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ON THE α-SPECTRAL RADIUS OF UNIFORM HYPERGRAPHS

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Abstract

For $0 \leq \alpha < 1$ and a uniform hypergraph G, the α-spectral radius of G is the largest H-eigenvalue of $\alpha\mathcal{D}(G)+(1-\alpha)\mathcal{A}(G)$, where $\mathcal{D}(G)$ and $\mathcal{A}(G)$ are the diagonal tensor of degrees and the adjacency tensor of G, respectively. We give upper bounds for the α-spectral radius of a uniform hypergraph, propose some transformations that increase the α-spectral radius, and determine the unique hypergraphs with maximum α-spectral radius in some classes of uniform hypergraphs.

Keywords: α-spectral radius, α-Perron vector, adjacency tensor, uniform hypergraph, extremal hypergraph.

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1. Introduction

Let G be a hypergraph on n vertices with vertex set $V(G)$ and edge set $E(G)$. If $|e| = k$ for each $e \in E(G)$, then G is said to be a k-uniform hypergraph. For a vertex $v \in V(G)$, the set of the edges containing v in G is denoted by $E_G(v)$, and the degree of v in G, denoted by $d_G(v)$ or d_v, is the size of $E_G(v)$. We say that G is regular if all vertices of G have equal degrees. Otherwise, G is irregular.

For $u, v \in V(G)$, a walk from u to v in G is defined to be an alternating sequence of vertices and edges $(v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ with $v_0 = u$ and $v_s = v$ such that edge e_i contains vertices v_{i-1} and v_i, and $v_{i-1} \neq v_i$ for $i = 1, \ldots, s$. The value s is the length of this walk. A path is a walk with all v_i distinct and all e_i

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distinct. A cycle is a walk containing at least two edges, all e_i are distinct and all v_i are distinct except $v_0 = v_s$. If there is a path from u to v for any $u, v \in V(G)$, then we say that G is connected. A hypertree is a connected hypergraph with no cycles. For $k \geq 2$, the number of vertices of a k-uniform hypertree with m edges is $1 + (k-1)m$.

The distance between vertices u and v in a connected hypergraph G is the length of a shortest path from u to v in G. The diameter of connected hypergraph G is the maximum distance between any two vertices of G.

For positive integers k and n, a tensor $\mathcal{T} = (\mathcal{T}_{i_1\cdots i_k})$ of order k and dimension n is a multidimensional array with entries $\mathcal{T}_{i_1\cdots i_k} \in \mathbb{C}$ for $i_j \in [n] = \{1, \dots, n\}$ and $j \in [k]$, where $\mathbb{C}$ is the complex field.

Let $\mathcal{M}$ be a tensor of order $k \geq 2$ and dimension n, and $\mathcal{N}$ a tensor of order $\ell \geq 1$ and dimension n. The product $\mathcal{M}\mathcal{N}$ is the tensor of order $(k-1)(\ell-1)+1$ and dimension n with entries [22]

$$(\mathcal{M}\mathcal{N})_{ij_1\cdots j_{k-1}} = \sum_{i_2,\dots,i_k \in [n]} \mathcal{M}_{ii_2\cdots i_k} \mathcal{N}_{i_2 j_1} \cdots \mathcal{N}_{i_k j_{k-1}},$$

with $i \in [n]$ and $j_1, \dots, j_{k-1} \in [n]^{\ell-1}$. Then for a tensor $\mathcal{T}$ of order k and dimension n and an n-dimensional vector $x = (x_1, \dots, x_n)^\top$, $\mathcal{T}x$ is an n-dimensional vector whose i-th entry is

$$(\mathcal{T}x)_i = \sum_{i_2,\dots,i_k=1}^{n} \mathcal{T}_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k},$$

where $i \in [n]$. For some complex λ, if there is a nonzero vector x such that

$$\mathcal{T}x = \lambda \left(x_1^{k-1}, \dots, x_n^{k-1} \right)^\top,$$

then λ is called an eigenvalue of $\mathcal{T}$, and x is called an eigenvector of $\mathcal{T}$ corresponding to λ. Moreover, if both λ and x are real, then we call λ an H-eigenvalue and x an H-eigenvector of $\mathcal{T}$. See [10, 18, 20] for more details. The spectral radius of $\mathcal{T}$ is the largest modulus of its eigenvalues, denoted by $\rho(\mathcal{T})$.

Let G be a k-uniform hypergraph with vertex set $V(G) = \{v_1, \dots, v_n\}$, where $k \geq 2$. The adjacency tensor of G is defined in [1] as the tensor $\mathcal{A}(G)$ of order k and dimension n whose $(i_1, \dots, i_k)$-entry is $\frac{1}{(k-1)!}$ if $\{v_{i_1}, \dots, v_{i_k}\} \in E(G)$, and 0 otherwise. The degree tensor of G is the diagonal tensor $\mathcal{D}(G)$ of order k and dimension n with $(i, \dots, i)$-entry to be the degree of vertex $v_i \in [n]$. Then $\mathcal{Q}(G) = \mathcal{D}(G) + \mathcal{A}(G)$ is the signless Laplacian tensor of G [20]. Motivated by work of Nikiforov [14] (see also [5, 15]), Lin *et al.* [11] proposed to study the convex linear combinations $\mathcal{A}_\alpha(G)$ of $\mathcal{D}(G)$ and $\mathcal{A}(G)$ defined by

$$\mathcal{A}_\alpha(G) = \alpha\mathcal{D}(G) + (1-\alpha)\mathcal{A}(G),$$

where $0 \le \alpha < 1$. The α-spectral radius of G is the spectral radius of $\mathcal{A}_\alpha(G)$, denoted by $\rho_\alpha(G)$. Note that $\rho_0(G)$ is the spectral radius of G, while $2\rho_{1/2}(G)$ is the signless Laplacian spectral radius of G.

For $k \ge 2$, let G be a k-uniform hypergraph with $V(G) = [n]$, and x a n-dimensional column vector. Let $x_V = \prod_{v \in V} x_v$ for $V \subseteq V(G)$. Then

$$x^\top(\mathcal{A}_\alpha(G)x) = \alpha \sum_{u \in V(G)} d_u x_u^k + (1-\alpha)k \sum_{e \in E(G)} x_e,$$

or equivalently,

$$x^\top(\mathcal{A}_\alpha(G)x) = \sum_{e \in E(G)} \left(\alpha \sum_{u \in e} x_u^k + (1-\alpha)k x_e \right).$$

For a uniform hypergraph G, bounds for the spectral radius $\rho_0(G)$ have been given in [1, 12, 13, 29], and bounds for the signless Laplacian spectral radius $2\rho_{1/2}(G)$ may be found in [6, 12, 21]. Recently, Lin *et al.* [11] gave upper bounds for α-spectral radius of connected irregular k-uniform hypergraphs, extending some known bounds for ordinary graphs. Some hypergraph transformations have been proposed to investigate the change of the 0-spectral radius, and the unique hypergraphs that maximize or minimize the 0-spectral radius have been determined among some classes of uniform hypergraphs (especially for hypertrees), see, e.g., [2, 4, 8, 16, 24, 25, 28, 31].

In this paper, we give upper bounds for the α-spectral radius of a uniform hypergraph, propose some hypergraph transformations that increase the α-spectral radius, and determine the unique hypergraphs with maximum α-spectral radius in some classes of uniform hypergraphs such as the class of k-uniform hypercacti with m edges and r cycles for $0 \le r \le \lfloor \frac{m}{2} \rfloor$, and the class of k-uniform hypertrees with m edges and diameter $d \ge 3$.

2. Preliminaries

A tensor $\mathcal{T}$ of order $k \ge 2$ and dimension n is said to be weakly reducible, if there is a nonempty proper subset J of $[n]$ such that for $i_1 \in J$ and $i_j \in [n] \setminus J$ for some $j = 2, \ldots, k$, $\mathcal{T}_{i_1 \cdots i_k} = 0$. Otherwise, $\mathcal{T}$ is weakly irreducible.

For $k \ge 2$, an n-dimensional vector x is said to be k-unit if $\sum_{i=1}^n x_i^k = 1$.

Lemma 1 [3, 27]**.** *Let $\mathcal{T}$ be a nonnegative tensor of order $k \ge 2$ and diminsion n. Then $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ and there is a k-unit nonnegative eigenvector corresponding to $\rho(\mathcal{T})$. If furthermore $\mathcal{T}$ is weakly irreducible, then there is a unique k-unit positive eigenvector corresponding to $\rho(\mathcal{T})$.*

If G is a k-uniform hypergraph with $k \geq 2$, then $\mathcal{A}_\alpha(G)$ is weakly irreducible if and only if G is connected (see [17, 20] for the treatment of $\mathcal{A}_0(G)$ and $2\mathcal{A}_{1/2}(G)$, respectively). Thus, if G is connected, then by Lemma 1, there is a unique k-unit positive H-eigenvector x corresponding to $\rho_\alpha(G)$, which is called the α-Perron vector of G.

For a nonnegative tensor $\mathcal{T}$ of order $k \geq 2$ and dimension n, let $r_i(\mathcal{T}) = \sum_{i_2\cdots i_k=1}^{n} \mathcal{T}_{ii_2\cdots i_k}$ for $i = 1, \ldots, n$.

Lemma 2 [7, 27]**.** *Let $\mathcal{T}$ be a nonnegative tensor of order $k \geq 2$ and dimension n. Then*

$$\rho(\mathcal{T}) \leq \max_{1\leq i\leq n} r_i(\mathcal{T})$$

with equality when $\mathcal{T}$ is weakly irreducible if and only if $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T})$.

For two tensors $\mathcal{M}$ and $\mathcal{N}$ of order $k \geq 2$ and dimension n, if there is an $n \times n$ nonsingular diagonal matrix U such that $\mathcal{N} = U^{-(k-1)}\mathcal{M}U$, then we say that $\mathcal{M}$ and $\mathcal{N}$ are diagonal similar.

Lemma 3 [22]**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two diagonal similar tensors of order $k \geq 2$ and dimension n. Then $\mathcal{M}$ and $\mathcal{N}$ have the same real eigenvalues.*

Let G be a connected k-uniform hypergraph on n vertices, where $k \geq 2$. Let $0 \leq \alpha < 1$. For an n-dimensional k-unit nonnegative vector x, by [19, Theorem 2] (and its proof) and Lemma 1, we have $\rho_\alpha(G) \geq x^\top(\mathcal{A}_\alpha(G)x)$ with equality if and only if x is the α-Perron vector of G. If x is the α-Perron vector of G, then for any $v \in V(G)$,

$$\rho_\alpha(G)x_v^{k-1} = \alpha d_v x_v^{k-1} + (1-\alpha)\sum_{e\in E_v(G)} x_{e\setminus\{v\}},$$

which is called the eigenequation of G at v.

For a hypergraph G with $\emptyset \neq X \subseteq V(G)$, let $G[X]$ be the subhypergraph induced by X, i.e., $G[X]$ has vertex set X and edge set $\{e \subseteq X : e \in E(G)\}$. If $E' \subseteq E(G)$, then $G - E'$ is the hypergraph obtained from G by deleting the edges in E'. If E' is set of subsets of $V(G)$ and no element of E' is an edge of G, then $G + E'$ is the hypergraph obtained from G by adding elements of E' as edges.

A k-uniform hypertree with m edges is a hyperstar, denoted by $S_{m,k}$, if all edges share a common vertex. A k-uniform loose path with $m \geq 1$ edges, denoted by $P_{m,k}$, is the k-uniform hypertree whose vertices and edges may be labelled as $(v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m)$ such that the vertices $v_1, \ldots, v_{m-1}$ are of degree 2, and all the other vertices of G are of degree 1.

If P is a path or a cycle of a hypergraph G, $V(P)$ denotes the vertex set of the hypergraph P.

3. Upper Bounds for α-Spectral Radius

For a connected irregular k-uniform hypergraph G with n vertices, maximum degree Δ and diameter D, where $2 \leq k < n$, it was shown in [11] that for $0 \leq \alpha < 1$,

$$\rho_\alpha(G) < \Delta - \frac{4(1-\alpha)}{((4D-1-2\alpha)(k-1)+1)\,n}.$$

For a k-uniform hypergraph G, upper bounds on $\rho_0(G)$ and $2\rho_{1/2}(G)$ have been given in [12, 29].

Theorem 4. *Let G be a k-uniform hypergraph on n vertices with maximum degree Δ and second maximum degree Δ', where $k \geq 2$. For $\alpha = 0$, let $\delta = \left(\frac{\Delta}{\Delta'}\right)^{\frac{1}{k}}$, and for $0 < \alpha < 1$, let $\delta = 1$ if $\Delta = \Delta'$ and δ be a root of $h(t) = 0$ in $((\frac{\Delta}{\Delta'})^{\frac{1}{k}}, +\infty)$ if $\Delta > \Delta'$, where $h(t) = (1-\alpha)\Delta' t^k + \alpha(\Delta' - \Delta)t^{k-1} - (1-\alpha)\Delta$ for $0 \leq \alpha < 1$. Then*

$$\rho_\alpha(G) \leq \alpha\Delta + (1-\alpha)\Delta\delta^{-(k-1)}. \tag{1}$$

Moreover, if G is connected, then equality holds in (1) if and only if G is a regular hypergraph or $G \cong G'$, where $V(G') = V(H) \cup \{v\}$, $E(G') = \{e \cup \{v\} : e \in E(H)\}$, and H is a regular $(k-1)$-uniform hypergraph on $n-1$ vertices with $v \notin V(H)$.

Proof. By Theorem 2.1 and Lemma 2.2 in [22], we may assume that $d_1 \geq \cdots \geq d_n$. Then $\Delta = d_1$ and $\Delta' = d_2$.

If $d_1 = d_2$, then $\delta = 1$, and by Lemma 2, we have

$$\rho_\alpha(G) \leq \max_{1 \leq i \leq n} r_i(\mathcal{A}_\alpha(G)) = \max_{1 \leq i \leq n} d_i = d_1 = \alpha d_1 + (1-\alpha) d_1 \delta^{-(k-1)},$$

and when G is connected, $\mathcal{A}_\alpha(G)$ is weakly irreducible, thus by Lemma 2, equality (1) holds if and only if $r_1(\mathcal{A}_\alpha(G)) = \cdots = r_n(\mathcal{A}_\alpha(G))$, i.e., G is a regular hypergraph.

Suppose in the following that $d_1 > d_2$. Let $U = \mathrm{diag}(t, 1, \ldots, 1)$ be an $n \times n$ diagonal matrix, where $t > 1$ is a variable to be determined later. Let $\mathcal{T} = U^{-(k-1)} \mathcal{A}_\alpha(G) U$. By Lemma 3, $\mathcal{A}_\alpha(G)$ and $\mathcal{T}$ have the same real eigenvalues. Obviously, both $\mathcal{A}_\alpha(G)$ and $\mathcal{T}$ are nonnegative tensors of order k and dimension n. By Lemma 1, $\rho(\mathcal{A}_\alpha(G))$ is an eigenvalue of $\mathcal{A}_\alpha(G)$ and $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$. Therefore $\rho_\alpha(G) = \rho(\mathcal{A}_\alpha(G)) = \rho(\mathcal{T})$. For $i \in [n] \setminus \{1\}$, let $d_{1,i} = |\{e : 1, i \in e \in E(G)\}|$. Obviously, $d_{1,i} \leq d_i$. Note that

$$\begin{aligned}
r_1(\mathcal{T}) &= \sum_{i_2,\dots,i_k\in[n]} \mathcal{T}_{1i_2\cdots i_k} \\
&= \alpha\mathcal{D}_{1\cdots 1} + (1-\alpha) \sum_{i_2,\dots,i_k\in[n]} U_{11}^{-(k-1)} \mathcal{A}_{1i_2\cdots i_k} U_{i_2i_2}\cdots U_{i_ki_k} \\
&= \alpha d_1 + (1-\alpha) \sum_{i_2,\dots,i_k\in[n]\setminus\{1\}} \frac{1}{t^{k-1}} \mathcal{A}_{1i_2\cdots i_k} = \alpha d_1 + \frac{(1-\alpha)d_1}{t^{k-1}},
\end{aligned}$$

and for $2 \le i \le n$,

$$\begin{aligned}
r_i(\mathcal{T}) &= \sum_{i_2,\dots,i_k\in[n]} \mathcal{T}_{ii_2\cdots i_k} = \alpha\mathcal{D}_{i\cdots i} + (1-\alpha) \sum_{i_2,\dots,i_k\in[n]} U_{ii}^{-(k-1)} \mathcal{A}_{ii_2\cdots i_k} U_{i_2i_2}\cdots U_{i_ki_k} \\
&= \alpha d_i + (1-\alpha) \sum_{\substack{i_2,\dots,i_k\in[n]\\ 1\in\{i_2,\dots,i_k\}}} U_{ii}^{-(k-1)} \mathcal{A}_{ii_2\cdots i_k} U_{i_2i_2}\cdots U_{i_ki_k} \\
&+ (1-\alpha) \sum_{\substack{i_2,\dots,i_k\in[n]\\ 1\notin\{i_2,\dots,i_k\}}} U_{ii}^{-(k-1)} \mathcal{A}_{ii_2\cdots i_k} U_{i_2i_2}\cdots U_{i_ki_k} \\
&= \alpha d_i + (1-\alpha) \sum_{\substack{i_2,\dots,i_k\in[n]\\ 1\in\{i_2,\dots,i_k\}}} \mathcal{A}_{ii_2\cdots i_k} t + (1-\alpha) \sum_{\substack{i_2,\dots,i_k\in[n]\\ 1\notin\{i_2,\dots,i_k\}}} \mathcal{A}_{ii_2\cdots i_k} \\
&= \alpha d_i + (1-\alpha) t d_{1,i} + (1-\alpha)(d_i - d_{1,i}) \\
&= d_i + (1-\alpha)(t-1)d_{1,i} \le (1+(1-\alpha)(t-1))d_i \le (1+(1-\alpha)(t-1))d_2
\end{aligned}$$

with equality if and only if $d_{1,i} = d_i = d_2$.

Note that $h\left(\left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}\right) = \alpha(d_2 - d_1)\left(\frac{d_1}{d_2}\right)^{\frac{k-1}{k}} \le 0$ with equality if and only if $\alpha = 0$, and that $h(+\infty) > 0$. Thus $h(t) = 0$ does have a root δ, as required. Let $t = \delta$. Then $t > 1$,

$$\alpha d_1 + \frac{(1-\alpha)d_1}{t^{k-1}} = (1+(1-\alpha)(t-1))d_2,$$

and thus for $1 \le i \le n$,

$$r_i(\mathcal{T}) \le \alpha d_1 + (1-\alpha)d_1\delta^{-(k-1)}.$$

Now by Lemma 2,

$$\rho_\alpha(G) = \rho(\mathcal{T}) \le \max_{1\le i\le n} r_i(\mathcal{T}) \le \alpha d_1 + (1-\alpha)d_1\delta^{-(k-1)}.$$

This proves (1).

Suppose that G is connected. Then $\mathcal{A}_\alpha$ is weakly irreducible, and so is $\mathcal{T}$.

Suppose that equality holds in (1). From the above arguments and by Lemma 2, we have $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T}) = \alpha d_1 + (1-\alpha)d_1\delta^{-(k-1)}$, and $d_{1,i} = d_i = d_2$ for $i = 2, \dots, n$. Then vertex 1 is contained in each edge of G. Let H be the hypergraph with $V(H) = V(G)\backslash\{1\} = \{2, \dots, n\}$ and $E(H) = \{e\backslash\{1\} : e \in E(G)\}$. Then H is a regular $(k-1)$-uniform hypergraph on vertices $2, \dots, n$ of degree d_2. Therefore $G \cong G'$, where $V(G') = V(H) \cup \{1\}$, $E(G') = \{e \cup \{v\} : e \in E(H)\}$, and H is a regular $(k-1)$-uniform hypergraph on vertices $2, \dots, n$ of degree d_2.

Conversely, if $G \cong G'$, where $V(G') = V(H) \cup \{1\}$, $E(G') = \{e \cup \{v\} : e \in E(H)\}$, and H is a regular $(k-1)$-uniform hypergraph on vertices $2, \dots, n$ of degree d_2, then by the above arguments, we have $r_i(\mathcal{T}) = \alpha d_1 + (1-\alpha)d_1(\frac{1}{\delta})^{k-1}$ for $1 \le i \le n$, and thus by Lemma 3, $\rho(\mathcal{A}_\alpha(G)) = \rho(\mathcal{T}) = \alpha d_1 + (1-\alpha)d_1\delta^{-(k-1)}$, i.e., (1) is an equality. ∎

As $\delta \ge \left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}$ with equality if and only if $d_1 = d_2$, we have by Theorem 4 that $\rho_\alpha(G) \le \alpha d_1 + (1-\alpha)d_1^{\frac{1}{k}} d_2^{1-\frac{1}{k}}$ with equality if and only if G is regular.

Letting $\alpha = 0$ in Theorem 4, we have $\delta = \left(\frac{d_1}{d_2}\right)^{\frac{1}{k}}$ and thus (1) becomes $\rho_0(G) \le d_1^{\frac{1}{k}} d_2^{1-\frac{1}{k}}$, see [29]. Letting $\alpha = \frac{1}{2}$ in Theorem 4, δ is the root of $d_2 t^k + (d_2 - d_1)t^{k-1} - d_1 = 0$, and (1) becomes $2\rho_{1/2}(G) \le d_1 + d_1\delta^{-(k-1)}$, see [12].

Let G be a connected k-uniform hypergraph with n vertices, m edges, maximum degree Δ and diameter D, where $k \ge 2$. For $0 \le \alpha < 1$, let $\overline{x}$ be the maximum entry of the α-Perron vector of G. From [11], we have

$$\rho_\alpha(G) \le \Delta - \frac{(1-\alpha)k(n\Delta - km)}{2(n\Delta - km)(k-1)D + (1-\alpha)k}\overline{x}^k,$$

and if $D = 1$ and $k \ge 3$, then

$$\rho_\alpha(G) \le \Delta - \frac{(1-\alpha)(n\Delta - km)n}{2(n\Delta - km)(k-1) + (1-\alpha)n}\overline{x}^k.$$

Theorem 5. *Let G be a connected k-uniform hypergraph on n vertices with m edges and maximum degree Δ, where $k \ge 2$. Let x be the α-Perron vector of G with maximum entry $\overline{x}$. For $0 \le \alpha < 1$, we have*

$$\rho_\alpha(G) \le \alpha\Delta + (1-\alpha)km\overline{x}^k$$

$$\rho_\alpha(G) \le \alpha\Delta + (1-\alpha)\left(\sum_{i \in V(G)} d_i^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}} \overline{x}^{k-1}$$

with either equality if and only if G is regular.

Proof. From the eigenequation of G at $i \in V(G)$, we have

$$(\rho_\alpha - \alpha\Delta)x_i^{k-1} \leq (\rho_\alpha - \alpha d_i)x_i^{k-1} = (1-\alpha)\sum_{e\in E_i(G)}\prod_{v\in e\setminus\{i\}} x_v \leq (1-\alpha)d_i\overline{x}^{k-1}$$

with equality if and only if for $v \in e \setminus \{i\}$ with $e \in E_i(G)$, $x_v = \overline{x}$. Then

$$(\rho_\alpha - \alpha\Delta)x_i^k \leq (1-\alpha)d_i\overline{x}^k,$$

and thus

$$\rho_\alpha - \alpha\Delta \leq (1-\alpha)\overline{x}^k \sum_{i\in V(G)} d_i = (1-\alpha)km\overline{x}^k$$

with equality if and only if all entries of x are equal, or equivalently, G is regular.

On the other hand, we have

$$(\rho_\alpha - \alpha\Delta)^{\frac{k}{k-1}} x_i^k \leq (1-\alpha)^{\frac{k}{k-1}} d_i^{\frac{k}{k-1}} \overline{x}^k,$$

and thus

$$(\rho_\alpha - \alpha\Delta)^{\frac{k}{k-1}} \leq (1-\alpha)^{\frac{k}{k-1}} \overline{x}^k \sum_{i\in V(G)} d_i^{\frac{k}{k-1}},$$

implying that

$$\rho_\alpha(G) \leq \alpha\Delta + (1-\alpha)\left(\sum_{i\in V(G)} d_i^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}} \overline{x}^{k-1}$$

with equality if and only if G is regular. ■

Let $\alpha = 0$ in Theorem 5, we have $\overline{x} \geq \frac{\rho_0^{\frac{1}{k-1}}}{\left(\sum_{i\in V(G)} d_i^{\frac{k}{k-1}}\right)^{\frac{1}{k}}}$, which has been reported in [9].

4. Transformations Increasing α-Spectral Radius

In the following, we propose several types of hypergraph transformations that increase the α-spectral radius.

Theorem 6. *For $k \geq 2$, let G be a k-uniform hypergraph with $u, v_1, \ldots, v_r \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ for $r \geq 1$ such that $u \notin e_i$ and $v_i \in e_i$ for $i = 1, \ldots, r$, where $v_1, \ldots, v_r$ are not necessarily distinct. Let $e_i' = (e_i \setminus \{v_i\}) \cup \{u\}$ for $i = 1, \ldots, r$. Suppose that $e_i' \notin E(G)$ for $i = 1, \ldots, r$. Let $G' = G - \{e_1, \ldots, e_r\} + \{e_1', \ldots, e_r'\}$. Let x the α-Perron vector of G. If $x_u \geq \max\{x_{v_1}, \ldots, x_{v_r}\}$, then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leq \alpha < 1$.*

Proof. Note that $\rho_\alpha(G) = x^\top(\mathcal{A}_\alpha(G)x)$ and $\rho_\alpha(G') \geq x^\top(\mathcal{A}_\alpha(G')x)$ with equality if and only if x is also the α-Perron vector of G'. Thus

$$\begin{aligned}\rho_\alpha(G') - \rho_\alpha(G) &\geq x^\top(\mathcal{A}_\alpha(G')x) - x^\top(\mathcal{A}_\alpha(G)x)\\ &= \alpha\left(rx_u^k - \sum_{i=1}^{r} x_{v_i}^k\right) + (1-\alpha)k\sum_{i=1}^{r}(x_u - x_{v_i})x_{e_i\setminus\{v_i\}} \geq 0,\end{aligned}$$

and thus $\rho_\alpha(G') \geq \rho_\alpha(G)$. Suppose that $\rho_\alpha(G') = \rho_\alpha(G)$. Then $\rho_\alpha(G') = x^\top(\mathcal{A}_\alpha(G')x)$, and thus x is the α-Perron vector of G'. From the eigenequations of G' and G at u and noting that $E_u(G') = E_u(G) \cup \{e'_1, \ldots, e'_r\}$, we have

$$\begin{aligned}\rho_\alpha(G')x_u^{k-1} &= \alpha(d_u + r)x_u^{k-1} + (1-\alpha)\sum_{e\in E_u(G')} x_{e\setminus\{u\}}\\ &> \alpha d_u x_u^{k-1} + (1-\alpha)\sum_{e\in E_u(G)} x_{e\setminus\{u\}} = \rho_\alpha(G)x_u^{k-1},\end{aligned}$$

a contradiction. It follows that $\rho_\alpha(G') > \rho_\alpha(G)$. ∎

We say that the hypergraph G' in Theorem 6 is obtained from G by moving edges $e_1, \ldots, e_r$ from $v_1, \ldots, v_r$ to u. Theorem 6 has been established in [8] for $\alpha \in \left\{0, \frac{1}{2}\right\}$.

Theorem 7. *Let G be a connected k-uniform hypergraph with $k \geq 2$, and e and f be two edges of G with $e \cap f = \emptyset$. Let x be the α-Perron vector of G. Let $U \subset e$ and $V \subset f$ with $1 \leq |U| = |V| \leq k-1$. Let $e' = U \cup (f \setminus V)$ and $f' = V \cup (e \setminus U)$. Suppose that $e', f' \notin E(G)$. Let $G' = G - \{e, f\} + \{e', f'\}$. If $x_U \geq x_V$, $x_{e\setminus U} \leq x_{f\setminus V}$ and one is strict, then $\rho_\alpha(G) < \rho_\alpha(G')$ for $0 \leq \alpha < 1$.*

Proof. Note that

$$\begin{aligned}\rho_\alpha(G') - \rho_\alpha(G) &\geq x^\top(\mathcal{A}_\alpha(G')x) - x^\top(\mathcal{A}_\alpha(G)x)\\ &= (1-\alpha)k\sum_{g\in E(G')} x_g - (1-\alpha)k\sum_{g\in E(G)} x_g\\ &= (1-\alpha)k\left(x_U x_{f\setminus V} + x_V x_{e\setminus U} - x_U x_{e\setminus U} - x_V x_{f\setminus V}\right)\\ &= (1-\alpha)k(x_U - x_V)\left(x_{f\setminus V} - x_{e\setminus U}\right) \geq 0.\end{aligned}$$

Thus $\rho_\alpha(G') \geq \rho_\alpha(G)$. Suppose that $\rho_\alpha(G') = \rho_\alpha(G)$. Then $\rho_\alpha(G') = x^\top(\mathcal{A}_\alpha(G')x)$ and thus x is the α-Perron vector of G'. Suppose without loss of generality that $x_{e\setminus U} < x_{f\setminus V}$. Then for $u \in U$

$$-x_{e\setminus\{u\}} + x_{e'\setminus\{u\}} = -x_{U\setminus\{u\}}\left(x_{e\setminus U} - x_{f\setminus V}\right) > 0.$$

From the eigenequations of G' and G at a vertex $u \in U$, we have

$$\begin{aligned}\rho_\alpha(G')x_u^{k-1} &= \alpha d_u x_u^{k-1} + (1-\alpha)\sum_{g\in E_u(G')} x_{g\setminus\{u\}}\\ &= \alpha d_u x_u^{k-1} + (1-\alpha)\left(\sum_{g\in E_u(G)} x_{g\setminus\{u\}} - x_{e\setminus\{u\}} + x_{e'\setminus\{u\}}\right)\\ &> \alpha d_u x_u^{k-1} + (1-\alpha)\sum_{g\in E_u(G)} x_{g\setminus\{u\}} = \rho_\alpha(G)x_u^{k-1},\end{aligned}$$

a contradiction. It follows that $\rho_\alpha(G') > \rho_\alpha(G)$. ■

The above result has been known for $k = 2$ in [5] and $\alpha = 0$ [25].

A path $P = (v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ in a k-uniform hypergraph G is called a pendant path at v_0, if $d_G(v_0) \geq 2$, $d_G(v_i) = 2$ for $1 \leq i \leq s-1$, $d_G(v) = 1$ for $v \in e_i \setminus \{v_{i-1}, v_i\}$ with $1 \leq i \leq s$, and $d_G(v_s) = 1$. If $s = 1$, then we call P or e_1 a pendant edge of G (at v_0). A pendant path of length 0 at v_0 is understood as the trivial path consisting of a single vertex v_0.

If P is a pendant path at u in a k-uniform hypergraph G, we say G is obtained from H by attaching a pendant path P at u with $H = G[V(G) \setminus (V(P) \setminus \{u\})]$. In this case, we write $G = H_u(s)$ if the length of P is s. Let $H_u(0) = H$.

For a k-uniform hypergraph G with $u \in V(G)$, and $p \geq q \geq 0$, let $G_u(p,q) = (G_u(p))_u(q)$.

Theorem 8. *For $k \geq 2$, let G be a connected k-uniform hypergraph with $|E(G)| \geq 1$ and $u \in V(G)$. For $p \geq q \geq 1$ and $0 \leq \alpha < 1$, we have $\rho_\alpha(G_u(p,q)) > \rho_\alpha(G_u(p+1,q-1))$.*

Proof. Let $(u, e_1, u_1, \ldots, u_p, e_{p+1}, u_{p+1})$ and $(u, f_1, v_1, \ldots, v_{q-2}, f_{q-1}, v_{q-1})$ be the pendant paths of $G_u(p+1,q-1)$ at u of lengths $p+1$ and $q-1$, respectively. Let $v_0 = u$. Let x be the α-Perron vector of $G_u(p+1,q-1)$.

Suppose that $\rho_\alpha(G_u(p,q)) < \rho_\alpha(G_u(p+1,q-1))$. We prove that $x_{u_{p-i}} > x_{v_{q-i-1}}$ for $i = 0, \ldots, q-1$.

Suppose that $x_{v_{q-1}} \geq x_{u_p}$. Let H be the k-uniform hypergraph obtained from $G_u(p+1,q-1)$ by moving e_{p+1} from u_p to v_{q-1}. By Theorem 6 and noting that $H \cong G_u(p,q)$, we have $\rho_\alpha(G_u(p,q)) = \rho_\alpha(H) > \rho_\alpha(G_u(p+1,q-1))$, a contradiction. Thus $x_{u_p} > x_{v_{q-1}}$.

Suppose that $q \geq 2$ and $x_{u_{p-i}} > x_{v_{q-i-1}}$, where $0 \leq i \leq q-2$. We want to show that $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$. Suppose that this is not true, i.e., $x_{v_{q-i-2}} \geq x_{u_{p-i-1}}$. Suppose that $x_{e_{p-i}\setminus\{u_{p-i-1},u_{p-i}\}} \leq x_{f_{q-i-1}\setminus\{v_{q-i-2},v_{q-i-1}\}}$. Then $x_{e_{p-i}\setminus\{u_{p-i}\}} \leq x_{f_{q-i-1}\setminus\{v_{q-i-1}\}}$. Let $H' = G_u(p+1,q-1) - \{e_{p-i}, f_{q-i-1}\} + \{e', f'\}$, where $e' = \{u_{p-i}\} \cup (f_{q-i-1} \setminus \{v_{q-i-1}\})$ and $f' = \{v_{q-i-1}\} \cup (e_{p-i} \setminus$

$\{u_{p-i}\})$. Obviously, $H' \cong G_u(p,q)$. By Theorem 7, we have $\rho_\alpha(G_u(p,q)) = \rho_\alpha(H') > \rho_\alpha(G_u(p+1,q-1))$, a contradiction. Thus $x_{e_{p-i}\setminus\{u_{p-i-1},u_{p-i}\}} > x_{f_{q-i-1}\setminus\{v_{q-i-2},v_{q-i-1}\}}$, and then $x_{e_{p-i}\setminus\{u_{p-i-1}\}} > x_{f_{q-i-1}\setminus\{v_{q-i-2}\}}$. Let $H'' = G_u(p+1,q-1)-\{e_{p-i},f_{q-i-1}\}+\{e'',f''\}$, where $e'' = (e_{p-i}\setminus\{u_{p-i-1}\})\cup\{v_{q-i-2}\}$ and $f''= (f_{q-i-1}\setminus\{v_{q-i-2}\})\cup\{u_{p-i-1}\}$. Obviously, $H'' \cong G_u(p,q)$. By Theorem 7, we have $\rho_\alpha(G_u(p,q)) = \rho_\alpha(H'') > \rho_\alpha(G_u(p+1,q-1))$, also a contradiction. It follows that $x_{u_{p-i-1}} > x_{v_{q-i-2}}$, i.e., $x_{u_{p-(i+1)}} > x_{v_{q-(i+1)-1}}$.

Therefore $x_{u_{p-i}} > x_{v_{q-i-1}}$ for $i = 0,\dots,q-1$. Particularly, $x_{u_{p-q+1}} > x_{v_0}$.

Now let H^* be the k-uniform hypergraph obtained from $G_u(p+1,q-1)$ by moving all the edges containing u except e_1 and f_1 from u to u_{p-q+1}. By Theorem 6 and noting that $H^* \cong G_u(p,q)$, we have $\rho_\alpha(G_u(p,q)) > \rho_\alpha(G_u(p+1,q-1))$, a contradiction. Therefore $\rho_\alpha(G_u(p,q)) > \rho_\alpha(G_u(p+1,q-1))$. ■

The above result has been reported for $k = 2$ in [5] and $\alpha = 0$ in [25].

Theorem 9. *Let G be a k-uniform hypergraph with $k \geq 2$, $e = \{v_1,\dots,v_k\}$ be an edge of G with $d_G(v_i) \geq 2$ for $i = 1,\dots,r$, and $d_G(v_i) = 1$ for $i = r+1,\dots,k$, where $3 \leq r \leq k$. Let G' be the hypergraph obtained from G by moving all edges containing $v_3,\dots,v_r$ but not containing v_1 from $v_3,\dots,v_r$ to v_1. Then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leq \alpha < 1$.*

Proof. Let x be the α-Perron vector of G, and $x_{v_t} = \max\{x_{v_i} : 3 \leq i \leq r\}$. If $x_{v_1} \geq x_{v_t}$, then by Theorem 6, $\rho_\alpha(G') > \rho_\alpha(G)$. Suppose that $x_{v_1} < x_{v_t}$. Let G'' be the hypergraph obtained from G by moving all edges containing v_i but not containing v_t from v_i to v_t for all $3 \leq i \leq r$ with $i \neq t$, and moving all edges containing v_1 but not containing v_t from v_1 to v_t. It is obvious that $G'' \cong G'$. By Theorem 6, we have $\rho_\alpha(G') = \rho_\alpha(G'') > \rho_\alpha(G)$. ■

5. Hypergraphs with Large α-Spectral Radius

A hypercactus is a connected k-uniform hypergraph in which any two cycles (viewed as two hypergraphs) have at most one vertex in common. Let $H_{m,r,k}$ be a k-uniform hypergraph consisting of r cycles of length 2 and $m-2r$ pendant edges with a vertex in common. If $r = 0$, then $H_{m,r,k} \cong S_{m,k}$.

Theorem 10. *For $k \geq 2$, let G be a k-uniform hypercactus with m edges and r cycles, where $0 \leq r \leq \lfloor \frac{m}{2} \rfloor$ and $m \geq 2$. For $0 \leq \alpha < 1$, we have $\rho_\alpha(G) \leq \rho_\alpha(H_{m,r,k})$ with equality if and only if $G \cong H_{m,r,k}$.*

Proof. Let G be a k-uniform hypercactus with maximum α-spectral radius among k-uniform hypercacti with m edges and r cycles.

Let x be the α-Perron vector of G.

Suppose first that $r = 0$, i.e., G is a hypertree with m edges. Let d be diameter of G. Obviously, $d \geq 2$. Suppose that $d \geq 3$. Let $(u_0, e_1, u_1, \ldots, e_d, u_d)$ be a diametral path of G. Choose $u \in e_{d-1}$ with $x_u = \max\{x_v : v \in e_{d-1}\}$. Let G_1 be the hypertree obtained from G by moving all edges (except e_{d-1}) containing a vertex of e_{d-1} different from u from these vertices to u. By Theorem 6, we have $\rho_\alpha(G_1) > \rho_\alpha(G)$, a contradiction. Thus $d = 2$, implying that $G \cong S_{m,k} = H_{m,0,k}$.

Suppose in the following that $r \geq 1$.

If there exists an edge e with at least three vertices of degree at least 2, then let $e = \{v_1, \ldots, v_k\}$ with $d_G(v_i) \geq 2$ for $i = 1, \ldots, \ell$, and $d_G(v_i) = 1$ for $i = \ell + 1, \ldots, k$, where $3 \leq \ell \leq k$. Let G' be the hypergraph obtained from G by moving all edges containing $v_3, \ldots, v_\ell$ except e from $v_3, \ldots, v_\ell$ to v_1. Obviously, G' is a k-uniform hypercactus with m edges and r cycles. By Theorem 9, $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. Thus, every edge in G has $k - 2$ vertices of degree 1.

Suppose that there exist two vertex-disjoint cycles. We choose two such cycles C_1 and C_2 by requiring that $d_G(C_1, C_2)$ is as small as possible, where $d_G(C_1, C_2) = \min\{d_G(u, v) : u \in V(C_1), v \in V(C_2)\}$. Let $u \in V(C_1)$ and $v \in V(C_2)$ with $d_G(C_1, C_2) = d_G(u, v)$. We may assume that $x_u \geq x_v$. Let G'' be the hypergraph obtained from G by moving edges containing v in C_2 from v to u. Obviously, G'' is a k-uniform hypercactus with m edges and r cycles. By Theorem 6, $\rho_\alpha(G'') > \rho_\alpha(G)$, a contradiction. Thus, if $r \geq 2$, then all cycles in G share a common vertex, which we denote by w. If $r = 1$, then w is a vertex of degree 2 of the unique cycle.

Let $(v_0, e_1, v_1, \ldots, v_{\ell-1}, e_\ell, v_0)$ be a cycle of G of length $\ell \geq 2$, where $v_0 = w$. Suppose that $\ell \geq 3$. Assume that $x_{v_0} \geq x_{v_2}$. Let G^* be the hypergraph obtained from G by moving the edge e_2 from v_2 to v_0. Obviously, G^* is a k-uniform hypercactus with m edges and r cycles. By Theorem 6, $\rho_\alpha(G^*) > \rho_\alpha(G)$, a contradiction. Thus, every cycle of G is of length 2, and there are exactly $m - 2r$ edges that are not on any cycle.

Suppose that $G \ncong H_{m,r,k}$. Then there exists a vertex z such that $d_G(w, z) = 2$. Let z' be the unique vertex such that $d_G(w, z') = d_G(z', z) = 1$. There are two cases. First suppose that z' lies on some cycle. Let e_1 and e_2 be the cycle containing w and z'. Let H be the hypergraph obtained from G by moving all edges containing z' except e_1 and e_2 from z' to w if $x_w \geq x_{z'}$, and the hypergraph obtained from G by moving all edges containing w except e_1 and e_2 from w to z otherwise. Now suppose that z' does not lie on any cycle. Let e be the edge containing w and z'. Let H be the hypergraph obtained from G by moving all edges containing z' except e from z' to w if $x_w \geq x_{z'}$, and the hypergraph obtained from G by moving all edges containing w except e from w to z otherwise. In either case, H is a k-uniform hypercactus with m edges and r cycles. By Theorem 6, $\rho_\alpha(H) > \rho_\alpha(G)$, a contradiction. It follows that $G \cong H_{m,r,k}$. ■

Corollary 11. *Suppose that $k \geq 2$.*

(i) *If G is a k-uniform hypertree with $m \geq 1$ edges, then $\rho_\alpha(G) \leq \rho_\alpha(S_{m,k})$ for $0 \leq \alpha < 1$ with equality if and only if $G \cong S_{m,k}$.*

(ii) *If G is a k-uniform unicyclic hypergraphs with $m \geq 2$ edges, then $\rho_\alpha(G) \leq \rho_\alpha(H_{m,1,k})$ for $0 \leq \alpha < 1$ with equality if and only if $G \cong H_{m,1,k}$.*

The cases when $\alpha = 0$ in Corollary 11 (i) and (ii) have been known in [8, 2].

For $2 \leq d \leq m$, let $S_{m,d,k}$ be the k-uniform hypertree obtained from the k-uniform loose path $P_{d,k} = (v_0, e_1, v_1, \ldots, v_{d-1}, e_d, v_d)$ by attaching $m-d$ pendant edges at $v_{\lfloor \frac{d}{2} \rfloor}$. Obviously, $S_{m,2,k} \cong S_{m,k}$.

Theorem 12. *For $k \geq 2$, let G be a k-uniform hypertree with m edges and diameter $d \geq 2$. For $0 \leq \alpha < 1$, we have $\rho_\alpha(G) \leq \rho_\alpha(S_{m,d,k})$ with equality if and only if $G \cong S_{m,d,k}$.*

Proof. It is trivial for $d = 2$. Suppose that $d \geq 3$.

Let G be a k-uniform hypertree with maximum α-spectral radius among hypertrees with m edges and diameter d.

Let $P = (v_0, e_1, v_1, \ldots, e_d, v_d)$ be a diametral path of G. Let x be the α-Perron vector of G.

Claim 1. *Every edge of G has at least $k-2$ vertices of degree 1.*

Proof. Suppose that there is at least one edge with at least three vertices of degree at least 2. Let $f = \{u_1, \ldots, u_k\}$ be such an edge. First suppose that f is not an edge on P. We may assume that $d_G(u_1, P) = d_G(u_i, P) - 1$ for $i = 2, \ldots, k$, where $d_G(w, P) = \min\{d_G(w, v) : v \in V(P)\}$. Then $d_G(u_1) \geq 2$. We may assume that $d_G(u_i) \geq 2$ for $i = 2, \ldots, r$ and $d_G(u_i) = 1$ for $i = r+1, \ldots, k$, where $3 \leq r \leq k$. Let G' be the hypertree obtained from G by moving all edges containing $u_3, \ldots, u_r$ except f from $u_3, \ldots, u_r$ to u_1. Obviously, G' is a hypertree with m edges and diameter d. By Theorem 9, $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. Thus f is an edge on P, i.e., $f = e_i$ for some i with $2 \leq i \leq d-1$. Let $e_i \setminus \{v_{i-1}, v_i\} = \{v_{i,1}, \ldots, v_{i,k-2}\}$. We may assume that $v_{i,1}, \ldots, v_{i,s}$ are precisely those vertices with degree at least 2 among $v_{i,1}, \ldots, v_{i,k-2}$, where $1 \leq s \leq k-2$. Let G'' be the hypertree obtained from G by moving all edges containing $v_{i,1}, \ldots, v_{i,s}$ except e_i from $v_{i,1}, \ldots, v_{i,s}$ to v_i. Obviously, G'' is a hypertree with m edges and diameter d. By Theorem 9, $\rho_\alpha(G'') > \rho_\alpha(G)$, also a contradiction. It follows that all edges of G have at most two vertices of degree at least 2. Claim 1 follows. □

Claim 2. *Any edge not on P is a pendant edge.*

Proof. Suppose that e is an edge not on P and it is not a pendant edge. Then there are two vertices, say u and v, in e such that $d_u \geq 2$ and $d_v \geq 2$. Suppose without loss of generality that $d_G(u,P) < d_G(v,P)$. Let w be the vertex on P with $d_G(u,P) = d_G(u,w)$. Let G^* be the hypertree obtained from G by moving all edges containing v except e from v to w if $x_w \geq x_v$, and the hypertree obtained from G by moving all edges containing w (except the edge in the path connecting w and v) from w to v otherwise. By Theorem 6, $\rho_\alpha(G^*) > \rho_\alpha(G)$, a contradiction. This proves Claim 2. □

Claim 3. *There is at most one vertex of degree greater than two in G.*

Proof. Suppose that there are two vertices, say s and t, on P with degree greater than two. We may assume that $x_s \geq x_t$. Let H be the hypertree obtained from G by moving all pendant edges containing t from t to s. By Theorem 6, we have $\rho_\alpha(H) > \rho_\alpha(G)$, a contradiction. Claim 3 follows. □

Combing Claims 1–3, G is a hypertree obtained from the path P by attaching $m-d$ pendant edges at some v_i with $1 \leq i \leq d-1$, and by Theorem 8, we have $G \cong S_{m,d,k}$. ■

The above result for $\alpha = 0$ has been proved in [25] by a relation between the 0-spectral radius of a power hypergraph and the 0-spectral radius of its graph. Recall that for $\alpha = 0$ and $k = 2$, Simić and one author of this paper [23] determined the tree on n vertices and diameter d with kth largest 0-spectral radius for $k = 1, \ldots, \lfloor \frac{d}{2} \rfloor + 1$ if $4 \leq d \leq n-4$ and for $k = 1, \ldots, \lfloor \frac{d}{2} \rfloor$ if $d = n-3$.

Suppose that $m \geq d \geq 3$. Let H be the hypergraph obtained from $S_{m,d,k}$ by moving edge e_d from v_{d-1} to $v_{\lfloor \frac{d}{2} \rfloor}$ if $x_{v_{\lfloor \frac{d}{2} \rfloor}} \geq x_{v_{d-1}}$, and the hypergraph obtained from $S_{m,d,k}$ by moving edges containing $v_{\lfloor \frac{d}{2} \rfloor}$ except $e_{\lfloor \frac{d}{2} \rfloor + 1}$ from $v_{\lfloor \frac{d}{2} \rfloor}$ to v_{d-1} otherwise. Obviously, $H \cong S_{m,d-1,k}$. By Theorem 6, $\rho_\alpha(S_{m,d,k}) < \rho_\alpha(S_{m,d-1,k})$. Now by Theorem 12, Corollary 11(i) follows. Moreover, if G is a k-uniform hypertree with $m \geq 3$ edges and $G \ncong S_{m,k}$, $\rho_\alpha(G) \leq \rho_\alpha(S_{m,3,k})$ with equality if and only if $G \cong S_{m,3,k}$, which has been known for $\alpha = 0$ in [8].

For $2 \leq t \leq m$, let $T_{m,t,k}$ be the k-uniform hypertree consisting of t pendant paths of almost equal lengths (i.e., $t - \left(m - t \lfloor \frac{m}{t} \rfloor\right)$ pendant paths of length $\lfloor \frac{m}{t} \rfloor$ and $m - t \lfloor \frac{m}{t} \rfloor$ pendant paths of length $\lfloor \frac{m}{t} \rfloor + 1$) at a common vertex. Particularly, $T_{m,2,k}$ is just the k-uniform loose path $P_{m,k}$.

Theorem 13. *Let G be a k-uniform hypertree with m edges and $t \geq 2$ pendant edges. For $0 \leq \alpha < 1$, we have $\rho_\alpha(G) \leq \rho_\alpha(T_{m,t,k})$ with equality if and only if $G \cong T_{m,t,k}$.*

Proof. Let G be a k-uniform hypertree with maximum α-spectral radius among hypertrees with m edges and t pendant edges. Let x be the α-Perron vector of G.

Suppose that there exists an edge $e = \{u_1, \ldots, u_k\}$ with at least three vertices of degree at least 2. Assume that $d_G(u_i) \geq d_G(u_{i+1})$ for $i = 1, \ldots, k-1$. Let G' be the hypertree obtained from G by moving all edges containing $u_3, \ldots, u_k$ except

e from these vertices to u_1. Obviously, G' is a hypertree with m edges and t pendant edges. By Theorem 9, $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. It follows that each edge of G has at most two vertices of degree at least 2.

Suppose that there are two vertices, say u, v with degree greater than 2. We may assume that $x_u \geq x_v$. Let H be the hypertree obtained from G by moving an edge not on the path connecting u and v containing v from v to u. By Theorem 6, we have $\rho_\alpha(H) > \rho_\alpha(G)$, a contradiction. Thus, there is at most one vertex of degree greater than 2 in G.

If there is no vertex of degree greater than 2, then $t = 2$, and G is the k-uniform loose path $P_{m,k}$. If there is exactly one vertex of degree greater than 2, then $t \geq 3$, G is a hypertree consisting of t pendant paths at a common vertex, and by Theorem 8, we have $G \cong T_{m,t,k}$. ∎

For $\alpha = 0$, this is known in [26, 30].

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