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A SPECTRAL CHARACTERIZATION OF THE s-CLIQUE EXTENSION OF THE TRIANGULAR GRAPHS

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This paper is dedicated to the memory of Prof. Slobodan Simić.

Abstract

A regular graph is co-edge regular if there exists a constant μ such that any two distinct and non-adjacent vertices have exactly μ common neighbors. In this paper, we show that for integers $s \geq 2$ and n large enough, any co-edge-regular graph which is cospectral with the s-clique extension of the triangular graph $T(n)$ is exactly the s-clique extension of the triangular graph $T(n)$.

Keywords: co-edge-regular graph, s-clique extension, triangular graph.

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1. Introduction

All graphs in this paper are simple and undirected. For definitions related to distance-regular graphs, see [1,11]. Before we state the main result, we give more definitions.

Let G be a simple connected graph on vertex set $V(G)$, edge set $E(G)$ and adjacency matrix A. The eigenvalues of G are the eigenvalues of A. Let $\lambda_0, \lambda_1, \ldots, \lambda_t$ be the distinct eigenvalues of G and m_i be the multiplicity of λ_i $(i = 0, 1, \ldots, t)$. Then the multiset $\{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_t^{m_t}\}$ is called the *spectrum* of G. Two graphs are called *cospectral* if they have the same spectrum. Note that a graph H cospectral with a k-regular graph G is also k-regular.

Recall that a regular graph is called *co-edge-regular*, if there exists a constant μ such that any two distinct and non-adjacent vertices have exactly μ common neighbors. Our main result in this paper is as follows.

Theorem 1. *Let Γ be a co-edge-regular graph with spectrum*

$$\left\{(2sn-3s-1)^1, (sn-3s-1)^{n-1}, (-s-1)^{\frac{n^2-3n}{2}}, (-1)^{\frac{(s-1)n(n-1)}{2}}\right\},$$

where $s \geq 2$ and $n \geq 1$ are integers. If $n \geq 48s$, then Γ is the s-clique extension of the triangular graph $T(n)$.

This paper is a follow-up paper of Hayat, Koolen and Riaz [4]. They showed a similar result for the square grid graphs. In that paper, they gave the following conjecture.

Conjecture 2 [4]**.** *Let Γ be a connected co-edge-regular graph with four distinct eigenvalues. Let $t \geq 2$ be an integer and $|V(\Gamma)| = n(\Gamma)$. Then there exists a constant n_t such that, if $\theta_{min}(\Gamma) \geq -t$ and $n(\Gamma) \geq n_t$ both hold, then Γ is the s-clique extension of a strongly regular graph for some $2 \leq s \leq t-1$.*

This conjecture is wrong as the $p \times q$-grids $(p > q \geq 2)$ show. So we would like to modify this conjecture as follows.

Conjecture 3. *Let Γ be a connected co-edge-regular graph with parameters (n, k, μ) having four distinct eigenvalues. Let $t \geq 2$ be an integer. Then there exists a constant n_t such that, if $\theta_{min}(\Gamma) \geq -t$, $n \geq n_t$ and $k < n - 2 - \frac{(t-1)^2}{4}$, then either Γ is the s-clique extension of a strongly regular graph for $2 \leq s \leq t-1$ or Γ is a $p \times q$-grid with $p > q \geq 2$.*

The reason for the valency condition is, that in [12], it was shown that for $\lambda \geq 2$, there exist constants $C(\lambda)$ such that a connected k-regular co-edge-regular graph with order v and smallest eigenvalue at least $-\lambda$ satisfies one of the following conditions.

(i) $v - k - 1 \leq \frac{(\lambda-1)^2}{4} + 1$, or;

(ii) Every pair of distinct non-adjacent vertices has at most $C(\lambda)$ common neighbours.

Koolen *et al.* [8] improved this result by showing that one can take $C(\lambda) = (\lambda - 1)\lambda^2$ if k is much larger than λ. This paper is part of the project to show the conjecture for $t = 3$.

Another motivation comes from the lecture notes [9]. In these notes, Terwilliger shows that any local graph of a thin Q-polynomial distance-regular graph is co-edge-regular and has at most five distinct eigenvalues. So it is interesting to study co-edge-regular graphs with a few distinct eigenvalues.

We mainly follow the method of Hayat *et al.* [4]. The main difference is that we simplify the method of Hayat *et al.* when we show that every vertex lies on exactly two lines. This leads to a better bound for which we can show this. This will also improve the bound given in the result of Hayat *et al.*

2. Preliminaries

2.1. Definitions

For two distinct vertices x and y, we write $x \sim y$ (respectively, $x \nsim y$) if they are adjacent (respectively, nonadjacent) to each other. For a vertex x of G, we define $N_G(x) = \{y \in V(G) \mid y \sim x\}$, and $N_G(x)$ is called the neighborhood of x. The graph induced by $N_G(x)$ is called the *local graph* of G with respect to x and is denoted by $G(x)$. We denote the number of common neighbors between two distinct vertices x and y by $\lambda_{x,y}$ (respectively, $\mu_{x,y}$) if $x \sim y$ (respectively, $x \nsim y$).

A graph is called *regular* if every vertex has the same valency. A regular graph G with n vertices and valency k is called *co-edge-regular* with parameters (n, k, μ) if any two nonadjacent vertices have exactly $\mu = \mu(G)$ common neighbors. In addition, if any two adjacent vertices have precisely $\lambda = \lambda(G)$ common neighbors, then G is called *strongly regular* with parameters (n, k, λ, μ). A graph G is called *walk-regular* if the number of closed walks of length r from a given vertex x is independent of the choice of x for all r, that is to say, for any x, A^r_{xx} is constant for all r, where A is the adjacency matrix of G.

Let X be a set of size t. The *Johnson graph* $J(t, d)$ $(t \geq 2d)$ is a graph with vertex set $\binom{X}{d}$, the set of d-subsets of X, where two d-subsets are adjacent whenever they have $d - 1$ elements in common. $J(t, 2)$ is the *triangular graph* $T(t)$. Recall that a *clique* (or a complete graph) is a graph in which every pair of vertices is adjacent. A *coclique* is a graph that any two distinct vertices are nonadjacent. A *t-clique* is a clique with t-vertices and is denoted by K_t. The line graph of K_t is also the triangular graph $T(t)$ which is strongly regular with parameters $(\binom{t}{2}, 2t-4, t-2, 4)$ and spectrum $\{(2t-4)^1, (t-4)^{t-1}, (-2)^{\frac{t^2-3t}{2}}\}$.

The Kronecker product $M_1 \otimes M_2$ of two matrices M_1 and M_2 is obtained by replacing the ij-entry of M_1 by $(M_1)_{ij}M_2$ for all i and j. Note that if τ and η are eigenvalues of M_1 and M_2, respectively, then $\tau\eta$ is an eigenvalue of $M_1 \otimes M_2$.

2.2. Interlacing

Lemma 4 ([6], Interlacing)**.** *Let N be a real symmetric $n \times n$ matrix with eigenvalues $\theta_1 \geq \cdots \geq \theta_n$ and R be a real $n \times m$ $(m < n)$ matrix with $R^T R = I$. Set $M = R^T NR$ with eigenvalues $\mu_1 \geq \cdots \geq \mu_m$. Then*

(i) *the eigenvalues of M interlace those of N, i.e.,*

$$\theta_i \geq \mu_i \geq \theta_{n-m+i}, \quad i = 1, 2, \ldots, m,$$

(ii) *if the interlacing is tight, that is, there exists an integer $j \in \{1, 2, \ldots, m\}$ such that $\theta_i = \mu_i$ for $1 \leq i \leq j$ and $\theta_{n-m+i} = \mu_i$ for $j+1 \leq i \leq m$, then $RM = NR$.*

In the case that R is permutation-similar to $\begin{pmatrix} I & O \\ O & O \end{pmatrix}$, then M is just a principal submatrix of N.

Let $\pi = \{V_1, \ldots, V_m\}$ be the partition of the index set of the columns of N and let N be partitioned according to π as

$$\begin{pmatrix} N_{1,1} & \ldots & N_{1,m} \\ \vdots & \ddots & \vdots \\ N_{m,1} & \ldots & N_{m,m} \end{pmatrix},$$

where $N_{i,j}$ denotes the block matrix of N formed by rows in V_i and columns in V_j. The *characteristic matrix* P is the $n \times m$ matrix whose jth column is the characteristic vector of V_j $(j = 1, \ldots, m)$. *The quotient matrix* of N with respect to π is the $m \times m$ matrix Q whose entries are the average row sum of the blocks N_{ij} of N, i.e.,

$$Q_{i,j} = \frac{1}{V_i} \left(P^T NP\right)_{i,j}.$$

The partition π is called *equitable* if each block $N_{i,j}$ of N has constant row (and column) sum, i.e., $PQ = NP$. The following lemma can be shown by using Lemma 4.

Lemma 5 [5]**.** *Let N be a real symmetric matrix with π as a partition of the index set of its columns. Suppose Q is the quotient matrix of N with respect to π, then the following hold.*

(i) *The eigenvalue of Q interlace the eigenvalues of N.*

(ii) *If the interlacing is tight (as defined in Lemma 4(ii)), then the partition π is equitable.*

By an equitable partition of a graph, we always mean an equitable partition of its adjacency matrix A.

2.3. Clique extensions of $T(n)$

In this subsection, we define s-clique extensions of graphs and we will give some specific results for the s-clique extension of triangular graphs.

Recall an s-clique is a clique with s vertices, where s is a positive integer. The *s-clique extension* of a graph G with $|V(G)|$ vertices is the graph $\widetilde{G}$ obtained from G by replacing each vertex $x \in V(G)$ by a clique $\widetilde{X}$ with s vertices, satisfying $\widetilde{x} \sim \widetilde{y}$ in $\widetilde{G}$ if and only if $x \sim y$ in G, where $\widetilde{x} \in \widetilde{X}, \widetilde{y} \in \widetilde{Y}$. If $\widetilde{G}$ is an s-clique extension of G, then the adjacency matrix of $\widetilde{G}$ is $(A + I_{|V(G)|}) \otimes J_s - I_{s|V(G)|}$, where J_s is the all-ones matrix of size s and $I_{|V(G)|}$ is the identity matrix of size $|V(G)|$. In particular, if G has $t+1$ distinct eigenvalues and its spectrum is

$$\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_t^{m_t}, \tag{2.1}$$

then the spectrum of $\widetilde{G}$ is

$$\begin{aligned} &\Big\{(s(\theta_0+1)-1)^{m_0}, (s(\theta_1+1)-1)^{m_1}, \ldots, \\ &(s(\theta_t+1)-1)^{m_t}, (-1)^{(s-1)(m_0+m_1+\cdots+m_t)}\Big\}. \end{aligned} \tag{2.2}$$

Note that if the adjacency matrix A of a connected regular graph G with $|V(G)|$ vertices and valency k has four distinct eigenvalues $\{\theta_0 = k, \theta_1, \theta_2, \theta_3\}$, then A satisfies the following equation (see [7]):

$$A^3 - \left(\sum_{i=1}^{3} \theta_i\right) A^2 + \left(\sum_{1 \le i < j \le 3} \theta_i \theta_j\right) A - \theta_1\theta_2\theta_3 I = \frac{\prod_{i=1}^{3}(k-\theta_i)}{|V(G)|} J. \tag{2.3}$$

This implies that G is walk-regular, see [10].

Now we assume Γ is a cospectral graph with the s-clique extension of the triangular graph $T(n)$, where $s \ge 2$ and $n \ge 4$ are integers. Then by (2.1) and (2.2), the graph Γ has spectrum

$$\begin{aligned} &\{\theta_0^{m_0}, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\} \\ &= \Big\{(s(2n-3)-1)^1, (s(n-3)-1)^{n-1}, (-s-1)^{\frac{n^2-3n}{2}}, (-1)^{(s-1)\frac{n(n-1)}{2}}\Big\}. \end{aligned} \tag{2.4}$$

Note that Γ is regular with valency k, where $k = (s-1)+2(n-2)s = s(2n-3)-1$. Using (2.3), we obtain

$$A^3 + (3 + 4s - sn)A^2 + ((3 - n)s^2 + (8 - 2n)s + 3)A + (1 - (n - 4)s - (n - 3)s^2)I = 4s^2(2n - 3)J.$$

Therefore,

$$A^3_{xy} = \begin{cases} 2s^2n^2 - 2s^2n - 6sn - 3s^2 + 9s + 2, & \text{if } x = y, \\ 9s^2n + 2sn - 15s^2 - 8s - 3 - (3 + 4s - sn)\lambda_{xy}, & \text{if } x \sim y, \\ 8s^2n - 12s^2 - (3 + 4s - sn)\mu_{xy}, & \text{if } x \nsim y. \end{cases} \tag{2.5}$$

The following result is known as the *Hoffman bound*.

Lemma 6 (Cf. [2], Theorem 3.5.2)**.** *Let X be a k-regular graph with least eigenvalue τ. Let $\alpha(X)$ be the size of maximum coclique in X. Then*

$$\alpha(X) \leq \frac{|X|(-\tau)}{k - \tau}.$$

If equality holds, then each vertex not in a coclique of size $\alpha(X)$ has exactly $-\tau$ neighbours in it.

Applying Lemma 6 to the complement of Γ, we obtain the following lemma.

Lemma 7. *For any clique C of Γ with order c, we have*

$$c \leq s(n - 1).$$

If equality holds, then every vertex $x \in V(\Gamma) \setminus V(C)$ has exactly $2s$ neighbors in C.

3. Lines in Γ

Recall that Γ is a graph that is cospectral with the s-clique extension of the triangular graph $T(n)$, where $s \geq 2$ and $n \geq 1$ are integers. This implies that Γ is walk-regular. Now we assume that Γ is also co-edge-regular, i.e., there exist precisely $\mu = \mu(\Gamma)$ common neighbors between any two distinct nonadjacent vertices of Γ. Note that for Γ, we have $\mu = 4s$ from the spectrum of the s-clique extension of $T(n)$.

Fix a vertex, denoted by ∞ and let $\Gamma(\infty)$ be the local graph of Γ at vertex ∞. Let $V(\Gamma(\infty)) = \{x_1, x_2, \ldots, x_k\}$, where $k = s(2n - 3) - 1$. Let x_i have valency d_i inside $\Gamma(\infty)$ for $i = 1, 2, \ldots, k$. Because Γ is walk-regular, the number of closed walks through a fixed vertex ∞ of length 3 and 4 only depends on the spectrum.

This means that the number of edges in $\Gamma(\infty)$ is determined by the spectrum and as Γ is co-edge-regular, we also see that the number of walks of length 2 in $\Gamma(\infty)$ is determined by the spectrum of Γ. This implies these numbers are the same as in a local graph of the s-clique extension of $T(n)$.

Let Δ be the s-clique extension of $T(n)$. Fix a vertex u of Δ. Then there are $s-1$ vertices with valency $(s-2)+2s(n-2)$ and $2s(n-2)$ vertices with valency $s(n-2)+2(s-1)$ in the local graph of $T(n)$ with respect to a fixed vertex. Using (2.5), this implies that the sum of valencies and the sum of square of valencies of vertices in $\Gamma(\infty)$ are constant, and are given by the following equations.

$$\sum_{i=1}^{k} d_i = 2\varepsilon = 2s^2n^2 - 2s^2n - 6sn - 3s^2 + 9s + 2, \tag{3.1}$$

$$\sum_{i=1}^{k} (d_i)^2 = 2sn(s^2n^2 - 6sn - 6s^2 + 10s + 8) + 9s^3 + 3s^2 - 24s - 4, \tag{3.2}$$

where ε is the number of edges inside $\Gamma(\infty)$. By (3.1) and (3.2), we obtain

$$\sum_{i=1}^{k} (d_i - (sn-2))^2 = (s-1)s^2(n-3)^2. \tag{3.3}$$

It turns out that (3.3) is of crucial importance in proving our main result. Now we show the following lemma that will be used later.

Lemma 8. *Fix a vertex ∞ of Γ and let $\Gamma(\infty)$ be the local graph of Γ at ∞. Define $E = \{y \sim \infty \mid d_y > \frac{3}{4}s(n-1)\}$ and let $e = |E|$. Let $F = \{y \sim \infty \mid d_y \le \frac{3}{4}s(n-1)\}$ and $f = |F|$. If $n \ge 55$, then the following hold.*

(1) *$f \le 16(s-1)$.*

(2) *The subgraph of Γ induced on E is not complete.*

(3) *The subgraph of Γ induced on E does not contain a coclique of order three.*

Proof. Note that $f = k - e$. As $\frac{3}{4}s(n-1) + 1 \le \frac{3}{4}(sn-2)$, by (3.3), we obtain

$$\begin{aligned}(s-1)s^2(n-3)^2 = \sum_{y\sim\infty} (d_y - (sn-2))^2 &\ge \sum_{y\in F} \big(d_y - (sn-2)\big)^2 \\ &\ge \sum_{y\in F} \left(\frac{1}{4}(sn-2)\right)^2 \\ &= \frac{1}{16} f(sn-2)^2 \ge \frac{1}{16} f(sn-s)^2.\end{aligned} \tag{3.4}$$

So

$$f \le 16(s-1),$$

which implies $f < \frac{1}{2}(sn-2)$ if $n \geq 55$ (and $s \geq 2$). This means

$$e = k - f > sn.$$

By Lemma 7, we obtain that e is greater than the order of a maximum size clique and hence the subgraph induced on E is not complete.

Now we show that E does not contain a coclique of order three. Suppose $X \subset E$ is a coclique in $\Gamma(\infty)$ with vertices $\{x_1, x_2, x_3\}$. Define A_i $(i = 1, 2, 3)$ such that

$$A_i = \{y \sim \infty \mid y \sim x_i, y \nsim x_j \text{ for all } x_j \in X, j \neq i\} \cup \{x_i\}.$$

Since Γ is co-edge-regular, the vertices x_i and x_j $(i \neq j)$ have at most $4s-1$ common neighbours. By the inclusion-exclusion principle, we have

$$\frac{3 \times (\frac{3}{4}s(n-1)+1) - k}{3} \leq 4s - 1.$$

This gives $n < 54$. This shows the lemma. ■

A maximal clique of Γ is called a *line* if it contains more than $\frac{3}{4}s(n-1)$ vertices. We show the existence of lines of Γ in the following.

Proposition 9. *If $n \geq 48s \geq 96$, then for every vertex ∞, there are exactly two lines through ∞, say C_1 and C_2. Denote $m = |V(C_1) \cap V(C_2) \setminus \{\infty\}|$ and $\ell = k + 1 - |V(C_1) \cup V(C_2)|$. Then $m \leq 4s - 1$ and $\ell \leq 16(s-1)$.*

Proof. Fix a vertex ∞ of Γ, let $E = \{y \sim \infty \mid d_y > \frac{3}{4}s(n-1)\}$. By Lemma 8, a maximum coclique in E has order two as $n \geq 48s \geq 55$. Let x_1, x_2 be distinct nonadjacent vertices in E and let $y \in E$. Then y has at least one neighbour in $\{x_1, x_2\}$.

Let $A_i = \{y \in E \mid y \sim x_i, y \nsim x_j \text{ for } j = 1, 2, j \neq i\}$ for $i = 1, 2$. Then the subgraph induced on A_i is complete for $i = 1, 2$. Let C_i be a maximal clique containing the vertex set $\{\infty\} \cup A_i$ for $i = 1, 2$. Note that $C_1 \neq C_2$ as $x_1 \nsim x_2$. Let $M = V(C_1) \cap V(C_2) \setminus \{\infty\}$ and $L = V(\Gamma(\infty)) \setminus (V(C_1) \cup V(C_2))$. Let $m = |M|$ and $\ell = |L|$. By the co-edge-regularity of Γ, we have $m \leq 4s - 1$. Let $F = \{y \sim \infty \mid d_y \leq \frac{3}{4}s(n-1)\}$ and $f = |F|$. We have, by Lemma 8, that $f \leq 16(s-1)$.

Suppose $x \in E \setminus (V(C_1) \cup V(C_2))$. Then x has at least $(\frac{3}{4}s(n-1) - (4s - 2) - 16(s-1))/2$ neighbours in at least one of C_1 and C_2. If $n \geq 48s \geq 96$, then this number is at least $4s$, which is a contradiction. Hence $E \subseteq V(C_1) \cup V(C_2)$. So, $L \subseteq F$ and hence $\ell \leq f \leq 16(s-1)$ by Lemma 8. This shows that $|V(C_1)| + |V(C_2)| \geq k - \ell \geq k - 16(s-1)$. Assume $|V(C_1)| \geq |V(C_2)|$, then we see that

$$|V(C_2)| \geq k - 16(s-1) - s(n-1) > \frac{3}{4}s(n-1),$$

as $n \geq 48s \geq 96$. This gives that there are exactly two lines through ∞. ■

Now we prove the following property for lines through a vertex.

Lemma 10. *Fix a vertex* ∞ *of* Γ *and let* C_1 *and* C_2 *be the two lines through* ∞ *with respective orders* c_1 *and* c_2. *Let* $L = V(\Gamma(\infty)) \setminus (V(C_1) \cup V(C_2))$ *and* $M = V(C_1) \cap V(C_2) \setminus \{\infty\}$, *and* $\ell = |L|$, $m = |M| \geq 0$. *If* $n \geq 48s \geq 96$, *then* $\ell + m = s - 1$ *and*

$$s(n-3) + 1 \leq c_i \leq s(n-1)$$

for $i = 1, 2$.

Proof. Let $Q = V(C_1)\Delta V(C_2)$, where Δ means "symmetric difference". Then, by Lemma 7, $|Q| \leq |V(C_1)| + |V(C_2)| \leq 2s(n-1)$.

Note that Q is the complement of $L \cup M$ inside $V(\Gamma(\infty))$.

For $y \in M$, we have

$$2sn - 19s \leq k - 1 - \ell \leq d_y \leq k - 1 = 2sn - 3s - 2, \tag{3.5}$$

by Proposition 9.

Now let $y \in L$. Then y has at least $4s - 1$ neighbors in each of C_1 and C_2. Hence, by Proposition 9, we obtain

$$d_y \leq 2 \times (4s-1) + \ell - 1 \leq 2(4s-1) + 16(s-1) - 1 \leq 24s. \tag{3.6}$$

By (3.3), we obtain

$$\begin{aligned}(s-1)s^2(n-3)^2 &= \sum_{y\sim\infty}(d_y - (sn-2))^2 \\ &\geq \sum_{y\in L}(d_y - (sn-2))^2 + \sum_{y\in M}(d_y - (sn-2))^2 \\ &\geq \ell((sn-s) - 24s)^2 + m((2sn-19s) - sn)^2 \\ &= \ell s^2(n-25)^2 + ms^2(n-19)^2 \geq (\ell+m)s^2(n-25)^2.\end{aligned} \tag{3.7}$$

So

$$\ell + m \leq \frac{(s-1)(n-3)^2}{(n-25)^2} < s$$

if $n \geq 48s$. Hence

$$\ell + m \leq s - 1. \tag{3.8}$$

This gives for $y \in L \cup M$, using (3.5), (3.6) and $l \leq s-1$, that

$$d_y - (sn-2) \leq k - 1 - (sn-2) = sn - 3s.$$

Note that by (3.8),

$$\begin{aligned} s(n-1) \geq |V(C_j)| &\geq 1 + k - s(n-1) - l \\ &\geq 2sn - 3s - s(n-1) - (s-1) = s(n-3) + 1 \end{aligned} \tag{3.9}$$

for $j = 1, 2$.

For $y \in V(\Gamma(\infty)) \setminus (L \cup M)$, we obtain

$$sn - 4s \leq |V(C_2)| - m - 2 \leq d_y \leq |V(C_2)| - 1 + 4s - 1 + \ell \leq sn + 4s - 3.$$

Hence $|d_y - (sn-2)| \leq 4s$.

Now (3.3) gives us

$$\begin{aligned} (s-1)s^2(n-3)^2 &= \sum_{y \sim \infty} (d_y - (sn-2))^2 \\ &\leq \sum_{y \in L \cup M} (d_y - (sn-2))^2 + \sum_{y \in Q} (d_y - (sn-2))^2 \\ &\leq (\ell + m)s^2 n^2 + 2s(n-1)(4s)^2. \end{aligned} \tag{3.10}$$

So

$$\ell + m \geq \frac{(s-1)(n-3)^2 - 32s(n-1)}{n^2} > s - 2,$$

if $n \geq 48s \geq 96$. This implies $\ell + m = s - 1$. This shows the lemma. ∎

We obtain the following lemma immediately.

Lemma 11. *Fix a vertex ∞ of Γ and let C_1 and C_2 be the two lines through ∞ with respective orders c_1 and c_2. Assume $m = |V(C_1) \cap V(C_2) \setminus \{\infty\}|$. If $n \geq 48s$, then $c_1 + c_2 = 2s(n-2) + 2(m+1)$.*

Proof. Let $\ell = |V(\Gamma(\infty)) \setminus (V(C_1) \cup V(C_2))|$. Then we have

$$(c_1 - m - 1) + (c_2 - m - 1) + m + \ell = k = 2sn - 3s - 1.$$

If $n \geq 48s$, then we have $\ell + m = s - 1$ by Lemma 10, hence $c_1 + c_2 = 2s(n-2) + 2(m+1)$. ∎

In the next two sections, we will follow the method as used in Hayat *et al.* [4].

4. The Order of Lines

In this section, we will show the following lemma on the order of lines.

Lemma 12. *Let $s \geq 2$ and $n \geq 1$ be integers. Let Γ be a co-edge-regular graph that is cospectral with the s-clique extension of the triangular graph $T(n)$. Let q_i be the number of lines with order $s(n-3)+i$ for $i = 1, \dots, 2s$ and $\delta = \sum_{i=1}^{2s} q_i$ be the number of lines in Γ. Assume $n \geq 48s$. Then*

$$\sum_{i=1}^{2s}(s(n-3)+i)q_i = sn(n-1) \tag{4.1}$$

holds, and the number δ satisfies

$$n \leq \delta \leq n+2. \tag{4.2}$$

If $\delta = n$, then $q_i = 0$ for all $i < 2s$, and $q_{2s} = n$.

Proof. Assume $n \geq 48s$. By Proposition 9, any vertex of Γ lies on exactly two lines. Now consider the set

$$W = \{(x, C) |\ x \in V(C),\ \text{where } C \text{ is a line}\}.$$

Then, by double counting, the cardinality of the set W, we see (4.1). Moreover, we see that

$$\delta = \sum_{i=1}^{2s} q_i < \sum_{i=1}^{2s} \frac{s(n-3)+i}{s(n-3)} q_i = n+2+\frac{6}{n-3}.$$

Thus, if $n > 10$, we obtain

$$\delta \leq n+2.$$

On the other hand, we have

$$\delta = \sum_{i=1}^{2s} q_i \geq \sum_{i=1}^{2s} \frac{s(n-3)+i}{s(n-1)} q_i = n.$$

This shows $\delta \geq n$, and $\delta = n$ implies that all lines have order $s(n-1)$, which means $q_i \neq 0$ if and only if $i = 2s$. This completes the proof. ■

5. THE NEIGHBORHOOD OF A LINE

In this section we will show the following proposition.

Proposition 13. *Let Γ be a co-edge-regular graph that is cospectral with the s-clique extension of the triangular graph $T(n)$, where $s \geq 2, n \geq 1$ are integers. If $n \geq 48s$, then Γ contains exactly n lines.*

Proof. In Lemma 12, we have seen that the number δ of lines satisfies $n \le \delta \le n+2$. Now we assume that $n+1 \le \delta \le n+2$, in order to obtain a contradiction. Let q_i be the number of lines of order $s(n-3)+i$ in Γ, where $i = 1, \ldots, 2s$. Let h be minimal such that $q_h \neq 0$. Then clearly, $1 \le h \le 2s$. Fix a line C with exactly $s(n-3)+h$ vertices. Let q_i' be the number of lines C' with $s(n-3)+i$ vertices that intersect C in at least one vertex. So $q_i \ge q_i'$. By Lemma 11, we obtain

$$|V(C) \cap V(C')| = \frac{h+i-2s}{2}. \tag{5.1}$$

By Proposition 9, every vertex lies on exactly two lines, and hence we obtain

$$\sum_{i=1}^{2s} q_i \binom{\frac{h+i-2s}{2}}{2} \ge \sum_{i=1}^{2s} q_i' \binom{\frac{h+i-2s}{2}}{2} = s(n-3)+h. \tag{5.2}$$

Now multiply (5.2) by 2 and subtract (4.1) from obtained equation, we find

$$\delta(h+s(1-n)) = \sum_{i=1}^{2s} q_i(h+s(1-n)) \quad \ge s(-n^2+3n-6)+2h \tag{5.3}$$

as $\delta = \sum_{i=1}^{2s} q_i$. This gives

$$h(\delta-2) \ge 2s(n-3) + (\delta-n)s(n-1).$$

As $n+1 \le \delta \le n+2$, we see

$$hn \ge h(\delta-2) \ge 2s(n-3)+(\delta-n)s(n-1) \ge 2s(n-3)+s(n-1) = 3sn-7s. \tag{5.4}$$

Since $n \ge 48s$, (5.4) implies that $h \ge 3s$. This contradicts to $h \le 2s$. This completes the proof. ∎

6. PROOF OF THE MAIN RESULT

In this section we show our main result, Theorem 1.

***Proof of Theorem* 1.** Assume $n \ge 48s$. By Propositions 9 and 13 and Lemma 12, we find that there are exactly n lines, each of order $s(n-1)$, and every vertex x in Γ lies on exactly two lines. Moreover, by Lemma 11, the two lines through any vertex x have exactly s vertices in common, and every neighbor of x lies in one of the two lines through x. Now consider the following equivalence relation $\mathcal{R}$ on the vertex set $V(\Gamma)$: $x\mathcal{R}x'$ if and only if $\{x\} \cup N_\Gamma(x) = \{x'\} \cup N_\Gamma(x')$, where $x, x' \in V(\Gamma)$.

Every equivalence class under $\mathcal{R}$ contains s vertices and it is the intersection of two lines. Let us define the graph $\hat{\Gamma}$ whose vertices are the equivalent classes and two classes, say S_1 and S_2, are adjacent in $\hat{\Gamma}$ if and only if any vertex in S_1 is adjacent to any vertex in S_2. Then $\hat{\Gamma}$ is a regular graph with valency $2n-4$, and Γ is the s-clique extension of $\hat{\Gamma}$. Note that the spectrum of $\hat{\Gamma}$ is equal to

$$\left\{(2n-4)^1, (n-4)^{n-1}, (-2)^{\frac{n^2-3n}{2}}\right\},$$

by the relation of the spectra of Γ and $\hat{\Gamma}$, see (2.1) and (2.2). Since $\hat{\Gamma}$ is a connected regular graph with valency $2n-4$, and it has exactly three distinct eigenvalues, it follows that $\hat{\Gamma}$ is a strongly regular graph with parameters $\left(\binom{n}{2}, 2n-4, n-2, 4\right)$.

As proved in [3], the triangular graphs are determined by the spectrum except when $n = 8$. Since we assume that n is large enough, the graph $\hat{\Gamma}$ is the triangular graph $T(n)$. This completes the proof. ∎

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