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BRANCH-WEIGHT UNIQUE TREES

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Abstract

A branch at a vertex x in a tree is a maximal subtree containing x as an endvertex. The branch-weight of x is the maximum number of edges in any branch at x. The branch-weight sequence of a tree is the multiset consisting of the branch-weights of all vertices arranged in nonincreasing order. Non-isomorphic trees may have the same branch-weight sequence. A tree T is said to be branch-weight unique in a family of trees if T is uniquely determined in the family by its branch-weight sequence. A spider is a tree in which exactly one vertex has degree exceeding two. It is known that spiders are branch-weight unique in the family of spiders but not in the family of all trees. In this study, a necessary and sufficient condition is obtained whereby a spider may be branch-weight unique in the family of all trees. Moreover, two types of trees are proposed to be branch-weight unique in the family of all trees.

Keywords: branch-weight, branch-weight sequence, branch-weight unique, tree, spider.

2010 Mathematics Subject Classification: 05C05, 05C60.

1. Introduction

Let T be a non-trivial tree and $x \in V(T)$. The degree of x is denoted by $deg_T(x)$. An *endvertex* of T is a vertex with degree one. A *branch* at x in T is a maximal subtree of T containing x as an endvertex. There are k branches at x if $deg_T(x) = k$. Let B be a branch at x. If $deg_T(x) \geq 2$, then x is the only endvertex of B that is not an endvertex of T. The *branch-weight* of x, denoted by $bw(x)$, is the maximum number of edges in any branch at x, or equivalently, the maximum number of vertices in any component of $T - x$. A *multiset* is a set in which an element may appear several times. A *sequence* is a multiset of ordered numbers. The *branch-weight sequence* of T is the sequence consisting of the branch-weights of all vertices of T arranged in nonincreasing order. A *caterpillar* is a tree of order at least three containing a path such that each vertex not on the path is adjacent to a vertex on the path. A *spider* is a tree in which exactly one vertex has degree exceeding two. Skurnick [8, 10] obtained necessary and sufficient conditions whereby a given finite sequence of positive integers may be realizable as the branch-weight sequence of a caterpillar, or more generally, of a non-trivial tree [8]. Non-isomorphic trees may have the same branch-weight sequence. Figure 1 shows a caterpillar and a non-caterpillar tree with the same branch-weight sequence $\{10, 10, 10, 10, 10, 10, 10, 8, 8, 7, 4\}$. Figure 2 shows caterpillars with the same branch-weight sequence $\{8, 8, 8, 8, 8, 7, 6, 5, 4\}$, and Figure 3 exhibits a spider and a non-spider tree with the same branch-weight sequence $\{9, 9, 9, 9, 9, 8, 8, 8, 7, 3\}$. We note that each number beside a vertex in these trees is the branch-weight of the vertex. Let $\mathcal{F}$ be a family of trees and $T \in \mathcal{F}$. Then T is said to be *branch-weight unique* in $\mathcal{F}$ if T is uniquely determined in $\mathcal{F}$ by its branch-weight sequence. That is, whenever $T' \in \mathcal{F}$, and T' and T have the same branch-weight sequence, then T' is isomorphic to T. According to Figures 1 and 2, caterpillars are neither branch-weight unique in the family of all trees, nor branch-weight unique in the family of caterpillars. By contrast, spiders are branch-weight unique in the family of spiders. In [7], it is proved that any two spiders with the same branch-weight sequence are isomorphic. However, Figure 3 shows that spiders are not branch-weight unique in the family of all trees. This study is concerned with branch-weight unique trees in the family of all trees. In the following section, a necessary and sufficient condition is obtained whereby a spider may be branch-weight unique in the family of all trees. Moreover, two types of trees that are branch-weight unique in the family of all trees are introduced.

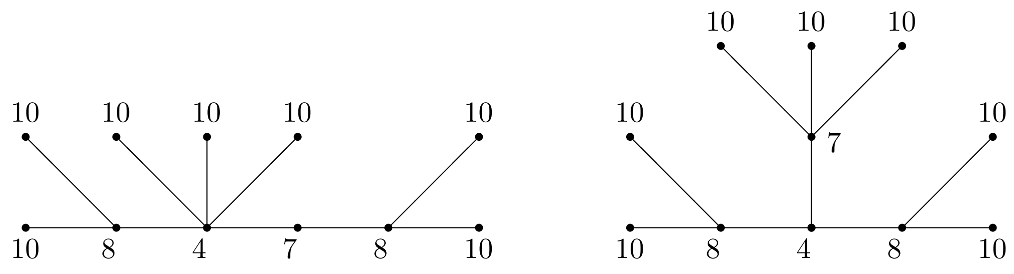


Figure 1

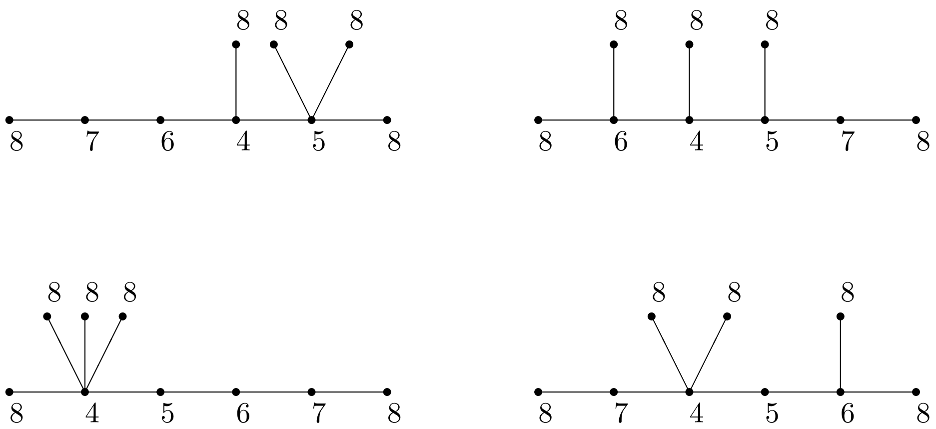


Figure 2

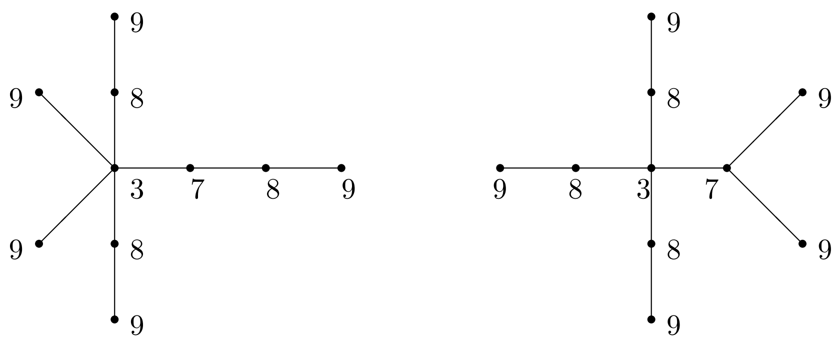


Figure 3

2. Main Results

Let T be a non-trivial tree of order n and $\{bw(x_i)\}_{i=1}^{n}$ be the branch-weight sequence of T, where $V(T) = \{x_1, x_2, \ldots, x_n\}$ and $bw(x_1) \geq bw(x_2) \geq \cdots \geq bw(x_n)$. The *complementary-weight* of x_i, denoted by $cw(x_i)$, is defined by $cw(x_i) = n - bw(x_i)$. The *complementary-weight sequence* of T is defined by $\{cw(x_i)\}_{i=1}^{n}$. We note that the complementary-weight sequence of T is in nondecreasing order. Clearly, T is uniquely determined by its branch-weight sequence in a family of trees if and only if it is uniquely determined by its complementary-weight sequence in the family. Complementary-weights and complementary-weight sequences were introduced in [8, 10]. In this study, they are used to clarify the proofs. The *centroid* of T, denoted by $Cen(T)$, is the set of vertices of T with minimum branch-weight, or equivalently, with maximum complementary-weight. Branch-weights and centroids were introduced and studied in [2], a comprehensive text on distance in graphs. These concepts were subsequently extended to connected graphs [5, 6, 9, 11]. Jordan proved the following well-known theorem about the centroid of a tree. It will be used later in the main results.

Theorem 1 [2, 3]**.** *The centroid of a tree consists of either a single vertex or a pair of adjacent vertices.*

We now state some preliminary results.

Proposition 2 [4]**.** (1) *Let $x_1x_2\cdots x_k$ $(k \geq 3)$ be a path in a tree T, where $bw(x_1) \leq bw(x_2)$. Then $bw(x_2) < bw(x_3) < \cdots < bw(x_k)$.*

(2) *Let $x_1x_2\cdots x_k$ $(k \geq 3)$ be a path in a tree T, where $cw(x_1) \geq cw(x_2)$. Then $cw(x_2) > cw(x_3) > \cdots > cw(x_k)$.*

We note that Theorem 1 follows immediately from Proposition 2.

Proposition 3 [1, 3]**.** *Let T be a tree of order n and $v \in V(T)$. Then the following holds.*

(1) *$v \in Cen(T)$ if and only if $bw(v) \leq \frac{n}{2}$.*

(2) *$Cen(T) = \{v\}$ if and only if $bw(v) < \frac{n}{2}$.*

(3) *$v \in Cen(T)$ if and only if $cw(v) \geq \frac{n}{2}$.*

(4) *$Cen(T) = \{v\}$ if and only if $cw(v) > \frac{n}{2}$.*

Proposition 4 [4]**.** *Let T be a tree and $x \in V(T)$. We assume that B is a component of $T - x$ such that $bw(x) = |V(B)|$. Then the following holds.*

(1) *$cw(y) < cw(x)$ for each $y \in V(T) \backslash (V(B) \cup \{x\})$.*

(2) *If $y \in V(T)\backslash\{x\}$ and $cw(y) \geq cw(x)$, then $y \in V(B)$.*

For a tree T and $x \in V(T)$, let $N(x)$ denote the set $\{v \in V(T) : vx \in E(T)\}$. It is seen that $deg_T(x) = |N(x)|$. Suppose $x \notin Cen(T)$. The following lemma shows that there exists exactly one vertex y in $N(x)$ with $cw(y) > cw(x)$, and the sum of the complementary-weights of the other vertices in $N(x)$ equals $cw(x) - 1$ if $deg_T(x) \geq 2$.

Lemma 5. *Let T be a tree and $x \in V(T)\backslash Cen(T)$. Then there exists $y \in N(x)$ such that $cw(y) > cw(x)$ and $\sum_{v \in N(x)\backslash\{y\}} cw(v) = cw(x) - 1$ if $deg_T(x) \geq 2$.*

Proof. Let $k = deg_T(x)$ and $N(x) = \{x_1, x_2, \ldots, x_k\}$, and let the components of $T - x$ be $B_1, B_2, \ldots, B_k$, with $x_i \in V(B_i)$ and $bw(x) = |V(B_1)|$. Let $u \in Cen(T)$. Then $cw(u) > cw(x)$. By Proposition 4(2), $u \in V(B_1)$. Now claim that we can take $y = x_1$ and obtain $cw(y) > cw(x)$. First if x and u are adjacent, then u is exactly the vertex x_1 and $cw(x_1) > cw(x)$. If x and u are not adjacent, there is a path $u \cdots x_1 x$. By Proposition 2(2), $cw(x_1) > cw(x)$ as $cw(u) \geq cw(x_1)$. Next, if $k \geq 2$ we claim that $cw(x_i) = |V(B_i)|$ for $i \geq 2$. As $bw(x) = |V(B_1)|$, we see that $|V(B_1)| \geq |V(B_i)|$. It is obvious that for $i \geq 2$ the maximum component of $T - x_i$ has vertex set $\{x\} \cup \left(\bigcup_{j \neq i} V(B_j)\right)$. That is, $bw(x_i) = 1 + \sum_{j \neq i} |V(B_j)|$. Then $cw(x_i) = |V(T)| - bw(x_i) = |V(B_i)|$ for $i \geq 2$, where $|V(T)| = 1 + \sum_{j=1}^{k} |V(B_j)|$. We conclude that $\sum_{i=2}^{k} cw(x_i) = \sum_{i=2}^{k} |V(B_i)| = |V(T)| - |V(B_1)| - 1 = |V(T)| - bw(x) - 1 = cw(x) - 1$. That is, $\sum_{v \in N(x) \setminus \{y\}} cw(v) = cw(x) - 1$ if $deg_T(x) \geq 2$. ■

The following lemma can be obtained from Lemma 5.

Lemma 6. *Let T be a tree and $x \in V(T)$. If x is a non-endvertex and $x \notin Cen(T)$, then the following holds.*

(1) $\sum_{\substack{v \in N(x), \\ cw(v) < cw(x)}} cw(v) = cw(x) - 1.$

(2) *$deg_T(x) = 2$ if and only if x is adjacent to a vertex with complementary-weight $cw(x) - 1$.*

In a spider, the vertex with degree exceeding two is called the *body* of the spider.

Proposition 7 [7]. *We assume that S is a spider of order n. Let $x_1 x_2 x_3 \cdots x_p x_{p+1}$ be a path in S, where x_1 is an endvertex of S and x_{p+1} is the body of S. Then the following holds.*

(1) *If $p \leq \left[\frac{n}{2}\right]$, then $cw(x_i) = i$ for $1 \leq i \leq p$.*

(2) *If $p \geq \left[\frac{n}{2}\right] + 1$, then*

$$cw(x_i) = \begin{cases} i & for \;\; 1 \leq i \leq \left[\frac{n}{2}\right], \\ n - i + 1 & for \;\; \left[\frac{n}{2}\right] + 1 \leq i \leq p + 1. \end{cases}$$

Proof. It is obvious that $bw(x_i) = \max\{i - 1, n - i\}$ for $1 \leq i \leq p$. We assume that $i \leq \left[\frac{n}{2}\right]$. Then $2i \leq n$, which implies $i - 1 < n - i$. Thus, $bw(x_i) = n - i$ and $cw(x_i) = i$. We now assume that $i \geq \left[\frac{n}{2}\right] + 1$. Then $2i > n$, which implies $i - 1 \geq n - i$. Thus, $bw(x_i) = i - 1$ and $cw(x_i) = n - i + 1$.

(1) As $p \leq \left[\frac{n}{2}\right]$ and $1 \leq i \leq p$, we have $i \leq \left[\frac{n}{2}\right]$. Hence, $cw(x_i) = i$.

(2) We now assume that $p \geq \left[\frac{n}{2}\right] + 1$. If $1 \leq i \leq \left[\frac{n}{2}\right]$, then $cw(x_i) = i$. If $\left[\frac{n}{2}\right] + 1 \leq i \leq p$, then $cw(x_i) = n - i + 1$. We evaluate $cw(x_{p+1})$. As $p \geq \left[\frac{n}{2}\right] + 1$, we see that among the components of $T - x_{p+1}$, the path $x_1 x_2 x_3 \cdots x_p$ has the largest order. Hence, $bw(x_{p+1}) = p$ and $cw(x_{p+1}) = n - p$. ■

Let S be a spider and b the body of S. Then, each component of $S - b$ is called a *leg* of S. Obviously, each leg is a path. If a spider has k legs, and these legs have orders $l_1, l_2, \ldots, l_k$, where $l_1 \leq l_2 \leq \cdots \leq l_k$, then the spider is denoted by $SP(l_1, l_2, \cdots, l_k)$. An *average spider* is a spider $SP(l_1, l_2, \ldots, l_k)$ with $l_k \leq l_1 + l_2$.

Proposition 8. *We assume that S is an average spider of order n. Let $x_1x_2x_3 \cdots x_px_{p+1}$ be a path in S, where x_1 is an endvertex of S and x_{p+1} is the body of S. Then $Cen(S) = \{x_{p+1}\}$ and $cw(x_i) = i$ for $i = 1, 2, \ldots, p$.*

Proof. Let $S = SP(l_1, l_2, \ldots, l_k)$. By assumption, $p = l_t$ for some $1 \leq t \leq k$. As S is an average spider, we have $2p \leq 2l_k \leq l_k + l_1 + l_2 \leq n - 1$. That is, $p \leq \frac{n-1}{2}$. This implies that every component of $S - x_{p+1}$ has order less than $\frac{n}{2}$; hence, $bw(x_{p+1}) < \frac{n}{2}$. By Proposition 3(2), $Cen(S) = \{x_{p+1}\}$. As $p \leq \frac{n-1}{2} \leq \left[\frac{n}{2}\right]$, by Proposition 7(1), $cw(x_i) = i$ for $i = 1, 2, \ldots, p$. ■

We will make use of the following notations. A nondecreasing sequence consisting of n multiplicities of a_i for $i = 1, 2, \ldots, r$, where $a_1 < a_2 < \cdots < a_r$, is denoted by $\{a_1, a_2, \ldots, a_r\}^n$. The superscript n can be omitted if $n = 1$. For two nondecreasing sequences A and B, we use $A \cup B$ to denote the nondecreasing sequence consisting of all the elements of A and B. Thus, if $c \in A \cup B$, then the multiplicities of c in $A \cup B$ equal the sum of the multiplicities of c in A and the multiplicities of c in B. For instance, using these notations, the nondecreasing sequence $\{1, 1, 1, 2, 2, 2, 5\}$ can also be written in the following forms: $\{1\}^3 \cup \{2\}^3 \cup \{5\}$, $\{1, 2\}^3 \cup \{5\}$, $\{1, 2\}^2 \cup \{1, 2, 5\}$, etc.

Let S be an average spider of order n with k legs. Assume that for $i = 1, 2, \ldots, r$ there are t_i legs of order ℓ_i in S, with $\ell_1 < \ell_2 < \cdots < \ell_r$ and $\sum_{i=1}^{r} t_i = k$, where $r, t_i \in N$. Let b be the body of S. We see that $bw(b) = \ell_r$ and $cw(b) = n - \ell_r$. By Proposition 8, it is seen that the complementary-weight sequence of S is $\{1, 2, \ldots, \ell_1\}^{t_1} \cup \{1, 2, \ldots, \ell_2\}^{t_2} \cup \cdots \cup \{1, 2, \ldots, \ell_r\}^{t_r} \cup \{n - \ell_r\}$. It is easy to see that this sequence can also be written in the following form. The proof is omitted.

Proposition 9. *The aforementioned average spider has complementary-weight sequence*
$\{1, 2, \ldots, \ell_1\}^k \cup \{n - \ell_1\}$ *if* $r = 1$, *or*
$\{1, 2, \ldots, \ell_1\}^k \cup \{\ell_1 + 1, \ell_1 + 2, \ldots, \ell_2\}^{k-t_1} \cup \{\ell_2 + 1, \ell_2 + 2, \ldots, \ell_3\}^{k-(t_1+t_2)}$
$\cup \cdots \cup \{\ell_{r-1} + 1, \ell_{r-1} + 2, \ldots, \ell_r\}^{k-\sum_{j=1}^{r-1} t_j} \cup \{n - \ell_r\}$ *if* $r \geq 2$.

Theorem 10. *An average spider is branch-weight unique in the family of all trees.*

Proof. Let S be an average spider assumed as in Proposition 9, and C be the complementary-weight sequence of S. It suffices to show that if we construct a tree T with complementary-weight sequence C, then T is isomorphic to S. We divide the construction of T into three parts. In each part, an assertion is made followed by its proof.

Part 1. For $j = 2, 3, \ldots, \ell_1$, a vertex with complementary-weight j is adjacent to a vertex with complementary-weight $j - 1$.

First from C we see that for $j = 1, 2, \ldots, \ell_1$, T has exactly k multiplicities of vertices with complementary-weight j. If $\ell_1 = 1$, there is nothing to prove. So let $\ell_1 \geq 2$. By Lemma 6(1), a vertex with complementary-weight 2 is adjacent to a vertex with complementary-weight 1. Next, if $\ell_1 \geq 3$, we consider a vertex with complementary-weight 3. As each vertex with complementary-weight 1 is an endvertex of degree one and is already adjacent to a vertex with complementary-weight 2, by Lemma 6(1) we see that a vertex with complementary-weight 3 is adjacent to a vertex with complementary-weight 2. Now suppose that for $j = 2, 3, \ldots, a$, some a, where $3 \leq a < \ell_1$, a vertex with complementary-weight j is adjacent to a vertex with complementary-weight $j - 1$. By Lemma 6(2), it is seen that each vertex with complementary-weight j, where $2 \leq j \leq a - 1$, is of degree two and has been adjacent to two vertices, one is with complementary-weight $j - 1$ and the other is with complementary-weight $j + 1$. So for a vertex with complementary-weight $a + 1$, by Lemma 6(1) it must be adjacent to a vertex with complementary-weight a. Thus by induction this part holds.

Part 2. For $j = \ell_1 + 1, \ell_1 + 2, \ldots, \ell_r$, a vertex with complementary-weight j is adjacent to a vertex with complementary-weight $j - 1$.

From C we see that there is no vertex with complementary-weight $\ell_1 + 1$ if $r = 1$. So let $r \geq 2$. Distinguish two cases: $t_1 \geq 2$ and $t_1 = 1$.

Case 1. $t_1 \geq 2$. We have $\ell_r \leq 2\ell_1$ as S is an average spider. Therefore, for any $j = \ell_1 + 1, \ell_1 + 2, \ldots, \ell_r$, no vertex with complementary-weight j is adjacent to two or more vertices with complementary-weights at least ℓ_1 by Lemma 6(1). Hence, the assertion of Part 2 follows from the assertion of Part 1.

Case 2. $t_1 = 1$. We have $\ell_r \leq \ell_1 + \ell_2$ as S is an average spider. Using the similar arguments as in Part 1, we see that a vertex with complementary-weight $j' = \ell_1 + 1, \ell_1 + 2, \ldots, \ell_2$ is adjacent to a vertex with complementary-weight $j' - 1$. Therefore, no vertex with complementary-weight $j = \ell_2 + 1, \ell_2 + 2, \ldots, \ell_r$ is adjacent to a vertex with complementary-weight $1, 2, \ldots, \ell_1 - 1$ or $\ell_1 + 1, \ell_1 + 2, \ldots, \ell_2 - 1$, from which the assertion of Part 2 follows by Lemma 6(1), as $j \leq \ell_r \leq \ell_1 + \ell_2$.

Part 3. T is taken to be a spider and T is isomorphic to S.

Till now, there are $k + 1$ vertices in T that have not been completed the adjacency. Among the $k + 1$ vertices, one is with complementary-weight $n - \ell_r$, the others are t_i vertices with complementary-weight ℓ_i, for $i = 1, 2, \ldots, r$. From the previous two parts, we see that each of these t_i vertices with complementary-weight ℓ_i is not yet adjacent to a vertex with complementary-weight greater than ℓ_i. We note that for $i = 2, 3, \ldots, r$, all these t_i vertices with complementary-weight ℓ_i are of degree two; and the t_1 vertices with complementary-weight ℓ_1 are either of degree two if $\ell_1 \geq 2$, or of degree one if $\ell_1 = 1$. By Lemma 5, it is seen that all the aforementioned t_i vertices with complementary-weight ℓ_i are adjacent to the vertex with complementary-weight $n - \ell_r$. Then the construction of T is complete. We see that T is a spider with t_i legs of order ℓ_i for $i = 1, 2, \ldots, r$. The assertion of Part 3 holds. ■

The following lemma provides the complementary-weight sequence of a non-average spider with three legs.

Lemma 11. *We assume that a spider $SP(a, b, p)$ has order n and $a + b < p$. Let C be the complementary-weight sequence of $SP(a, b, p)$. Then*
$C = \{1, 2, \ldots, a\}^3 \cup \{a+1, a+2, \ldots, b\}^2 \cup \{b+1, b+2, \ldots, b+a\} \cup \big\{b+a+1, b+a+2, \ldots, \left[\frac{n}{2}\right]\big\}^2$ *if n is even; or*
$C = \{1, 2, \ldots, a\}^3 \cup \{a+1, a+2, \ldots, b\}^2 \cup \{b+1, b+2, \ldots, b+a\} \cup \big\{b+a+1, b+a+2, \ldots, \left[\frac{n}{2}\right]\big\}^2 \cup \big\{\left[\frac{n}{2}\right]+1\big\}$ *if n is odd. We note that if $a = b$, the subsequence $\{a+1, a+2, \ldots, b\}^2$ of C is the empty set.*

Proof. Let the three legs of $SP(a, b, p)$ be paths $x_1 x_2 \cdots x_a$, $y_1 y_2 \cdots y_b$ and $z_1 z_2 \cdots z_p$, where x_1, y_1, z_1 are endvertices and x_a, y_b, z_p are vertices adjacent to the body. We denote the body by z_{p+1}. Then $bw(z_{p+1}) = p$ and $cw(z_{p+1}) = n - p$. If $b \geq \left[\frac{n}{2}\right] + 1$, then $n > 2b > n$, which is a contradiction. Hence, $a \leq b \leq \left[\frac{n}{2}\right]$. By Proposition 7(1), $cw(x_i) = i$ and $cw(y_j) = j$ for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, b$. By the assumption $a + b < p$, we have $n < 2p + 1$. Then $p > \frac{n-1}{2}$. Distinguish two cases.

Case 1. $p = \left[\frac{n}{2}\right]$, where n is even. By Proposition 7(1), $cw(z_k) = k$ for $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$. We have $C = \{1, 2, \ldots, a\} \cup \{1, 2, \ldots, b\} \cup \{1, 2, \ldots, b+a\} \cup \{b+a+1\}^2$, where $b + a + 1 = \left[\frac{n}{2}\right]$.

Case 2. $p \geq \left[\frac{n}{2}\right] + 1$. By Proposition 7(2),

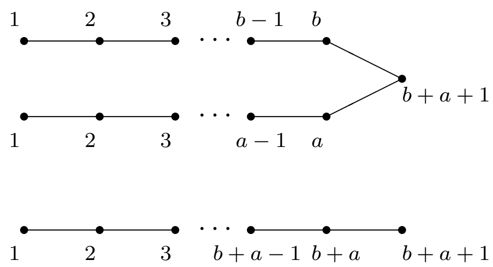
$$cw(z_i) = \begin{cases} i & \text{for } 1 \leq i \leq \left[\frac{n}{2}\right], \\ n - i + 1 & \text{for } \left[\frac{n}{2}\right] + 1 \leq i \leq p + 1. \end{cases}$$

Then $C = \{1, 2, \ldots, a\} \cup \{1, 2, \ldots, b\} \cup \{1, 2, \ldots, b+a\} \cup \big\{b+a+1, b+a+2, \ldots, \left[\frac{n}{2}\right]\big\}^2$ if n is even; or $C = \{1, 2, \ldots, a\} \cup \{1, 2, \ldots, b\} \cup \{1, 2, \ldots, b+a\} \cup \big\{b+a+1, b+a+2, \ldots, \left[\frac{n}{2}\right]\big\}^2 \cup \big\{\left[\frac{n}{2}\right]+1\big\}$ if n is odd.

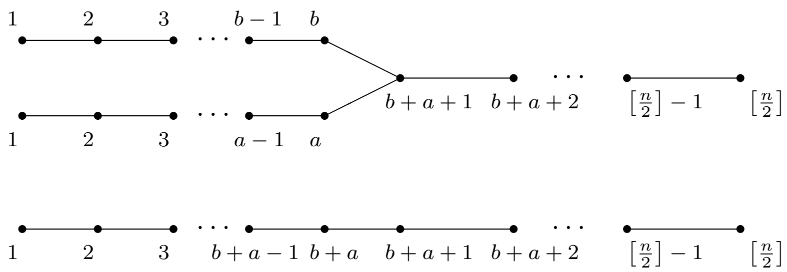
From Cases 1 and 2, we see that this lemma holds. ■

Theorem 12. *Every spider with three legs is branch-weight unique in the family of all trees.*

Proof. Let $S = SP(a, b, p)$, where $a \leq b \leq p$. It suffices to consider the case $a + b < p$, as S is an average spider if $p \leq a + b$ and, by Theorem 10, is branch-weight unique in the family of all trees. Thus, we assume that $a + b < p$. Let $n = a+b+p+1$ and C be the complementary-weight sequence of S. We complete the proof by showing that if we construct a tree T with complementary-weight sequence C, then T and S are isomorphic. First, using arguments similar to those in the proof of Theorem 10, we see that a vertex in T with complementary-weight j is adjacent to a vertex with complementary-weight $j - 1$ for $j = 2, 3, \ldots, b + a$. Now consider the two vertices with complementary-weight $b + a + 1$. By Lemma 6(1), we see that one is adjacent to the two vertices with complementary-weights a and b, the other is adjacent to the vertex with complementary-weight $b + a$. As seen in the following figure.



Next consider the two vertices with complementary weight $b + a + 2$. As all vertices with complementary-weights less than $b + a + 1$ have been completed the adjacency, by Lemma 6(1) we see that each of the two vertices is adjacent to a vertex with complementary-weight $b + a + 1$. Using this argument several times, we see that for $j = b + a + 3, b + a + 4, \ldots, \left[\frac{n}{2}\right]$, each vertex with complementary-weight j is adjacent to a vertex with complementary-weight $j - 1$. As seen in the following figure.



If n is an even number, by Theorem 1, both vertices with complementary-weight $\left[\frac{n}{2}\right]$ are in the centroid and are adjacent by an edge; if n is an odd number, by Lemmas 6(2) and 5, the two vertices are of degree two, and each one is adjacent to a vertex with complementary-weight larger than $\left[\frac{n}{2}\right]$. As there is the only vertex, with complementary-weight $\left[\frac{n}{2}\right]+1$, outside the exhibition as shown in the above figure, so, both vertices with complementary-weight $\left[\frac{n}{2}\right]$ are adjacent to the vertex with complementary-weight $\left[\frac{n}{2}\right]+1$. Hence, T is constructed to be a spider with three legs of orders a, b, and p. That is, T is isomorphic to S. ■

Lemma 13. *Let S be a spider with at least four legs. If S is not an average spider, then there exists a tree T such that T and S have the same branch-weight sequence and T is not isomorphic to S.*

Proof. Let $S = SP(l_1, l_2, \dots, l_k)$, where $l_1 \le l_2 \le \cdots \le l_k$ and $|V(S)| = n$. By assumption, $l_1 + l_2 < l_k$ and $k \ge 4$. Let b be the body of S, and let the three legs of S with orders l_1, l_2 and l_k be the paths $x_1x_2\cdots x_{l_1}$, $y_1y_2\cdots y_{l_2}$, and $z_1z_2\cdots z_{l_k}$, respectively, where x_1, y_1, z_1 are endvertices and x_{l_1}, y_{l_2} and z_{l_k} are vertices adjacent to b. We now construct a tree T by deleting the three edges $x_{l_1}b$, $y_{l_2}b$, and $z_{l_1+l_2}z_{l_1+l_2+1}$ from S, and adding three new edges $x_{l_1}z_{l_1+l_2+1}$, $y_{l_2}z_{l_1+l_2+1}$, and $z_{l_1+l_2}b$ to the remaining graph. We now prove that any two vertices in both T and S with the same notation have the same branch-weight. In the following, we use $bw_S(v)$ and $bw_T(v)$ to denote the branch-weights of a vertex v in S and T, respectively. Clearly, $bw_S(b) = l_k$. Let B be the component of $T - b$ with $V(B) = \{x_1, x_2, \dots, x_{l_1}, y_1, y_2, \dots, y_{l_2}, z_{l_1+l_2+1}, z_{l_1+l_2+2}, \dots, z_{l_k}\}$. It is seen that $|V(B')| \le |V(B)|$ for every component B' of $T - b$. Hence $bw_T(b) = l_k$. Next claim that for each vertex in $V(S)\backslash\{b, z_{l_1+l_2+1}\}$, with the same notation v, we have $bw_S(v) = bw_T(v)$. For $i = 1, 2, \dots, k$, let $v_{i,1}v_{i,2}\cdots v_{i,l_i}$ be the leg of S with order l_i, where $v_{i,1}$ is the endvertex and v_{i,l_i} is the vertex adjacent to b. We note that $v_{1,1}v_{1,2}\cdots v_{1,l_1}$, $v_{2,1}v_{2,2}\cdots v_{2,l_2}$ and $v_{k,1}v_{k,2}\cdots v_{k,l_k}$ are the legs $x_1x_2\cdots x_{l_1}$, $y_1y_2\cdots y_{l_2}$ and $z_1z_2\cdots z_{l_k}$, respectively. For $i = 1, 2, \dots, k$, in both S and T we have $deg(v_{i,1}) = 1$, and $deg(v_{i,j}) = 2$ for all $j \ge 2$ and $v_{i,j} \ne v_{k,l_1+l_2+1} = z_{l_1+l_2+1}$. Then for $i = 1, 2, \dots, k$, $bw_S(v_{i,1}) = bw_T(v_{i,1}) = n - 1$; and the graphs $S - v_{i,j}$ and $T - v_{i,j}$ both have two components, where $j \ge 2$ and $v_{i,j} \ne z_{l_1+l_2+1}$. It is seen that the two components of $S - v_{i,j}$ have orders $j - 1$ and $n - j$, and the two components of $T - v_{i,j}$ also have orders $j - 1$ and $n - j$, where $j \ge 2$ and $v_{i,j} \ne z_{l_1+l_2+1}$. So we have $bw_S(v_{i,j}) = bw_T(v_{i,j})$ for $j \ge 2$ and $v_{i,j} \ne z_{l_1+l_2+1}$. Now claim that $bw_S(z_{l_1+l_2+1}) = bw_T(z_{l_1+l_2+1})$. It is seen that $S - z_{l_1+l_2+1}$ have two components with orders $l_1 + l_2$ and $n - l_1 - l_2 - 1$, respectively; $T - z_{l_1+l_2+1}$ have three components with orders l_1, l_2 and $n - l_1 - l_2 - 1$, respectively. As $n - l_1 - l_2 - 1 \ge l_3 + l_k > l_k > l_1 + l_2$, we have $bw_S(z_{l_1+l_2+1}) = bw_T(z_{l_1+l_2+1}) = n - l_1 - l_2 - 1$. Thus for each vertex in $V(S)$ with the same notation v, we have $bw_S(v) = bw_T(v)$. That is, T and S have the same branch-weight sequence.

Clearly, T is not isomorphic to S because T has two vertices (b and $z_{l_1+l_2+1}$) with degrees greater than two. ■

In fact, an example of Lemma 13 has already been exhibited in Figure 3.

Theorem 14. *A spider with k legs is branch-weight unique in the family of all trees if and only if either $k = 3$ or the spider is an average spider and $k \geq 4$.*

Proof. ($\Leftarrow$) By Theorems 12 and 10.
($\Rightarrow$) Let $S = SP(l_1, l_2, \ldots, l_k)$. We assume that $k \geq 4$ and S is not an average spider. By Lemma 13, there exists a tree T such that T and S have the same branch-weight sequence and T is not isomorphic to S. That is, S is not branch-weight unique in the family of all trees. ■

We now introduce two types of trees that are branch-weight unique in the family of all trees. For two vertices u and v in a graph, we use $d(u, v)$ to denote the distance between u and v. A *rooted tree* is a tree with a specific vertex designated as the root. Let T be a rooted tree with root r. Then, T is called a *perfect symmetry tree of type* I if $deg_T(x) = deg_T(y)$ whenever $d(x, r) = d(y, r)$, where $x, y \in V(T)$. Figure 4 shows a perfect symmetry tree of type I. Therein, the degree of a vertex u equals four if $d(u, z) = 1$, and the degree of a vertex v equals one if $d(v, z) = 2$, where z is the root. Let now T_1 be a perfect symmetry tree of type I with root z_1 and T_2 be a copy of T_1 with the root renamed z_2. Let T' be a tree constructed from T_1 and T_2 by joining the two vertices z_1 and z_2 with a new edge. We call such a tree T' a *perfect symmetry tree of type* II. Figure 5 shows a perfect symmetry tree of type II that is formed by joining with an edge the two roots of two copies of the tree in Figure 4. Using arguments similar to those in the proof of Theorems 10 and 12, it is not difficult to see that the following theorem holds. The proof is left to the reader.

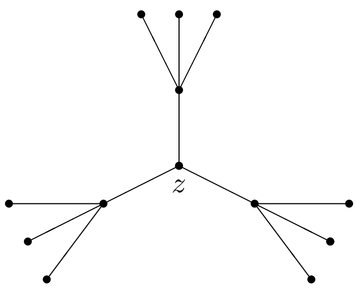


Figure 4

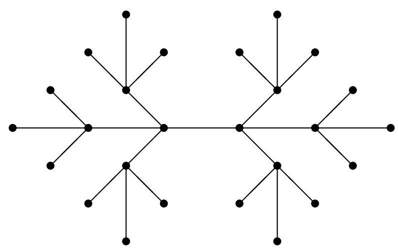


Figure 5

Theorem 15. *Perfect symmetry trees of type* I *and type* II *are branch-weight unique in the family of all trees.*

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