEBRAIC FRUSTRATION

Let $\Phi=(\Gamma,\varphi)$ be a $\mathbb{T}_4$-gain graph. For each set of edges $E'\subseteq E(\Gamma)$, $\Phi-E'$ denotes the $\mathbb{T}_4$-gain subgraph of Φ obtained by deletion of edges in E'. More precisely, once defined $\Gamma-E'$ as the graph with $V(\Gamma-E')=V(\Gamma)$ and $E(\Gamma-E')=E(\Gamma)-E'$, by definition

$$\Phi - E' = \left(\Gamma - E', \varphi|_{\vec{E}(\Gamma)-\vec{E}'}\right).$$

Proposition 3.1. *Let $\Phi=(\Gamma,\varphi)$ be a $\mathbb{T}_4$-gain graph of order n. Then*

$$\lambda_n(\Phi) \leq \epsilon(\Phi). \tag{4}$$

Proof. If Φ is balanced, then $\lambda_n(\Phi)=\epsilon(\Phi)=0$ and the assertion holds. We assume for the rest of the proof that Φ is unbalanced and Γ is connected.

Case 1. $n\geq 4$. Let $F\subset E(\Gamma)=E$, with $|F|=\epsilon(\Gamma)>0$, be a minimum set of edges for which $\Phi-F$ is balanced. By Proposition 2.1, there exists a potential function

$$\theta : V(\Gamma - F) = V(\Gamma) \longrightarrow \mathbb{T}_4$$

for $\varphi|_{\vec{E}(\Gamma)-\vec{F}}$.

Consider now the n-complex vector $\mathbf{x}\in\mathbb{C}^n$ defined as follows

$$\mathbf{x}^T = (x_1,\ldots,x_n) = (\theta(v_1),\ldots,\theta(v_n)).$$

It is easy to see that $\mathbf{x}^*\mathbf{x}=n$. By Proposition 2.2, it follows that

$$\lambda_n(\Phi) \leq \frac{\mathbf{x}^* L(\Phi)\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \frac{1}{n}\cdot\left(\sum_{\substack{e_{ij}\in\vec{E}(\Gamma)\\ i<j}} |x_i - \varphi(e_{ij})x_j|^2\right). \tag{5}$$

We first claim that

$$\sum_{\substack{e_{ij}\in\vec{E}(\Gamma)-\vec{F}\\ i<j}} \left|x_i-\varphi(e_{ij})x_j\right|^2=0. \tag{6}$$

In fact, for each $e_{ij}\in\vec{E}(\Gamma)-\vec{F}$, by definition of $\mathbf{x}$ and the defining property of the potential function we get

$$\left|x_i-\varphi(e_{ij})x_j\right|^2=\left|\theta(v_i)-\theta(v_i)\theta(v_j)^{-1}\theta(v_j)\right|^2=0.$$

Suppose now $e_{ij}\in\vec{F}$. In this case

$$\begin{aligned}\left|x_i-\varphi(e_{ij})x_j\right|^2 &= \left|\theta(v_i)-\varphi(e_{ij})\theta(v_j)\right|^2\\ &= 2-2\operatorname{Re}(\overline{\theta(v_i)}\theta(v_j)\varphi(e_{ij}))\le 4.\end{aligned} \tag{7}$$

By gathering (5), (6) and (7), we finally get

$$\begin{aligned}\lambda_n(\Phi)\le\frac{\mathbf{x}^*L(\Phi)\mathbf{x}}{\mathbf{x}^*\mathbf{x}} &= \frac{1}{n}\cdot\left(\sum_{\substack{e_{ij}\in\vec{E}(\Gamma)-\vec{F}\\ i<j}}|x_i-\varphi(e_{ij})x_j|^2+\sum_{\substack{e_{ij}\in\vec{F}\\ i<j}}|x_i-\varphi(e_{ij})x_j|^2\right)\\ &= \frac{1}{n}\cdot\left(0+\sum_{\substack{e_{ij}\in\vec{F}\\ i<j}}|x_i-\varphi(e_{ij})x_j|^2\right)\\ &\le\frac{4|F|}{n}=\frac{4\epsilon(\Phi)}{n}\le\epsilon(\Phi).\end{aligned}$$

Case 2. $n\le 3$. Since $\Phi=(\Gamma,\varphi)$ is unbalanced, then $n=3$ and Γ is a triangle. Being an unoriented cycle, Γ has a gain $\varphi(\Gamma)=\varphi(e_{12})\varphi(e_{23})\varphi(e_{31})$ defined up to complex conjugation. A direct computation shows that

$$\lambda_3(\Phi)=\begin{cases}0 & \text{if } \varphi(\Gamma)=1,\\ 2-\sqrt{3} & \text{if } \varphi(\Gamma)\in\{\pm i\},\\ 1 & \text{if } \varphi(\Gamma)=-1.\end{cases} \tag{8}$$

In all cases $\lambda_3(\Phi)\le 1=\epsilon(\Phi)$. This completes the proof. ∎

Let C_n be the (unoriented) cycle of order n. Dealing with complex unit gain graphs, in [10, Section 2] the authors distinguish balanced, real unbalanced, and imaginary unbalanced cycles, depending whether $\varphi(C_n) = 1$, $\varphi(C_n) = -1$ or $\varphi(C_n) \in \{\pm i\}$. According to [10, Corollary 2.3],

$$\det(L(C_n, \varphi)) = \begin{cases} 2 & \text{if } C_n \text{ is imaginary unbalanced,} \\ 4 & \text{if } C_n \text{ is real unbalanced.} \end{cases}$$

Such result suggests that an imaginary unbalanced gain cycle lies somehow between a balanced cycle and a real unbalanced one. Our Equation (8), confirms such conceptual claim. More generally, by [7, Theorem 6.1] it follows that

$$\lambda_n(C_n, \varphi) = \begin{cases} 0 & \text{if } C_n \text{ is balanced,} \\ 2 - \cos(\pi/2n) & \text{if } C_n \text{ is imaginary unbalanced,} \\ 2 - \cos(\pi/n) & \text{if } C_n \text{ is real unbalanced,} \end{cases}$$

and obviously $0 < 2 - \cos(\pi/2n) < 2 - \cos(\pi/n)$.

The inequality

$$\nu(\Phi) \leq \epsilon(\Phi) \tag{9}$$

is not hard to prove. In fact, let F be a subset of edges such that $\Phi - F$ is balanced. By removing from Γ one ending point of each edge in F, we obtain a gain graph Φ' which is surely balanced, being a gain subgraph of $\Phi - F$. This argument proves (9), where the equality holds if and only if F is a matching.

In the next theorem, we prove that $\lambda_n(\Phi) \leq \nu(\Phi)$ for every $\mathbb{T}_4$-gain graph Φ. It is surely instructive to compare the proof of Proposition 3.1 with those of [2, Lemma 3.1] and [9, Corollary 2.3], where Inequality (4) is proved to hold for signed graphs. Alternatively, Inequality (4) could be deduced from (9) and Theorem 3.2.

Before stating our two main theorems, we now describe a particularly convenient way to label vertices of any unbalanced gain graph $\Phi = (\Gamma, \varphi)$ with $\nu(\Phi) = k > 0$.

We assume that the graph $\Lambda = \Gamma[v_1, \ldots, v_k]$ induced by the 'first' k vertices of Γ is such that relative complement

$$\Phi - \left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) = \left(\Gamma - \Lambda, \varphi|_{\vec{E}(\Gamma - \Lambda)}\right)$$

is balanced. Denoted by $B_1, B_2, \ldots, B_m$ the connected components of $\Gamma - \Lambda$, the remaining $n - k$ vertices of Γ lies in $B_1 + \cdots + B_m$. We label them in such a way that

$$v_i \in B_p, \quad v_j \in B_q, \quad p < q \Longrightarrow i < j. \tag{10}$$

Let $n_p = |B_p|$. According to (10), the integers corresponding to vertices in B_p are

$$k + n_1 + \cdots + n_{p-1} + 1, \ldots, k + n_1 + \cdots + n_p.$$

We introduce a lighter notation for elements in $V(B_p)$, by setting

$$w_j^{(p)} = v_{k+n_1+\cdots+n_{p-1}+j} \qquad \text{for all } j = 1, \ldots, n_p.$$

Therefore, $V(\Lambda) = \{v_1, \ldots, v_k\}$ and $V(B_p) = \left\{w_1^{(p)}, \ldots, w_{n_p}^{(p)}\right\}$.

Throughout the rest of the paper, we tacitly assume to have chosen for any unbalanced gain graph Φ a vertex labeling satisfying the above rules. Given two gain subgraphs $\mathcal{S}$ and $\mathcal{T}$ of Γ, we will sometimes consider the following subset of oriented edges

$$\vec{E}_\Gamma(\mathcal{S}, \mathcal{T}) = \left\{e_{ij} \in \vec{E}(\Gamma) \mid v_i \in \mathcal{S},\ v_j \in \mathcal{T}\right\},$$

where $\mathcal{S}$ or $\mathcal{T}$ could possibly have a single vertex.

Theorem 3.2. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathbb{T}_4$-gain graph of order n. Then $\lambda_n(\Phi) \leq \nu(\Phi)$.*

Proof. To avoid trivial cases, we assume that $\nu(\Gamma) = k > 0$. We are considering a vertex labeling for Γ such that

$$\Phi - \left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) = \left(\Gamma - \Lambda, \varphi|_{\vec{E}(\Gamma-\Lambda)}\right)$$

is balanced, where $\Lambda = \Gamma[v_1, \ldots, v_k]$. Let $B_1, B_2, \ldots, B_m$ be the connected components of $\Gamma - \Lambda$.

In the sequel of the proof, it will come in handy the n-by-m complex matrix $Y = (y_{jp})$, where

$$y_{jp} = \begin{cases} 1 & \text{if the vertex } v_j \text{ belongs to } V(B_p), \\ 0 & \text{otherwise.} \end{cases}$$

Note that the columns of Y, being mutually orthogonal, are linearly independent.

Up to possibly replacing φ with a switching equivalent gain function, we can assume that all edges in $\vec{E}(\Gamma - \Lambda)$ are neutral.

Let $\tilde{\Phi} = (\tilde{\Gamma}, \tilde{\varphi})$ be the $\mathbb{T}_4$-gain graph defined as follows. $\tilde{\Gamma}$ is the join $\Lambda \vee (\Gamma - \Lambda)$, and $\tilde{\varphi} : \vec{E}(\tilde{\Gamma}) \to \mathbb{T}_4$ is any fixed map such that $\tilde{\varphi}|_{\vec{E}(\Gamma)} = \varphi$. By construction, Φ is a gain subgraph of $\tilde{\Phi}$. Let now $\mathbf{y}_h$ be the h-th column of the matrix Y. Observe that

$$\left|y_{ih} - \tilde{\varphi}(e_{ij}) y_{jh}\right|^2 = \begin{cases} 1 & \text{if } e_{ij} \in \vec{E}_{\tilde{\Gamma}}(\Lambda, B_h) \cup \vec{E}_{\tilde{\Gamma}}(B_h, \Lambda), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Hence, we have

$$\lambda_n(\Phi) \leq \frac{\mathbf{y}_h^* L(\Phi)\mathbf{y}_h}{\mathbf{y}_h^*\mathbf{y}_h} \leq \frac{\mathbf{y}_h^* L(\tilde{\Phi})\mathbf{y}_h}{\mathbf{y}_h^*\mathbf{y}_h} = \frac{1}{|B_h|} \cdot \left(\sum_{\substack{e_{ij} \in \vec{E}(\tilde{\Gamma}) \\ i<j}} |y_{ih} - \varphi(e_{ij}) y_{jh}|^2 \right)$$

$$(12) \qquad = \frac{1}{|B_h|} \cdot \left(\sum_{e_{ij} \in \vec{E}_{\tilde{\Gamma}}(\Lambda, B_h)} |y_{ih} - \varphi(e_{ij}) y_{jh}|^2 \right) = \frac{|\Lambda||B_h|}{|B_h|} = k,$$

where the first equality on the third row comes from (11). ■

Our Theorem 3.2 extends [2, Theorem 3.2] to all $\mathbb{T}_4$-gain graphs. Since, for every $\mathbb{T}_4$-gain graph $\Phi = (\Gamma, \varphi)$, $\lambda_n(\Phi)$ is a lower bound for both the frustration number $\nu(\Phi)$ and the frustration index $\epsilon(\Phi)$, in such larger context too it makes sense to call *algebraic frustration* the least Laplacian eigenvalue.

A subgraph $\mathcal{T}$ of Γ, a gain $z \in \mathbb{T}_4$, and a vertex v_h of Γ determine the following subsets of oriented edges

$$\vec{E}^z_\Phi(v_h, \mathcal{T}) = \vec{E}_\Gamma(v_h, \mathcal{T}) \cap \varphi^{-1}(z).$$

Clearly

$$\vec{E}_\Gamma(v_h, \mathcal{T}) = \bigcup_{z \in \mathbb{T}_4} \vec{E}^z_\Phi(v_h, \mathcal{T}).$$

The next theorem identifies conditions required on a $\mathbb{T}_4$-gain graph Φ to achieve $\lambda_n(\Phi) = \nu(\Phi)$. The proof of [3, Theorem 2.1] served as source of inspiration.

Theorem 3.3. *Let $\Phi = (\Gamma, \varphi)$ be an unbalanced $\mathbb{T}_4$-gain graph of order n with $\nu(\Phi) = k$; suppose $\Lambda = \Gamma[v_1, \dots, v_k]$ is such that $\Phi - \left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right)$ is balanced.*

Assuming that all edges in $\vec{E}(\Gamma - \Lambda)$ are neutral, the equality

$$\lambda_n(\Phi) = \nu(\Phi)$$

holds if and only if the following conditions are satisfied.

(i) *$\Gamma = \Lambda \vee (B_1 + B_2 + \cdots + B_m)$, where $B_1, B_2, \dots, B_m$ are the connected components of $\Gamma - \Lambda$,*

(ii) *for any $v_q \in V(\Lambda)$ and any $h = 1, \dots, m$,*

$$(13) \qquad \left|\vec{E}^1_\Phi(v_q, B_h)\right| = \left|\vec{E}^{-1}_\Phi(v_q, B_h)\right| \quad \textit{and} \quad \left|\vec{E}^i_\Phi(v_q, B_h)\right| = \left|\vec{E}^{-i}_\Phi(v_q, B_h)\right|,$$

(iii) $K(\Phi, \Lambda) := L\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) + (n - 2k)I - \sum_{p=1}^m S_p^* \, L(B_p, 1)^\dagger \, S_p \succeq O,$

where $L(B_p,1)^\dagger$ denotes the Moore-Penrose pseudo-inverse of $L(B_p,1)$, and S_p is the $|B_p| \times k$ matrix whose rs-th entry is $-\varphi\big(w_r^{(p)} v_s\big)$, with $w_r^{(p)} \in V(B_p)$ and $v_s \in V(\Lambda)$.

Proof. Assume first that $\lambda_n(\Phi) = \nu(\Phi) = k$. Then in (12) all inequalities are equalities, in particular $\Gamma = \Lambda \vee (B_1 + B_2 + \cdots + B_m)$. In addition, columns $\mathbf{y}_h$'s of the matrix Y defined along the proof of Theorem 3.2 form a set of independent eigenvectors for the eigenvalue k. On the q-th row, the eigenvalue equation $L(\Phi)\mathbf{y}_h = k\mathbf{y}_h$ says

$$k\, y_{qh} = \deg(v_q)\, y_{qh} - \sum_{e_{qj} \in \vec{E}(\Gamma)} \varphi(e_{qj}) y_{jh} \qquad (q \text{ is fixed}). \tag{14}$$

If v_q belongs to $V(\Lambda)$, then $y_{qh} = 0$ for all h's, and Equation (14) becomes

$$\sum_{e_{qj} \in \vec{E}_\Gamma(v_q, B_h)} \varphi(e_{qh}) = 0,$$

and this is only possible if Equation (13) holds.

We now use the hypothesis that k is the least eigenvalue. Let us consider the matrix $L' = L(\Phi) - kI_n$. Clearly

$$\mathrm{Spec}(L') = \{\lambda - k \mid \lambda \in \mathrm{Spec}(L(\Phi))\}.$$

In particular, $\lambda_n(L') = 0$ and L' is positive semidefinite. According to the chosen vertex labeling, the matrix L' can be described as follows in terms of blocks

$$L' = \left(\begin{array}{c|ccc} L(\Lambda) + (n-2k)I_k & S_1^* & \cdots & S_m^* \\ \hline S_1 & L(B_1) & & \\ \vdots & & \ddots & \\ S_m & & & L(B_m) \end{array}\right),$$

where $L(\Lambda)$ and $L(B_p)$ stand for $L\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right)$ and $L(B_p,1)$, respectively. Note that S_p is the $n_p \times k$ matrix whose j-th column S_p^j contains the opposite of gains of oriented edges connecting the several vertices of B_p to the vertex $v_j \in V(\Lambda)$; namely $S_p = \big(S_p^1 | S_p^2 | \cdots | S_p^k\big)$, with

$$S_p^j = -\left(\varphi\left(w_1^{(p)} v_j\right), \varphi\left(w_2^{(p)} v_j\right), \ldots, \varphi\left(w_{n_p}^{(p)} v_j\right)\right)^\top.$$

We now look for a nonsingular matrix C such that $C^* L' C$ is block diagonal. To this purpose, we have to find suitable matrices Z_p, for $p = 1, \ldots, m$, such that $S_p - L(B_p) Z_p = O$.

Let $\mathbf{1}_q \in \mathbb{C}^q$ be the all-ones vector. Since $\varphi|_{B_p} = 1$, then $\mathbf{1}_{n_p}$ belongs to the kernel of $L(B_p)$. The already proved Condition (ii) implies that S_p^j is orthogonal to $\mathbf{1}_{n_p}$. That is why S_p^j surely belongs to the column space of $L(B_p)$, and consequently there exists a non-zero vector Z_p^j such that $S_p^j = L(B_p)Z_p^j$. Hence, the matrix $Z_p = \left(Z_p^1|Z_p^2|\cdots|Z_p^k\right)$ satisfies $S_p - L(B_p)Z_p = O$, as wanted.

Let $\operatorname{Spec}(L(B_p)) = \left\{\mu_1, \ldots, \mu_{n_p-1}, \mu_{n_p} = 0\right\}$. The Hermitian matrix $L(B_p)$ is unitarily diagonalizable; in other words

$$L(B_p) = U \text{ diag}\left(\mu_1, \ldots, \mu_{n_p-1}, 0\right) U^*,$$

where U is a unitary matrix whose j-th column consists of a normalized eigenvector of μ_i. We can suppose that the last column of U is $\frac{1}{\sqrt{n_p}}\mathbf{1}_{n_p}$. The Moore-Penrose pseudo-inverse of $L(B_p)$ has the following form

$$L(B_i)^\dagger = U \operatorname{diag}\left(\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_{n_p-1}^{-1}, 0\right) U^*.$$

Thus,

$$\begin{aligned} L(B_p)^\dagger L(B_p) = U \operatorname{diag}(1, \ldots, 1, 0)U^* &= U\left(I_{n_p} - \operatorname{diag}(0, \ldots, 0, 1)\right) U^* \\ &= I_{n_p} - \frac{1}{n_p} J_{n_p}. \end{aligned}$$

Recall now that $\left(S_p^r\right)^* \cdot J_{n_p} = O_{n_p}$. It follows that the complex number $(S_p^r)^* \cdot Z_p^s$ is equal to

$$\left(S_p^r\right)^* \left(I_{n_p} - \frac{1}{n_p} J_{n_p}\right) Z_p^s = \left(S_p^r\right)^* L(B_p)^\dagger L(B_p) Z_p^s = \left(S_p^r\right)^* L(B_p)^\dagger S_p^s. \tag{15}$$

Consider the following block defined matrix

$$C = \left(\begin{array}{c|ccc} I_k & O & \cdots & O \\ \hline -Z_1 & I_{n_1} & & \\ \vdots & & \ddots & \\ -Z_m & & & I_{n_m} \end{array}\right).$$

It is straightforward to check that $C^\top L' C$ is a block diagonal matrix given by

$$C^* L' C = \left[L(\Lambda) + (n - 2k)I - \sum_{p=1}^m S_p^* Z_p\right] \oplus L(B_1) \oplus \cdots \oplus L(B_m).$$

Since L' is positive semidefinite, then $C^\top L' C$ is positive semidefinite as well. In particular, we get

$$L(\Lambda) + (n - 2k)I - \sum_{p=1}^m S_p^* Z_p \succeq O,$$

which is equivalent to Condition (iii) by (15).

The 'if' part of the proof is much shorter. Assume that Φ is a $\mathbb{T}_4$-gain graph satisfying (i)–(iii). Since, for every $h = 1, \dots, m$, $L(\Phi)\mathbf{y}_h = k\mathbf{y}_h$, surely $k \in \mathrm{Spec}(L(\Phi))$ with at least multiplicity m. Condition (iii) ensures that $k = \min \mathrm{Spec}(L(\Phi))$.

It remains to prove that $\nu(\Gamma) = k$. According to Theorem 3.2, we have that $k \leq \nu(\Gamma)$. However, by deleting the vertices of Λ we obtain a balanced gain graph, hence we also have $\nu(\Gamma) \leq k$. This completes the proof. ∎

Remark 3.4. In the statement of Theorem 3.3, the assumption $\varphi|_{B_h} = 1$ for each $j = 1, \dots, m$ is not really restrictive. In fact, because of balancedness of the several B_h's, given any gain function $\varphi : V(\Gamma) \to \mathbb{T}_4$ there always exists a map φ' such that $\varphi' \sim \varphi$ and $\varphi'|_{B_h} = 1$ for each $j = 1, \dots, m$.

The matrices S_p's defined in the statement of Theorem 3.3 deserve a comment. If all oriented edges in $\vec{E}_\Gamma(\Lambda, \Gamma - \Lambda)$ sharing a same ending vertex have the same gain, we have $S_p^1 = \cdots = S_p^k := \mathcal{E}_p$, and the matrix $K(\Phi, \Lambda)$ assumes a slightly more treatable form. Namely,

$$K(\Phi, \Lambda) = L\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) + (n - 2k)I - \left(\sum_{p=1}^{m} \mathcal{E}_p^* \ L(B_p, 1)^\dagger \ \mathcal{E}_p\right) J_k.$$

The following corollary is useful to quickly exclude the possibility for a $\mathbb{T}_4$-gain graphs to satisfy Condition (ii) of Theorem 3.3.

Corollary 3.5. *Let $\Phi = (\Gamma, \varphi)$ be an unbalanced $\mathbb{T}_4$-gain graph of order n with $\nu(\Phi) = k$; let $\Lambda \subset \Gamma$ with $|\Lambda| = k$ such that $\Phi - \left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right)$ is balanced. If $\lambda_n(\Phi) = \nu(\Phi)$, then each connected component of $\Gamma - \Lambda$ has an even number of vertices.*

Proof. Let B a connected component of $\Gamma - \Lambda$. From Conditions (i) and (ii) in Theorem 3.3, it immediately follows that

$$|B| = 2 \cdot \sum_{v_q \in V(\Lambda)} \left(|\vec{E}^1_\Phi(v_q, B)| + |\vec{E}^i_\Phi(v_q, B)|\right).$$

∎

Example 3.6. Consider the gain graph Φ depicted in Figure 1. Each continuous (respectively, dashed) thick undirected line represents two opposite oriented edges with gain 1 (respectively, gain -1); whereas the arrows detect the oriented edges uv such that $\varphi(uv) = i$.

We have that

$$L(\Phi) = \begin{pmatrix} L(S) + 8I_2 & S_1^* & S_2^* \\ S_1 & L(B_1) + 2I_4 & O \\ S_2 & O & L(B_2) + 2I_4 \end{pmatrix}.$$

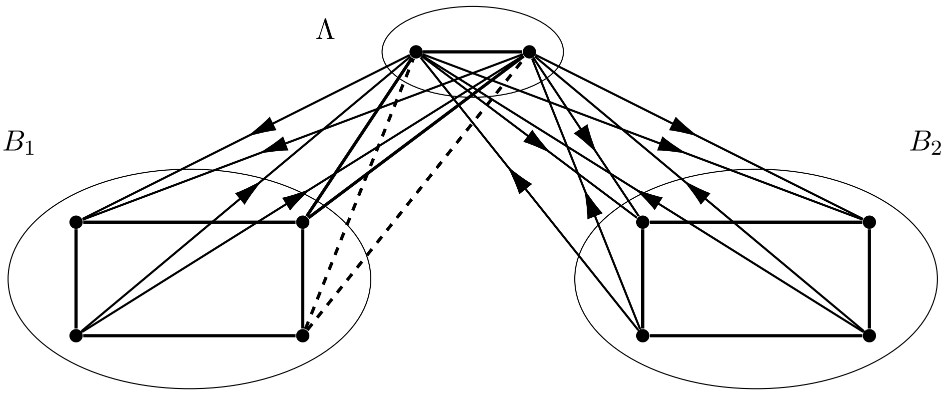

Figure 1. A gain graph Φ with $\lambda_n(\Phi) = \nu(\Phi) = 2$.

Since Conditions (i) and (ii) of Theorem 3.3 hold, then $2 \in \mathrm{Spec}(L(\Phi))$ with multiplicity at least 2. Now $\min(\mathrm{Spec}(L(\Phi))) = 2$ if and only if Condition (iii) of Theorem 3.3 holds as well. Now, we must check whether the matrix $K(\Phi, \lambda)$ is positive semidefinite. A suitable vertex labeling for B_1 and B_2 gives

$$(S_1^1)^* = (S_1^2)^* := (\mathcal{E}_1)^* = (-1, 1, i, -i),$$

and

$$(S_2^1)^* = (S_2^2)^* := (\mathcal{E}_2)^* = (-i, i, -i, i).$$

After some computations we find that

$$L(B_1)^\dagger = L(B_2)^\dagger = \frac{1}{16} \cdot \begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix},$$

Hence, $\mathcal{E}_1^* \; L(B_1)^\dagger \; \mathcal{E}_1 = 3/2$ and $\mathcal{E}_2^* \; L(B_1)^\dagger \; \mathcal{E}_2 = 1$.

Finally, we compute

$$K(\Phi, \Lambda) = L\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) + 6I - \left(\sum_{p=1}^{2} \mathcal{E}_p^* \; L(B_p, 1)^\dagger \; \mathcal{E}_p\right) J_2 = \frac{1}{2} \cdot \begin{pmatrix} 9 & -7 \\ -7 & 9 \end{pmatrix},$$

which is positive definite. Through a MATLAB computation we get

$$\mathrm{Spec}(L(\Phi)) = \left\{2^{(3)};\ 4^{(3)};\ 8 - \sqrt{8};\ 6;\ 10;\ 8 + \sqrt{8}\right\},$$

confirming that $\lambda_{10}(\Phi) = 2$.

The next example will convince the reader that Condition (iii) in Theorem 3.3 is really independent from Conditions (i) and (ii).

Example 3.7. We consider the two non-switching equivalent gain graphs $\Phi = (\Gamma, \varphi)$ and $\Phi' = (\Gamma, \varphi')$ depicted in Figure 2, where Γ is the wheel graph $\{s\} \vee C_6$. Once again, each continuous (respectively, dashed) thick undirected line represents two opposite oriented edges with gain 1 (respectively, gain -1); arrows identify instead the oriented edges uv such that $\varphi(uv) = i$.

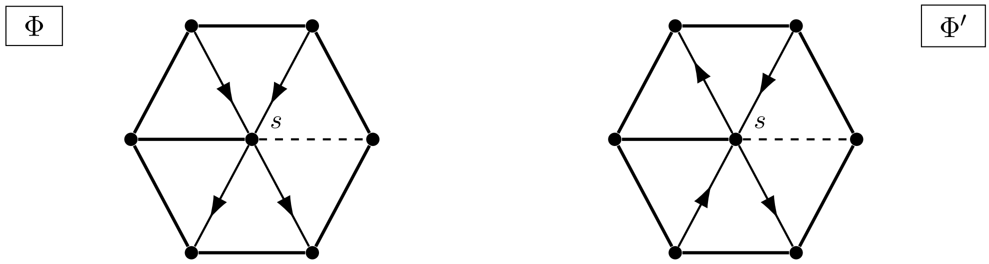

Figure 2. Two gain graphs Φ and Φ' of order 7 such that $\lambda_7(\Phi) < \nu(\Phi) = 1 = \nu(\Phi') = \lambda_7(\Phi')$.

Note first that $\varphi|_{C_6} = \varphi'|_{C_6} = 1$, hence $\nu(\Phi) = \nu(\Phi') = 1$. In order to make our notation consistent with Theorem 3.3, we set $\{s\} = \Lambda$ and $C_6 = B_1 = B$. In particular

$$\left|\vec{E}^{1}_{\Phi}(s,B)\right| = \left|\vec{E}^{-1}_{\Phi}(s,B)\right| = 1 = \left|\vec{E}^{1}_{\Phi'}(s,B)\right| = \left|\vec{E}^{-1}_{\Phi'}(s,B)\right|,$$

and

$$\left|\vec{E}^{i}_{\Phi}(s,B)\right| = \left|\vec{E}^{-i}_{\Phi}(s,B)\right| = 2 = \left|\vec{E}^{i}_{\Phi'}(s,B)\right| = \left|\vec{E}^{-i}_{\Phi'}(s,B)\right|.$$

In the cases at hand, it turns out that

$$K(\Phi, \Lambda) = -0.5 \not\succeq 0 \qquad \text{and} \qquad K(\Phi', \Lambda) = 3.5 \succeq 0.$$

Hence, Theorem 3.3 predicts that

$$\lambda_7(\Phi) < \nu(\Phi) = 1 = \nu(\Phi') = \lambda_7(\Phi'),$$

which is confirmed by MATLAB computations. In fact, the Laplacian spectra of the two gain graphs turn out to be

$$\mathrm{Spec}(L(\Phi)) = \{0.9161;\ 1;\ 2;\ 4^{(2)};\ 4.7868;\ 7.2971\},$$

and

$$\mathrm{Spec}(L(\Phi')) = \{1;\ 1.506;\ 2;\ 3;\ 4;\ 4.8901;\ 7.6039\}.$$

Let Φ and Φ' be the two gain graphs considered in Example 3.7. Sharp-eyed readers may already spotted that, for all $z \in \mathbb{T}_4$, endpoints of oriented edges in $\vec{E}^{z}_{\Phi'}(s,B)$ are never adjacent. On the contrary

$$\vec{E}^{i}_{\Phi}(s,B) = \{su, sv\} \qquad \text{and} \qquad uv \in E(\Gamma).$$

The authors believe that, in the set $\{\Phi, \Phi'\}$, only Φ' fulfills Condition (iii) of Theorem 3.3 precisely for the just mentioned structural difference. In fact we state the following conjecture.

Conjecture 3.8. *Let B be a regular graph of order $2q$, and let Λ consist of a single vertex s. Suppose $\varphi : \Gamma \to \mathbb{T}_4$ is a gain function defined on the cone graph $\Gamma = \Lambda \vee B$ satisfying*

(i) *all oriented edges in $\vec{E}(B)$ are neutral;*

(ii) $\left|\vec{E}^{1}_{(\Gamma,\varphi)}(s,B)\right| = \left|\vec{E}^{-1}_{(\Gamma,\varphi)}(s,B)\right|$ *and* $\left|\vec{E}^{i}_{(\Gamma,\varphi)}(s,B)\right| = \left|\vec{E}^{-i}_{(\Gamma,\varphi)}(s,B)\right|$;

(iii) *for all $z \in \mathbb{T}_4$, the endpoints of oriented edges in $\vec{E}^{z}_{(\Gamma,\varphi)}(s,B)$ form an independent set.*

Then, the algebraic frustration $\lambda_{2q+1}(\Gamma,\varphi)$ is equal to the frustration number $\nu(\Gamma,\varphi) = 1$.

Anyway Conjecture 3.8 claims that Conditions (i)–(iii) are sufficient to get $\lambda_{2q+1}(\Gamma,\varphi) = \nu(\Phi)$, but certainly they are not necessary. There are gain graphs of type $\Phi = (\{s\} \vee B; \varphi)$, the graph B being regular of order $2q$, for which Condition (iii) of Conjecture 3.8 fails, but nevertheless $\lambda_{2q+1}(\Phi) = \nu(\Phi) = 1$. Examples of such gain graphs can be provided with the aid of our last theorem.

Theorem 3.9. *Let K_{4t} and Λ denote the complete graph of order $4t$ and any graph of order $k \le 2t$, respectively. Consider a $\mathbb{T}_4$-gain graph $\Phi = (\Gamma,\varphi)$ such that*

- $\Gamma = \Lambda \vee K_{4t}$,
- *all oriented edges in $\vec{E}(K_{4t})$ are neutral,*
- $\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right)$ *is the smallest gain subgraph such that the relative complement $\Phi - \left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right)$ is balanced,*
- *for every fixed vertex $w \in V(K_{4t})$, and for every pair $v, v' \in V(\Lambda)$, we have $\varphi(vw) = \varphi(v'w)$ and*

$$\left|\vec{E}^{1}_{\Phi}(v,K_{4t})\right| = \left|\vec{E}^{-1}_{\Phi}(v,K_{4t})\right) = \left|\vec{E}^{i}_{\Phi}(v,K_{4t})\right)\right| = \left|\vec{E}^{-i}_{\Phi}(v,K_{4t})\right)\right| = t.$$

Then, the algebraic frustration $\lambda_{4t+k}(\Phi)$ is equal to the frustration number $\nu(\Phi) = k$.

Proof. Under the hypotheses of the statement, a gain graph Φ clearly satisfies Conditions (i) and (ii) of Theorem 3.3. We now label the vertices $w_1, \dots, w_{4t}$ of K_{4t} in such a way that, for every $v \in V(\Lambda)$,

$$\varphi(vw_{ct+d}) = e^{\frac{c\pi}{2}} \qquad (c \in \{0,1,2,3\} \quad \text{and} \quad d = 1,\dots,t).$$

We have now to check that Condition (iii) of Theorem 3.3 holds. In our case

$$K(\Phi,\Lambda) = L\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) + (4t-k)I_k - \left(\mathcal{E}^* L(K_{4t},1)^\dagger \mathcal{E}\right) J_k,$$

where $\mathcal{E}^* = (-1,\ldots,-1,i,\ldots,i,1,\ldots,1,-i,\ldots,-i)$.

It can be proved that $L(K_{4t},1)^\dagger = (l_{ij})$, where

$$16t^2 \cdot l_{ij} = \begin{cases} 4t-1 & \text{if } i=j, \\ \quad -1 & \text{if } i=j. \end{cases}$$

Hence, $\mathcal{E}^* L(K_{4t},1)^\dagger \mathcal{E} = 1$.

Note now that $(4t-k)I_k - J_k$ is symmetric diagonally dominant, since we are supposing $k \leq 2t$. This implies that

$$K(\Phi,\Lambda) = L\left(\Lambda, \varphi|_{\vec{E}(\Lambda)}\right) + ((4t-k)I_k - J_k) \succeq 0.$$

Since the gain graph Φ satisfies all hypotheses of Theorem 3.3, we have $\lambda_{4t+k}(\Phi) = \nu(\Phi) = k$. ■

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