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ON REGULAR SIGNED GRAPHS WITH THREE EIGENVALUES

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Abstract

In this paper our focus is on regular signed graphs with exactly 3 (distinct) eigenvalues. We establish certain basic results; for example, we show that they are walk-regular. We also give some constructions and determine all the signed graphs with 3 eigenvalues, under the constraint that they are either signed line graphs or have vertex degree 3. We also report our result of computer search on those with at most 10 vertices.

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1. Introduction

A *signed graph* $\dot{G}$ is obtained from an (unsigned) graph G by accompanying each edge e by the sign $\sigma(e) \in \{1, -1\}$. We say that G is the *underlying graph* of $\dot{G}$. The set of vertices of $\dot{G}$ is denoted by $V(\dot{G})$; we also write n for $|V(\dot{G})|$. Obviously, every graph can be interpreted as a signed graph with all edges being positive.

The $n \times n$ adjacency matrix A (or $A_{\dot{G}}$) of $\dot{G}$ is obtained from the $(0, 1)$-adjacency matrix of G by reversing the sign of all 1's which correspond to negative edges. The characteristic polynomial of $\dot{G}$ is $\Phi_{\dot{G}}(x) = \det(xI - A_{\dot{G}})$; its roots form the spectrum of $\dot{G}$. The minimal polynomial $\phi_{\dot{G}}$ is the unique monic polynomial of minimal degree such that $\phi_{\dot{G}}(A_{\dot{G}}) = O$. The polynomial $\Phi_{\dot{G}}$ has integral coefficients; a method for their computation is given in [1]. Moreover, $\phi_{\dot{G}}$ also has integral coefficients, and the proof is exactly the same as that for the minimal polynomial of a graph — see, for example, [3].

Signed graphs with 2 (here and after, distinct) eigenvalues are studied in [4, 8, 13]. It has been shown that they must be regular. Here we continue by the next natural step, i.e., by considering regular signed graphs with 3 eigenvalues.

The paper is organized as follows. Section 2 is reserved for terminology and notation. Some basic results on signed graphs with exactly 3 eigenvalues are given in Section 3. Certain constructions are presented in Section 4. We also determine all the signed graphs with 3 eigenvalues, which are either signed line graphs or have vertex degree 3. Computational results concerning those with at most 10 vertices are given in Section 5.

2. Preliminaries

Our notation is standard; in particular, we write I, J and O for an identity matrix, an all-1 matrix and an all-0 matrix, respectively. Occasionally, the size of a matrix will be given in a subscript.

If the vertices i and j of a signed graph are adjacent, we write $i \sim j$; in particular the existence of a positive (respectively, negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ (respectively, $i \overset{-}{\sim} j$).

A signed graph is complete, bipartite or r-regular if the same holds for its underlying graph. In particular, a 3-regular signed graph is called *cubic*.

We say that a signed graph is *homogeneous* if its edge set is empty or all the edges have the same sign. Otherwise, it is *inhomogeneous*. The *negation* $-\dot{G}$ of $\dot{G}$ is obtained by reversing the sign of all edges of $\dot{G}$.

A walk in a signed graph is defined in the same way as the walk in a graph. A walk is *positive* if the number of negative edges contained (counted with their repetition) is even; otherwise, it is *negative*. Accordingly, a cycle in a signed graph is *positive* if the number of negative edges contained is even; otherwise, it is *negative*. The difference between the numbers of positive and negative walks of length k starting at vertex i and terminating at j is denoted by $w_k(i,j)$; for $i=j$, we simply write $w_k(i)$. In particular, we use t_i (respectively, q_i) to denote the difference between the numbers of positive and negative triangles (respectively, quadrangles) passing through i.

A signed graph is called *walk-regular* if, for each vertex i, $w_k(i)$ is constant for every non-negative integer k. Setting $k=2$, we arrive at an unsurprising conclusion that every walk-regular signed graph is regular.

For U a subset of the vertex set of a signed graph $\dot{G}$, let $\dot{G}^U$ be the signed graph obtained from $\dot{G}$ by reversing the sign of every edge between a vertex in U and a vertex outside U. The signed graph $\dot{G}^U$ is said to be *switching equivalent* to $\dot{G}$. Similarly, we say that the signed graphs $\dot{G}$ and $\dot{H}$ are *switching isomorphic* if $\dot{H}$ is isomorphic to a signed graph which is switching equivalent to $\dot{G}$. Switching equivalence and switching isomorphism preserve the eigenvalues.

In this paper we use the concept of signed line graphs that can be found in [1, 13]. We repeat the basic notation and definition, since there is an alternative version of signed line graphs, which differs from our up to sign [14]. Introduce the vertex-edge orientation $\eta\colon V(\dot{G})\times E(\dot{G}) \longrightarrow \{1,0,-1\}$ formed by obeying the following rules: (1) $\eta(i,jk)=0$ if $i\notin\{j,k\}$, (2) $\eta(i,ij)=1$ or $\eta(i,ij)=-1$ and (3) $\eta(i,ij)\eta(j,ij)=-\sigma(ij)$. In fact, every edge gets two orientations, so η is also called a *bi-orientation*. The vertex-edge incidence matrix B_η is the matrix whose rows and columns are indexed by $V(\dot{G})$ and $E(\dot{G})$, respectively, such that its (i,e)-entry is equal to $\eta(i,e)$. Then, even in the case that multiple edges exist, we have

$$B_\eta^T B_\eta = 2I + A_{L(\dot{G})}\,,$$

where $L(\dot{G})$ is taken to be a *signed line graph* of $\dot{G}$. Note that $L(\dot{G})$ depends on η, so one may observe that we should write $L(\dot{G}_\eta)$, but since different orientations give switching equivalent line graphs, we simplify the notation and use $L(\dot{G})$ to denote a representative of the entire switching equivalence class. We remark that this concept does not generalize the concept of line graphs; for example, the line graph of a positive triangle is a negative triangle.

3. Basic Results

We start with a straightforward result.

Theorem 1. *An r-regular signed graph $\dot{G}$ has at most 3 eigenvalues if and only if there exist $\alpha,\beta,\gamma\in\mathbb{Z}$ such that*

$$
(1)\qquad
\begin{aligned}
2t_i &= \alpha r + \gamma, && \textit{for all } i,\\
w_3(i,j) &= \alpha w_2(i,j), && \textit{for all } i \nsim j,\\
w_3(i,j) &= \alpha w_2(i,j) - \beta, && \textit{for all } i \overset{+}{\sim} j,\\
w_3(i,j) &= \alpha w_2(i,j) + \beta, && \textit{for all } i \overset{-}{\sim} j,
\end{aligned}
$$

where t_i is defined in the previous section.

If $\dot{G}$ has exactly 3 eigenvalues, say λ, μ and ν, then

$$
(2)\qquad \lambda + \mu + \nu = \alpha,\ \ \lambda\mu + \lambda\nu + \mu\nu = \beta,\ \ \lambda\mu\nu = \gamma.
$$

Proof. If $\dot{G}$ has at most 3 eigenvalues, then there exist integral coefficients α, β and γ such that

$$
(3)\qquad A^3 - \alpha A^2 + \beta A - \gamma I = O
$$

holds for its adjacency matrix. Equating the corresponding entries, we arrive at the equalities (1). (For example, the (i,i)-entries of A^3, A^2, A and I are $2t_i$, r, 0 and 1, respectively, and similarly in the remaining cases.)

Conversely, if the equalities (1) hold, then the identity (3) also holds, and thus $\dot{G}$ has at most 3 eigenvalues.

Finally, if $\dot{G}$ has exactly 3 eigenvalues, considering the minimal polynomial of its adjacency matrix, we conclude the proof. ■

Note that if a regular signed graph has at most 3 eigenvalues, then

$$
\sum_{j \overset{+}{\sim} i} w_2(i,j) - \sum_{j \overset{-}{\sim} i} w_2(i,j)
$$

is a constant for all i. Indeed, using $A^3 = A^2A$, we get that the above difference is equal to the (i,i)-entry of A^3 which is constant by the first equality of (1).

The definition of a walk-regular signed graph, given in the previous section, extends the notion of walk-regularity of a graph. Moreover, the result of Godsil and McKay [5] – stating that a graph G is walk-regular if and only if the diagonal entries of A^k ($k \geq 1$) are mutually equal or if and only if the characteristic polynomials Φ_{G-i} are identical for all $i \in V(G)$ — can directly be transferred to the field of signed graphs. The proof, established in the mentioned reference and [6], remains unchanged. What is interesting is the following result.

Theorem 2. *A regular signed graph with at most 3 eigenvalues is walk-regular.*

Proof. We prove that A^k has a constant diagonal for $k \geq 1$. By Theorem 1, we already know that this holds for $k \leq 3$.

Fix k ($k \geq 4$) and assume that the diagonal entries are equal for all powers as of A^{k-1}. Using (3), we get

$$A^{k-3}\left(A^3 - \alpha A^2 + \beta A - \gamma I\right) = O,$$

giving

$$A^k = \alpha A^{k-1} - \beta A^{k-2} + \gamma A^{k-3}.$$

Since the right-hand side has a constant diagonal, the result follows by induction argument. ∎

Now, when do our signed graphs have exactly 2 eigenvalues? We already mentioned that such signed graphs must be regular, which in fact follows easily by considering their characteristic polynomial. In the same way, we conclude that a signed graph has 2 eigenvalues if and only if there exists an integer α such that

$$\begin{aligned} w_2(i,j) &= 0, && \text{for all } i \nsim j, \\ w_2(i,j) &= \alpha, && \text{for all } i \overset{+}{\sim} j, \\ w_2(i,j) &= -\alpha, && \text{for all } i \overset{-}{\sim} j. \end{aligned}$$

These eigenvalues are discussed in [13]; they are either irrational square roots of an integer (with equal multiplicities) or integral. Here is a similar result when a signed graph has 3 eigenvalues.

Theorem 3. *If a regular signed graph has exactly* 3 *eigenvalues and their multiplicities are not all mutually equal, then either one eigenvalue is integral and the remaining two are irrational algebraic conjugates with equal multiplicities or all the eigenvalues are integral.*

Proof. Let the eigenvalues be denoted as in Theorem 1, let m_λ, m_μ and m_ν denote their multiplicities, and assume (without loss of generality) that $m_\lambda \leq m_\mu \leq m_\nu$. Observe that the polynomial

$$\frac{\Phi(x)}{\phi(x)^{m_\lambda}} = (x-\mu)^{m_\mu - m_\lambda}(x-\nu)^{m_\nu - m_\lambda} \tag{4}$$

has integral coefficients, as follows, say, by the polynomial long division scheme, since both Φ and ϕ have integral coefficients, and the latter one is monic.

For $m_\lambda < m_\mu$, comparing the coefficients of the polynomial (4) with those of the binomials it consists of, we immediately conclude that if $m_\mu < m_\nu$, then $\mu, \nu \in \mathbb{Z}$, while if $m_\mu = m_\nu$, then $\mu, \nu \in \mathbb{Z}$ or they are irrational algebraic conjugates. In all cases $(x-\mu)(x-\nu)$ has integral coefficients, and consequently the same holds for $\phi(x)/((x-\mu)(x-\nu))$, so λ is integral.

For $m_\lambda = m_\mu$, the polynomial (4) reduces to $(x-\nu)^{m_\nu - m_\lambda}$, and since it has integral coefficients, ν must be integral. Considering $\phi(x)/(x-\nu)$, we get that λ and μ are integral or irrational algebraic conjugates. ∎

We express the multiplicities in terms of the remaining parameters. So, using the notation from the above and considering the first three spectral moments of the adjacency matrix, we get

$$m_\lambda = \frac{(r+\mu\nu)n}{(\lambda-\mu)(\lambda-\nu)},\ m_\mu = \frac{(r+\lambda\nu)n}{(\mu-\lambda)(\mu-\nu)},\ m_\nu = \frac{(r+\lambda\mu)n}{(\nu-\lambda)(\nu-\mu)}. \tag{5}$$

Using the spectral moments of order 3 and 4, and Theorem 2, we get a pair of feasibility conditions

$$\begin{aligned} m_\lambda\lambda^3 + m_\mu\mu^3 + m_\nu\nu^3 &= 2t_i n,\\ m_\lambda\lambda^4 + m_\mu\mu^4 + m_\nu\nu^4 &= (r(2r-1)+2q_i)n. \end{aligned} \tag{6}$$

4. Constructions and Determinations

Let $\dot{G} \times \dot{H}$ denote the *tensor product* of $\dot{G}$ and $\dot{H}$, i.e., the signed graph whose adjacency matrix is identified with the Kronecker product $A_{\dot{G}} \otimes A_{\dot{H}}$. It is known that every eigenvalue (with repetition) of $\dot{G} \times \dot{H}$ is obtained as a product of an eigenvalue of $\dot{G}$ and an eigenvalue of $\dot{H}$. If $\dot{G}$ and $\dot{H}$ are connected and at least one of them is non-bipartite, then their tensor product is connected.

Various signed graphs with 3 eigenvalues can be obtained as tensor products of existing signed graphs with 2 or 3 eigenvalues. Similarly, for an all-1 matrix J, $A_{\dot{G}} \otimes J$ is the adjacency matrix of a signed graph with 3 eigenvalues whenever $\dot{G}$ has either 2 eigenvalues or 3 eigenvalues such that one of them is 0. One can arrive at similar conclusions by considering the matrix $(A_{\dot{G}}+I) \otimes J - I$ (this idea can be found in [3]).

Here is another construction. By $\ddot{G}$ we denote the signed multigraph obtained by inserting a negative (parallel) edge between every pair of adjacent vertices of a graph G.

Proposition 4. *There is a connected regular signed graph with $(n, r) = (4k, 2k)$ and eigenvalues $\pm k\sqrt{2}$ and 0, for $k \geq 2$. There also exists a connected regular signed graph with $(n, r) = (ik, 4k)$ and eigenvalues $\pm 2k$ and 0, for $i \geq 3, k \geq 2$.*

Proof. For the first family, we consider the adjacency matrix A of a negative quadrangle (with eigenvalues $\pm\sqrt{2}$) and the $k \times k$ all-1 matrix J_k. The Kronecker product $A \otimes J_k$ determines a desired complete bipartite signed graph.

For the second, recall from [13] that the signed line graph $L(\ddot{C}_i), i \geq 3$, is 4-regular with eigenvalues ± 2. If A is its adjacency matrix, then a desired signed graph is obtained by the same product (with J_k). Observe that it is connected by the way of construction. ■

By [13], if G is an r-regular signed graph with n vertices, then the characteristic polynomial of its line graph is given by

$$\Phi_{L(\dot{G})}(x) = -(x+2)^{\frac{(r-2)n}{2}}\Phi_{\dot{G}}(r-x-2). \tag{7}$$

Theorem 5. *The line graph $L(\dot{G})$ of a connected r-regular signed graph has* 3 *eigenvalues if and only if one of the following holds.*

(i) *$\dot{G}$ is a positive quadrangle, any pentagon or a negative hexagon,*

(ii) *$\dot{G}$ has* 2 *eigenvalues, but it is not switching equivalent to K_n, or*

(iii) *$\dot{G}$ is switching equivalent to a strongly regular graph.*

Proof. First, if $r = 2$, then $L(\dot{G})$ is also a cycle, and the result follows easily, giving (i).

For $r \geq 3$, we recall, say from [10], that the vertex degree occurs in the spectrum of a connected signed graph if and only if it is switching equivalent to its underlying graph. Now, if $L(\dot{G})$ has 3 eigenvalues then, by (7), either $\dot{G}$ has 2 eigenvalues, where r is not one of them, or $\dot{G}$ has 3 eigenvalues, where r is one of them. Indeed, the eigenvalues of $L(\dot{G})$ are -2 and the solutions of $\Phi_{\dot{G}}(r-x-2) = 0$, and the conclusion follows since the existence of r in the spectrum of $\dot{G}$ implies that -2 is one of the solutions of the previous equation.

In the former case, we immediately arrive at (ii), as a connected signed graph which has 2 eigenvalues and is switching equivalent to its underlying graph must be $\dot{K}_n$ ($n \geq 2$).

In the latter case, $\dot{G}$ has 3 eigenvalues and it is switching equivalent to its underlying graph. The underlying graph must be strongly regular, since every connected regular graph with 3 eigenvalues is strongly regular [12, Theorem 3.4.7].

The opposite implication is verified directly. ■

We also determine cubic signed graphs with 3 eigenvalues.

Theorem 6. *Every connected cubic signed graph with* 3 *eigenvalues is switching equivalent to its underlying graph or to the negation of its underlying graph.*

Proof. Assume to the contrary, and denote the signed graph in question by $\dot{G}$. Throughout the proof we use the notation from Section 3.

If $\dot{G}$ has no integral eigenvalue, then by Theorem 3, they are of the same multiplicity equal to $\frac{n}{3}$. By the second equality of (6), we have

$$\lambda^4 + \mu^4 + \nu^4 = 45 + 6q_i.$$

Using $\lambda + \mu + \nu = 0$ and $\lambda^2 + \mu^2 + \nu^2 = 9$ (the spectral moments), we compute $\lambda^4 + \mu^4 + \nu^4 = \frac{81}{2}$, which implies $q_i = -\frac{3}{4}$. Since this parameter must be integral, we arrive at contradiction.

Assume now that at least one eigenvalue of $\dot{G}$, say λ, is integral. If an eigenvalue of $\dot{G}$ is 3 or -3, then $\dot{G}$ is switching equivalent to G or to its negation, so assume further that the spectrum of $\dot{G}$ lies in $(-3, 3)$. Since λ is integral, it follows that

$$z = \mu\nu \quad \text{and} \quad s = \mu + \nu \tag{8}$$

are also integral; if necessary, see (2). Moreover, we have $z \in [-8,\ 8]$ and $s \in [-5,\ 5]$. By (8), $-\mu^2 + s\mu - z = 0$, and since μ is real and $\mu \neq \nu$, we obtain $s^2 - 4z > 0$.

Now, we need to consider all the integral possibilities for z and s (belonging to the given segments and satisfying the last inequality) in conjunction with the equalities (5)–(6). In fact, all the possibilities are resolved easily giving no feasible solutions. We skip the details, but for example, if $z \in \{7, 8\}$ then there is no possibility for s. For $z \in \{5, 6\}$, given inequality is satisfied only for $s = 5$, but then at least one of μ or ν does not belong to the given interval, and so on. A critical case passing all the (in)equalities (with λ, t_i, q_i integral) occurs for $(z, s) = (-5, 0)$, giving $(\lambda, \mu, \nu, t_i, q_i) = \big(0, \sqrt{5}, -\sqrt{5}, 0, 0\big)$. Since the last two parameters are zero, we have $A_{\dot{G}}^2 = 3I$, and thus the eigenvalues of $\dot{G}$ are $\pm\sqrt{3}$, contradicting the obtained values. ∎

Clearly, all connected homogeneous cubic signed graphs with 3 eigenvalues are cubic strongly regular graphs or their negations. The former ones are the Petersen graph and the complete bipartite graph $K_{3,3}$.

So far, we restricted ourselves only to signed graphs that are regular. Of course, the non-regular ones can also have 3 eigenvalues. In fact, there is an extensive literature concerning homogeneous non-regular signed graphs with 3 eigenvalues (see [9] and references therein). Our contribution includes specified signed graphs derived from block designs.

Recall that a *balanced incomplete block design* (a *BIBD*) is an arrangement of p points into b blocks of size k in such a way that every point is contained in r blocks and every pair of points occurs together in l blocks. It holds $p \leq b$ (the Fisher inequality), and in the case of equality, a BIBD is said to be symmetric. Since $b = \frac{p(p-1)}{k(k-1)}l$ and $r = \frac{p-1}{k-1}l$ (see [12, p. 107]), the integers (p, k, l) are usually taken as the basic parameters of a BIBD.

If P is the $p \times b$ point-block incidence matrix of a BIBD, then

$$A = \begin{pmatrix} O & 2P - J \\ (2P - J)^T & O \end{pmatrix}$$

is the adjacency matrix of a complete bipartite signed graph, say $\dot{G}$. We say that $\dot{G}$ is associated with the corresponding BIBD. Determine now whether such a signed graph has 3 eigenvalues.

Theorem 7. *A signed graph $\dot{G}$ associated with a BIBD with parameters (p,k,l) has 3 eigenvalues if and only if $k=\frac{p}{2}$, $k=p$ or a design is asymmetric with $k=\frac{1}{2}\left(p\pm\sqrt{p}\right)$.*

Proof. By using the notation introduced upon the theorem, we get

$$A^2=\begin{pmatrix}(2P-J)(2P-J)^T & O\\ O & (2P-J)^T(2P-J)\end{pmatrix}.$$

Since the diagonal blocks of A^2 share the same non-zero eigenvalues, we conclude that the eigenvalues of A^2 are the eigenvalues of the top-left block and, if $p<b$, zero.

Since $PP^T=lJ_p+(r-l)I_p$, $PJ^T=rJ_p$, $JJ^T=bJ_p$ and $JP^T=rJ_p$, the eigenvalues of $(2P-J)(2P-J)^T$ are

$$pb-4(p-1)(r-l)\ \text{ and }\ 4(r-l).$$

Therefore, A has 3 eigenvalues if and only one of the above eigenvalues is zero or they are equal, along with $p<b$. Now, both implications follow by direct algebraic computation. Indeed, equating the former eigenvalue with zero and expressing b and r in terms of the basic parameters, we get

$$\frac{p^2(p-1)}{k(k-1)}l-4(p-1)\left(\frac{p-1}{k-1}l-l\right)=0,$$

which, after a short transformation, gives $k=\frac{p}{2}$. Similarly, $r-l=0$ gives $k=p$, while in the last case we arrive at $k=\frac{1}{2}\left(p\pm\sqrt{p}\right)$. ■

For $k=p$, we get a homogeneous complete bipartite signed graph. Examples of BIBDs satisfying the remaining conditions of the previous theorem can be obtained by inspecting the list of BIBDs of small order given in [2, Section II.1]. Observe that if a BIBD has $k=\frac{1}{2}\left(p+\sqrt{p}\right)$, then the complementary BIBD has $k=\frac{1}{2}\left(p-\sqrt{p}\right)$. Finally the assumption that, in the last case, a BIBD is asymmetric cannot be omitted, because there exist symmetric BIBDs which satisfy the condition for k and which produce regular signed graphs without 3 eigenvalues. An example is a symmetric BIBD with parameters $(4s^2, s(2s+1), s(s+1))$, for $s\in\mathbb{Z}\backslash\{0\}$, known as a Menon design, giving a regular signed graph with eigenvalues $\pm 2s$.

5. COMPUTATIONAL RESULTS

We conclude our research by the exhaustive computer search on connected non-complete regular signed graphs with 3 eigenvalues, at most 10 vertices and vertex

·	·	+	+	+	+
·	·	+	−	+	−
+	+	·	·	+	+
+	−	·	·	−	+
+	+	+	−	·	·
+	−	+	+	·	·

1. $2.73^2, -0.73^2, -2^2$

·	·	·	·	+	+	+	+
·	·	·	·	+	+	+	+
·	·	·	·	+	+	−	−
·	·	·	·	+	+	−	−
+	+	+	+	·	·	·	·
+	+	+	+	·	·	·	·
+	+	−	−	·	·	·	·
+	+	−	−	·	·	·	·

2. $2.83^2, 0^4, -2.83^2$

·	·	+	·	+	+	+	+
·	·	·	−	+	−	+	−
+	·	·	·	+	+	+	+
·	−	·	·	−	+	−	+
+	+	+	−	·	·	+	·
+	−	+	+	·	·	·	+
+	+	+	−	+	·	·	·
+	−	+	+	·	+	·	·

3. $3.83^2, -1^4, -1.83^2$

·	·	·	−	+	·	·	−	+
·	·	·	+	·	−	+	·	−
·	·	·	·	−	+	−	+	·
−	+	·	·	·	·	+	−	·
+	·	−	·	·	·	−	·	+
·	−	+	·	·	·	·	+	−
·	+	−	+	−	·	·	·	·
−	·	+	−	·	+	·	·	·
+	−	·	·	+	−	·	·	·

4. $3^2, 0^5, -3^2$

·	·	·	+	−	−	+	−	−
·	·	·	−	+	−	−	+	−
·	·	·	−	−	+	−	−	+
+	−	−	·	·	·	+	−	−
−	+	−	·	·	·	−	+	−
−	−	+	·	·	·	−	−	+
+	−	−	+	−	−	·	·	·
−	+	−	−	+	−	·	·	·
−	−	+	−	−	+	·	·	·

5. $4^2, 1^2, -2^5$

·	·	·	·	·	+	+	+	+	·
·	·	·	·	·	+	+	−	·	+
·	·	·	·	·	+	−	·	+	−
·	·	·	·	·	+	·	−	−	−
·	·	·	·	·	·	+	+	−	−
+	+	+	+	·	·	·	·	·	·
+	+	−	·	+	·	·	·	·	·
+	−	·	−	+	·	·	·	·	·
+	·	+	−	−	·	·	·	·	·
·	+	−	−	−	·	·	·	·	·

6. $2.24^4, 0^2, -2.24^4$

·	·	·	·	+	·	+	+	+	·
·	·	·	·	+	·	−	−	·	+
·	·	·	·	·	+	+	·	−	+
·	·	·	·	·	−	·	+	−	+
+	+	·	·	·	·	+	−	·	·
·	·	+	−	·	·	·	·	−	−
+	−	+	·	+	·	·	·	·	·
+	−	·	+	−	·	·	·	·	·
+	·	−	−	·	−	·	·	·	·
·	+	+	+	·	−	·	·	·	·

7. $2.24^4, 0^2, -2.24^4$

·	·	·	+	·	+	·	+	·	+
·	·	·	·	+	·	+	·	−	+
·	·	·	·	·	+	+	−	+	·
+	·	·	·	·	+	·	+	·	−
·	+	·	·	·	·	−	·	+	+
+	·	+	+	·	·	·	−	·	·
·	+	+	·	−	·	·	·	−	·
+	·	−	+	·	−	·	·	·	·
·	−	+	·	+	·	−	·	·	·
+	+	·	−	+	·	·	·	·	·

8. $2.24^4, 0^2, -2.24^4$

·	·	+	+	+	+	+	+	+	+
·	·	+	−	+	−	−	+	+	−
+	+	·	·	+	+	−	−	+	+
+	−	·	·	+	−	+	−	+	−
+	+	+	+	·	·	+	+	−	−
+	−	+	−	·	·	−	+	−	+
+	−	−	+	+	−	·	·	−	+
+	+	−	−	+	+	·	·	−	−
+	+	+	+	−	−	−	−	·	·
+	−	+	−	−	+	+	−	·	·

9. $3.16^4, 0^2, -3.16^4$

·	·	+	+	+	+	+	+	+	+
·	·	+	+	−	−	+	+	−	−
+	+	·	·	+	+	−	+	−	+
+	+	·	·	−	−	+	−	−	+
+	−	+	−	·	·	+	+	−	+
+	−	+	−	·	·	−	−	+	−
+	+	−	+	+	−	·	·	−	−
+	+	+	−	+	−	·	·	+	+
+	−	−	−	−	+	−	+	·	·
+	−	+	+	+	−	−	+	·	·

10. $4^2, 1.24^4, -3.24^4$

·	·	−	−	−	−	+	+	+	+
·	·	−	−	−	−	+	+	+	+
−	−	·	·	+	+	−	−	+	+
−	−	·	·	+	+	−	−	+	+
−	−	+	+	·	·	+	+	−	−
−	−	+	+	·	·	+	+	−	−
+	+	−	−	+	+	·	·	−	−
+	+	−	−	+	+	·	·	−	−
+	+	+	+	−	−	−	−	·	·
+	+	+	+	−	−	−	−	·	·

11. $4.47^2, 0^6, -4.47^2$

Table 1. Data on signed graphs of Theorem 8. We use a schematic representation of the adjacency matrix (dot for 0, plus for 1 and minus for -1) and also give the corresponding spectrum.

degree at least 4. (We omit those with $r = 2$, while those with $r = 3$ are considered in Theorem 6.) Complete signed graphs are excluded, since the spectrum of such a signed graph coincides with the Seidel spectrum of a graph induced by negative edges, and such graphs with exactly 3 Seidel eigenvalues are already considered in literature – for a review and a recent progress, see the work of Greaves [7]. We also exclude in the presentation those that are switching isomorphic to homogeneous signed graphs, since they are determined easily.

The search method is described in [11] (where one can find a report on search on switching non-isomorphic signed graphs with at most 8 vertices). Accordingly, here we determine a spanning tree of a given regular underlying graph, then consider the number of eigenvalues of all possible signed graphs obtained by reversing the signs of all edges outside the tree, and simultaneously eliminate switching isomorphic ones. Here is a theorem.

Theorem 8. *Apart from those that are switching isomorphic to the homogeneous ones, there are exactly* 15 *switching non-isomorphic connected non-complete signed graphs with* 3 *eigenvalues, at most* 10 *vertices and vertex degree at least* 4*. The data on* 11 *signed graphs is given in Table* 1*, while the remaining* 4 *are the negations of those with asymmetric spectrum.*

The reader may observe that some of signed graphs presented in Table 1 are obtained by the Kronecker products considered at the beginning of Section 4. For example, the 2nd signed graph is $A_{\dot{C}_4} \otimes J_2$ ($\dot{C}_4$ being the quadrangle with one negative edge), the 5th is $A_{K_3} \otimes (2I_3 - J_3)$ and the 11th is $S_{C_5} \otimes J_2$ (S_{C_5} being the Seidel matrix of the pentagon).

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