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FORBIDDEN SUBGRAPHS FOR COLLAPSIBLE GRAPHS AND SUPEREULERIAN GRAPHS

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Abstract

In this paper, we completely characterize the connected forbidden subgraphs and pairs of connected forbidden subgraphs that force a 2-edge-connected (2-connected) graph to be collapsible. In addition, the characterization of pairs of connected forbidden subgraphs that imply a 2-edge-connected graph of minimum degree at least three is supereulerian will be considered. We have given all possible forbidden pairs. In particular, we prove that every 2-edge-connected noncollapsible (or nonsupereulerian) graph of minimum degree at least three is Z_3-free if and only if it is K_3-free, where Z_i is a graph obtained by identifying a vertex of a K_3 with an end-vertex of a P_{i+1}.

Keywords: forbidden subgraph, supereulerian, collapsible.

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1. Introduction

For the notation or terminology not defined here, see [1]. A graph is called *trivial* if it has only one vertex, *nontrivial* otherwise. All graphs involved in the conclusion considered in this paper are simple graphs. Let G be a connected graph. We use $\kappa(G)$, $\kappa'(G)$ and $g(G)$ to denote the *connectivity*, *edge-connectivity* and *girth* of G, respectively. Let u be a vertex of G and S be a subset of $V(G)$ (or $E(G)$). The *induced subgraph* of G is denoted by $G[S]$. We use $N_G(u)$ to denote the neighborhood and $d_G(u)$ to denote the degree of u. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and maximum degree of G, respectively. The neighbors of S in G is defined as $N_G(S) = \bigcup_{x\in S} N_G(x)\backslash S$ and $N_G[S] = N_G(S)\cup S$. Define $N_T(S) = N_G(S) \cap T$ for $T \subseteq V(G)$. Let $V_i(G) = \{u \in V(G) : d_G(u) = i\}$ and $V_{\geq i}(G) = \{u \in V(G) : d_G(u) \geq i\}$. If F, G are graphs, we write $F \subseteq G$ if F is a subgraph of G and $F \cong G$ if F and G are isomorphic. For $x, y \in V(G)$ and $H \subseteq G$, let $E(u, H) = \{uv \in E(G) : v \in V(H)\}$, $d_G(x, y) = |E(P(x, y))|$, where $P(x, y)$ is the shortest path between x and y, $d_G(x, H) = \min\{d_G(x, y) : y \in V(H)\}$ and $N^i(H) = \{x : d_G(x, H) = i\}$. Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be $\mathcal{H}$-free if G does not contain H as an induced subgraph for all $H \in \mathcal{H}$, and we call each graph H of $\mathcal{H}$ a *forbidden subgraph*. In particular, if $\{H\} = \mathcal{H}$, then we simply say that G is H-free and we call $\mathcal{H}$ a *forbidden pair* if $|\mathcal{H}|$ =2. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for every graph H_2 in $\mathcal{H}_2$, there is a graph H_1 in $\mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. By the definition of the relation "$\preceq$", if $\mathcal{H}_1 \preceq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

Let K_n denote the complete graph of order n, and $K_{m,n}$ denote the complete bipartite graph with partition sets of size m and n, and P_n denote the path of order n, and C_n denote the cycle of order n. Use $T_{i,j,k}$ to denote the tree of three paths of length i, j, k with one common vertex. The graphs $Z_i, B_{i,j}, N_{i,j,k}, H_1, M_1, M_2$ are depicted in Figure 1.

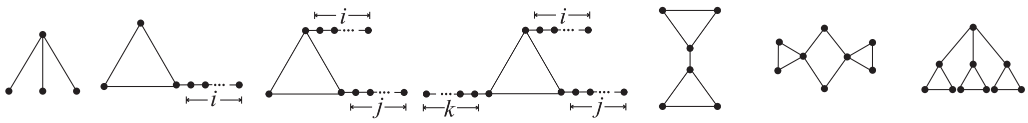

Figure 1. The common induced subgraphs.

A graph is *supereulerian* if it has a connected spanning subgraph such that each vertex has even degree. Lv and Xiong characterized all forbidden pairs for a 2-connected graph to be supereulerian.

Theorem 1 (Lv and Xiong [9, 10]). *Let R, S ($\neq P_3$) be connected graphs of order at least* 3 *and let G be a* 2*-connected graph of order at least* 7*. Then $\{R, S\}$-free graph G is supereulerian if and only if $\{R, S\} \preceq \{K_{1,4}, P_5\}, \{K_{1,3}, N_{1,1,3}\}, \{K_{1,3}, Z_4\}, \{K_{1,3}, P_7\}$, $\{C_4, P_5\}$.*

Afterwards, Čada *et al.* [3] revealed how the forbidden subgraphs change when the minimum degree was increased slightly. They characterized two forbidden subgraphs forcing a 2-connected $K_{1,3}$-free graph G with $\delta(G) \geq 3$ excepting two families of counterexamples to be supereulerian. We may restate their results as follows. In fact, they gave more general results with some exceptions.

Theorem 2 (Čada *et al.* [3]). *If a* 2*-connected* $K_{1,3}$*-free graph* G *with* $\delta(G) \geq 3$ *is* R*-free for* $R \in \{N_{2,2,4}, Z_8\}$*, respectively, then* G *is supereulerian.*

Motivated by these two results above, in this paper, we shall consider the forbidden pairs that force a 2-edge-connected graph of minimum degree at least three to be supereulerian.

Theorem 3. *Let* G *be a* 2*-edge-connected graph with* $\delta(G) \geq 3$ *such that it satisfies one of the following.*

(1) G *is* $\{T_{2,2,1}, S\}$*-free for any* $S \in \{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8\}$*,*

(2) G *is* $\{P_7, T_{2,2,2}\}$*-free,*

(3) G *is* $\{P_6, S\}$*-free other than the Petersen graph for any* $S \in \{M_1, M_2\}$*,*

(4) G *is* $\{T_{2,1,1}, H_1\}$*-free.*

Then G *is supereulerian.*

Comparing Theorems 2 and 3(1), we know that if we keep $S \in \{N_{2,2,4}, Z_8\}$, then we may extend the other one of the pair, the $K_{1,3}$ (i.e., $T_{1,1,1}$), to $T_{2,2,1}$ (a little wider). For a graph G, let $O(G)$ denote the set of odd degree vertices in G. In [4], Catlin defined collapsible graphs. Given a subset $R \subseteq V(G)$ with $|R|$ even, a subgraph Γ of G is an *R-subgraph* if both $O(\Gamma) = R$ and $G - E(\Gamma)$ are connected. A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph. Catlin [4] shows that every collapsible graph is supereulerian. We then study a characterization of connected forbidden graphs to assure collapsibility.

Theorem 4. *Let* $r \geq 4$ *be an integer and* $\mathcal{H}$ *be a connected forbidden pair. Then*

(1) *every* 2*-edge-connected* $\mathcal{H}$*-free graph* G *of order at least* $r+3$ *implies that it is collapsible if and only if* $\mathcal{H} \preceq \{K_{1,3}, P_5\}$*,* $\{K_{1,r}, P_4\}$ *or* $\{C_4, P_5\}$*,*

(2) *every* 2*-connected* $\mathcal{H}$*-free graph* G *of order at least* $r+3$ *implies that it is collapsible if and only if* $\mathcal{H} \preceq \left\{K_{2,\lceil r/2 \rceil}, P_5\right\}$*.*

Theorem 5. *Let* H *be a connected graph of order at least* 3*. Then* H*-free* 2*-edge-connected graph* G *with* $\delta(G) \geq 3$ *implies* G *is collapsible* (*supereulerian*) *if and only if* H *is an induced subgraph of* P_5*.*

The proofs of Theorems 4, 5 and Theorem 3 are placed in Sections 2 and 3, respectively. In the last section, we will exhibit a theorem to show the forbidden pairs in Theorem 3 except the pair $\{P_7, T_{2,2,2}\}$ are sharp and leave a conjecture.

2. Forbidden Subgraphs Guaranteeing a 2-Edge-Connected Graph To Be Collapsible

For a subset $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. Note that the edges in $E(G/X)$ can be regarded as edges in $E(G)$. If H is a subgraph, then we use G/H for $G/E(H)$. Note that by this definition, if H is a connected subgraph of G, then $G/H = G/G[V(H)]$.

Catlin showed in [4] that every vertex of G lies in an unique maximal collapsible subgraph of G and C_2, K_3 are collapsible. The *reduction* of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is *reduced* if it is the reduction of some graph.

The following result will be used to verify whether a graph is supereulerian.

Theorem 6 (Catlin [4]). *Let G be a connected graph and let H be a collapsible subgraph of G and let G' be the reduction of G. Then each of the following holds.*

(a) *G is collapsible (supereulerian) if and only if G/H is collapsible (supereulerian). In particular, G is collapsible if and only if G' is K_1.*

(b) *G is supereulerian if and only if G' is supereulerian.*

For two disjoint subsets V_1, V_2 and a 4-cycle $C_4 = x_1x_2x_3x_4x_1$ of graph G, define $G/\pi(V_1, V_2)$ to be the graph obtained from $G - E(G[V_1 \cup V_2])$ by identifying V_1 to form a vertex v_1, by identifying V_2 to form a vertex v_2, and by adding a new edge v_1v_2 and define $G/\pi(C_4) = G/\pi(\{x_1, x_3\}, \{x_2, x_4\})$.

Theorem 7 (Catlin [5]). *For the graphs G and $G/\pi(C_4)$ defined above. If $G/\pi(C_4)$ is collapsible, then G is collapsible.*

In [8], the authors give a method to verify whether a subgraph of G is collapsible. They construct a C-subpartition (X_1, X_2) of G starting with a 4-cycle $x_1x_2x_3x_4x_1 \subseteq G$ as follows.

1. $X_1 := \{x_1, x_3\}$, $X_2 := \{x_2, x_4\}$, $\{i, j\} = \{1, 2\}$.

2. **While** $u \in N_G(X_1 \cup X_2) \neq \emptyset$, $N_G(X_1) \cap N_G(X_2) = \emptyset$ and $N_G(u) \cap N_G[X_1 \cup X_2] \neq \emptyset$ **do**

$\{X_i := X_i \cup \{u\}, X_j := X_j$, if $|E(u, X_i)| \geq 2$; $X_i := X_i \cup (N_G(X_i) \cap N_G[u])$, $X_j := X_j$, else if $N_G(X_i) \cap N_G[u] \neq \emptyset$; $X_i := X_i \cup (N_G(X_j) \cap N_G(u)), X_j := X_j \cup \{u\}$, else. $\}$.

Although the C-subpartition of G is not unique, the following result is true and would play an important role in the proofs in Section 3.

Lemma 8 (Liu *et al.* [8]). *For a C-subpartition (X_1, X_2) of a graph G and any nonempty set $X_{12} \subseteq N_G(X_1) \cap N_G(X_2)$, $G[X_1 \cup X_2 \cup X_{12}]$ is collapsible.*

Before presenting the proofs of Theorems 4 and 5, we need some preparations.

Lemma 9. *Let G be a 2-edge-connected noncollapsible graph which has a maximal nontrivial collapsible subgraph H. Then*

(1) *$|E(u, H)| = 1$ for any $u \in N_G(H)$,*

(2) *$N_G(H)$ is an independent set of G,*

(3) *if G is P_5-free, then $|N_G(H') \cap V(H)| = 1$ for any component H' of $G - H$.*

Proof. (1) Let $G^* = G/H$ and $v_H \in V(G^*)$ be the contraction image of H. If $|E(u, H)| \geq 2$ for some $u \in N_G(H)$, then G^* has a collapsible subgraph C_2 containing vertices v_H and u. Whence $G[V(H) \cup \{u\}]$ is collapsible by Theorem 6(a), contradicting the maximality of H.

(2) Let $G^* = G/H$ and $v_H \in V(G^*)$ be the contraction image of H. If there is an edge $uv \in E(G[N_G(H)])$, then G^* has a collapsible subgraph C_3 containing vertices v_H, u, v. Whence $G[V(H) \cup \{u, v\}]$ is collapsible by Theorem 6(a), contradicting the maximality of H.

(3) By contradiction, assume that $G - H$ has a component H' with $|N_G(H') \cap V(H)| \geq 2$. Then H' has an induced path $P(u, v)$ such that $N_G(H) \cap V(P(u, v)) = \{u, v\}$ and there are two vertices $u' \in N_G(u) \cap V(H)$, $v' \in N_G(v) \cap V(H)$. By (1) and (2), $|E(P(u, v))| \geq 2$. Note that $|V(H)| \geq 3$. Then there is a vertex $v'' \in N_H(v)$ such that $uP(u, v)vv'v''$ is an induced path of order at least 5. ■

The following results imply the sufficiency of Theorem 4(1).

Theorem 10. *Every 2-edge-connected graph G is collapsible if it satisfies one of the following.*

(1) *G is P_4-free other than $K_{2,t}$ for any $t \geq 2$,*

(2) *G is $\{K_{1,3}, P_5\}$-free that is neither C_4 nor C_5,*

(3) *G is $\{C_4, P_5\}$-free other than C_5.*

Proof. By contradiction, assume that G is not collapsible. Choose a collapsible subgraph H of G such that $|V(H)|$ is maximized (possibly H is trivial). Then $N_G(H) \neq \emptyset$.

(1) We claim that G is reduced. Suppose otherwise. Then $|V(H)| \geq 3$. By Lemma 9(1), (2) and since $\kappa'(G) \geq 2$, there is an edge h_1h_2 such that $E(h_1, H) = \emptyset$ and $|E(h_2, H)| = 1$. Then $G[\{h_1, h_2, h_3, h_4\}] \cong P_4$ for some $h_3 \in N_G(h_2) \cap V(H)$ and $h_4 \in N_H(h_3)$, a contradiction. Choose a vertex $u \in V(G)$ such that $d(u) = \Delta(G) \geq 2$. Let $N_G(u) = \{x_1, \ldots, x_{\Delta(G)}\}$. Since $N_G(u)$ is an independent set of G and $\kappa'(G) \geq 2$, there is a vertex $v \in N_G(x_1) \backslash \{u\}$. Then $vx_i \in E(G)$ for $i \in \{2, \ldots, \Delta(G)\}$ since $G[\{v, x_1, u, x_i\}] \not\cong P_4$. If there is a vertex $w \in N_G(\{x_1, \ldots, x_{\Delta(G)}\}) \backslash \{u, v\}$, by symmetry, then $wx_i \in E(G)$ and $\Delta(G) \geq d_G(x_i) \geq 3$ for $i \in \{1, \ldots, \Delta(G)\}$, and hence $G[\{u, v, w, x_1, \ldots, x_{\Delta(G)}] \cong K_{3,\Delta(G)}$ is collapsible, a contradiction. This implies that $N_G(\{x_1, \ldots, x_{\Delta(G)}\}) = \{u, v\}$, and hence $G \cong K_{2,\Delta(G)}$, a contradiction.

(2) Then G is reduced; for otherwise, there are two edges $uw, vw \in E(G)$ such that $w \in V(H)$ and $u, v \in N_G(H)$ by Lemma 9(3) and then $G[\{u, v, w, w'\}] \cong$

$K_{1,3}$ for some $w' \in N_H(w)$, a contradiction. Furthermore, $\Delta(G) = 2$ since G is $\{C_3, K_{1,3}\}$-free. Hence $G \in \{C_4, C_5\}$ since G is P_5-free, a contradiction.

(3) If G is reduced, then $g(G) = 5$ since G is $\{C_4, P_5\}$-free and then $G = C_5$ since G is P_5-free, a contradiction. Thus $|V(H)| \geq 3$ and there is a vertex $v \in V(H)$ and two vertices $u_1, u_2 \in N_{G-H}(v)$ by Lemma 9(3). Then u_1, u_2 have no common neighbor in $G - H$ since G is C_4-free. Let $w_1 \in N_{G-H}(u_1)$ and $w_2 \in N_{G-H}(u_2)$. Then $E(\{w_1, w_2\}, H) = \emptyset$ and hence $w_1w_2 \in E(G)$ since $G[\{w_1, u_1, v, u_2, w_2\}] \ncong P_5$. However, $G[\{w_1, w_2, u_2, v, v'\}] \cong P_5$ for some $v' \in N_H(v)$, a contradiction. ■

Let t_1, t_2 be two positive integers and let u_1, v_1 be two nonadjacent vertices of degree t_1 in K_{2,t_1} and let u_2, v_2 be two nonadjacent vertices of degree t_2 in K_{2,t_2}. Define S_{t_1,t_2} be the graph obtained from K_{2,t_1} and K_{2,t_2} by identifying v_1 and v_2, and by adding a new edge u_1u_2. Let $K^-_{3,3} = K_{3,3} - e$ for any $e \in E(K_{3,3})$. Catlin shows that $K^-_{3,3}$ is collapsible. The following result implies the sufficiency of Theorem 4(2).

Theorem 11. *Every 2-connected P_5-free graph G is either collapsible or $G \in \{K_{2,t} : t \geq 2\} \cup \{S_{t_1,t_2} : t_2 \geq t_1 \geq 1\}$.*

Proof. Assume that G is not collapsible. Then G is reduced. If not, then by Lemma 9(3), G has a maximal non-trivial collapsible subgraph H such that $G-H$ has a component H' with $N_G(H') \cap V(H) = \{u_0\}$ for some $u_0 \in V(H)$, and hence u_0 is a cut vertex of G, a contradiction. Then $g(G) \geq 4$ since K_3 is collapsible. If $\Delta(G) = 2$, then $G \in \{C_4, C_5\}$ since G is P_5-free. Therefore, assume that $\Delta(G) \geq 3$. Let $u \in V(G)$ with $d(u) = \Delta(G)$ and $V_i = N^i(u)$. Then $|V_1| \geq 3$ and $E(G[V_1]) = \emptyset$. If $V_3 \neq \emptyset$, then G has an induced path $uu_1u_2u_3$ such that $u_i \in V_i$. For any $u' \in V_1 \backslash \{u_1\}$, by the definition of V_i, $u'u_3 \notin E(G)$. Then $u_2u' \in E(G)$ since $G[\{u, u_1, u_2, u_3, u'\}] \ncong P_5$. Since $\kappa(G) \geq 2$, $|V_2| \geq 2$ and there is a vertex $u'_2 \in V_2$ such that $E(u'_2, V_3) \neq \emptyset$. By symmetry, $u'_2u' \in E(G)$ for any $u' \in V_1$, and so $K^-_{3,3} \subseteq G[\{u, u_2, u'_2\} \cup V_1]$ is collapsible, a contradiction.

Then $V_3 = \emptyset$, and let $t = \Delta(G)$. If $|V_2| = 1$, then $G \cong K_{2,t}$. So assume that $|V_2| \geq 2$. Let $V_1 = \{u_1, \ldots, u_t\}$. Since $\kappa(G) \geq 2$, there are two vertices $v_1, v_2 \in V_2$ such that $u_1v_1, u_2v_2 \in E(G)$. If $E(G[V_2]) = \emptyset$, then $v_1v_2 \notin E(G)$. Since $G[\{v_1, u_1, u, u_2, v_2\}] \ncong P_5$, $\{u_1v_2, u_2v_1\} \cap E(G) \neq \emptyset$. By symmetry, assume that $u_1v_2 \in E(G)$. Then $v_1u_2 \notin E(G)$. Suppose otherwise. Since $K^-_{3,3} \nsubseteq G[\{u_i, u, u_1, u_2, v_1, v_2\}]$ for any $i \in \{3, \ldots, t\}$, $E(u_i, \{v_1, v_2\}) = \emptyset$, and so u_i has a neighbor v_i in V_2. Since $G[\{v_i, u_i, u, u_1, v_1\}] \ncong P_5$, $u_1v_i \in E(G)$. Then $d_G(u_1) \geq t + 1$, a contradiction. So $v_1u_j \in E(G)$ for some $j \in \{3, \ldots, t\}$ and either $G[\{v_1, u_j, u, u_2, v_2\}] \cong P_5$ if $u_jv_2 \notin E(G)$ or $K^-_{3,3} \subseteq G[\{v_1, u_j, u, u_2, v_2\}]$ if $u_jv_2 \in E(G)$, a contradiction.

Hence $E(G[V_2]) \neq \emptyset$. Assume that $v_1v_2 \in E(G)$. Since $g(G) \geq 4$, $u_1v_2, u_2v_1 \notin E(G)$. Then for $i \in \{3, \ldots, t\}$, $E(u_i, \{v_1, v_2\}) \neq \emptyset$ since $G[\{u_i, u, u_1, v_1, v_2\}] \ncong P_5$.

Without loss of generality, assume that $\{v_1u_1, \ldots v_1u_{t_1}, v_2u_{t_1+1}, \ldots, v_2u_{t_1+t_2}\} \subseteq E(G)$ for some integers t_1, t_2 with $t_1 + t_2 = t$. If $|V_2| \geq 3$, then there is a vertex $v_3 \in V_2$ such that $E(v_3, V_1) \neq \emptyset$. Furthermore, $E(v_3, \{v_1, v_2\}) \neq \emptyset$. Suppose otherwise. Assume that $v_3u_i \in E(G)$ for some $i \in \{1, \ldots, t_1\}$. Then for any $j \in \{t_1 + 1, \ldots, t\}$ and $k \in \{1, \ldots, t_1\}\backslash\{i\}$, $v_3u_j \in E(G)$ since $G[\{v_3, u_i, v_1, v_2, u_j\}] \ncong P_5$, and hence $v_3u_k \in E(G)$ since $G[\{u_k, v_1, v_2, u_j, v_3\}] \ncong P_5$. Then either $K^-_{3,3} \subseteq G[\{u, u_1, u_2, u_3, v_1, v_3\}]$ if $t_1 \geq 2$ or $K^-_{3,3} \subseteq G[\{u, u_1, u_2, u_3, v_2, v_3\}]$ if $t_1 = 1$, a contradiction. By symmetry, assume that $v_2v_3 \in E(G)$. Since $G[\{v_3, v_2, v_1, u_i, u\}] \ncong P_5$ for any $i \in \{1, \ldots, t_1\}$, $\{v_3u_1, \ldots, v_3u_{t_1}\} \subseteq E(G)$. Then $t_1 \geq 2$, since otherwise, $d_G(v_2) \geq t_2 + 2 > t_1 + t_2$, a contradiction. Then there is a C-subpartition $(X_1, X_2) = (\{u, v_1, v_2, u_3, \ldots, u_t\}, \{u_1, u_2\})$ with $v_3 \in N_G(X_1) \cap N_G(X_2)$. By Lemma 8, $G[X_1 \cup X_2 \cup \{v_3\}]$ is collapsible, a contradiction. Therefore, $|V_2| = 2$ and $G = S_{t_1,t_2}$. ■

Corollary 12. *Every 2-connected P_5-free graph G of order at least $\Delta(G) + 4$ is collapsible.*

We construct some graphs as follows. The graph L_1 is obtained from a complete graph K_n and a path $x_1x_2x_3$ by adding the edges x_1y_1, x_3y_3 for some $y_1, y_3 \in V(K_n)$. The graph L_2 is obtained from a complete graph K_n and a path $x_1x_2x_3$ by adding the edges x_1y_1, x_3y_1 for some $y_1 \in V(K_n)$. Since the reduction of L_1 and L_2 are isomorphic to C_4 which is noncollapsible, L_1 and L_2 are noncollapsible.

***Proof of Theorem* 4.** By Theorems 10 and 11, the sufficiency clearly holds. It remains to show the necessity. Let $\mathcal{H} = \{R, S\}$. Note that $K_{2,t}$ $(t \geq 2)$, C_k $(k \geq 4)$ and S_{t_1,t_2} $(t_2 \geq t_1 \geq 1)$ are noncollapsible.

(1) Since $L_1, L_2, K_{2,t}, C_k$ are 2-edge-connected, each graph contains at least one of R, S as an induced subgraph. Without loss of generality, assume that $K_{2,t}$ contains R as an induced subgraph. If R contains cycle C_4 as a subgraph, note that L_1 and C_k are C_4-free and their maximal common induced subgraph is P_5, then $S \subseteq P_5$. On the other hand, L_2 is $\{P_5, K_{2,3}\}$-free, then $R \subseteq C_4$ and $\{R, S\} \preceq \{C_4, P_5\}$.

If R contains $K_{1,3}$ as a subgraph, note that L_1 and C_k are $K_{1,3}$-free, then $S \subseteq P_5$. On the other hand, L_2 is $\{P_5, K_{1,4}, T_{2,1,1}\}$-free, then $\{R, S\} \preceq \{K_{1,3}, P_5\}$.

If R contains $K_{1,4}$ as a subgraph, then $S \subseteq P_5$. On the other hand, L_2 is $\{P_5, K_{1,4}\}$-free, and then $S \subseteq P_4$. Note that $K_{2,t}$ is $\{K_{1,r}, P_4\}$-free for $r \geq |V(K_{2,t})| - 3$, $\{R, S\} \preceq \{K_{1,|V(K_{2,t})|-2}, P_4\}$.

(2) Note that $L_1, K_{2,k}, C_k, S_{t_1,t_2}$ are 2-connected. Without loss of generality, assume that L_1 contains R as an induced subgraph. If R contains C_5 as a subgraph, then $K_{2,t}$ and C_k are C_5-free and their maximal common induced subgraph are P_3, a contradiction. Thus $R \subseteq P_5$. Since S_{t_1,t_2} is P_5-free

and there maximal common induced subgraph is $K_{2,\lfloor(|V(S_{t_1,t_2})|-3)/2\rfloor}$, $\{R,S\} \preceq \{K_{2,\lfloor(|V(S_{t_1,t_2})|-3)/2\rfloor}, P_5\}$. ■

Theorem 13 (Lai [6]). *If every edge of a* 2*-connected graph* G *lies in a cycle of length at most* 4 *in* G *and* $\delta(G) \geq 3$*, then* G *is collapsible.*

Define kK_1 be an empty graph with k vertices.

Theorem 14. *Every* 2*-edge-connected* Z_3*-free graph* G *with* $\delta(G) \geq 3$ *is either collapsible or* K_3*-free.*

Proof. Assume that G has a triangle K. Choose an induced collapsible subgraph H containing K with $|V(H)|$ maximized. If $H = G$, then G is collapsible. We then assume that $V(G)\backslash V(H) \neq \emptyset$. Let $G^* = G/H$ and $v_H \in V(G^*)$ be the contraction image of H, and let $V_i = N^i(v_H)$. Then $\kappa'(G^*) \geq 2$, $V_2(G^*) \subseteq \{v_H\}$. By Lemma 9(2), $G^*[V_1] \cong kK_1$ for $k = |V_1| \geq 2$. Since $\delta(G) \geq 3$, $l \geq 2$ for $l = |V_2|$. Since G is Z_3-free, v_H is not in an induced P_4 and $V_3 = \emptyset$.

If there is an edge $u_1u_2 \in E(G[V_2])$, then $u_1v_{12} \in E(G^*)$ for some $v_{12} \in V_1$. Since $G^*[\{u_2, u_1, v_{12}, v_H\}] \ncong P_4$, $v_{12}u_1 \in E(G^*)$. Furthermore, $N_{G^*}(\{u_1, u_2\}) \cap V_1 = \{v_{12}\}$. Suppose otherwise. Assume that $v_2u_1 \in E(G)$ for some $v_2 \in V_1$. By symmetry, $u_2v_2 \in E(G^*)$, and so $G^*[\{v_H, v_{12}, v_2, u_1, u_2\}]$ is collapsible, contracting the choice of H. Note that there is a vertex $v_{34} \in V_2$ such that $E(v_{34}, \{u_1, u_2\}) = \emptyset$. Then there are two edges $v_{34}u_3, v_{34}u_4$ for some $u_3, u_4 \in V_2$. By symmetry, $E(\{u_3, u_4\}, \{u_1, u_2\}) = \emptyset$. Since $G^*[\{u_iv_{34}, v_H, v_{12}\}] \ncong P_4$ for $i \in \{3, 4\}$, $\{u_3v_{12}, u_4v_{12}\} \subseteq E(G^*)$. Then $N_{G^*}(u_3) \subseteq V_1$ and hence there is a vertex $v_0 \in N_{V_1}(u_3)$. However, $K_{3,3}^- \subseteq G^*[v_H, v_{12}, v_{34}, v_0, u_3, u_4]$ is collapsible, contradicting the choice of H.

This implies that $G^*[V_2] \cong lK_1$. Then $k \geq d_{G^*}(u_1) \geq 3$. Hence G^* is a 2-connected bipartite graph such that $\delta(G^*) \geq 3$ and each edge lies in an induced C_4. Then G^* is collapsible by Theorem 13, which implies G is collapsible by Theorem 6(a). ■

Corollary 15. *Let* G *be a* 2*-edge-connected noncollapsible* (*or nonsupereulerian*) *graph with* $\delta(G) \geq 3$*. Then* G *is* Z_3*-free if and only if* G *is* K_3*-free.*

An edge of G is said to be *subdivided* when it is deleted and replaced by a path of length 2 connecting its ends, the internal vertex of this path being a new vertex. A *subdivision* of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions. We use $\theta(i, j, k) = \theta(t_1, t_2, x_1 \cdots x_i, y_1 \cdots y_j, z_1 \cdots z_k)$ to denote the graph obtained from the *Theta graph* with 3-multiple edges and two vertices t_1, t_2 by replacing the 3-multiple edges with three internal vertex-disjoint paths $t_1x_1 \cdots x_it_2$, $t_1y_1 \cdots y_jt_2$ and $t_1z_1 \cdots z_kt_2$, respectively.

***Proof of Theorem* 5.** The sufficiency. It suffices to prove that G is collapsible. Let H be an induced collapsible subgraph of G with $|V(H)|$ maximized. On the contrary, we may choose a vertex $v \in N_G(H)$. Then $|E(v, H)| = 1$ and $E(w, H) = \emptyset$ for any $w \in N_G(v)\backslash V(H)$. Since G is P_5-free and by Theorem 14, $4 \leq g(G) \leq 5$ and $|V(H)| = 1$, which means G is reduced. If $g(G) = 5$, let $C = x_1x_2x_3x_4x_5x_1$ be an induced cycle of G, then $|E(y_i, C)| = 1$ for $y_i \in N_G(x_i)\backslash V(C)$ and $G[y_1, x_1, x_2, x_3, x_4] \cong P_5$, a contradiction. Hence $g(G) = 4$. Let $C = x_1x_2x_3x_4x_1$ be an induced cycle of G and $y_i \in N_G(x_i)\backslash V(C)$. If $y_1x_3 \in E(G)$, note that $H' = G[\{x_1, x_2, x_3, x_4, y_1\}] \cong \theta(1, 1, 1)$, then $|E(y_2, H')| = |E(y_4, H')| = |E(y', H')| = 1$ and $\{y_2y_4, y_2y', y_4y'\} \nsubseteq E(G)$ for $y' \in N_G(y_1)\backslash V(C)$ since $G[V(H) \cup \{y_1, y_2, y_4, y'\}]$ is not collapsible. By symmetry, assume $y_2y_4 \notin E(G)$, then $G[\{y_2, x_2, x_3, x_4, y_4\}] \cong P_5$, a contradiction. This implies $|E(y_i, C)| = 1$ for $i \in \{1, 2, 3, 4\}$. Then $y_1y_3, y_2y_4 \in E(G)$ since $G[\{y_1, x_1, x_2, x_3, y_3\}] \ncong P_5$ and $G[\{y_2, x_2, x_3, x_4, y_4\}] \ncong P_5$. Since $G[\{y_1, y_3, x_3, x_2, y_2\}] \ncong P_5$, $E(y_2, \{y_1, y_3\} \neq \emptyset$. By symmetry, assume that $y_1y_2 \in E(G)$. Then there is a C-subpartition $(X_1, X_2) = (\{x_1, x_4, y_2, y_4\}, \{x_2, y_1\})$ with $x_3 \in N_G(X_1) \cap N_G(X_2)$. By Lemma 8, $G[\{x_1, x_2, x_3, x_4, y_1, y_2, y_4\}]$ is collapsible, a contradiction.

The necessity. All graphs in Figure 4 are 2-edge-connected noncollapsible (nonsupereulerian). Especially, each graph in $\{G_1, G_6, G_{11}\}$ contains H as its induced subgraph. Note that G_6 and G_{11} have no common induced C_k for any $k \geq 3$ and G_6 is $K_{1,3}$-free, H should be the subgraph of path. Since G_1 is P_6-free, $H \subseteq P_5$. ∎

3. Forbidden Subgraphs Guaranteeing a 2-Edge-Connected Graph To Be Supereulerian

Before presenting the proofs, we need to prepare some results. A graph H is a *minor* of G if H is isomorphic to the contraction image of a subgraph of G. We call H an *induced minor* of G if H is isomorphic to the contraction image of an induced subgraph of G.

If a graph G has an induced minor H with $V(H) = \{v_1, v_2, \ldots, v_t\}$, then for pair of $\{i, j\} \subseteq \{1, 2, \ldots, t\}$, v_i is the contraction image of an induced subgraph G_{v_i} of G. Let X_{v_i} be the minimal subset of $V(G_{v_i})$ such that $G[X_{v_i}]$ is connected and $\left|E\big(X_{v_i}, \bigcup_{k\in\{1,2,\ldots,t\}\backslash\{i\}} V(G_{v_k})\big)\right| = d_H(v_i)$. Then $\left|E\big(X_{v_i}, X_{v_j}\big)\right| = 1$ if $v_iv_j \in E(H)$ and $\left|E\big(X_{v_i}, X_{v_j}\big)\right| = 0$ otherwise. Note that $H' = G[X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_t}]$ is an induced subgraph of G, called the H*-subgraph.*

Theorem 16 (Wang and Xiong [12]). *$H'[N_{H'}[X_{v_i}]]$ has either an induced $T_{i,j,k}$ or an induced $N_{i',j',k'}$ for some i, j, k, i', j', k' if $d_H(v_i) \geq 3$.*

A *wheel* W_n is the graph obtained from the n-cycle $C_n = v_1v_2 \cdots v_nv_1$, where $n \geq 2$, by adding an extra vertex v and new edges $\{vv_i : 1 \leq i \leq n\}$. The *subdivided wheel* W_n^* is the graph obtained from W_n by replacing v_iv_{i+1} by a path $v_iv_i'v_{i+1}$ with $\{v_1', \ldots, v_n'\} \cap V(W_n) = \emptyset$ $(1 \le i \le n)$. Let $W^* = \{W_n^* : n \ge 2\}$.

Theorem 17 (Lai [7]). *If G is 2-edge-connected and does not have an induced minor isomorphic to a member in W^*, then G is supereulerian.*

For a cycle C, let C^+ denote the graph obtained from C by adding one edge between one pair non-adjacent vertices in C.

Lemma 18. *Let G be a 2-edge-connected graph. If every W_i^*-subgraph $(i \ge 2)$ H of G is in a subgraph $\bar{H} \subseteq G$ such that the reduction of $\bar{H}$ is K_1, C_k or C_k^+ for some integer $k \ge 4$, then G is supereulerian.*

Proof. Let G' be the reduction of G. If $G' \cong K_1$, then G is supereulerian by Theorem 6(a). We then assume that $\kappa'(G') \ge \kappa'(G) \ge 2$. If G' has an induced minor $W_{i_0}^*$ for some $i_0 \ge 2$, then G' has an induced $W_{i_0}^*$-subgraph H' and G has a corresponding induced $W_{i_0}^*$-subgraph H such that H' is the reduction of H. By hypothesis, G has a subgraph $\bar{H}$ such that $H \subseteq \bar{H}$ and the reduction of $\bar{H}$, say $\bar{H}'$, satisfies that $\bar{H}' \in \{K_1, C_k, C_k^+\}$ for some integer $k \ge 4$. Note that H' is an induced subgraph of $\bar{H}'$ and H' has three vertex-disjoint paths with length at least 2 between any two vertices of degree 3. This is impossible. Therefore, G' has no induced minor W_i^* for any $i \ge 2$. By Theorem 17, G' is supereulerian, and hence G is supereulerian by Theorem 6(b). ∎

Theorem 19 (Liu *et al.* [8]). *Every 2-connected P_7-free graph G with $\delta(G) \ge 3$ is supereulerian or $P(10)$.*

The following result extends Theorem 19 and serves for the proofs of Theorem 3(2), (3).

Theorem 20. *Every 2-connected P_7-free graph G with $|V_2(G)| \le 1$ is supereulerian or $P(10)$.*

Proof. Let G' be the graph obtained from G by contracting all collapsible subgraph L of G such that $g(G/L) \ge 3$ and then either $|N_G(L)| \ge 3$ or $V(L) \cap V_2(G) \ne \emptyset$. Then G' is a $N_{1,1,1}$-free simple graph such that $|V_2(G')| \le 1$ and the vertex of degree 2 of G' is not in a collapsible subgraph of G'. Since any induced path of G' can be extended to an induced path of G, G' is P_7-free. By Theorem 6(a), it suffices to prove that G' is supereulerian or $G = G' \cong P(10)$. By contradiction, assume that G' is nonsupereulerian and $G' \ncong P(10)$. Then G' is nontrivial and $\kappa'(G') \ge \kappa'(G) \ge 2$. In the proof below, we need a set of 2-connected nonsupereulerian graphs $\mathcal{F} = \{F_1, F_2, \ldots, F_{10}\}$ (see Figure 2).

Claim 21. *Every induced W_i^*-subgraph $(i \ge 2)$ of G' is isomorphic to a member of $\mathcal{F}$.*

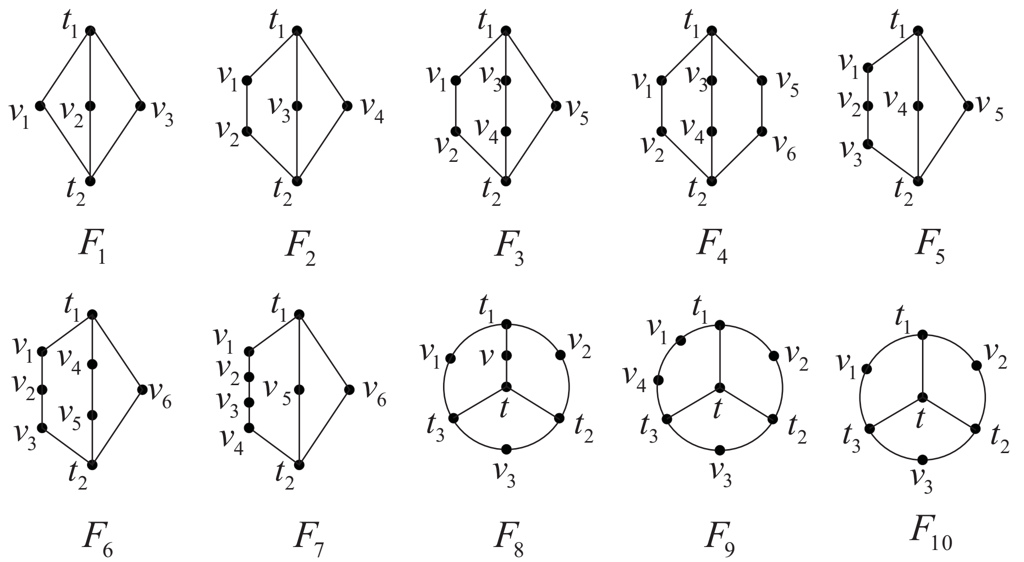

Figure 2. The graphs that are nonsupereulerian.

Proof. Let F be the induced W_i^*-subgraph of G'. Then G' has an induced minor W_i^*. For any vertex $v \in V(W_i^*)$, if $d_{W_i^*}(v) = 2$, then $F[N_F[X_v]]$ is isomorphic to a path. By Theorem 16 and since G' is $N_{1,1,1}$-free, if $d_{W_i^*}(v) = 3$, then $F[N_F[X_v]]$ is isomorphic to a $T_{l,m,n}$ for some integers l, m, n. As F is P_7-free, $i \in \{2, 3\}$. This implies that F is isomorphic to the subdivision of W_i^*. Then $F \in \mathcal{F}$. $\square$

Claim 22. *If G' has a subgraph $F \cong F_i$ for $i \in \{1, 2, 3\}$, then either $G'[V(F)] \cong F$ or $G'[V(F)]$ is collapsible.*

Proof. If $F \in \{F_1, F_2\}$, then we can verify that adding any edge between any pair of nonadjacent vertices of F results a collapsible graph.

Therefore, suppose that $F \cong F_3$. Then $E(G'[V(F)])\backslash E(L) \neq \{v_1v_3\}$, since otherwise, $|N_{G'}(t_1v_1v_5t_1)| \geq 3$ and $g(G'/t_1v_1v_5t_1) \geq 3$, contradicting the construction of G'. By symmetry, $E(G'[V(F)])\backslash E(L) \neq \{v_2v_4\}$. If $v_1v_4 \in E(G')$, then there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_4, v_5\}, \{v_1, v_3\})$ such that $v_2 \in N_{G'}(X_1) \cap N_{G'}(X_2)$, and hence $G'[V(F)]$ is collapsible by Lemma 8. By symmetry, $G'[V(F)]$ is collapsible if $v_2v_3 \in E(G')$. If $E(v_5, \{v_1, v_2, v_3, v_4\}) \neq \emptyset$, $E(t_1, \{v_2, v_4\}) \neq \emptyset$, $E(t_2, \{v_1, v_3\}) \neq \emptyset$, $t_1t_2 \in E(G')$ or $\{v_1v_3, v_2v_4\} \subseteq E(G')$, then we can verify that $G'[V(F)]$ is collapsible. Hence either $G'[V(F)] \cong F$ or $G'[V(F)]$ is collapsible. $\square$

Claim 23. *If G' has two induced subgraphs $\bar{F}$ and $F \cong F_i$ for $i \in \{1,2,3\}$ such that $F \subseteq \bar{F}$ and the reduction of $\bar{F}$ is a cycle C_k or C_k^+ for some $k \geq 4$, then F is in a collapsible subgraph of G'.*

Proof. Since the reduction of $\bar{F}$ is C_k or C_k^+, G' has a collapsible subgraph $L \subseteq \bar{F}$ such that either $t_1, t_2 \in V(L)$ or $t_1t_2 \in E(G'/L)$. Since $i \in \{1,2,3\}$, $G'[V(L) \cup V(F)]$ is collapsible. □

Since G' is nonsupereulerian and by Lemma 18, we can choose an induced $W_{i_0}^*$-subgraph H of G' such that for any integer $k \geq 4$ and any graph $\bar{H}$ with $H \subseteq \bar{H} \subseteq G'$, $\bar{H}$ is not collapsible and the reduction of $\bar{H}$ is not C_k or C_k^+. By Claim 21, we assume that $H \cong F_{j_0}$ for some $j_0 \in \{1, \ldots, 10\}$ with j_0 minimized. Then every induced subgraph $F \cong F_j$ of G' with $j < j_0$ is in a subgraph $\bar{F}$ of G' such that the reduction of $\bar{F}$ is K_1 or a cycle C or C^+; in addition, by Claims 22 and 23, if $j \leq \min\{j_0 - 1, 3\}$, then F is not necessary induced in G' and F is in a collapsible subgraph.

Claim 24. $j_0 \leq 3$.

Proof. By contradiction, assume that $4 \leq j_0 \leq 10$. Suppose that $H \cong F_{10}$. Then $|\{v_1, v_2, v_3\} \cap V_{\geq 3}(G')| \geq 2$. By symmetry, assume that v_1, v_2 have neighbors u_1, u_2 outside $V(H)$, respectively. Note that H has two C-subpartitions $(\{t, v_1, v_3\}, \{t_1, t_2, t_3, v_2\})$, $(\{v_1, t_1, t_2, t_3\}, \{t, v_2, v_3\})$. Then $|E(u_1, H)| = 1$, since otherwise, $G'[V(H) \cup \{u_1\}]$ is collapsible by Lemma 8, contradicting the choice of H. This implies that $u_1 \neq u_2$ and $|E(u_2, H)| = 1$. Thus either $u_1v_1t_3v_3t_2v_2u_2$ (if $u_1u_2 \notin E(G')$) or $u_1u_2v_2t_1tt_3v_3$ (if $u_1u_2 \in E(G')$) would be an induced P_7 in G', a contradiction.

Suppose that $H \cong F_9$. Then $\{v_2, v_3\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_2 has a neighbor u_2 outside $V(H)$. Note that there is a C-subpartition $(\{t, v_2, v_3\}, \{t_1, t_2, t_3, v_1, v_4\})$. Then $E(u_2, \{t_1, t_2, t_3, v_1, v_4\}) = \emptyset$, since otherwise, $G'[V(H) \cup \{u_2\}]$ is collapsible by Lemma 8, a contradiction. As $G'[\{u_2, v_2, t_2, t, t_3, v_4, v_1\}] \ncong P_7$, $u_2t \in E(G')$. Then $F_1 \subseteq G'[\{t_1, t_2, t, v_2, u_2\}]$ is in a collapsible subgraph of G' by the choice of H, and hence $G'[V(H) \cup \{u_2\}]$ is in a collapsible subgraph of G', a contradiction.

Suppose that $H \cong F_8$. Then $\{v_1, v_2\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_1 has a neighbor u_1 outside $V(H)$. Note that there is a C-subpartition $(\{v_1, v_2\}, \{t, v_3, t_1, t_2, t_3, v\})$. By Lemma 8, $E(u_1, \{t, v_3, t_1, t_2, t_3, v\}) = \emptyset$ and hence $G'[\{u_1, v_1, t_1, v, t, t_2, v_3\}] \cong P_7$, a contradiction.

Suppose that $H \cong F_7$. Then $\{v_5, v_6\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_6 has a neighbor u_6 outside $V(H)$. By the choice of H, $G'[\{t_1, t_2, v_5, v_6, u_6\}]$ is not in a collapsible subgraph, and hence $E(u_6, \{t_1, t_2, v_5\}) = \emptyset$. In addition, $E(u_6, \{v_2, v_3\}) = \emptyset$. Suppose otherwise. By symmetry, assume that $u_6v_2 \in E(G')$. Since $G'[V(H) \cup \{u_6\}]$ is not collapsible, $\{u_6v_1, u_6v_3\} \not\subseteq E(G')$. Since

$G'[\{u_6, v_2, v_1, t_1, v_5, t_2, v_4\}] \not\cong P_7$ and $G'[\{u_6, v_2, v_3, v_4, t_2, v_5, t_1\}] \not\cong P_7$, $u_6v_4 \in E(G')$. Note that there is a C-subpartition $(X_1, X_2) = (\{v_2, v_4, v_5, v_6\}, \{t_1, t_2, v_3, u_6\})$ such that $v_1 \in N_{G'}(X_1) \cap N_{G'}(X_2)$. Then $G'[V(H) \cup \{u_6\}]$ is collapsible by Lemma 8, a contradiction. Then one of $u_6v_6t_1v_1v_2v_3v_4$ (if $\{u_6v_1, u_6v_4\} \cap E(G') = \emptyset$), $v_2v_3v_4u_6v_6t_1v_5$ (if $\{u_6v_1, u_6v_4\} \subseteq E(G')$), $u_6v_1v_2v_3v_4t_2v_5$ (if $\{u_6v_1, u_6v_4\} \cap E(G') = \{u_6v_1\}$) and $u_6v_4v_3v_2v_1t_1v_5$ (if $\{u_6v_1, u_6v_4\} \cap E(G') = \{u_6v_4\}$) would be an induced P_7 in G', a contradiction.

Suppose that $H \cong F_6$. Then $\{v_4, v_5\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_4 has a neighbor u_4 outside $V(H)$. By the choice of H, $G'[\{t_1, t_2, v_4, v_5, v_6, u_4\}]$ is not in a collapsible subgraph, then $\{u_4t_1, u_4v_5\} \not\subseteq E(G')$ and $F_2 \not\subseteq G'[\{t_1, t_2, v_4, v_5, v_6, u_4\}]$, and hence $E(u_4, \{t_2, v_6\}) = \emptyset$. Then $u_4v_5 \notin E(G')$, since otherwise, $u_4t_1 \notin E(G')$, $d_{G'}(u_4) \geq 3$, $|N_{G'}(u_4v_4v_5u_4)| \geq 3$ and $g(G'/(u_4v_4v_5u_4)) \geq 3$, contracting the construction of G'. By symmetry, $u_4t_1 \notin E(G)$. In addition, $\{u_4v_1, u_4v_3\} \not\subseteq E(G')$, since otherwise, there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_2, u_4\}, \{v_1, v_3, v_4, v_6\})$ such that $v_5 \in N_{G'}(X_1) \cap N_{G'}(X_2)$, then $G'[V(H) \cup \{u_4\}]$ is collapsible by Lemma 8, a contradiction. Then $u_4v_2 \in E(G')$; for otherwise, one of $u_4v_4t_1v_6t_2v_3v_2$ (if $\{u_4v_1, u_4v_3\} \cap E(G') = \emptyset$), $v_4u_4v_1v_2v_3t_2v_6$ (if $u_4v_1 \in E(G')$) and $v_6t_2v_3u_4v_4t_1v_1$ (if $u_4v_3 \in E(G')$) would be an induced P_7 in G', a contradiction. Thus $E(u_4, \{v_1, v_3\}) = \emptyset$ by the construction of G', and then $u_4v_2v_1t_1v_6t_2v_5$ would be an induced P_7 in G', a contradiction.

Suppose that $H \cong F_5$. Then $\{v_4, v_5\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_4 has a neighbor u_4 outside $V(H)$. By the construction of G' and by the choice of H, $E(u_4, \{t_1, t_2, v_5\}) = \emptyset$. In addition, $u_4v_2 \notin E(G')$; for otherwise, $F_3 \subseteq G'[\{t_1, t_2, v_1, v_2, v_3, v_4, u_4\}]$ is in a collapsible subgraph of G' by the choice of H, and then $G'[V(H) \cup \{u_4\}]$ is in a collapsible subgraph, a contradiction. And $\{u_4v_1, u_4v_3\} \not\subseteq E(G')$; for otherwise, there is a C-subpartition $(X_1, X_2) = (\{t_1, v_1, v_3, v_4\}, \{t_2, v_2, u_4\})$ with $v_5 \in N_{G'}(X_1) \cap N_{G'}(X_2)$, then $G'[V(H) \cup \{u_4\}]$ is collapsible by Lemma 8, a contradiction. Then $\{u_4v_1, u_4v_3\} \cap E(G') = \emptyset$. Suppose otherwise. Without loss of generality, assume that $u_4v_1 \in E(G')$. Then $\{v_5, u_4\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_5 has a neighbor u_5 outside $V(H) \cup \{u_4\}$ with $E(u_5, \{t_1, t_2, v_2, v_4\}) = \emptyset$. Then $u_5u_4 \notin E(G')$; for otherwise $F_2 \subseteq G'[\{v_4, v_5, t_1, t_2, u_4, u_5\}]$ is in a collapsible subgraph of G' and then $G'[V(H) \cup \{u_4, u_5\}]$ is in a collapsible subgraph of G', a contradiction. In addition, $u_5v_1 \notin E(G')$; for otherwise, $F_3 \subseteq G'[\{v_1, v_4, u_4, u_5, v_2, v_3, v_4\}]$ is in a collapsible subgraph of G' and then $G'[V(H) \cup \{u_4, u_5\}]$ is in a collapsible subgraph of G', a contradiction. Thus $u_5v_5t_2v_4t_1v_1v_2$ would be an induced P_7 in G', a contradiction. This implies $|E(u_4, H)| = 1$ and u_4 has a neighbor u'_4 outside $V(H) \cup \{u_4\}$. Since $G'[V(H) \cup \{u_4, u'_4\}]$ is not collapsible and by the construction of G', $u'_4v_4 \notin E(G')$, $\{u'_4v_2, u'_4v_i\} \not\subseteq E(G')$ for $i \in \{1, 3\}$ and $G'[\{t_1, t_2, v_4, v_5, u_4, u'_4\}]$ is not collapsible. Then $u'_4v_5 \notin E(G')$ since $F_2 \not\subseteq G'[\{t_1, t_2, v_4, v_5, u_4, u'_4\}]$ and $\{u'_4t_1, u'_4t_2\} \not\subseteq E(G')$ since $F_1 \not\subseteq G'[\{t_1, t_2, v_4, v_5, u'_4\}]$. By symmetry, assume that $u'_4t_1 \notin E(G')$. Then

$\{u_4'v_1, u_4'v_3\} \not\subseteq E(G')$ since $F_3 \not\subseteq G'[\{t_1, t_2, v_1, v_3, v_4, u_4, u_4'\}]$ and $\{u_4'v_1, u_4't_2\} \not\subseteq E(G')$ since $F_3 \not\subseteq G'[\{v_1, t_2, v_2, v_3, t_1, u_4'\}]$. Thus $\{u_4'v_2, u_4't_2\} \subseteq E(G')$; for otherwise, one of $u_4u_4'v_1v_2v_3t_2v_5$ (if $u_4'v_1 \in E(G')$), $u_4u_4'v_3v_2v_1t_1v_5$ (if $u_4'v_3 \in E(G')$) and $u_4u_4'v_2v_3t_2v_5t_1$ (if $u_4'v_2 \in E(G')$ and $u_4't_2 \notin E(G')$) would be an induced P_7 in G', a contradiction. Note that $\{v_3, v_5\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that $u_5 \in N_{G'}(v_5)\backslash V(H)$ with $|E(u_5, H)| = 1$. Since $F_2 \not\subseteq G'[\{t_1, t_2, v_4, v_5, u_4, u_5\}]$, $u_4u_5 \notin E(G')$. Since $G'[\{u_5, v_5, t_1, v_1, v_2, u_4', u_4\}] \ncong P_7$, $u_5u_4' \in E(G')$. Note that $\{u_4, u_5\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that there is a vertex $u_5' \in N_{G'}(u_5)\backslash(V(H) \cup \{u_4'\})$ with $u_5'v_2 \in E(G)$. However there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_1, v_2, u_4\}, \{v_3, v_4, v_5, u_5, u_4'\})$ with $u_5' \in N_{G'}(X_1) \cap N_{G'}(X_2)$, then $G'[V(H) \cup \{u_4, u_5, u_4', u_5'\}]$ is collapsible by Lemma 8, a contradiction.

Suppose finally that $H \cong F_4$. Then $|\{v_1, \ldots, v_6\} \cap V_{\geq 3}(G')| \geq 5$. By symmetry, assume that $\{v_1, \ldots, v_5\} \subseteq V_{\geq 3}(G')$. Let $u_1 \in N_{G'}(v_1)\backslash V(H)$. Then $u_1t_1 \notin E(G')$. In addition, $u_1v_2 \notin E(G')$. Suppose otherwise. If $u_1t_2 \notin E(G')$, then either $|N_{G'}(u_1v_1v_2u_1)| \geq 3$ or $d_{G'}(u_1) = 2$; if $u_1t_2 \in E(G')$, then $G'[\{u_1, v_1, v_2, t_2\}]$ is collapsible and $|N_{G'}(\{u_1, v_1, v_2, t_2\})| \geq 3$, contracting the construction of G'. Furthermore, $\{u_1v_3, u_1v_5\} \not\subseteq E(G')$; for otherwise, $E(u_1, \{v_2, v_4, v_6\}) \neq \emptyset$ and $F_1 \subseteq G'[\{t_1, v_1, v_3, v_5, u_1\}]$ is in a collapsible subgraph of G', and then $G'[V(H) \cup \{u_1\}]$ is in a collapsible subgraph of G', a contradiction. By symmetry, assume that $u_1v_3 \notin E(G')$. Since $F_3 \not\subseteq G'[\{t_1, t_2, v_1, u_1, v_2, v_i, v_{i-1}\}]$ for $i \in \{4, 6\}$, $E(u_1, \{v_4, v_6\}) = \emptyset$. Then $u_1t_2 \in E(G')$ since $G'[\{u_1, v_1, t_1, v_3, v_4, t_2, v_6\}] \ncong P_7$. By symmetry, v_j has a neighbor u_j outside $V(H)$ and $u_jt_{j(\mathrm{mod}\ 2)+1} \in E(G')$ for $j \in \{2, \ldots, 5\}$. Note that $\{u_1, u_2\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u_1 has a neighbor u_1' outside $V(H)$. Since there is a C-subpartition $(\{t_2, v_1, v_3, v_5, u_1, u_4\}, \{t_1, v_2, v_4, v_6, u_3\})$, by Lemma 8, $E(u_1', \{t_1, v_2, v_4, v_6, u_3\}) = \emptyset$. By symmetry, $E(u_1', \{u_1, t_2\}) = \emptyset$. Then $u_1'v_3 \notin E(G')$; for otherwise, $F_2 \subseteq G'[\{t_2, v_3, v_4, u_3, u_1, u_1'\}]$ is in a collapsible subgraph of G' and then $G'[V(H) \cup \{u_1, u_1'\}]$ is in a collapsible subgraph of G', a contradiction. Since $G'[\{u_1', u_1, t_2, v_6, v_5, t_1, v_3\}] \ncong P_7$, $u_1'v_5 \in E(G')$. However, there is a C-subpartition $(X_1, X_2) = (\{t_1, v_2, v_4, v_5, u_1, u_1'\}, \{t_2, v_1, v_3, u_2\})$ with $v_6 \in N_{G'}(X_1) \cap N_{G'}(X_2)$, then $G'[V(H) \cup \{u_1, u_2, u_1'\}]$ is collapsible by Lemma 8, a contradiction. □

For $j_0 \leq 3$, we shall distinguish the following three cases.

Case 1. $H \cong F_1$. Since $G'[V(H) \cup \{u\}]$ is not collapsible, $|E(u, H)| = 1$ for any $u \in N_{G'}(\{v_1, v_2, v_3\}) \setminus V(H)$ by Lemma 8.

Claim 25. *There is an induced path $P(v_i, v_j)$ between v_i and v_j outside H for some $\{i, j\} \subseteq \{1, 2, 3\}$.*

Proof. Note that $|\{v_1, v_2, v_3\} \cap V_{\geq 3}(G')| \geq 2$. By symmetry, v_1, v_2 have neighbors u_1, u_2 outside $V(H)$ with $|E(u_1, H)| = |E(u_2, H)| = 1$. Then u_1, u_2 have neighbors u_1', u_2' outside $V(H) \cup \{u_1, u_2\}$ with $N_{G'}\{u_1', u_2'\} \subseteq \{t_1, t_2\}$. In addition,

$|N_{G'}(u_i') \cap \{t_1, t_2\}| \leq 1$; for otherwise $G'[V(H) \cup \{u_i, u_i'\}]$ is collapsible for $i \in \{1, 2\}$. Then either $u_1'u_1v_1t_1(t_2)v_2u_2u_2'$ (if $N_{G'}(u_1') \cup N_{G'}(u_2') = \{t_2\}(\{t_1\})$) or $u_1u_1't_1v_3t_2u_2'u_2$ (if $N_{G'}(u_1') \cup N_{G'}(u_2') = \{t_1, t_2\}$ and $\{u_1't_1, u_2't_2\} \subseteq E(G')$) would be an induced P_7 in G', a contradiction. $\square$

By Claim 25, we may choose a longest induced path $P(v_1, v_2)$ satisfying $|E(V(P(v_1, v_2)), V(H))| = 2$. Then $3 \leq |E(P(v_1, v_2))| \leq 4$ since G' is P_7-free.

Suppose firstly that $P(v_1, v_2) = v_1u_1u_2v_2$. Note that $\{u_1, u_2\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u_2 has a neighbor u_2' outside $V(H) \cup \{u_1, u_2\}$. Note that there are two C-subpartitions $(\{t_1, t_2, v_3\}, \{v_1, v_2, u_1, u_2\})$ and $(\{t_1, t_2, v_1, u_2\}, \{v_2, v_3, u_1\})$. Then $E(u_2', \{t_1, t_2, v_2, v_3\}) = \emptyset$ since $G'[V(H) \cup \{u_1, u_2, u_2'\}]$ is not collapsible and by Lemma 8. In addition, $u_2'u_1 \notin E(G)$, since otherwise, either $u_2'v_1 \in E(G')$, $G'[\{v_1, v_2, u_1, u_2, u_2'\}]$ is collapsible and $|N_{G'}(\{v_1, v_2, u_1, u_2, u_2'\})| \geq 3$ or $u_2'v_1 \notin E(G')$, contracting the construction of G'. We then claim that $u_2'v_1 \notin E(G')$. Suppose otherwise. Note that $\{u_1, u_2'\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u_1 has a neighbor u_1' outside $V(H) \cup \{u_1, u_2, u_2'\}$. Then $E(u_1', \{v_1, u_2\}) = \emptyset$. Since $G'[V(H) \cup \{u_1, u_2, u_1', u_2'\}]$ is not collapsible, $E(u_1', \{t_1, t_2, v_2, v_3, u_2'\}) = \emptyset$. Thus u_1' has a neighbor u_1'' outside $V(H) \cup \{u_1, u_2, u_1', u_2'\}$. By the choice of $P(v_1, v_2)$, $E(u_1'', \{v_2, v_3\}) = \emptyset$. Since $G'[V(H) \cup \{u_1''\}]$ is not collapsible, $\{u_1''t_1, u_1''t_2\} \not\subseteq E(G')$. Assume that $u_1''t_1 \notin E(G')$. Then $u_1''u_2 \notin E(G')$ since $G'[V(H) \cup \{u_1, u_1', u_2, u_2', u_1''\}]$ is not collapsible, and hence $u_1''u_1'u_1u_2v_2t_1v_3$ would be an induced P_7 in G', a contradiction.

Then u_2' has a neighbor u_2'' outside $V(H) \cup \{u_1, u_2, u_2'\}$. By the choice of $P(v_1, v_2)$, $E(u_2'', \{v_1, v_2, u_2, u_2'\}) = \emptyset$. Since $G'[V(H) \cup \{u_2''\}]$ is not collapsible, $\{u_2''t_1, u_2''t_2\} \not\subseteq E(G')$. Assume that $u_2''t_1 \notin E(G')$. Then $u_2''u_1 \in E(G')$ since $G'[\{u_2'', u_2', u_2, u_1, v_1, t_1, v_3\}] \ncong P_7$. Note that $\{u_2', u_2''\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u_2'' has a neighbor u_2''' outside $V(H) \cup \{u_1, u_2, u_2', u_2''\}$. Then $E(u_2''', V(H) \cup \{u_1, u_2, u_2', u_2''\}) = \emptyset$ and $u_2'''u_2''u_2'u_2v_2t_1v_1$ would be an induced P_7 in G', a contradiction.

It remains to consider the case when $|E(P(v_1, v_2))| = 4$. Assume that $P(v_1, v_2) = v_1u_1uu_2v_2$. Note that $\{u_1, u_2\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u_2 has a neighbor u_2' outside $V(H) \cup \{u_1, u_2\}$. Then $E(u_2', \{v_2, u\}) = \emptyset$. Since $G'[V(H) \cup \{u_2'\}]$ is not collapsible, $\{u_2't_1, u_2't_2\} \not\subseteq E(G')$. Assume that $u_2't_1 \notin E(G')$. Then $\{u_2'v_1, u_2'v_3, u_2'u_1\} \cap E(G') \neq \emptyset$ since $G'[\{u_2', u_2, u, u_1, v_1, t_1, v_3|] \ncong P_7$. We claim that $u_2'v_1 \notin E(G')$. Suppose otherwise. If v_3 has a neighbor u_3 outside $V(H) \cup \{u_1, u_2, u, u_2'\}$, then $|E(u_3, V(H) \cup \{u_1, u_2, u, u_2'\})| = 1$ and then $u_3v_3t_2v_1u_1uu_2$ would be an induced P_7 in G', a contradiction. Hence $d_{G'}(v_3) = 2$. Then u has a neighbor u' outside $V(H) \cup \{u_1, u_2, u, u_2'\}$ such that $E(u', \{t_1, t_2, v_2, v_3, u_1, u_2, u\}) = \emptyset$. Since $G'[\{u', u, u_2, u_2', v_1, t_1, v_3\}] \ncong P_7$, $u'v_1 \in E(G')$. Then u' has a neighbor u'' outside $V(H) \cup \{u_1, u_2, u, u'\}$ such that $E(u'', V(H) \cup \{u_2, u\}) = \emptyset$ and $u''u'uu_2v_2t_1v_3$ would be an induced P_7 in G', a contradiction. We then claim that $u_2'v_3 \notin E(G')$. Suppose otherwise. Note that

$\{u_1, u_2'\} \cap V_{\geq 3}(G') \neq \emptyset$. If u_2' has a neighbor u_2'' outside $V(H) \cup \{u_1, u_2, u, u_2'\}$, then $E(u_2'', V(H) \cup \{u, u_2\}) = \emptyset$ since $G'[V(H) \cup \{u, u_2, u_2''\}]$ is not collapsible, and hence either $u_2''u_2'u_2uu_1v_1t_1$ ($u_2''u_1 \notin E(G')$) or $u_2''u_1uu_2v_2t_1v_3$ ($u_2''u_1 \in E(G')$) would be an induced P_7 in G', a contradiction. Then $d_{G'}(u_2') = 2$ and u_1 has a neighbor u_1' outside $V(H) \cup \{u_1, u_2, u, u_2'\}$ such that $E(u_1', \{t_1, t_2, v_1, v_2, u\}) = \emptyset$ and $\{u_1'v_3, u_1'u_2'\} \not\subseteq E(G')$ since $G'[V(H) \cup \{u_1, u_2, u, u_1', u_2'\}]$ is not collapsible. Then $u_1'u_2 \in E(G')$; for otherwise, either $u_1'u_1uu_2v_2t_1v_3$ or $u_1'u_1v_1t_1v_2u_2u_2'$ would be an induced P_7 in G', a contradiction. Note that u has a neighbor u' outside $V(H) \cup \{u_1, u_2, u_1', u_2'\}$ such that $E(u', V(H) \cup \{u_1, u_2, u_1', u_2'\}) = \emptyset$ since $G'[V(H) \cup \{u, u_1, u_2, u', u_1', u_2'\}]$ is not collapsible. Then $u'uu_2u_2'v_3t_1v_1$ would be an induced P_7 in G', a contradiction.

Then $u_2'u_1 \in E(G')$, $\{u, u_2'\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u has a neighbor u' outside $V(H) \cup \{u_1, u_2, u_2', u\}$. Then $|E(u', V(H) \cup \{u_1, u_2, u_2', u\})| = 1$ and u' has a neighbor u'' outside $V(H) \cup \{u_1, u_2, u_2', u, u'\}$. Since $G'[V(H) \cup \{u_1, u_2, u, u_2', u''\}]$ is not collapsible, $\{u''t_1, u''t_2\} \not\subseteq E(G')$ and $\{u''u_1, u''u_2\} \not\subseteq E(G')$. By symmetry, assume that $\{u''t_1, u''u_1\} \cap E(G') = \emptyset$. Then $u''u \notin E(G')$. Since $G'[\{u'', u', u, u_1, v_1, t_1, v_2\}] \ncong P_7$ and $G'[\{u'', u', u, u_1, v_1, t_1, v_3\}] \ncong P_7$, $u''v_1 \in E(G')$. However, $u_2uu'u''v_1t_1v_3$ would be an induced P_7 in G', a contradiction.

Case 2. $H \cong F_2$. Note that $\{v_3, v_4\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that v_4 has a neighbor u_4 outside $V(H)$. By the choice of H, $u_4v_3 \notin E(G')$. Since there is a C-subpartition $(\{t_1, t_2, v_1, v_2\}, \{v_3, v_4\})$, $E(u_4, \{t_1, t_2, v_1, v_2\}) = \emptyset$ by Lemma 8. This implies $|E(u_4, H)| = 1$ and then $N_{G'}(u_4) \setminus V(H) \neq \emptyset$.

Claim 26. *u_4 has a neighbor outside $V(H)$ that is adjacent to exactly one of $\{v_1, v_2\}$.*

Proof. By contradiction, assume that $\{u_4'v_1, u_4'v_2\} \cap E(G') = \emptyset$ for any $u_4' \in N_{G'}(u_4) \setminus V(H)$. Since $\{v_1, v_2\} \cap V_{\geq 3}(G') \neq \emptyset$, by symmetry, assume that v_2 has a neighbor u_2 outside $V(H) \cup \{u_4, u_4'\}$. By the choice of H, $E(u_2, \{t_2, v_1, v_3, v_4, u\}) = \emptyset$ and $u'v_4 \notin E(G')$. Since $G'[V(H) \cup \{u_2, u_4, u_4'\}]$ is not collapsible, $\{u_4't_1, u_4'u_2\} \not\subseteq E(G')$ and $\{u_4't_1, u_4'v_3\} \not\subseteq E(G')$. Then $u_2t_1 \in E(G')$; for otherwise, exactly one of $u_4'u_4v_4t_1v_1v_2u_2$ (if $\{u_2u_4', u_4't_1\} \cap E(G') = \emptyset$), $v_2u_2u_4'u_4v_4t_1v_3$ (if $u_2u_4' \in E(G')$) and $u_2v_2t_2v_3t_1u_4'u_4$ (if $u_4't_1 \in E(G')$) would be an induced P_7 in G', a contradiction. Since $\{v_1, u_2\} \cap V_{\geq 3}(G') \neq \emptyset$, by symmetry, assume that v_1 has a neighbor u_1 outside $V(H) \cup \{u_2, u_4, u_4'\}$. By symmetry, $u_1t_2 \in E(G')$. Then there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_1, v_3, v_4\}, \{u_1, v_2\})$ with $u_2 \in N_{G'}(X_1) \cap N_{G'}(X_2)$, and then $G'[V(H) \cup \{u_1, u_2\}]$ is collapsible, a contradiction. $\square$

By symmetry, assume that $u_4'v_2 \in E(G')$ for some $u_4' \in N_{G'}(u_4) \setminus V(H)$. Then $d_{G'}(v_3) = 2$. Suppose otherwise. Assume that v_3 has a neighbor u_3 outside $V(H) \cup \{u_4, u_4'\}$. Since there is a C-subpartition $(\{t_1, t_2, v_1, v_2, u_4\}, \{v_3, v_4, u_4'\})$,

$E(u_3, \{t_1, t_2, v_1, v_2, u_4\}) = \emptyset$. Then $u_3v_4 \notin E(G')$. Since $G'[\{u_3, v_3, t_1, v_1, v_2, u'_4, u_4\}] \ncong P_7$, $u_3u'_4 \in E(G')$. Note that $\{u_3, u_4\} \cap V_{\geq 3}(G') \neq \emptyset$. By symmetry, assume that u_3 has a neighbor u'_3 outside $V(H) \cup \{u_3, u_4, u'_4\}$. Then $|E(u'_3, V(H) \cup \{u_3, u_4, u'_4\})| = 1$ and $u'_3u_3u'_4v_2v_1t_1v_4$ would be an induced P_7 in G', a contradiction. Hence $d(u_4) \geq 3$ and u_4 has a neighbor u''_4 outside $V(H) \cup \{u_4, u'_4\}$ with $E(u''_4, \{u'_4, v_3, v_4\}) = \emptyset$. Then $u''_4t_1 \notin E(G')$. Suppose otherwise. Then $E(u', \{t_1, t_2, v_1, v_2, u_4, u'_4\}) = \emptyset$ for $u' \in N_{G'}(u''_4)$ since there is a C-subpartition $(\{t_1, t_2, v_1, v_2, u_4, u'_4\}, \{v_3, v_4, u''_4\})$. Since $G'[\{u', u''_4, u_4, u'_4, v_2, t_2, v_3\}] \ncong P_7$, $u'v_3 \in E(G')$. However, there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_1, v_2, u'\}, \{v_3, v_4, u_4, u''_4\})$ with $u'_4 \in N_{G'}(X_1) \cap N_{G'}(X_2)$, and then $G'[V(H) \cup \{u_u, u'_4, u''_4, u'\}]$ is collapsible, a contradiction. Since $G'[V(H) \cup \{u_4, u'_4, u''_4\}]$ is not collapsible, $\{u''_4v_1, u''_4t_2\} \nsubseteq E(G')$. Then $u''_4v_2 \in E(G')$, since otherwise, either $u''_4u_4u'_4v_2v_1t_1v_3$ (if $u''_4v_1 \notin E(G')$) or $u''_4u_4u'_4v_2t_2v_3t_1$ (if $u''_4t_2 \notin E(G')$) would be an induced P_7 in G', a contradiction. Note that $G'[\{v_2, u_4, t_2, v_4, u'_4, u''_4\}] \cong F_2$, by symmetry, u''_4 has a neighbor w and w has two neighbors w_1, w_2 such that $\{w_1v_4, w_2v_4\} \subseteq E(G')$. Then $ww_1v_4u_4u'_4v_2v_1$ would be an induced P_7 in G', a contradiction.

Case 3. $H \cong F_3$.

Claim 27. *Either v_1 and v_4 or v_2 and v_3 (or both) have a common neighbor.*

Proof. Note that $|\{v_1, v_2, v_3, v_4\} \cap V_{\geq 3}(G')| \geq 3$. By contradiction, without loss of generality, assume that v_1, v_4 have neighbors u_1, u_4 outside $V(H)$, respectively. Then $E(\{u_1, u_4\}, \{t_1, t_2, v_1, v_4, v_5\}) = \emptyset$. Since $G'[\{u_1, v_1, t_1, v_5, t_2, v_4, u_4\}] \ncong P_7$, $u_1u_4 \in E(G')$. If v_5 has a neighbor u_5 outside $V(H) \cup \{u_1, u_4\}$, then $|E(u_5, H)| = 1$. Since $G'[V(H) \cup \{u_1, u_4, u_5\}]$ is not collapsible, $\{u_4u_5, u_1u_5\} \nsubseteq E(G')$, $\{u_1u_5, u_1v_3\} \nsubseteq E(G')$ and $\{u_4u_5, u_4v_2\} \nsubseteq E(G')$. Then exactly one of $u_5v_5t_1u_1u_4v_4t_2$ (if $\{u_1u_5, u_4u_5\} \cap E(G') = \emptyset$), $u_5u_1v_1v_2t_2v_4v_3$ (if $u_1u_5 \in E(G')$) and $u_5u_4v_4t_2v_2v_1t_1$ (if $u_1u_4 \in E(G')$) would be an induced P_7 in G', a contradiction. Hence $d_G(v_5) = 2$. Then at least one of $\{v_2, v_3\}$ has a neighbor outside $V(H) \cup \{u_1, u_4\}$; for otherwise, $\{u_1v_3, u_4v_2\} \subseteq E(G')$ and $G'[V(H) \cup \{u_1, u_4\}]$ is collapsible, a contradiction. By symmetry, let u_3 be a neighbor of v_3 outside $V(H) \cup \{u_1, u_4\}$. By symmetry, $E(u_3, \{t_1, t_2, v_2, v_4, v_5\}) = \emptyset$. Since $G'[V(H) \cup \{u_1, u_3, u_4\}]$ is not collapsible, $\{u_3u_1, u_3v_1\} \nsubseteq E(G')$ and $\{u_3v_1, u_3u_4\} \nsubseteq E(G')$. Then exactly one of $u_3v_3v_4t_2v_2v_1u_1$ (if $\{u_3u_1, u_3v_1\} \cap E(G') = \emptyset$), $u_3u_1u_4v_4t_2v_5t_1$ (if $u_3u_1 \in E(G')$) and $u_3v_1u_1u_4v_4t_2v_5$ (if $u_3v_1 \in E(G')$) would be an induced P_7 in G', a contradiction. □

Without loss of generality, let $u_{1,4}$ be a common neighbor of v_1, v_4. Since $G'[\{t_1, t_2, v_1, v_2, v_3, v_4, u_{1,4}\}] \cong F_3$, either v_2 and v_3 or v_4 and $u_{1,4}$ (or both) have a common neighbor by Claim 29. By symmetry, let $u_{2,3}$ be a common neighbor of v_2, v_3. Furthermore, we claim that two of $\{v_5, u_{1,4}, u_{2,3}\}$ have a common neighbor. Suppose otherwise. By symmetry, assume that $u_{1,4}, u_{2,3}$ have neighbors $u'_{1,4}, u'_{2,3}$ outside $V(H) \cup \{u_{1,4}, u_{2,3}\}$, respectively. By the choice of H, $E(\{u'_{1,4}, u'_{2,3}\}, V(H)) = \emptyset$. Then either $u'_{2,3}u_{2,3}v_3t_1v_1u_{1,4}u'_{1,4}$ (if $u'_{1,4}u'_{2,3} \notin E(G')$) or $u_{2,3}u'_{2,3}u'_{1,4}u_{1,4}v_4t_2v_5$ (if $u'_{1,4}u'_{2,3} \in E(G')$) would be an induced P_7 in G', a contradiction. By symmetry, let u' be a common neighbor of $\{u_{1,4}, u_{2,3}\}$. Then $u'v_5 \in E(G')$ since $G'[\{t_1, v_5, t_2, v_4, u_{1,4}, u', u_{2,3}\}] \ncong P_7$. Note that $\tilde{H} = G'[V(H) \cup \{u_{1,4}, u', u_{2,3}\}] \cong P(10)$ and each vertex of $\tilde{H}$ is in $V(G)$. Furthermore, there is no other vertex outside $V(\tilde{H})$. Suppose otherwise. Since $\kappa'(G') \geq 2$, there is an induced path P whose internal vertices are all not in $V(\tilde{H})$, connecting two vertices of $\tilde{H}$. Then either$|E(P)| \geq 3$ and it would produce an induced P_7 in G' or $|E(P)| \leq 2$ and $G'[V(\tilde{H}) \cup V(P)]$ would be collapsible, a contradiction. Thus $G = G' \cong P(10)$. ■

Corollary 28. *If* 2*-connected* P_7*-free graph* G *other than* $P(10)$ *satisfies that* $G[V_2(G)]$ *is a path, then* G *is supereulerian.*

Proof. Let $G^* = G/V_2(G)$. Then G^* is 2-connected, P_7-free with $|V_2(G^*)| \leq 1$ and G is supereulerian if and only if G^* is supereulerian. By Theorem 20, G^* is supereulerian or isomorphic to $P(10)$. If $G^* \cong P(10)$, then either G has an induced P_7 (if $V_2(G) \neq \emptyset$) or $G = G^* = P(10)$ (if $V_2(G) = \emptyset$), a contradiction. Thus G^* is supereulerian and then G is supereulerian. ■

***Proof of Theorem* 3.** By contradiction, let G be a 2-edge-connected nonsupereulerian graph with $\delta(G) \geq 3$.

(1) Then we will obtain an induced subgraph of G isomorphic to any one in $\{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8\}$ when G is $T_{2,2,1}$-free, contradicting that G is $\mathcal{H}$-free. By Lemma 18, G has an induced $W^*_{i_0}$-subgraph H satisfy the following property.

Property P: the reduction of $\bar{H}$ *is not a cycle* C *or* C^+ *for any* $\bar{H}$ *with* $H \subseteq \bar{H}$.

Claim 29. *Every subgraph* L *of* G *isomorphic to one of* $\{\theta(1,1,1), \theta(2,1,1)\}$ *is in a collapsible subgraph.*

Proof. Suppose that $L = \theta(t_1, t_2, v_1, v_2, v_3)$. For $i \in \{1,2,3\}$, let $u_i \in N_G(v_i) \backslash V(L)$. If $|E(u_i, H)| \geq 2$, note that there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_j\}, \{v_i, v_k\})$ with $u_i \in N_G(X_1) \cap N_G(X_2)$ for any $j \neq k \in \{1,2,3\} \backslash \{i\}$, then $G[V(L) \cup \{u_i\}]$ is collapsible by Lemma 8; we are done. In the case when $|E(u_i, H)| = 1$, $u_1 \neq u_2 \neq u_3$. Then $\{u_1u_2, u_1u_3, u_2u_3\} \subseteq E(G)$ since $G[\{t_1, v_i, u_i, v_j, u_j, v_k\}] \ncong T_{2,2,1}$. Therefore, there is a C-subpartition $(X_1, X_2) = (\{t_1, t_2, v_1\}, \{v_2, v_3, u_2, u_3\})$ with $u_1 \in N_G(X_1) \cap N_G(X_2)$, then $G[V(L) \cup \{u_1, u_2, u_3\}]$ is collapsible by Lemma 8.

Suppose that $L = \theta(t_1, t_2, v_1v_2, v_3, v_4)$ and $u_3 \in N_G(v_3) \backslash V(L)$. Note that there is a C-subpartition $(\{t_1, t_2, v_1, v_2\}, \{v_3, v_4\})$. If $E(u_3, \{t_1, t_2, v_1, v_2\}) \neq \emptyset$, then $G[V(L) \cup \{u_3\}]$ is collapsible; we are done. If not, then $u_3v_4 \in E(G)$ since

$G[\{t_1, v_1, v_2, v_3, v_4, u_3\}] \ncong T_{2,2,1}$ and then $G[\{t_1, t_2, v_3, v_4, u_3\}] \cong \theta(1, 1, 1)$ is in a collapsible subgraph L' by above discussion. Hence $G[V(L) \cup \{u_3\}]$ is collapsible since $G[V(L') \cup \{v_1, v_2\}]/L'$ is isomorphic to one of $\{K_1, C_2, C_3\}$. □

Claim 30. *H is not isomorphic to $\theta(i, j, k)$ for any $i \geq j \geq k \geq 1$.*

Proof. By contradiction, then $j = k = 1$ and assume that $H = \theta(t_1, t_2, x_1 \cdots x_i, y_1, z_1)$ since G is $T_{2,2,1}$-free. By Property P, t_1, t_2 are not in a collapsible subgraph of G. By Claim 29, $i \geq 3$, $N_G(y_1) \cap N_G(z_1) = \emptyset$ and $E(N_G(y_1), N_G(z_1)) = \emptyset$. Note that z_1 has a neighbor z_1' outside $V(H)$ with $E(z_1', \{t_1, t_2, y_1\}) = \emptyset$. Since $G[\{t_1, t_2, x_1, x_i, z_1, z_1'\}] \ncong T_{2,2,1}$, $E(z_1', \{x_1, x_i\}) \neq \emptyset$. By symmetry, assume that $z_1'x_1 \in E(G)$. Then $z_1'x_2 \notin E(G)$ since $G[\{t_1, t_2, x_1, x_2, y_1, z_1, z_1'\}]$ is not collapsible. Since $G[\{x_1, x_2, x_3, t_1, y_1, z_1'\}] \ncong T_{2,2,1}$, $z_1'x_3 \in E(G)$. By symmetry, $z_1'x_l \in E(G)$ and $z_1'x_{l+1} \notin E(G)$ for $l \in \{1, 3, \dots\}$. Then either $G[\{z_1', x_1, t_1, x_3, x_5, x_6\}] \cong T_{2,2,1}$ (if $i \geq 6$) or $G[V(H) \cup \{z_1'\}]$ is collapsible (if $i \leq 5$) by Lemma 8 since there is a C-subpartition $(X_1, X_2) = (\{x_1, y_1, z_1\}, \{t_1, t_2, x_3, \dots, x_i\})$ with $x_2 \in N_G(X_1) \cap N_G(X_2)$, a contradiction. □

Now, we use $\theta'(i, j, k) = \theta'(t_0, xyzx, x_1 \cdots x_i, y_1 \cdots y_j, z_i \cdots z_k)$ to denote the graph obtained from the complete graph K_4 with the vertex set $\{t_0, x, y, z\}$ by replacing the edges t_0x, t_0y, t_0z by the paths $t_0x_1 \cdots x_ix, t_0y_1 \cdots y_jy, t_0z_1 \cdots z_kz$, respectively

Claim 31. *H is not isomorphic to $\theta'(i, j, k)$ for any $i \geq j \geq k \geq 1$.*

Proof. By contradiction, then $H = \theta'(t_0, xyzx, x_1, y_1, z_1)$ since G is $T_{2,2,1}$-free. Note that $H/xyzx \cong \theta(1, 1, 1)$, $|E(x_1', H)| = 1$ for $x_1' \in N_G(x_1) \backslash V(H)$ since $G[V(H) \cup \{x_1'\}]$ is not collapsible. Then $G[\{t_0, x_1, x_1', y_1, y_0, z_1\}] \cong T_{2,2,1}$, a contradiction. □

Claim 32. $i_0 \geq 3$.

Proof. By contradiction, assume that $i_0 = 2$. By Theorem 16 and Claims 30, 31, $H = P(i, j, k) = P(xyz, x'y'z', x_1 \cdots x_i, y_1 \cdots y_j, z_1 \cdots z_k)$ which is obtained from two vertex-disjoint triangles $xyzx$ and $x'y'z'x'$ by adding three vertex-disjoint paths $xx_1 \cdots x_ix'$, $yy_1 \cdots y_jy'$ and $zz_1 \cdots z_kz'$ for $i \geq j \geq k \geq 1$. Then $k \geq 2$; for otherwise, z_1 has a neighbor z_1' outside $V(H)$ with $|E(z_1', H)| = 1$ since $G[\{x, y, z, x', y', z', z_1, z_1'\}]$ is not collapsible by Property P, and hence $G[\{x, z, x', z', z_1, z_1'\}] \cong T_{2,2,1}$, a contradiction. Furthermore, any one of $\{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8\}$ is an induced subgraph of H if $j \geq 4$ or $i \geq 4$ and $j = 3$ or $i \geq 5$ and $j = 2$ and we are done. It remains to consider the case when $H \in \{P(2, 2, 2), P(3, 2, 2), P(3, 3, 2), P(4, 2, 2), P(3, 3, 3)\}$.

First assume that $H \in \{P(2, 2, 2), P(3, 2, 2), P(3, 3, 2), P(4, 2, 2)\}$. If there are two vertices $z_i' \in N_G(z_i) \backslash V(H)$ for $i \in \{1, 2\}$ with $\{z_1'z_2, z_2'z_1\} \cap E(G) = \emptyset$, then $\{z_1'z, z_2'z\} \nsubseteq E(G)$, $\{z_1'z', z_2'z'\} \nsubseteq E(G)$ and $\{z_1'z, z_2'z'\} \nsubseteq E(G)$ by Property P. Furthermore, $\{z_1'z', z_2'z\} \subseteq E(G)$ since $G[\{z_1, z, x, z_2, z'\}] \ncong T_{2,2,1}$ and $G[\{z_2, z', x', z_1, z, z_2'\}] \ncong T_{2,2,1}$. Then $z''z_2 \in E(G)$ for any $z'' \in N_G(z_1')$ since $G[\{z', z_2, z_2', z_1', z'', x'\}] \ncong T_{2,2,1}$ and $G[\{x, y, z, z_1, z_2, x', y', z', z_1', z_2', z''\}]$ is not collapsible and hence $G[\{z_2, z', x', z_1, z, z''\}] \cong T_{2,2,1}$, a contradiction. This implies that z_1, z_2 have at least one common neighbor z_0 outside $V(H)$. Since $G[V(H) \cup \{z_0\}]$ is not collapsible, $|E(z_0, H)| = 2$. Hence any of $\{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8\}$ is an induced subgraph of $G[V(H) \cup \{z_0\}]$, a contradiction.

Thus $H = P(3,3,3)$. Note that any of $\{N_{2,2,4}, B_{2,6}, B_{3,4}\}$ is an induced subgraph of H, it suffices to obtain an induced subgraph of G isomorphic to Z_8. We first claim that at least one pair of $\{x_1, x_3\}, \{y_1, y_3\}, \{z_1, z_3\}$ has at least one common neighbour. Suppose otherwise. Let x_i', y_i', z_i' be the neighbours of x_i, y_i, z_i outside H for $i \in \{1, 3\}$, respectively. Then $\{z_1'z_3, z_3'z_1\} \cap E(G) = \emptyset$. Since $G[\{z_1, z, x, z_2, z_3, z_1'\}] \ncong T_{2,2,1}$ and $G[\{z_3, z', x', z_2, z_1, z_3'\}] \ncong T_{2,2,1}$, $E(z_1', \{z_2, z, x\}) \neq \emptyset$ and $E(z_3', \{z_2, z', x'\}) \neq \emptyset$. By symmetry, $E(x_1', \{x_2, x, y\}) \neq \emptyset$, $E(x_3', \{x_2, x', y'\}) \neq \emptyset$, $E(y_1', \{y_2, y, z\}) \neq \emptyset$ and $E(y_3', \{y_2, y', z'\}) \neq \emptyset$. Note $\theta(1,1,1) \subseteq G[V(H) \cup \{x_1', x_3', y_1', y_3', z_1', z_3'\}]/(xyzx, x'y'z'x')$. Then $|E(x_1', H)| = |E(x_3', H)| = |E(y_1', H)| = |E(y_3', H)| = |E(z_1', H)| = |E(z_3', H)| = 1$ and $\{x_1'y, x_1'z\} \cap E(G) = \emptyset$ since G is $T_{2,2,1}$-free. Hence $E(x_1', \{x_2, x\}) \neq \emptyset$ and there is an induced subgraph isomorphic to Z_8, a contradiction. Without loss of generality, we may assume that z_1, z_3 has a common neighbour z_{13}. By Property P, $E(z_{13}, \{x, y, z, x', y', z'\}) = \emptyset$. Then $E(z_{13}, V(H) \backslash \{z_2\}) = \emptyset$ since G is $T_{2,2,1}$-free. If $z_{13}z_2 \notin E(G)$, then there is a vertex $z_{13}' \in N_G(Z_{13}) \backslash \{z_1, z_3\}$ such that $E(z_{13}', \{z_1, z_3, zz'\}) \neq \emptyset$ since $G[z_{13}', z_{13}, z_1, z_3, z, z'] \ncong T_{2,2,1}$. By Property P, $|E(z_{13}', \{x, y, z, z_1\})| \leq 1$. Then $z_{13}'z \notin E(G)$, since otherwise, $E(z_{13}', \{x_1, z_2\} = \emptyset$ by Property P and $G[\{z, x, x_1, z_1, z_2, z_{13}'\}] \cong T_{2,2,1}$. By symmetry, $z_{13}'z' \notin E(G)$. Then $E(z_{13}', \{z_1, z_3\} \neq \emptyset$. By Property P, $E(z_{13}', \{x_1, y_1, x_3, y_3\}) = \emptyset$ and then $E(z_{13}', \{x_2, y_2\}) = \emptyset$, and hence there is an induced subgraph isomorphic to Z_8, a contradiction. This implies that $z_{13}z_2 \in E(G)$. By symmetry, we can prove that x_1, x_2, x_3 have a common neighbor x_{13} and y_1, y_2, y_3 have a common neighbor y_{13}. Hence either there is an induced subgraph isomorphic to Z_8 or $G[V(H) \cup \{x_{13}, y_{13}, z_{13}\}]$ is collapsible, a contradiction. □

Then $W_{i_0}^*$ has a center vertex v and spoke-vertices $V_3(W_{i_0}^*) \cup V_2(W_{i_0}^*) = \{v_1, \ldots, v_{i_0}\} \cup \{v_1', \ldots, v_{i_0}'\}$ with $i_0 \geq 3$. If $H[N_H[X_v]]$ has an induced $N_{i,j,k}$ for some i, j, k, then $H[N_H[X_{v_t}]]$ has an induced N_{i_t,j_t,k_t} for some i_t, j_t, k_t and $t \in \{1, \ldots, i_0\}$ since G is $T_{2,2,1}$-free. Therefore, any of $\{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8\}$ is a subgraph of G, a contradiction. Then $H[N_H[X_v]]$ has an induced $T_{i,j,k}$ for some i, j, k by Theorem 16. Furthermore, $H[N_H[X_{v_t}]]$ has an induced T_{i_t,j_t,k_t} for some integers i_t, j_t, k_t and $t \in \{1, \ldots, i_0\}$ since G is $T_{2,2,1}$-free, which means H is isomorphic to the subdivision of $W_{i_0}^*$. Since G is $T_{2,2,1}$-free, $H \in \{F_9, F_{10}\}$ depicted in Figure 2. For $H \cong F_{10}$, as our discussion in Theorem 20, v_i has a neighbor u_i

outside $V(H)$ with $|E(u_i, H)| = 1$ for $i \in \{1, 2, 3\}$. Furthermore, at least two of $\{u_1, u_2, u_3\}$ are nonadjacent in G since $G[V(H) \cup \{u_1, u_2, u_3\}]$ is not collapsible. By symmetry, assume that $u_1u_2 \notin E(G)$. Thus $G[\{t_1, v_1, v_2, u_1, u_2, t\}] \cong T_{2,2,1}$, a contradiction. For $H \cong F_9$, we have that $G[\{t_1, v_1, v_4, v_2, u_2, t\}] \cong T_{2,2,1}$ for $u_2 \in N_G(v_2)\backslash V(H)$, a contradiction. This completes the proof of (1).

If G satisfies (2) or (3), especially G is P_7-free, then G is supereulerian or $P(10)$ if $\kappa(G) = 2$ by Theorem 20, a contradiction. Then assume that $\kappa(G) = 1$. By Corollary 28, there is a block B_0 of G and an induced path $P(v_1, v_2) \subseteq B_0$ with $V_2(B_0) \cap V(P(v_1, v_2)) = \{v_1, v_2\}$ and $|E(P(v_1, v_2))| \geq 2$. Note that v_1, v_2 are cut-vertices, there are two blocks B_1, B_2 such that $V(B_1) \cap V(B_0) = \{v_1\}$ and $V(B_2) \cap V(B_0) = \{v_2\}$.

(2) Then v_1, v_2 are not in collapsible subgraphs of G. Suppose otherwise. Replace G by the graph G^* obtained by adding vertex set $\{x_i, y_i\}$ and edge set $X_i = \{v_ix_i, v_iy_i, u_ix_i, u_iy_i, x_iy_i\}$ for $u_i \in N_{B_0}(v_i)$ if v_i is in a collapsible subgraph of B_0 for $i \in \{1, 2\}$ since G^* is 2-edge-connected $\{T_{2,2,2}, P_7\}$-free, $|V_{\leq 3}(G^*)| \leq |V_{\leq 3}(G)|$ and G is supereulerian if and only if G^* is supereulerian. Let $x_1 \in N_{B_1}(v_1)$ and $x_2 \in N_{B_2}(v_2)$. Since $\kappa(B_0) \geq 2$, there is an induced cycle $C \subseteq B_0$ with vertices v_1, v_2. Since G is P_7-free, $4 \leq l(C) \leq 6$. Then we claim that $d_C(v_1, v_2) = 3$. Suppose otherwise. Assume that $u_1u_{12}u_2 \subseteq C$ and $u \in N_G(u_{12})$ with $E(u, \{v_1, v_2\}) = \emptyset$. Let $u_1, u_2 \in N_G(u)$. By symmetry, $\{u_1v_1, u_2v_2\} \subseteq E(G)$ since $G[\{u_{12}, v_1, x_1, v_2, x_2, uu_i\}] \ncong T_{2,2,2}$ for $i \in \{1, 2\}$. However $G[\{x_1, v_1, u_1, u, u_2, v_2, x_2\}] \cong P_7$, a contradiction. Hence, $l(C) = 6$ and assume that $C = v_1u_1u_2v_2w_2w_1v_1$. Then $E(w_1', \{u_1, u_2\}) \neq \emptyset$ for $w_1' \in N_G(w_1)$ since G is P_7-free. Furthermore, $|E(w_1', C)| = 2$ since $G[V(C) \cup \{w_1'\}]$ is not collapsible, and $w_1''w_2 \in E(G)$ since $G[\{w_1, v_1, x_1, w_2, v_2, w_1', w_1''\}] \ncong T_{2,2,2}$ for $w_1'' \in N_G(w_1')\backslash V(C)$.

Then $w_1'u_2 \in E(G)$; for otherwise, $w_1'u_1 \in E(G)$ and $|E(w_1'', C)| = 1$ since $G[V(C) \cup \{w_1', w_1''\}]$ is not collapsible, and then $G[\{w_1'', w_2, v_2, u_2, u_1, v_1, x_1\}] \cong P_7$, a contradiction. Furthermore, $w_1''u_2 \notin E(G)$ since $G[V(C) \cup \{w_1', w_1''\}]$ is not collapsible and $w_1''u_1 \in E(G)$ since $G[\{w_1'', w_2, v_2, u_2, u_1, v_1, x_1\}] \ncong P_7$. Then $G[\{x_2, v_2, w_2, w_1'', u_1, v_1, x_1\}] \cong P_7$, a contradiction.

(3) We will obtain an induced subgraph of G isomorphic to any one in $\{M_1, M_2\}$. Since G is P_6-free, $V(B_i) \subseteq N_G(v_i)$ for $i \in \{1, 2\}$ and v_1, v_2 have at least two common neighbors u_1, u_2 in B_0 with degree 3. Then there is an induced M_1 of $G[V(B_1) \cup V(B_2) \cup \{u_1, u_2\}]$.

Furthermore, we claim that u_1, u_2 have a common neighbor. Suppose otherwise. Let $w_1 \in N_G(u_1) \setminus \{v_1, v_2\}$. Then w_1 has two neighbors z_1, z_2 such that $E(\{z_1, z_2\}, \{v_1, v_2, u_1\}) = \emptyset$. In addition, $z_1z_2 \notin E(G)$; for otherwise, there is an induced M_2 of $G[V(B_1) \cup V(B_2) \cup \{u_1, w_1, z_1, z_2\}]$, a contradiction. Furthermore, $E(u_2, \{z_1, z_2\}) = \emptyset$; for otherwise, $\{z_1u_2, z_2u_2\} \subseteq E(G)$, and then z_1 has a neighbor z_1' with $E(z_1', \{v_1, v_2, u_1, u_2, w_1\}) = \emptyset$ and hence it would produce an induced P_6, a contradiction. However u_2 has a neighbor w_2 such that either $z_1w_1u_1v_1u_2w_2$ (if $z_1w_2 \notin E(G)$) or $w_1z_1w_2u_2v_1v_1'$ (if $z_1w_2 \in E(G)$) would be an induced P_6 in G, a contradiction. Let v_0 be one common neighbor of u_1, u_2 other than $\{v_1, v_2\}$. Then v_0 has a neighbor u_0 other than $\{u_1, u_2\}$ and $E(u_0, \{v_1, v_2, u_1, u_2\}) = \emptyset$. Therefore u_0 has a neighbor u_0' such that $E(u_0', \{v_1, v_2\}) = \emptyset$ and $\{u_0'u_1, u_0'u_2\} \nsubseteq E(G)$ since $G[v_1, v_2, v_0, u_1, u_2, u_0, u_0']$ is not collapsible. However there is either an induced P_6 (if $u_0'v_0 \notin E(G)$) or an induced M_2 (if $u_0'v_0 \in E(G)$) in $G[V(B_1) \cup V(B_2) \cup \{u_1, v_0, u_0, u_0'\}]$, a contradiction.

(4) Suppose that $S = x_1x_2 \cdots x_{i-1}x_ix_{i+1} \cdots x_lx_1$ is a closed trail of G with a fixed orientation such that $|V(S)|$ is maximized and $V(G)\backslash V(S) \neq \emptyset$. Let $x_i^+ = x_{i+1}, x_i^- = x_{i-1}, x_i^{h+} = x_{i+h}$ and $x_i^{h-} = x_{i-h}$ (all subscribes are taken module by (l)). Then there is a vertex v outside S such that v is adjacent to a vertex x_1 of S since G is connected. Since $G[\{x_1, x_1^+, x_1^-, v, v'\}] \ncong T_{2,1,1}$ for some $v' \in N_G(v)$, $\{x_1^+x_1^-, vx_1^{2+}, vx_1^{2-}\} \cap E(G) \neq \emptyset$. We claim that $E(v, \{x_1^{2+}, x_1^{2-}\}) \neq \emptyset$. Suppose otherwise. Then $x_1^+x_1^- \in E(G)$. Note that there are two vertices v', v'' in $N_G(v)$ with $E(\{v', v''\}, \{x_1, x_1^+, x_1^-\}) = \emptyset$. Since $G[\{v, x_1, x_1^+, v', v''\}] \ncong T_{2,1,1}$, $v'v'' \in E(G)$. Then $G[\{x_1, x_1^+, x_1^-, v, v', v''\}] \cong H_1$, a contradiction. By symmetry, assume that $vx_1^{2+} \in E(G)$. Since $[\{v, x_1, x_1^-, x_1^{2+}, v'\}] \ncong T_{2,1,1}$ for $v' \in N_G(v)$, $vx_1^{2-} \in E(G)$. However $G[\{x_1, x_1^+, x', x_1^-, v\}] \cong T_{2,1,1}$ for $x' \in N_G(x_1^+)$, a contradiction. This completes the proof of (4) and the whole theorem. ■

4. Conclusion

The following theorem here indicates that the forbidden pairs in Theorem 3 except the pair $\{P_7, T_{2,2,2}\}$ are sharp and it remains to prove 2-edge-connected $\{R, S\}$-free graph G with $\delta(G) \geq 3$ is supereulerian for $\{R, S\} = \{Z_5, T_{2,2,2}\}$ or one of them is Z_3 to completely characterize forbidden pairs. Here H_2 (H_3) is the graph obtained from H_1 by contracting (subdividing) the edge which is not in any triangle of H_1 and H_4 is the graph obtained from H_1 by adding a pendant edge to the vertex of a triangle of H_1.

Theorem 33. *Let R, S be two connected graphs. If every 2-edge-connected $\{R, S\}$-free graph G with order at least 11 and $\delta(G) \geq 3$ implies that it is supereulerian, then $\{R, S\} \preceq \{T_{2,1,1}, H_1\}$, $\{T_{2,2,1}, N_{2,2,4}\}$, $\{T_{2,2,1}, B_{2,6}\}$, $\{T_{2,2,1}, B_{3,4}\}$, $\{T_{2,2,1}, Z_8\}$, $\{P_6, M_1\}$, $\{P_6, M_2\}$, $\{T_{2,2,2}, Z_5\}$, $\{Z_3, T_1\}$, $\{Z_3, T_2\}$, $\{Z_3, T_3\}$, $\{Z_3, T_4\}$, $\{Z_3, T_5\}$, $\{Z_3, T_6\}$, where $T_1, \ldots, T_6$ are depicted in Figure 3.*

Proof. All graphs in Figure 4 are 2-edge-connected nonsupereulerian graphs with minimum degree at least three. Then each graph contains at least one of R, S as an induced subgraph. Without loss of generality, assume that G_1 contains R

as an induced subgraph. Then R either is a tree with maximum degree at most 3 or contains a cycle as an induced subgraph. Now we distinguish the following two cases.

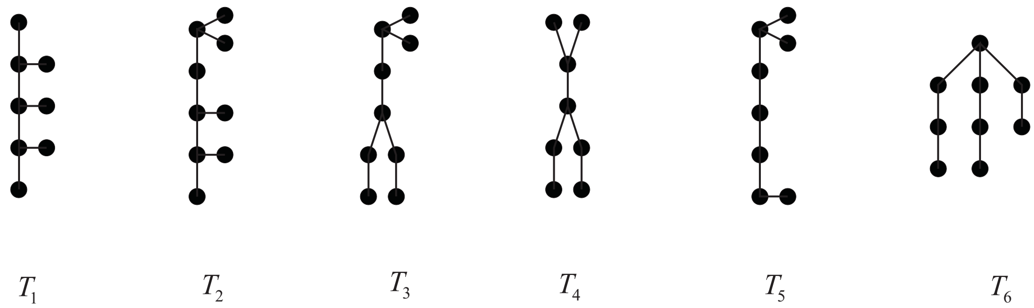

Figure 3. Some common induced subgraphs.

Case 1. R is a tree. If $\Delta(R) = 2$, then R is an induced subgraph of P_5, a contradiction. If $\Delta(R) = 3$, then $|V_3(R)| = 1$ and R is an induced subgraph of $T_{2,2,2}$. Since G_6, G_7, G_8, G_9 are $K_{1,3}$-free, S is an induced subgraph of G_6, G_7, G_8, G_9. Note that G_8 is K_4-free. Then R is K_4-free. Since G_7 is both H_2-free and H_3-free and G_9 is H_4-free, any maximal common induced subgraph of G_7, G_9 contains at most two triangles, and hence it is isomorphic to H_1 if it contains exactly two triangles. Since G_6 is $B_{4,4}$-free and G_7 is $\{N_{2,2,5}, B_{2,7}, B_{3,5}, Z_9\}$-free, the maximal common induced subgraphs containing exactly one triangle of G_6, G_7 are $N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8$. Since the common induced cycle of G_7, G_8 is C_3 and G_7 is P_{11}-free, the maximal common induced subgraph containing no triangle of G_7, G_8 is P_{10}. Therefore, S is an induced subgraph of a graph in $\{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8, H_1\}$.

On the other hand, since G_{10} is $\{H_1, T_{2,2,1}\}$-free, $\{R, S\} \preceq \{H_1, T_{2,1,1}\}$. Since G_4 is R-free and $T_{2,2,2}$-free for R is an induced subgraph of a graph in $\{N_{2,2,4}, B_{2,6}, B_{3,4}, Z_8\}$, $\{R, S\} \preceq \{N_{2,2,4}, T_{2,2,1}\}, \{B_{2,6}, T_{2,2,1}\}, \{B_{3,4}, T_{2,2,1}\}$ or $\{Z_8, T_{2,2,1}\}$. Since G_4 is $\{B_{1,1}, Z_6, P_8\}$-free, $\{R, S\} \preceq \{Z_5, T_{2,2,2}\}$.

Case 2. R is not a tree. Then R contains only a C_3 or C_4 as an induced subgraph since G_1 is C_k-free for $k \geq 5$.

Subcase 2.1. R contains C_4 as an induced subgraph. If R contains $K_{2,3}$ as an induced subgraph, then G_2, G_8, G_{11} are $K_{2,3}$-free, and G_2 is P_6-free, which means S is an induced subgraph of P_5, a contradiction. Then R is an induced subgraph of M_1. Note that G_8, G_5, G_{11} are C_4-free and G_5 is P_7-free, which means S is an induced subgraph of P_6. Therefore, $\{R, S\} \preceq \{M_1, P_6\}$.

Subcase 2.2. R contains K_r $(r \geq 4)$. Note that G_3, G_8, G_{11} are K_4-free, G_8 is $\{K_{1,3}, C_4\}$-free and G_{11} is C_3-free. Then R is an induced subgraph of path. Since G_3 is P_6-free, $S \subseteq P_5$, a contradiction.

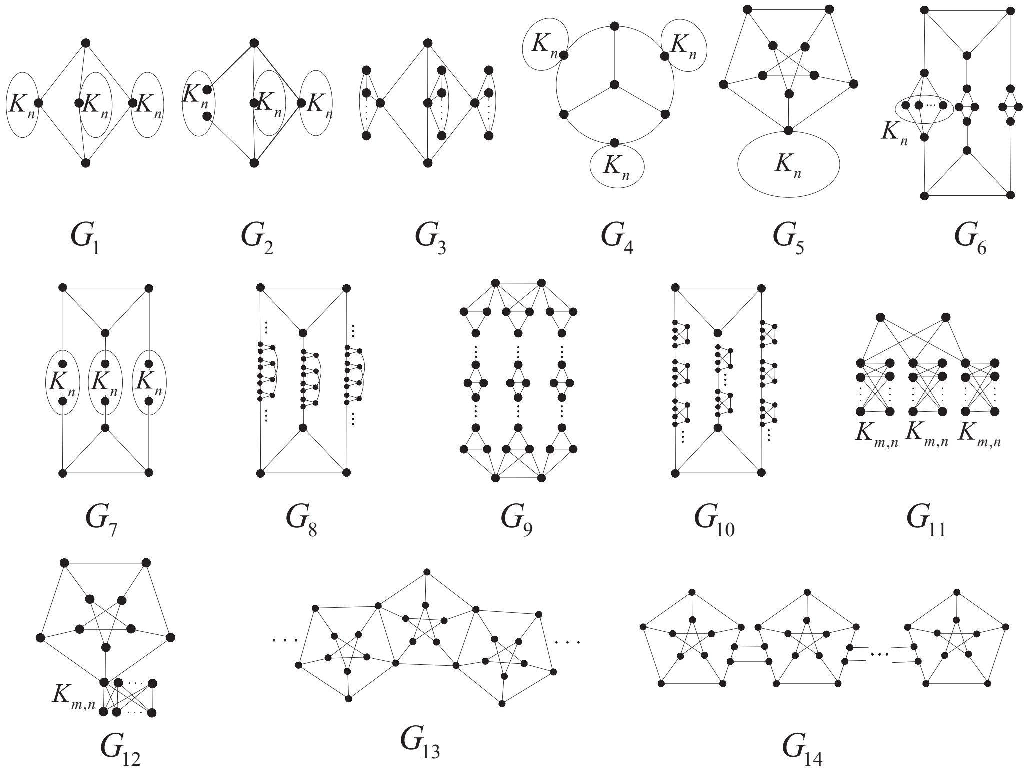

Figure 4. The graphs that are nonsupereulerian.

Subcase 2.3. R is K_4-free and C_4-free. Then R contains C_3 as an induced subgraph and R is $4C_3$-free. If R contains $3C_3$ as an induced subgraph, then R is an induced subgraph of M_2. Since G_5, G_8, G_{11} are M_2-free, $\{R, S\} \preceq \{M_2, P_6\}$.

If R contains $2C_3$ as an induced subgraph, then R is an induced subgraph of M_1. If R is $2C_3$-free, then R is an induced subgraph of Z_3. Note that $G_{11}, G_{12}, G_{13}, G_{14}$ are C_3-free. Since G_{11} is C_k-free for $k \geq 5$, G_{13} is C_4-free and $\Delta(G_{14}) = 3$, S is a tree with $\Delta(S) \leq 3$ and $|V_3(S)| \leq 3$. Then S is an induced subgraph of the common induced subgraphs of G_{11} and G_{12} which is one of $\{T_1, \ldots, T_6\}$. ∎

Considering the proof idea of Theorem 3(1), we believe the existence of a connected spanning even subgraph in a 2-edge-connected $\{Z_5, T_{2,2,2}\}$-free graph G with $\delta(G) \geq 3$.

Conjecture 34. *Every* 2*-edge-connected* $\{Z_5, T_{2,2,2}\}$*-free graph* G *with* $\delta(G) \geq 3$ *is supereulerian.*

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