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SIGNED COMPLETE GRAPHS WITH MAXIMUM INDEX

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In the Memorial of Professor Slobodan Simić

Abstract

Let $\Gamma = (G, \sigma)$ be a signed graph, where G is the underlying simple graph and $\sigma : E(G) \longrightarrow \{-, +\}$ is the sign function on the edges of G. The adjacency matrix of a signed graph has -1 or $+1$ for adjacent vertices, depending on the sign of the edges. It was conjectured that if Γ is a signed complete graph of order n with k negative edges, $k < n - 1$ and Γ has maximum index, then negative edges form $K_{1,k}$. In this paper, we prove this conjecture if we confine ourselves to all signed complete graphs of order n whose negative edges form a tree of order $k + 1$. A $[1, 2]$-subgraph of G is a graph whose components are paths and cycles. Let Γ be a signed complete graph whose negative edges form a $[1, 2]$-subgraph. We show that the eigenvalues of Γ satisfy the following inequalities:

$$-5 \leq \lambda_n \leq \cdots \leq \lambda_2 \leq 3.$$

Keywords: signed graph, complete graph, index.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* and *size* of G are defined as $|V(G)|$ and $|E(G)|$, respectively. The degree of a vertex u in G is denoted by $\deg_G(u)$. We denote the set of all neighbors of u in G by $N_G(u)$. If $\deg_G(u) = 1$, then u is called a *pendant vertex*. The complement of G is denoted by $\overline{G}$. A subgraph H with $|V(H)| = |V(G)|$ is called a *spanning subgraph* of G. An eigenvalue of G is called *main* if it has an eigenvector whose entries sum up to a non-zero value. Otherwise, it is called *non-main* eigenvalue. Let K_n denote the complete graph of order n. We denote the path and the cycle of order r by P_r and C_r, respectively. A tree containing exactly two non-pendant vertices is called a *double-star*. A double-star with degree sequence $(s+1, t+1, 1, \ldots, 1)$ is denoted by $D_{s,t}$. The matrix $J_{r\times s}$ is an all-one matrix of size $r \times s$, and when $r = s$ it is denoted by J_r. Also we use $j_k = (1, \ldots, 1)^t \in \mathbb{R}^k$. Let $\lambda_n(M) \leq \cdots \leq \lambda_1(M)$ be the eigenvalues of a symmetric real matrix M of order n.

A *signed graph* Γ is an ordered pair (G, σ), where $G = (V(G), E(G))$ is a simple graph (called the *underlying graph*), and $\sigma : E(G) \longrightarrow \{-, +\}$ is a mapping defined on the edge set of G. Signed graphs were introduced by Harary [7] in connection with the study of theory of social balance in social psychology proposed by Heider [8]. If all edges of a signed graph are positive (respectively, negative), then we denote it by $G = (G, +)$ (respectively, $(G, -)$). The *adjacency matrix* of a signed graph $\Gamma = (G, \sigma)$ is a square matrix $A(\Gamma) = A(G, \sigma) = (a^{\sigma}_{ij})$, where $a^{\sigma}_{ij} = \sigma(v_i v_j) a_{ij}$ and $A(G) = (a_{ij})$ is the adjacency matrix of G. The nullity of a graph G is the nullity of its adjacency matrix and is denoted by $\mathrm{null}(G)$. If Γ is a signed graph, then $\varphi(\Gamma, \lambda)$ is the characteristic polynomial of $A(\Gamma)$ which is referred to as the characteristic polynomial of Γ. The eigenvalues of the adjacency matrix of a graph are often referred to as the eigenvalues of the graph. The spectrum of a signed graph Γ is the set of all eigenvalues of Γ along with their multiplicities. The spectrum of graphs, in particular, signed graphs, has been studied extensively by many authors, for instance see [2, 3, 4]. Let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues of signed graph Γ with the corresponding multiplicities $m(\lambda_1), \ldots, m(\lambda_s)$. We denote the spectrum of Γ by

$$\mathrm{Spec}\,(\Gamma) = \begin{pmatrix} \lambda_1 & \ldots & \lambda_s \\ m(\lambda_1) & \ldots & m(\lambda_s) \end{pmatrix}.$$

In particular, the largest eigenvalue of Γ is called the *index* of Γ. In [9, 12] the authors have studied the largest eigenvalue of signed graphs. Let $\Gamma = (G, \sigma)$ be a signed graph and $v \in V(\Gamma)$. We obtain a new graph Γ' from Γ if we change the signs of all edges incident with v. We call v a *switching vertex*. A *switching* of a signed graph Γ is a graph that can be obtained by applying finitely many

switching operations. We call two graphs Γ_1 and Γ_2 *switching equivalent* if Γ_2 is a switching of Γ_1 and we write $\Gamma_1 \sim \Gamma_2$. It can be easily seen that if two signed graphs Γ_1 and Γ_2 on the same underlying graph G are switching equivalent, then their adjacency matrices are similar. Thus two switching equivalent signed graphs have the same spectrum. Let (K_n, H^-) be a signed complete graph whose negative edges induce a subgraph H. A $[1,2]$-*subgraph* of G is a subgraph whose components are paths and cycles. In [9], the connected signed graphs of fixed order, size, and number of negative edges with maximum index have been studied. The following conjecture on signed complete graphs was proposed in [9].

Conjecture 1. *Let Γ be a signed complete graph of order n and k negative edges that maximizes the index. If $k < n-1$, then negative edges form $K_{1,k}$.*

In this paper, we show this conjecture holds for signed complete graphs whose negative edges form a tree. Also, we find some sharp bounds for the eigenvalues of a signed complete graph. We prove that if H is a $[1,2]$-subgraph of K_n, then the eigenvalues of (K_n, H^-) satisfy the following inequalities:

$$-5 \leq \lambda_n \leq \cdots \leq \lambda_2 \leq 3.$$

Also, we study the spectrum of a signed complete graph whose negative edges form a $[1,2]$-subgraph. We show that if λ is a non-main eigenvalue of an arbitrary subgraph H or it is main with multiplicity greater than 1, then $-1-2\lambda$ is an eigenvalue of (K_n, H^-).

2. Signed Complete Graphs with Maximum Index

In [9], the connected signed graphs of fixed order, size, and number of negative edges with maximum index have been studied. In this section, we prove that among all signed complete graphs of order n whose negative edges form a tree of order $k+1$ and maximizes the index, the negative edges form $K_{1,k}$. Before stating the main theorem we need the following results.

Lemma 2 [3, Theorem 1]**.** *Let (K_n, H^-) be a signed complete graph and $|V(H)| = k < n$. Then $m(-1) = n-k-1+\mathrm{null}(H)$.*

Let $S(G) = J - I - 2A(G)$ be the *Seidel matrix* of the graph G. Clearly, $S(K_{r,s,n-r-s})$ can be seen as the adjacency matrix of $(K_n, K^-_{r,s,n-r-s})$ which is switching equivalent to $(K_n, K^-_{r,s})$. Hence the characteristic polynomials of $S(K_{r,s,n-r-s})$ and $(K_n, K^-_{r,s})$ are the same. Thus by [11, Theorem 1], we have the following result.

Lemma 3. *Let $(K_n, K^-_{1,k})$ be a signed complete graph. Then*

$$\varphi(K_n, K^-_{1,k}) = (\lambda+1)^{n-3}\Big(\lambda^3 + (3-n)\lambda^2 + (3-2n)\lambda + 4(n-k-1)k + 1 - n\Big).$$

Here, we need to introduce an additional notation. Assume that A is a symmetric real matrix of order n and $\{X_1, \ldots, X_m\}$ is a partition of $X = \{1, \ldots, n\}$. Let $\{X_1, \ldots, X_m\}$ partition the rows and columns of A, as follows,

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix},$$

where $A_{i,j}$ denotes the submatrix of A formed by rows in X_i and the columns in X_j. Then the $m \times m$ matrix $B = (b_{ij})$ is called the *quotient matrix* related to that partition, where b_{ij} denotes the average row sum of $A_{i,j}$. If the row sum of each $A_{i,j}$ is constant, then the partition is called *equitable.*

Theorem 4. *Let $\Gamma = (K_n, D^-_{s,t})$ be a signed complete graph and $n > s + t + 2$. Then*

$$\varphi(\Gamma, \lambda) = (\lambda + 1)^{n-5}\Big(\lambda^5 + (5 - n)\lambda^4 + (10 - 4n)\lambda^3 + (4ku + 8st + 10 - 6n)\lambda^2 + (8ku + 16st + 5 - 4n)\lambda + 4ku + 8st - 16stu + 1 - n\Big),$$

where $k = s + t + 1$ and $u = n - s - t - 2$.

Proof. Let $\Gamma = (K_n, D^-_{s,t})$. Since the adjacency matrix of P_4 is non-singular, $\mathrm{rank}(D_{s,t}) \geq 4$. On the other hand, since the adjacency matrix of $D_{s,t}$ has 4 distinct rows, $\mathrm{rank}(D_{s,t}) \leq 4$. Hence, $\mathrm{rank}(D_{s,t}) = 4$ and $\mathrm{null}(D_{s,t}) = s + t - 2$. Thus by Lemma 2, we have $m(-1) = n - 5$.

Suppose that $v, w \in V(D_{s,t})$ and $\deg_{D_{s,t}}(v) = s+1$, $\deg_{D_{s,t}}(w) = t+1$. Now, assume that the set of vertices of $D_{s,t}$ is partitioned into parts S, X, Y, T, such that $S = N_{D_{s,t}}(v) \setminus \{w\}$, $X = \{v\}$, $Y = \{w\}$ and $T = N_{D_{s,t}}(w) \setminus \{v\}$. Let

$$U = V(K_n) \setminus (S \cup X \cup Y \cup T).$$

Clearly, $\{S, X, Y, T, U\}$ is an equitable partition of $V(\Gamma)$. Now, the quotient matrix of $A(\Gamma)$ is as follows

$$B = \begin{bmatrix} s-1 & -1 & 1 & t & u \\ -s & 0 & -1 & t & u \\ s & -1 & 0 & -t & u \\ s & 1 & -1 & t-1 & u \\ s & 1 & 1 & t & u-1 \end{bmatrix}.$$

Thus the characteristic polynomial of B is

$$f(\lambda) = \lambda^5 + (5 - n)\lambda^4 + (10 - 4n)\lambda^3 + (4ku + 8st + 10 - 6n)\lambda^2 + (8ku + 16st + 5 - 4n)\lambda + 4ku + 8st - 16stu + 1 - n.$$

The characteristic polynomial of B divides the characteristic polynomial of $A(\Gamma)$, see [5, Lemma 2.3.1]. Also, it is easily seen that if -1 is a root of $f(\lambda)$, then $stu = 0$, a contradiction. Hence the proof is complete. ■

Now, we state two remarks.

Remark 5. Let $\Gamma = \big(K_n, D^-_{s,t}\big)$ and $n = s + t + 2$. We have

$$\big(K_n, D^-_{s,t}\big) \sim \big(K_n, K^-_{2,t}\big).$$

We know that the characteristic polynomials of $S(K_{2,t,n-t-2})$ and $\big(K_n, K^-_{2,t}\big)$ are the same. Then by [11, Theorem 1], we have the following result.

$$\varphi(\Gamma, \lambda) = (\lambda + 1)^{n-3}\Big(\lambda^3 + (3 - n)\lambda^2 + (3 - 2n)\lambda + 8st + 1 - n\Big).$$

Remark 6. Let $\Gamma = \big(K_n, D^-_{s,t}\big)$ and $\Gamma' = \big(K_n, K^-_{1,k}\big)$. If the number of negative edges of Γ and Γ' are equal (i.e., $k = s+t+1$), then we show that $\lambda_1(\Gamma') > \lambda_1(\Gamma)$. By Lemma 3, Theorem 4 and Remark 5, the following holds

$$\varphi(\Gamma', \lambda) - \varphi(\Gamma, \lambda) = (\lambda + 1)^{n-5}\big(- 8st\lambda^2 - 16st\lambda - 8st(3 + 2k - 2n)\big).$$

Note that the largest root of $\varphi(\Gamma', \lambda) - \varphi(\Gamma, \lambda)$ is $-1 + \sqrt{2(n - k - 1)}$. Let $n - k - 1 > 2$. Clearly, $(K_{n-k-1}, +)$ is an induced subgraph of Γ. Hence by the interlacing theorem, see [6, p.17], we deduce that

$$-1 + \sqrt{2(n - k - 1)} < n - k - 2 \leq \lambda_1(\Gamma).$$

Now, assume that $n-k-1 \leq 2$. Obviously, there exists a subgraph of Γ which is switching equivalent to $(K_3, +)$. Thus $-1+\sqrt{2(n - k - 1)} < 2 \leq \lambda_1(\Gamma)$. Therefore $\varphi(\Gamma', \lambda_1(\Gamma)) < 0$. Since $\lim_{\lambda\to+\infty} \varphi(\Gamma', \lambda) = +\infty$, so $\varphi(\Gamma', \lambda)$ has a root greater than $\lambda_1(\Gamma)$. This implies that $\lambda_1(\Gamma') > \lambda_1(\Gamma)$.

Also, we need the following result.

Lemma 7 [9, Lemma 5.1(i)]**.** *Let r, s and t be distinct vertices of a signed graph Γ and let $\mathbf{x} = (x_1, \ldots, x_n)^t$ be an eigenvector which corresponds to $\lambda_1(\Gamma)$. Let Γ' be obtained by reversing the sign of the positive edge rs and the negative edge rt. If*

$$\begin{cases} x_r \geq 0, & x_s \leq x_t, \text{ or} \\ x_r \leq 0, & x_s \geq x_t, \end{cases}$$

then $\lambda_1(\Gamma') \geq \lambda_1(\Gamma)$. If at least one inequality for the entries of $\mathbf{x}$ is strict, then $\lambda_1(\Gamma') > \lambda_1(\Gamma)$.

Here, we introduce the following short notation. If r, s, and t are the vertices considered in Lemma 7, then $R(r, s, t)$ is reversing the sign of the positive edge rs and the negative edge rt.

Theorem 8. *Among all signed complete graphs Γ of order n whose negative edges form a tree of order $k+1$ and maximizes the index, the negative edges form $K_{1,k}$.*

Proof. Suppose that k negative edges form a tree T and maximizes the index, such that it is not a star graph. Hence, there exists an induced path of order 4 in T, say $uvwz$. Let $\mathbf{x} = (x_1, \ldots, x_n)^t$ be an eigenvector which corresponds to $\lambda_1(\Gamma)$. Now, eight cases can be considered.

Case 1. $x_v = x_w = 0$. Suppose that $x_z \neq 0$, then $R(z, v, w)$ contradicts the maximality of $\lambda_1(\Gamma)$, see Lemma 7. Thus $x_z = 0$. Now, if there exists $z' \in V(T)$ such that $z'z \in E(T)$ and $x_{z'} \neq 0$, then we find a contradiction in the same way. Hence one can see that $x_a = 0$, for any $a \in V(T)$. If T is a spanning subgraph of K_n, then $\mathbf{x}$ is a zero vector which is impossible. Otherwise, there exists $t \in V(K_n) \setminus V(T)$ such that $x_t \neq 0$. Also, there exists a pendant vertex s of T such that $ss' \in E(T)$ for a vertex $s' \in V(T)$. Thus $R(s', t, s)$ contradicts the maximality of $\lambda_1(\Gamma)$.

Case 2. $x_w = 0$ and $x_v \neq 0$. If $x_u \neq x_v$ (or $x_u = x_v$), then by $R(w, u, v)$ (or $R(u, w, v)$) we get a signed complete graph having a negative tree with a greater index.

Case 3. $x_v = 0$ and $x_w \neq 0$. The proof is similar to Case 2.

Case 4. $x_u = 0$ and $x_v, x_w \neq 0$. If $x_v \neq x_w$ (or $x_v = x_w$), then $R(u, w, v)$ (or $R(w, u, v)$) contradicts the maximality of $\lambda_1(\Gamma)$.

Case 5. $x_u, x_v, x_w < 0$ or $x_u, x_v, x_w > 0$. Without loss of generality, assume that $x_u, x_v, x_w < 0$ (otherwise, we may consider $-\mathbf{x}$ instead). If $x_w \geq x_v$, then $R(u, w, v)$ contradicts the maximality of $\lambda_1(\Gamma)$. Suppose that $x_w < x_v$. If $x_z \geq x_w$ (or $x_z < x_w$), then $R(v, z, w)$ (or $R(z, v, w)$) results a signed complete graph having a negative tree with a greater index.

Case 6. $x_v, x_w < 0$ and $x_u > 0$ or $x_v, x_w > 0$ and $x_u < 0$. Without loss of generality, assume that $x_v, x_w < 0$ and $x_u > 0$. Thus $R(w, u, v)$ contradicts the maximality of $\lambda_1(\Gamma)$.

Case 7. $x_u, x_v < 0$ and $x_w > 0$ or $x_u, x_v > 0$ and $x_w < 0$. Without loss of generality, assume that $x_u, x_v < 0$ and $x_w > 0$, then by applying $R(u, w, v)$ we get a signed complete graph having a negative tree with a greater index.

Case 8. $x_u, x_w < 0$ and $x_v > 0$ or $x_u, x_w > 0$ and $x_v < 0$. We show the first one, then by considering $-\mathbf{x}$ instead of $\mathbf{x}$, the second one is proved. Hence assume that $x_u, x_w < 0$ and $x_v > 0$. If $x_z \leq 0$, then $R(z, v, w)$ contradicts the

maximality of $\lambda_1(\Gamma)$. Assume that $x_z > 0$. If $x_w \geq x_u$, then $R(z, u, w)$ contradicts the maximality of $\lambda_1(\Gamma)$. So $x_w < x_u$. If $x_z \geq x_v$, then $R(u, z, v)$ contradicts the maximality of $\lambda_1(\Gamma)$. Therefore one can see that $x_w < x_u < 0 < x_z < x_v$. Now, if there exists $z' \in V(T)$ such that $z'z \in E(T)$, then $x_{z'} \geq 0$ or $x_{z'} < 0$, and so $R(z', w, z)$ or $R(z', v, z)$ results a signed complete graph having a negative tree with a greater index. Hence we deduce that the vertex z is a pendant vertex of T. Now, if there exists $u' \in V(T)$ such that $u'u \in E(T)$, then $x_{u'} \leq 0$ or $x_{u'} > 0$, and so $R(u', v, u)$ or $R(u', w, u)$ contradicts the maximality of $\lambda_1(\Gamma)$. It follows that the vertex u is a pendant vertex of T. Clearly, if there exists $v' \in V(T)$ such that $v'v \in E(T)$ and $x_{v'} \geq 0$, then $R(v', u, v)$ contradicts the maximality of $\lambda_1(\Gamma)$. So $x_{v'} < 0$. Thus we deduce that T is a double-star. By Remark 6, we have a contradiction. Therefore T is a star graph, as claimed. ■

3. Bounds for the Eigenvalues of (K_n, H^-)

In this section, first we find some sharp bounds for the eigenvalues of a signed complete graph. Next, we study the spectrum of a signed complete graph whose negative edges form a $[1, 2]$-subgraph. Before stating the main theorem, we need the following inequalities which are well-known as Courant-Weyl inequalities, see [10, Lemma 3.1].

Lemma 9. *Let B and C be two $n \times n$ Hermitian matrices. Then*

$$\lambda_i(B + C) \leq \lambda_j(B) + \lambda_{i-j+1}(C) \ (n \geq i \geq j \geq 1),$$

$$\lambda_i(B + C) \geq \lambda_j(B) + \lambda_{i-j+n}(C) \ (1 \leq i \leq j \leq n).$$

Theorem 10. *Let H be a subgraph of K_n of order $k < n$ and let t (respectively, s) be the number of positive (respectively, negative) eigenvalues of H. If the eigenvalues of (K_n, H^-) and H are $\lambda_n \leq \cdots \leq \lambda_1$ and $\mu_k \leq \cdots \leq \mu_1$, respectively, then the following hold:*

$$\begin{aligned}
&-1 - 2\mu_1 \leq \lambda_n \leq -1 - 2\mu_2 \leq \lambda_{n-1} \leq -1 - 2\mu_3 \\
&\leq \cdots \leq -1 - 2\mu_t \leq \lambda_{n-t+1} < \underbrace{\lambda_{n-t} = \cdots = \lambda_{s+2}}_{n-k-1+\mathrm{null}(H)} < \lambda_{s+1} \leq -1 - 2\mu_{t+1+\mathrm{null}(H)} \\
&\leq \cdots \leq -1 - 2\mu_{k-2} \leq \lambda_3 \leq -1 - 2\mu_{k-1} \leq \lambda_2 \leq -1 - 2\mu_k,
\end{aligned}$$

where $\lambda_{n-t} = -1$. Moreover, the following holds:

$$n - 1 - 2\mu_1 \leq \lambda_1 \leq n - 1.$$

Proof. By a suitable labeling of the vertices of (K_n, H^-), one can see that

$$A(K_n, H^-) = B + C,$$

where

$$A(K_n, H^-) = \begin{bmatrix} A(K_k, H^-) & J_{k\times(n-k)} \\ J_{(n-k)\times k} & (J-I)_{n-k} \end{bmatrix}, \ B = \begin{bmatrix} -2A(H) & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{n-k} \end{bmatrix},$$

and $C = J_n - I_n$. Hence by Lemma 9, one can find the following inequalities for $2 \le i \le n$,

$$\lambda_i(B) - 1 = \lambda_i(B) + \lambda_n(C) \le \lambda_i(B+C) \le \lambda_{i-1}(B) + \lambda_2(C) = \lambda_{i-1}(B) - 1.$$

It is obvious that the spectrum of B is as follows:

$$-2\mu_1 \le -2\mu_2 \le \cdots \le -2\mu_t < \underbrace{0 = \cdots = 0}_{n-k+\mathrm{null}(H)} < -2\mu_{t+1+\mathrm{null}(H)} \le \cdots \le -2\mu_k.$$

This yields that

$$\begin{aligned} &-1 - 2\mu_1 \le \lambda_n \le -1 - 2\mu_2 \le \lambda_{n-1} \le -1 - 2\mu_3 \\ &\le \cdots \le -1 - 2\mu_t \le \lambda_{n-t+1} \le \underbrace{\lambda_{n-t} = \cdots = \lambda_{s+2}}_{n-k-1+\mathrm{null}(H)} \le \lambda_{s+1} \le -1 - 2\mu_{t+1+\mathrm{null}(H)} \\ &\le \cdots \le -1 - 2\mu_{k-2} \le \lambda_3 \le -1 - 2\mu_{k-1} \le \lambda_2 \le -1 - 2\mu_k, \end{aligned}$$

where $\lambda_{n-t} = -1$. Note that if $\lambda_i(B) = \lambda_{i-1}(B) = 0$, then one can see that $\lambda_i(B+C) = -1$ and this shows that the multiplicity of eigenvalue -1 is at least $n - k - 1 + \mathrm{null}(H)$. By Lemma 2, we have $m(-1) = n - k - 1 + \mathrm{null}(H)$, for $k < n$. We conclude that $\lambda_{n-t+1}, \lambda_{s+1} \ne -1$. Hence inequalities of the first part are proved. Moreover, by Lemma 9, we have

$$\lambda_1(B+C) \ge \lambda_n(B) + \lambda_1(C) = -2\mu_1 + n - 1.$$

Clearly, $\lambda_1 \le n - 1$. This completes the proof. ∎

In Lemma 2, we determined the multiplicity -1 of (K_n, H^-), where H is a non-spanning subgraph of K_n. If H is a spanning subgraph of K_n, then the following lemma provides a bound for the multiplicity -1 of (K_n, H^-).

Lemma 11 [3, Theorem 3]**.** *Let H be a spanning subgraph of K_n and (K_n, H^-) be a signed complete graph. Then the following statements hold:*

(i) $\mathrm{null}(\overline{H}) - 1 \le m(1) \le \mathrm{null}(\overline{H}) + 1$.

(ii) $\mathrm{null}(H) - 1 \leq m(-1) \leq \mathrm{null}(H) + 1$.

If H is a spanning subgraph of K_n, then $A(K_n, H^-) = J_n - I_n - 2A(H)$. By a similar argument as we did in the proof of Theorem 10, and by part (ii) of Lemma 11, one can obtain the following corollary.

Corollary 12. *Let H be a spanning subgraph of K_n. If the eigenvalues of (K_n, H^-) and H are $\lambda_n \leq \cdots \leq \lambda_1$ and $\mu_n \leq \cdots \leq \mu_1$, respectively, then the following hold:*

$$-1 - 2\mu_1 \leq \lambda_n \leq -1 - 2\mu_2 \leq \lambda_{n-1} \leq \cdots \leq \lambda_3 \leq -1 - 2\mu_{n-1} \leq \lambda_2 \leq -1 - 2\mu_n.$$

Moreover, the following holds:

$$n - 1 - 2\mu_1 \leq \lambda_1 \leq n - 1.$$

Corollary 13. *Let H be a $[1, 2]$-subgraph of K_n. If the eigenvalues of (K_n, H^-) are $\lambda_n \leq \cdots \leq \lambda_1$, then*

$$-5 \leq \lambda_n \leq \cdots \leq \lambda_2 \leq 3,$$

$$n - 5 \leq \lambda_1 \leq n - 1.$$

Proof. Let $\mu_k \leq \cdots \leq \mu_1$ be the eigenvalues of H. Then one can see that $-2 \leq \mu_i \leq 2$, for $i = 1, \ldots, k$. Hence by Theorem 10 and Corollary 12, the proof is complete. ■

Example 14. Let nC_3 be the disjoint union of n copies of C_3. Then by [2, Remark 1], one can see that,

$$\mathrm{Spec}(K_{3n}, nC_3^-) = \begin{pmatrix} 3n-5 & 1 & -5 \\ 1 & 2n & n-1 \end{pmatrix}.$$

So $3n - 5$ is a sharp bound for the largest eigenvalue of (K_{3n}, nC_3^-).

Let H be a $[1, 2]$-subgraph. In Corollary 13, we found some bounds for the eigenvalues of (K_n, H^-). Now, we study the spectrum of (K_n, H^-). For this aim, we need the following result.

Lemma 15 [1, Proposition 9]**.** *Consider a graph G and $\lambda \in \mathrm{Spec}(G)$. Then the following statements are equivalent:*

(i) *The eigenvalue λ is non-main or it is main with multiplicity greater than* 1,

(ii) *There is some eigenvector v of G associated to λ such that $j^t v = 0$,*

(iii) *The scalar $-1 - \lambda$ belongs to* $\mathrm{Spec}(\overline{G})$.

Now, we have the following result.

Theorem 16. *Let H be a subgraph of K_n and $|V(H)| = k < n$. If λ is a non-main eigenvalue of H or it is main with multiplicity greater than 1, then $-1-2\lambda$ is an eigenvalue of (K_n, H^-).*

Proof. By a suitable labeling of the vertices of (K_n, H^-), one can see that

$$A(K_n, H^-) = C - B,$$

where

$$C = \begin{bmatrix} A(\overline{H}) & J_{k\times(n-k)} \\ J_{(n-k)\times k} & (J-I)_{n-k} \end{bmatrix}, \ B = \begin{bmatrix} A(H) & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & \mathbf{0}_{n-k} \end{bmatrix}.$$

Obviously, $C + B = J_n - I_n$. Now, we obtain

$$A(K_n, H^-) = J_n - I_n - 2B.$$

Now, assume that λ is a non-main eigenvalue of H or it is main with multiplicity greater than 1. By Lemma 15, one can see that λ has an eigenvector α such that $j_k^t\alpha = 0$. Define

$$\beta = \begin{bmatrix} \alpha \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^n.$$

Then $B\beta = \lambda\beta$ and $j_n^t\beta = 0$, and so one can see that

$$A(K_n, H^-)\beta = (J_n - I_n - 2B)\beta = (-1-2\lambda)\beta.$$

Hence $-1-2\lambda$ is an eigenvalue of (K_n, H^-), as desired. ■

Lemma 17 [1, Theorem 16]**.** *For $n \geq 2$ and $1 \leq j \leq n$, λ_j is a non-main eigenvalue of P_n if and only if j is even. In particular, the least eigenvalue of P_n is non-main if and only if n is even.*

Note that the eigenvalues of P_n are $\lambda_j(P_n) = 2\cos\frac{j\pi}{n+1}$, $1 \leq j \leq n$, see [1, Lemma 15]. Also, all eigenvalues of C_n, except 2, are non-main and they are as, $\lambda_j(C_n) = 2\cos\frac{2j\pi}{n}$, $0 \leq j \leq n-1$. Hence, by Theorem 16 and Lemma 17, we have the following result.

Corollary 18. *Let $\Gamma = \left(K_n, \bigcup_{i=1}^m P_{r_i}^- \cup \bigcup_{i=1}^t C_{s_i}^-\right)$ be a signed complete graph whose negative edges induce the disjoint union of m distinct paths and t distinct cycles. Then $-1-4\cos\frac{j\pi}{r_i+1}$ is an eigenvalue of Γ if j is even, for $j = 1, \ldots, r_i$ and $i = 1, \ldots, m$. Also, $-1-4\cos\frac{2j\pi}{s_i}$ is an eigenvalue of Γ for $j = 1, \ldots, s_i - 1$ and $i = 1, \ldots, t$. Moreover, -5 is an eigenvalue of Γ with multiplicity at least $t-1$.*

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References

[1] N. Abreu, D.M. Cardoso, F.A.M. Franca and C.T.M. Vinagre, *On main eigenvalues of certain graphs*, Comput. Appl. Math., to appear.

[2] S. Akbari, S. Dalvandi, F. Heydari and M. Maghasedi, *On the eigenvalues of signed complete graphs*, Linear Multilinear Algebra **67** (2019) 433–441. doi:10.1080/03081087.2017.1403548

[3] S. Akbari, S. Dalvandi, F. Heydari and M. Maghasedi, *On the multiplicity of* -1 *and* 1 *in signed complete graphs*, Util. Math., to appear.

[4] F. Belardo, P. Petecki and J. Wang, *On signed graphs whose second largest Laplacian eigenvalue does not exceed* 3, Linear Multilinear Algebra **64** (2016) 1785–1799. doi:10.1080/03081087.2015.1120701

[5] A.E. Brouwer and W.H. Haemers, Spectra of Graphs (Springer, 2011). doi:10.1007/978-1-4614-1939-6

[6] D. Cvetković, P. Rowlinson and S.K. Simić, An Introduction to the Theory of Graph Spectra (Cambridge University Press, Cambridge, 2010).

[7] F. Harary, *On the notion of balance in a signed graph*, Michigan Math. J. **2** (1953) 143–146. doi:10.1307/mmj/1028989917

[8] F. Heider, *Attitude and cognitive organization*, J. Psychology **21** (1946) 107–112. doi:10.1080/00223980.1946.9917275

[9] T. Koledin and Z. Stanić, *Connected signed graphs of fixed order, size, and number of negative edges with maximal index*, Linear Multilinear Algebra **65** (2017) 2187–2198. doi:10.1080/03081087.2016.1265480

[10] H. Lin, R. Liu and J. Shu, *Some results on the largest and least eigenvalues of graphs*, Electron. J. Linear Algebra **27** (2014) 670–682. doi:/10.13001/1081-3810.1928

[11] S.M. Lv, L. Wei and H.X. Zhao, *On the Seidel integral complete multipartite graphs*, Acta Math. Appl. Sin. Engl. Ser. **28** (2012) 705–710. doi:10.1007/s10255-012-0126-x

[12] Z. Stanić, *Perturbations in a signed graph and its index*, Discuss. Math. Graph Theory **38** (2018) 841–852. doi:10.7151/dmgt.2035

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