# BRANCH-WEIGHT UNIQUE TREES 

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#### Abstract

A branch at a vertex $x$ in a tree is a maximal subtree containing $x$ as an endvertex. The branch-weight of $x$ is the maximum number of edges in any branch at $x$. The branch-weight sequence of a tree is the multiset consisting of the branch-weights of all vertices arranged in nonincreasing order. Non-isomorphic trees may have the same branch-weight sequence. A tree $T$ is said to be branch-weight unique in a family of trees if $T$ is uniquely determined in the family by its branch-weight sequence. A spider is a tree in which exactly one vertex has degree exceeding two. It is known that spiders are branch-weight unique in the family of spiders but not in the family of all trees. In this study, a necessary and sufficient condition is obtained whereby a spider may be branch-weight unique in the family of all trees. Moreover, two types of trees are proposed to be branch-weight unique in the family of all trees.


Keywords: branch-weight, branch-weight sequence, branch-weight unique, tree, spider.
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## 1. Introduction

Let $T$ be a non-trivial tree and $x \in V(T)$. The degree of $x$ is denoted by $d e g_{T}(x)$. An endvertex of $T$ is a vertex with degree one. A branch at $x$ in $T$ is a maximal subtree of $T$ containing $x$ as an endvertex. There are $k$ branches at $x$ if $\operatorname{deg}_{T}(x)=k$. Let $B$ be a branch at $x$. If $\operatorname{deg}_{T}(x) \geq 2$, then $x$ is the only endvertex of $B$ that is not an endvertex of $T$. The branch-weight of $x$, denoted by $b w(x)$, is the maximum number of edges in any branch at $x$, or equivalently, the maximum number of vertices in any component of $T-x$. A multiset is a
set in which an element may appear several times. A sequence is a multiset of ordered numbers. The branch-weight sequence of $T$ is the sequence consisting of the branch-weights of all vertices of $T$ arranged in nonincreasing order. A caterpillar is a tree of order at least three containing a path such that each vertex not on the path is adjacent to a vertex on the path. A spider is a tree in which exactly one vertex has degree exceeding two. Skurnick [8, 10] obtained necessary and sufficient conditions whereby a given finite sequence of positive integers may be realizable as the branch-weight sequence of a caterpillar, or more generally, of a non-trivial tree [8]. Non-isomorphic trees may have the same branch-weight sequence. Figure 1 shows a caterpillar and a non-caterpillar tree with the same branch-weight sequence $\{10,10,10,10,10,10,10,8,8,7,4\}$. Figure 2 shows caterpillars with the same branch-weight sequence $\{8,8,8,8,8,7,6,5,4\}$, and Figure 3 exhibits a spider and a non-spider tree with the same branch-weight sequence $\{9,9,9,9,9,8,8,8,7,3\}$. We note that each number beside a vertex in these trees is the branch-weight of the vertex. Let $\mathcal{F}$ be a family of trees and $T \in \mathcal{F}$. Then $T$ is said to be branch-weight unique in $\mathcal{F}$ if $T$ is uniquely determined in $\mathcal{F}$ by its branch-weight sequence. That is, whenever $T^{\prime} \in \mathcal{F}$, and $T^{\prime}$ and $T$ have the same branch-weight sequence, then $T^{\prime}$ is isomorphic to $T$. According to Figures 1 and 2, caterpillars are neither branch-weight unique in the family of all trees, nor branch-weight unique in the family of caterpillars. By contrast, spiders are branch-weight unique in the family of spiders. In [7], it is proved that any two spiders with the same branch-weight sequence are isomorphic. However, Figure 3 shows that spiders are not branch-weight unique in the family of all trees. This study is concerned with branch-weight unique trees in the family of all trees. In the following section, a necessary and sufficient condition is obtained whereby a spider may be branch-weight unique in the family of all trees. Moreover, two types of trees that are branch-weight unique in the family of all trees are introduced.


Figure 1


Figure 2


Figure 3

## 2. Main Results

Let $T$ be a non-trivial tree of order $n$ and $\left\{b w\left(x_{i}\right)\right\}_{i=1}^{n}$ be the branch-weight sequence of $T$, where $V(T)=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ and $b w\left(x_{1}\right) \geq b w\left(x_{2}\right) \geq \cdots \geq$ $b w\left(x_{n}\right)$. The complementary-weight of $x_{i}$, denoted by $c w\left(x_{i}\right)$, is defined by $c w\left(x_{i}\right)=n-b w\left(x_{i}\right)$. The complementary-weight sequence of $T$ is defined by $\left\{c w\left(x_{i}\right)\right\}_{i=1}^{n}$. We note that the complementary-weight sequence of $T$ is in nondecreasing order. Clearly, $T$ is uniquely determined by its branch-weight sequence in a family of trees if and only if it is uniquely determined by its complementaryweight sequence in the family. Complementary-weights and complementaryweight sequences were introduced in $[8,10]$. In this study, they are used to clarify
the proofs. The centroid of $T$, denoted by $\operatorname{Cen}(T)$, is the set of vertices of $T$ with minimum branch-weight, or equivalently, with maximum complementary-weight. Branch-weights and centroids were introduced and studied in [2], a comprehensive text on distance in graphs. These concepts were subsequently extended to connected graphs [5, 6, 9, 11]. Jordan proved the following well-known theorem about the centroid of a tree. It will be used later in the main results.

Theorem 1 [2, 3]. The centroid of a tree consists of either a single vertex or a pair of adjacent vertices.

We now state some preliminary results.
Proposition 2 [4]. (1) Let $x_{1} x_{2} \cdots x_{k}(k \geq 3)$ be a path in a tree $T$, where $b w\left(x_{1}\right) \leq b w\left(x_{2}\right)$. Then $b w\left(x_{2}\right)<b w\left(x_{3}\right)<\cdots<b w\left(x_{k}\right)$.
(2) Let $x_{1} x_{2} \cdots x_{k}(k \geq 3)$ be a path in a tree $T$, where $c w\left(x_{1}\right) \geq c w\left(x_{2}\right)$. Then $c w\left(x_{2}\right)>c w\left(x_{3}\right)>\cdots>c w\left(x_{k}\right)$.

We note that Theorem 1 follows immediately from Proposition 2.
Proposition 3 [1, 3]. Let $T$ be a tree of order $n$ and $v \in V(T)$. Then the following holds.
(1) $v \in C e n(T)$ if and only if $b w(v) \leq \frac{n}{2}$.
(2) $\operatorname{Cen}(T)=\{v\}$ if and only if $b w(v)<\frac{n}{2}$.
(3) $v \in C e n(T)$ if and only if $c w(v) \geq \frac{n}{2}$.
(4) $\operatorname{Cen}(T)=\{v\}$ if and only if $c w(v)>\frac{n}{2}$.

Proposition 4 [4]. Let $T$ be a tree and $x \in V(T)$. We assume that $B$ is a component of $T-x$ such that $b w(x)=|V(B)|$. Then the following holds.
(1) $c w(y)<c w(x)$ for each $y \in V(T) \backslash(V(B) \cup\{x\})$.
(2) If $y \in V(T) \backslash\{x\}$ and $c w(y) \geq c w(x)$, then $y \in V(B)$.

For a tree $T$ and $x \in V(T)$, let $N(x)$ denote the set $\{v \in V(T): v x \in E(T)\}$. It is seen that $\operatorname{deg}_{T}(x)=|N(x)|$. Suppose $x \notin C e n(T)$. The following lemma shows that there exists exactly one vertex $y$ in $N(x)$ with $c w(y)>c w(x)$, and the sum of the complementary-weights of the other vertices in $N(x)$ equals $c w(x)-1$ if $\operatorname{deg}_{T}(x) \geq 2$.

Lemma 5. Let $T$ be a tree and $x \in V(T) \backslash C e n(T)$. Then there exists $y \in N(x)$ such that $c w(y)>c w(x)$ and $\sum_{v \in N(x) \backslash\{y\}} c w(v)=c w(x)-1$ if $\operatorname{deg}_{T}(x) \geq 2$.
Proof. Let $k=\operatorname{deg}_{T}(x)$ and $N(x)=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$, and let the components of $T-x$ be $B_{1}, B_{2}, \ldots, B_{k}$, with $x_{i} \in V\left(B_{i}\right)$ and $b w(x)=\left|V\left(B_{1}\right)\right|$. Let $u \in C e n(T)$. Then $c w(u)>c w(x)$. By Proposition $4(2), u \in V\left(B_{1}\right)$. Now claim that we can
take $y=x_{1}$ and obtain $c w(y)>c w(x)$. First if $x$ and $u$ are adjacent, then $u$ is exactly the vertex $x_{1}$ and $c w\left(x_{1}\right)>c w(x)$. If $x$ and $u$ are not adjacent, there is a path $u \cdots x_{1} x$. By Proposition $2(2), c w\left(x_{1}\right)>c w(x)$ as $c w(u) \geq c w\left(x_{1}\right)$. Next, if $k \geq 2$ we claim that $c w\left(x_{i}\right)=\left|V\left(B_{i}\right)\right|$ for $i \geq 2$. As $b w(x)=\left|V\left(B_{1}\right)\right|$, we see that $\left|V\left(B_{1}\right)\right| \geq\left|V\left(B_{i}\right)\right|$. It is obvious that for $i \geq 2$ the maximum component of $T-x_{i}$ has vertex set $\{x\} \cup\left(\bigcup_{j \neq i} V\left(B_{j}\right)\right)$. That is, $b w\left(x_{i}\right)=1+\sum_{j \neq i}\left|V\left(B_{j}\right)\right|$. Then $c w\left(x_{i}\right)=|V(T)|-b w\left(x_{i}\right)=\left|V\left(B_{i}\right)\right|$ for $i \geq 2$, where $|V(T)|=1+\sum_{j=1}^{k}\left|V\left(B_{j}\right)\right|$. We conclude that $\sum_{i=2}^{k} c w\left(x_{i}\right)=\sum_{i=2}^{k}\left|V\left(B_{i}\right)\right|=|V(T)|-\left|V\left(B_{1}\right)\right|-1=|V(T)|-$ $b w(x)-1=c w(x)-1$. That is, $\sum_{v \in N(x) \backslash\{y\}} c w(v)=c w(x)-1$ if $d e g_{T}(x) \geq 2$.

The following lemma can be obtained from Lemma 5.
Lemma 6. Let $T$ be a tree and $x \in V(T)$. If $x$ is a non-endvertex and $x \notin$ Cen $(T)$, then the following holds.
(1) $\sum_{\substack{v \in N(x), c w(v)<c w(x)}} c w(v)=c w(x)-1$.
(2) $\operatorname{deg}_{T}(x)=2$ if and only if $x$ is adjacent to a vertex with complementaryweight $c w(x)-1$.

In a spider, the vertex with degree exceeding two is called the body of the spider.

Proposition 7 [7]. We assume that $S$ is a spider of order n. Let $x_{1} x_{2} x_{3} \cdots x_{p} x_{p+1}$ be a path in $S$, where $x_{1}$ is an endvertex of $S$ and $x_{p+1}$ is the body of $S$. Then the following holds.
(1) If $p \leq\left[\frac{n}{2}\right]$, then $c w\left(x_{i}\right)=i$ for $1 \leq i \leq p$.
(2) If $p \geq\left[\frac{n}{2}\right]+1$, then

$$
c w\left(x_{i}\right)= \begin{cases}i & \text { for } 1 \leq i \leq\left[\frac{n}{2}\right] \\ n-i+1 & \text { for }\left[\frac{n}{2}\right]+1 \leq i \leq p+1\end{cases}
$$

Proof. It is obvious that $b w\left(x_{i}\right)=\max \{i-1, n-i\}$ for $1 \leq i \leq p$. We assume that $i \leq\left[\frac{n}{2}\right]$. Then $2 i \leq n$, which implies $i-1<n-i$. Thus, $b w\left(x_{i}\right)=n-i$ and $c w\left(x_{i}\right)=i$. We now assume that $i \geq\left[\frac{n}{2}\right]+1$. Then $2 i>n$, which implies $i-1 \geq n-i$. Thus, $b w\left(x_{i}\right)=i-1$ and $c w\left(x_{i}\right)=n-i+1$.
(1) As $p \leq\left[\frac{n}{2}\right]$ and $1 \leq i \leq p$, we have $i \leq\left[\frac{n}{2}\right]$. Hence, $c w\left(x_{i}\right)=i$.
(2) We now assume that $p \geq\left[\frac{n}{2}\right]+1$. If $1 \leq i \leq\left[\frac{n}{2}\right]$, then $c w\left(x_{i}\right)=i$. If $\left[\frac{n}{2}\right]+1 \leq i \leq p$, then $c w\left(x_{i}\right)=n-i+1$. We evaluate $c w\left(x_{p+1}\right)$. As $p \geq\left[\frac{n}{2}\right]+1$, we see that among the components of $T-x_{p+1}$, the path $x_{1} x_{2} x_{3} \cdots x_{p}$ has the largest order. Hence, $b w\left(x_{p+1}\right)=p$ and $c w\left(x_{p+1}\right)=n-p$.

Let $S$ be a spider and $b$ the body of $S$. Then, each component of $S-b$ is called a leg of $S$. Obviously, each leg is a path. If a spider has $k$ legs, and
these legs have orders $l_{1}, l_{2}, \ldots, l_{k}$, where $l_{1} \leq l_{2} \leq \cdots \leq l_{k}$, then the spider is denoted by $S P\left(l_{1}, l_{2}, \cdots, l_{k}\right)$. An average spider is a spider $S P\left(l_{1}, l_{2}, \ldots, l_{k}\right)$ with $l_{k} \leq l_{1}+l_{2}$.

Proposition 8. We assume that $S$ is an average spider of order $n$. Let $x_{1} x_{2} x_{3} \cdots x_{p} x_{p+1}$ be a path in $S$, where $x_{1}$ is an endvertex of $S$ and $x_{p+1}$ is the body of $S$. Then $\operatorname{Cen}(S)=\left\{x_{p+1}\right\}$ and $c w\left(x_{i}\right)=i$ for $i=1,2, \ldots, p$.

Proof. Let $S=S P\left(l_{1}, l_{2}, \ldots, l_{k}\right)$. By assumption, $p=l_{t}$ for some $1 \leq t \leq k$. As $S$ is an average spider, we have $2 p \leq 2 l_{k} \leq l_{k}+l_{1}+l_{2} \leq n-1$. That is, $p \leq \frac{n-1}{2}$. This implies that every component of $S-x_{p+1}$ has order less than $\frac{n}{2}$; hence, $b w\left(x_{p+1}\right)<\frac{n}{2}$. By Proposition 3(2), $\operatorname{Cen}(S)=\left\{x_{p+1}\right\}$. As $p \leq \frac{n-1}{2} \leq\left[\frac{n}{2}\right]$, by Proposition $7(1), c w\left(x_{i}\right)=i$ for $i=1,2, \ldots, p$.

We will make use of the following notations. A nondecreasing sequence consisting of $n$ multiplicities of $a_{i}$ for $i=1,2, \ldots, r$, where $a_{1}<a_{2}<\cdots<a_{r}$, is denoted by $\left\{a_{1}, a_{2}, \ldots, a_{r}\right\}^{n}$. The superscript $n$ can be omitted if $n=1$. For two nondecreasing sequences $A$ and $B$, we use $A \cup B$ to denote the nondecreasing sequence consisting of all the elements of $A$ and $B$. Thus, if $c \in A \cup B$, then the multiplicities of $c$ in $A \cup B$ equal the sum of the multiplicities of $c$ in $A$ and the multiplicities of $c$ in $B$. For instance, using these notations, the nondecreasing sequence $\{1,1,1,2,2,2,5\}$ can also be written in the following forms: $\{1\}^{3} \cup\{2\}^{3} \cup\{5\},\{1,2\}^{3} \cup\{5\},\{1,2\}^{2} \cup\{1,2,5\}$, etc.

Let $S$ be an average spider of order $n$ with $k$ legs. Assume that for $i=$ $1,2, \ldots, r$ there are $t_{i}$ legs of order $\ell_{i}$ in $S$, with $\ell_{1}<\ell_{2}<\cdots<\ell_{r}$ and $\sum_{i=1}^{r} t_{i}=$ $k$, where $r, t_{i} \in N$. Let $b$ be the body of $S$. We see that $b w(b)=\ell_{r}$ and $c w(b)=n-\ell_{r}$. By Proposition 8, it is seen that the complementary-weight sequence of $S$ is $\left\{1,2, \ldots, \ell_{1}\right\}^{t_{1}} \cup\left\{1,2, \ldots, \ell_{2}\right\}^{t_{2}} \cup \cdots \cup\left\{1,2, \ldots, \ell_{r}\right\}^{t_{r}} \cup\left\{n-\ell_{r}\right\}$. It is easy to see that this sequence can also be written in the following form. The proof is omitted.

Proposition 9. The aforementioned average spider has complementary-weight sequence

$$
\begin{aligned}
& \left\{1,2, \ldots, \ell_{1}\right\}^{k} \cup\left\{n-\ell_{1}\right\} \text { if } r=1 \text {, or } \\
& \left\{1,2, \ldots, \ell_{1}\right\}^{k} \cup\left\{\ell_{1}+1, \ell_{1}+2, \ldots, \ell_{2}\right\}^{k-t_{1}} \cup\left\{\ell_{2}+1, \ell_{2}+2, \ldots, \ell_{3}\right\}^{k-\left(t_{1}+t_{2}\right)} \\
& \cup \cdots \cup\left\{\ell_{r-1}+1, \ell_{r-1}+2, \ldots, \ell_{r}\right\}^{k-\sum_{j=1}^{r-1} t_{j}} \cup\left\{n-\ell_{r}\right\} \text { if } r \geq 2 .
\end{aligned}
$$

Theorem 10. An average spider is branch-weight unique in the family of all trees.

Proof. Let $S$ be an average spider assumed as in Proposition 9, and $C$ be the complementary-weight sequence of $S$. It suffices to show that if we construct a tree $T$ with complementary-weight sequence $C$, then $T$ is isomorphic to $S$. We
divide the construction of $T$ into three parts. In each part, an assertion is made followed by its proof.

Part 1. For $j=2,3, \ldots, \ell_{1}$, a vertex with complementary-weight $j$ is adjacent to a vertex with complementary-weight $j-1$.

First from $C$ we see that for $j=1,2, \ldots, \ell_{1}, T$ has exactly $k$ multiplicities of vertices with complementary-weight $j$. If $\ell_{1}=1$, there is nothing to prove. So let $\ell_{1} \geq 2$. By Lemma $6(1)$, a vertex with complementary-weight 2 is adjacent to a vertex with complementary-weight 1 . Next, if $\ell_{1} \geq 3$, we consider a vertex with complementary-weight 3 . As each vertex with complementary-weight 1 is an endvertex of degree one and is already adjacent to a vertex with complementaryweight 2 , by Lemma 6(1) we see that a vertex with complementary-weight 3 is adjacent to a vertex with complementary-weight 2 . Now suppose that for $j=2,3, \ldots, a$, some $a$, where $3 \leq a<\ell_{1}$, a vertex with complementary-weight $j$ is adjacent to a vertex with complementary-weight $j-1$. By Lemma 6(2), it is seen that each vertex with complementary-weight $j$, where $2 \leq j \leq a-1$, is of degree two and has been adjacent to two vertices, one is with complementaryweight $j-1$ and the other is with complementary-weight $j+1$. So for a vertex with complementary-weight $a+1$, by Lemma 6(1) it must be adjacent to a vertex with complementary-weight $a$. Thus by induction this part holds.

Part 2. For $j=\ell_{1}+1, \ell_{1}+2, \ldots, \ell_{r}$, a vertex with complementary-weight $j$ is adjacent to a vertex with complementary-weight $j-1$.

From $C$ we see that there is no vertex with complementary-weight $\ell_{1}+1$ if $r=1$. So let $r \geq 2$. Distinguish two cases: $t_{1} \geq 2$ and $t_{1}=1$.

Case 1. $t_{1} \geq 2$. We have $\ell_{r} \leq 2 \ell_{1}$ as $S$ is an average spider. Therefore, for any $j=\ell_{1}+1, \ell_{1}+2, \ldots, \ell_{r}$, no vertex with complementary-weight $j$ is adjacent to two or more vertices with complementary-weights at least $\ell_{1}$ by Lemma 6(1). Hence, the assertion of Part 2 follows from the assertion of Part 1.

Case 2. $t_{1}=1$. We have $\ell_{r} \leq \ell_{1}+\ell_{2}$ as $S$ is an average spider. Using the similar arguments as in Part 1, we see that a vertex with complementary-weight $j^{\prime}=\ell_{1}+1, \ell_{1}+2, \ldots, \ell_{2}$ is adjacent to a vertex with complementary-weight $j^{\prime}-1$. Therefore, no vertex with complementary-weight $j=\ell_{2}+1, \ell_{2}+2, \ldots, \ell_{r}$ is adjacent to a vertex with complementary-weight $1,2, \ldots, \ell_{1}-1$ or $\ell_{1}+1, \ell_{1}+$ $2, \ldots, \ell_{2}-1$, from which the assertion of Part 2 follows by Lemma 6(1), as $j \leq \ell_{r} \leq \ell_{1}+\ell_{2}$.
Part 3. $T$ is taken to be a spider and $T$ is isomorphic to $S$.
Till now, there are $k+1$ vertices in $T$ that have not been completed the adjacency. Among the $k+1$ vertices, one is with complementary-weight $n-\ell_{r}$, the others are $t_{i}$ vertices with complementary-weight $\ell_{i}$, for $i=1,2, \ldots, r$. From
the previous two parts, we see that each of these $t_{i}$ vertices with complementaryweight $\ell_{i}$ is not yet adjacent to a vertex with complementary-weight greater than $\ell_{i}$. We note that for $i=2,3, \ldots, r$, all these $t_{i}$ vertices with complementaryweight $\ell_{i}$ are of degree two; and the $t_{1}$ vertices with complementary-weight $\ell_{1}$ are either of degree two if $\ell_{1} \geq 2$, or of degree one if $\ell_{1}=1$. By Lemma 5 , it is seen that all the aforementioned $t_{i}$ vertices with complementary-weight $\ell_{i}$ are adjacent to the vertex with complementary-weight $n-\ell_{r}$. Then the construction of $T$ is complete. We see that $T$ is a spider with $t_{i}$ legs of order $\ell_{i}$ for $i=1,2, \ldots, r$. The assertion of Part 3 holds.

The following lemma provides the complementary-weight sequence of a nonaverage spider with three legs.

Lemma 11. We assume that a spider $S P(a, b, p)$ has order $n$ and $a+b<p$. Let $C$ be the complementary-weight sequence of $S P(a, b, p)$. Then $C=\{1,2, \ldots, a\}^{3} \cup\{a+1, a+2, \ldots, b\}^{2} \cup\{b+1, b+2, \ldots, b+a\} \cup\{b+a+1$, $\left.b+a+2, \ldots,\left[\frac{n}{2}\right]\right\}^{2}$ if $n$ is even; or $C=\{1,2, \ldots, a\}^{3} \cup\{a+1, a+2, \ldots, b\}^{2} \cup\{b+1, b+2, \ldots, b+a\} \cup\{b+a+1$, $\left.b+a+2, \ldots,\left[\frac{n}{2}\right]\right\}^{2} \cup\left\{\left[\frac{n}{2}\right]+1\right\}$ if $n$ is odd. We note that if $a=b$, the subsequence $\{a+1, a+2, \ldots, b\}^{2}$ of $C$ is the empty set.

Proof. Let the three legs of $S P(a, b, p)$ be paths $x_{1} x_{2} \cdots x_{a}, y_{1} y_{2} \cdots y_{b}$ and $z_{1} z_{2} \cdots z_{p}$, where $x_{1}, y_{1}, z_{1}$ are endvertices and $x_{a}, y_{b}, z_{p}$ are vertices adjacent to the body. We denote the body by $z_{p+1}$. Then $b w\left(z_{p+1}\right)=p$ and $c w\left(z_{p+1}\right)=n-p$. If $b \geq\left[\frac{n}{2}\right]+1$, then $n>2 b>n$, which is a contradiction. Hence, $a \leq b \leq\left[\frac{n}{2}\right]$. By Proposition $7(1), c w\left(x_{i}\right)=i$ and $c w\left(y_{j}\right)=j$ for $i=1,2, \ldots, a$ and $j=1,2, \ldots, b$. By the assumption $a+b<p$, we have $n<2 p+1$. Then $p>\frac{n-1}{2}$. Distinguish two cases.

Case 1. $p=\left[\frac{n}{2}\right]$, where $n$ is even. By Proposition $7(1), c w\left(z_{k}\right)=k$ for $k=1,2, \ldots,\left[\frac{n}{2}\right]$. We have $C=\{1,2, \ldots, a\} \cup\{1,2, \ldots, b\} \cup\{1,2, \ldots, b+a\} \cup$ $\{b+a+1\}^{2}$, where $b+a+1=\left[\frac{n}{2}\right]$.

Case 2. $p \geq\left[\frac{n}{2}\right]+1$. By Proposition 7(2),

$$
c w\left(z_{i}\right)= \begin{cases}i & \text { for } 1 \leq i \leq\left[\frac{n}{2}\right] \\ n-i+1 & \text { for }\left[\frac{n}{2}\right]+1 \leq i \leq p+1\end{cases}
$$

Then $C=\{1,2, \ldots, a\} \cup\{1,2, \ldots, b\} \cup\{1,2, \ldots, b+a\} \cup\{b+a+1, b+a+$ $\left.2, \ldots,\left[\frac{n}{2}\right]\right\}^{2}$ if $n$ is even; or $C=\{1,2, \ldots, a\} \cup\{1,2, \ldots, b\} \cup\{1,2, \ldots, b+a\} \cup$ $\left\{b+a+1, b+a+2, \ldots,\left[\frac{n}{2}\right]\right\}^{2} \cup\left\{\left[\frac{n}{2}\right]+1\right\}$ if $n$ is odd.

From Cases 1 and 2, we see that this lemma holds.

Theorem 12. Every spider with three legs is branch-weight unique in the family of all trees.

Proof. Let $S=S P(a, b, p)$, where $a \leq b \leq p$. It suffices to consider the case $a+b<p$, as $S$ is an average spider if $p \leq a+b$ and, by Theorem 10, is branchweight unique in the family of all trees. Thus, we assume that $a+b<p$. Let $n=a+b+p+1$ and $C$ be the complementary-weight sequence of $S$. We complete the proof by showing that if we construct a tree $T$ with complementary-weight sequence $C$, then $T$ and $S$ are isomorphic. First, using arguments similar to those in the proof of Theorem 10, we see that a vertex in $T$ with complementary-weight $j$ is adjacent to a vertex with complementary-weight $j-1$ for $j=2,3, \ldots, b+a$. Now consider the two vertices with complementary-weight $b+a+1$. By Lemma $6(1)$, we see that one is adjacent to the two vertices with complementary-weights $a$ and $b$, the other is adjacent to the vertex with complementary-weight $b+a$. As seen in the following figure.


Next consider the two vertices with complementary weight $b+a+2$. As all vertices with complementary-weights less than $b+a+1$ have been completed the adjacency, by Lemma $6(1)$ we see that each of the two vertices is adjacent to a vertex with complementary-weight $b+a+1$. Using this argument several times, we see that for $j=b+a+3, b+a+4, \ldots,\left[\frac{n}{2}\right]$, each vertex with complementaryweight $j$ is adjacent to a vertex with complementary-weight $j-1$. As seen in the following figure.


If $n$ is an even number, by Theorem 1, both vertices with complementaryweight $\left[\frac{n}{2}\right]$ are in the centroid and are adjacent by an edge; if $n$ is an odd number, by Lemmas $6(2)$ and 5 , the two vertices are of degree two, and each one is adjacent to a vertex with complementary-weight larger than $\left[\frac{n}{2}\right]$. As there is the only vertex, with complementary-weight $\left[\frac{n}{2}\right]+1$, outside the exhibition as shown in the above figure, so, both vertices with complementary-weight $\left[\frac{n}{2}\right]$ are adjacent to the vertex with complementary-weight $\left[\frac{n}{2}\right]+1$. Hence, $T$ is constructed to be a spider with three legs of orders $a, b$, and $p$. That is, $T$ is isomorphic to $S$.

Lemma 13. Let $S$ be a spider with at least four legs. If $S$ is not an average spider, then there exists a tree $T$ such that $T$ and $S$ have the same branch-weight sequence and $T$ is not isomorphic to $S$.

Proof. Let $S=S P\left(l_{1}, l_{2}, \ldots, l_{k}\right)$, where $l_{1} \leq l_{2} \leq \cdots \leq l_{k}$ and $|V(S)|=n$. By assumption, $l_{1}+l_{2}<l_{k}$ and $k \geq 4$. Let $b$ be the body of $S$, and let the three legs of $S$ with orders $l_{1}, l_{2}$ and $l_{k}$ be the paths $x_{1} x_{2} \cdots x_{l_{1}}, y_{1} y_{2} \cdots y_{l_{2}}$, and $z_{1} z_{2} \cdots z_{l_{k}}$, respectively, where $x_{1}, y_{1}, z_{1}$ are endvertices and $x_{l_{1}}, y_{l_{2}}$ and $z_{l_{k}}$ are vertices adjacent to $b$. We now construct a tree $T$ by deleting the three edges $x_{l_{1}} b$, $y_{l_{2}} b$, and $z_{l_{1}+l_{2}} z_{l_{1}+l_{2}+1}$ from $S$, and adding three new edges $x_{l_{1}} z_{l_{1}+l_{2}+1}, y_{l_{2}} z_{l_{1}+l_{2}+1}$, and $z_{l_{1}+l_{2}} b$ to the remaining graph. We now prove that any two vertices in both $T$ and $S$ with the same notation have the same branch-weight. In the following, we use $b w_{S}(v)$ and $b w_{T}(v)$ to denote the branch-weights of a vertex $v$ in $S$ and $T$, respectively. Clearly, $b w_{S}(b)=l_{k}$. Let $B$ be the component of $T-b$ with $V(B)=\left\{x_{1}, x_{2}, \ldots, x_{l_{1}}, y_{1}, y_{2}, \ldots, y_{l_{2}}, z_{l_{1}+l_{2}+1}, z_{l_{1}+l_{2}+2}, \ldots, z_{l_{k}}\right\}$. It is seen that $\left|V\left(B^{\prime}\right)\right| \leq|V(B)|$ for every component $B^{\prime}$ of $T-b$. Hence $b w_{T}(b)=l_{k}$. Next claim that for each vertex in $V(S) \backslash\left\{b, z_{l_{1}+l_{2}+1}\right\}$, with the same notation $v$, we have $b w_{S}(v)=b w_{T}(v)$. For $i=1,2, \ldots, k$, let $v_{i, 1} v_{i, 2} \cdots v_{i, l_{i}}$ be the leg of $S$ with order $l_{i}$, where $v_{i, 1}$ is the endvertex and $v_{i, l_{i}}$ is the vertex adjacent to $b$. We note that $v_{1,1} v_{1,2} \cdots v_{1, l_{1}}, \quad v_{2,1} v_{2,2} \cdots v_{2, l_{2}}$ and $v_{k, 1} v_{k, 2} \cdots v_{k, l_{k}}$ are the legs $x_{1} x_{2} \cdots x_{l_{1}}$, $y_{1} y_{2} \cdots y_{l_{2}}$ and $z_{1} z_{2} \cdots z_{l_{k}}$, respectively. For $i=1,2, \ldots, k$, in both $S$ and $T$ we have $\operatorname{deg}\left(v_{i, 1}\right)=1$, and $\operatorname{deg}\left(v_{i, j}\right)=2$ for all $j \geq 2$ and $v_{i, j} \neq v_{k, l_{1}+l_{2}+1}=z_{l_{1}+l_{2}+1}$. Then for $i=1,2, \ldots, k, b w_{S}\left(v_{i, 1}\right)=b w_{T}\left(v_{i, 1}\right)=n-1$; and the graphs $S-v_{i, j}$ and $T-v_{i, j}$ both have two components, where $j \geq 2$ and $v_{i, j} \neq z_{l_{1}+l_{2}+1}$. It is seen that the two components of $S-v_{i, j}$ have orders $j-1$ and $n-j$, and the two components of $T-v_{i, j}$ also have orders $j-1$ and $n-j$, where $j \geq 2$ and $v_{i, j} \neq z_{l_{1}+l_{2}+1}$. So we have $b w_{S}\left(v_{i, j}\right)=b w_{T}\left(v_{i, j}\right)$ for $j \geq 2$ and $v_{i, j} \neq z_{l_{1}+l_{2}+1}$. Now claim that $b w_{S}\left(z_{l_{1}+l_{2}+1}\right)=b w_{T}\left(z_{l_{1}+l_{2}+1}\right)$. It is seen that $S-z_{l_{1}+l_{2}+1}$ have two components with orders $l_{1}+l_{2}$ and $n-l_{1}-l_{2}-1$, respectively; $T-z_{l_{1}+l_{2}+1}$ have three components with orders $l_{1}, l_{2}$ and $n-l_{1}-l_{2}-1$, respectively. As $n-l_{1}-l_{2}-1 \geq l_{3}+l_{k}>l_{k}>l_{1}+l_{2}$, we have $b w_{S}\left(z_{l_{1}+l_{2}+1}\right)=b w_{T}\left(z_{l_{1}+l_{2}+1}\right)=$ $n-l_{1}-l_{2}-1$. Thus for each vertex in $V(S)$ with the same notation $v$, we have $b w_{S}(v)=b w_{T}(v)$. That is, $T$ and $S$ have the same branch-weight sequence.

Clearly, $T$ is not isomorphic to $S$ because $T$ has two vertices ( $b$ and $z_{l_{1}+l_{2}+1}$ ) with degrees greater than two.

In fact, an example of Lemma 13 has already been exhibited in Figure 3.
Theorem 14. A spider with $k$ legs is branch-weight unique in the family of all trees if and only if either $k=3$ or the spider is an average spider and $k \geq 4$.

Proof. $(\Leftarrow)$ By Theorems 12 and 10.
$(\Rightarrow)$ Let $S=S P\left(l_{1}, l_{2}, \ldots, l_{k}\right)$. We assume that $k \geq 4$ and $S$ is not an average spider. By Lemma 13 , there exists a tree $T$ such that $T$ and $S$ have the same branch-weight sequence and $T$ is not isomorphic to $S$. That is, $S$ is not branchweight unique in the family of all trees.

We now introduce two types of trees that are branch-weight unique in the family of all trees. For two vertices $u$ and $v$ in a graph, we use $d(u, v)$ to denote the distance between $u$ and $v$. A rooted tree is a tree with a specific vertex designated as the root. Let $T$ be a rooted tree with root $r$. Then, $T$ is called a perfect symmetry tree of type I if $d e g_{T}(x)=d e g_{T}(y)$ whenever $d(x, r)=d(y, r)$, where $x, y \in V(T)$. Figure 4 shows a perfect symmetry tree of type I. Therein, the degree of a vertex $u$ equals four if $d(u, z)=1$, and the degree of a vertex $v$ equals one if $d(v, z)=2$, where $z$ is the root. Let now $T_{1}$ be a perfect symmetry tree of type I with root $z_{1}$ and $T_{2}$ be a copy of $T_{1}$ with the root renamed $z_{2}$. Let $T^{\prime}$ be a tree constructed from $T_{1}$ and $T_{2}$ by joining the two vertices $z_{1}$ and $z_{2}$ with a new edge. We call such a tree $T^{\prime}$ a perfect symmetry tree of type II. Figure 5 shows a perfect symmetry tree of type II that is formed by joining with an edge the two roots of two copies of the tree in Figure 4. Using arguments similar to those in the proof of Theorems 10 and 12, it is not difficult to see that the following theorem holds. The proof is left to the reader.


Figure 4


Figure 5

Theorem 15. Perfect symmetry trees of type I and type II are branch-weight unique in the family of all trees.

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