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ON THE 12-REPRESENTABILITY OF INDUCED SUBGRAPHS OF A GRID GRAPH

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Abstract

The notion of a 12-representable graph was introduced by Jones, Kitaev, Pyatkin and Remmel in [*Representing graphs via pattern avoiding words*, Electron. J. Combin. 22 (2015) #P2.53]. This notion generalizes the notions of the much studied permutation graphs and co-interval graphs. It is known that any 12-representable graph is a comparability graph, and also that a tree is 12-representable if and only if it is a double caterpillar. Moreover, Jones *et al.* initiated the study of 12- representability of induced subgraphs of a grid graph, and asked whether it is possible to characterize such graphs. This question of Jones *et al.* is meant to be about induced subgraphs of a grid graph that consist of squares, which we call square grid graphs. However, an induced subgraph in a grid graph does not have to contain entire squares, and we call such graphs line grid graphs.

In this paper we answer the question of Jones *et al.* by providing a complete characterization of 12-representable square grid graphs in terms of forbidden induced subgraphs. Moreover, we conjecture such a characterization for the line grid graphs and give a number of results towards solving this challenging conjecture. Our results are a major step in the direction of characterization of all 12-representable graphs since beyond our characterization, we also discuss relations between graph labelings and 12-representability, one of the key open questions in the area.

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1. Introduction

Let $\mathbb{P} = \{1, 2, \ldots\}$ and $\mathbb{P}^*$ be the set of all words over $\mathbb{P}$. Given a word $w = w_1 w_2 \cdots w_n \in \mathbb{P}$, denote by $A(w)$ the set of integers occurring in w. For example, $A(315353) = \{1, 3, 5\}$. For $B \subset A(w)$, let w_B be the word obtained from w by removing all the letters in $A(w)\backslash B$. For example, if $B = \{2, 3\}$ and $w = 12315251$ then $w_B = 232$. Let $\mathrm{red}(w)$ be the word that results from w by replacing each occurrence of the i-th smallest letter that occurs in w by i. For example, $\mathrm{red}(3729) = 2314$.

Let $u = u_1 u_2 \cdots u_m \in \mathbb{P}^*$ with $\mathrm{red}(u) = u$. Then we say that a word $w = w_1 w_2 \cdots w_n \in \mathbb{P}^*$ contains an occurrence of the *p*attern u if there exist integers $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\mathrm{red}(w_{i_1} w_{i_2} \cdots w_{i_m}) = u$. For example, the word 624635 contains two occurrences of the pattern 4231, namely the subsequences 6452 and 6352. The pattern u is *consecutive*, if in each of its occurrences $i_{t+1} - i_t = 1$ for all $1 \leq t \leq m - 1$.

Given a labeled graph $G = (V, E)$ and a pattern u, we say that G is *u-pattern representable* if there is a word $w \in \mathbb{P}^*$ such that $A(w) = V$, and for all $x, y \in V$, $xy \notin E$ if and only if $w_{\{x,y\}}$ contains an occurrence of u. In such a situation, we say that w *u-pattern represents* G, and w is called a *u-pattern-representant* of G. An unlabeled graph H is u-pattern representable if it admits a labeling resulting in a u-pattern representable labeled graph H'. We say that H' *realizes u-pattern representability* of H.

Requiring from u to be a consecutive pattern, we obtain the notion of a *u-representable graph* introduced in [2]. In this case, similarly to the above, we can define u-representability and u-representants, or just *representants* if u is clear from the context. The class of u-representable graphs generalizes the much studied class of word-representable graphs [3, 5], which is precisely 11-representable graphs. It was shown in [3] that if a consecutive pattern u is of length at least 3, then any graph can be u-represented. Also, note that a word avoids the pattern 12 if and only if it avoids the consecutive pattern 12, and thus the notion of a 12-pattern representable graph is equivalent to that of a 12-representable graph, which is the subject of interest in this paper.

Jones *et al.* [2] showed that the notion of a 12-representable graph generalizes the notions of the much studied *permutation graphs* (e.g. see [1, 6] and references therein) and *co-interval graphs* (e.g. see [7, 8]). Also, Jones *et al.* [2] showed that any 12-representable graph is a *comparability graph* (i.e., such a graph admits a *transitive orientation*), and also that a tree is 12-representable if and only if it is a *double caterpillar* (see [2] for definition). More relevant to this paper, Jones *et al.* [2] initiated the study of 12-representability of induced subgraphs of a *grid graph*, and asked whether it is possible to characterize such graphs. Examples of a grid graph and some of its possible induced subgraphs are given in Figure 1.1 which also appears in [2]. Jones *et al.* [2] showed that corner and skew ladder graphs, and thus ladder graphs, are 12-representable, while any graph with an induced cycle of size at least 5 is not 12-representable, so for example, the first two, and the last graphs in Figure 1.1 are not 12-representable.

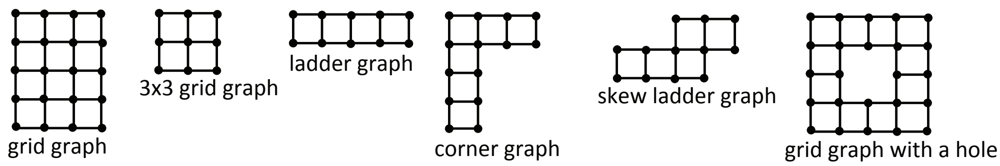

Figure 1.1. Induced subgraphs of a grid graph.

Even though it was not stated explicitly, the concern of Jones *at al.* in [2] was induced subgraphs of a grid graph that consist of a number of squares, which we call *square grid graphs*. In Section 2 we provide a complete characterization in terms of forbidden induced subgraphs of 12-representable square grid graphs (see Theorem 2.12). However, induced subgraphs of a grid graph may also contain edges (called by us "lines") that do not belong to any squares, for example, as in the graph in Figure 1.2. We refer to such subgraphs as *line grid graphs* (not to be confused with taking the line graph operation!) and think of the set of square grid graphs be disjoint with the set of line grid graphs. In Section 3 we give a number of results on 12-representation of line grid graphs and state a conjecture on the complete characterization related to 12-representability in this case (see Conjecture 3.6).

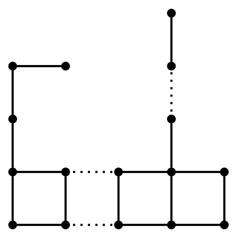

Figure 1.2. An example of a line grid graph.

We conclude the introduction by reviewing some basic definitions and results given in [2], which will be used frequently in our paper.

A graph $H = (V', E')$ is an *induced subgraph* of $G = (V, E)$ if $V' \subset V$ and for all $x, y \in V'$, $xy \in E'$ if and only if $xy \in E$. Also, similarly to the definition of $\mathrm{red}(w)$ for a word w, the reduced form $\mathrm{red}(H)$ of H is obtained from the graph H by replacing the i-th smallest label by i.

Observation 1.1 [2]**.** *If G is* 12*-representable and $H = (V', E')$ is an induced subgraph of G, then H is* 12*-representable.*

Theorem 1.2 [2]**.** *For a labeled* 12*-representable graph G, there exists a word-representant w in which each letter occurs at most twice. Also, G can be represented by a permutation if and only if G is a permutation graph.*

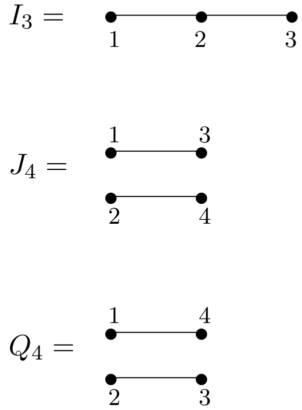

Figure 1.3. The graphs I_3, J_4 and Q_4.

Lemma 1.3 [2]**.** *Let $G = (V, E)$ be a labeled graph. If G has an induced subgraph H such that* $\mathrm{red}(H)$ *is equal to one of I_3, J_4 or Q_4 in Figure* 1.3*, then G is not* 12*-representable.*

Definition 1.4. A labeling of a graph is *good* if it contains no induced subgraphs equal to I_3, J_4 or Q_4 in the reduced form.

For two sets of integers A, B, we write $A < B$ if every element of A is less than each element in B. Also, a subset U of V is called a *cutset* of $G = (V, E)$ if $G \setminus U$ is disconnected.

Lemma 1.5 [2]**.** *Let $G = (V, E)$ be a labeled graph and U be a cutset of G. Assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two components of $G \setminus U$. If G is* 12*-representable, $|V_1| \geq 2$, $|V_2| \geq 2$, and the smallest element of $V_1 \cup V_2$ is in V_1, then $V_1 < V_2$.*

Theorem 1.6 [2]**.** *The cycle graph of length larger than* 4 *is not* 12*-representable.*

Finally, throughout this paper we assume that the graphs in question are connected since a graph is 12-representable if and only if each of its connected components is 12-representable. Indeed, if G is 12-representable, then clearly each of its connected componets is 12-representable using the hereditary nature of 12-representation. Conversely, label the connected components $G_1, G_2, \ldots$ of a graph G in a proper way, respectively, by $\{1, \ldots, |G_1|\}$, $\{|G_1|+1, \ldots, |G_1|+|G_2|\}, \ldots$, and then use the respective word-representants w_1, $w_2, \ldots$ to obtain the word $w_1 w_2 \cdots$ 12-representing G.

2. 12-Representability of Square Grid Graphs

Because the grid graph given in Figure 2.4 can be 12-represented by 3412, in all our arguments in this section, we always assume that the grid graphs are of size larger that 4. Note that the labeling of the square graph in Figure 2.4 is the only good labeling up to rotation and swapping 1 and 2, and 3 and 4.

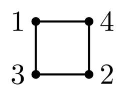

Figure 2.4. A labeled square grid graph with 4 nodes.

Let $F = \{X\} \cup \{C_{2n}\}_{n \geq 4}$, where X is given in Figure 2.5 and C_{2n} is the cycle graph on $2n$ nodes. Then we have the following lemma.

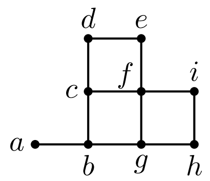

Figure 2.5. The non-12-representable graph X.

Lemma 2.1. *If a graph $G = (V, E)$ has an induced subgraph H in F, then G is not 12-representable.*

Proof. In view of Theorem 1.6, it suffices to show that X is not 12-representable.

We prove this by showing that there is no good labeling for X. If $1 \in \{a, b\}$, then viewing $\{c, f\}$ as a cutset, we have $\{d, e\} > \{h, i\}$ by Lemma 1.5. While, choosing $\{f, g\}$ as a cutset, we have $\{h, i\} > \{d, e\}$, a contradiction. Hence, $1 \notin \{a, b\}$. If $1 \in \{h, i\}$, then viewing $\{c, f\}$ as a cutset, we have $\{d, e\} > \{a, b\}$. While, choosing $\{c, g\}$ as a cutset, we have $\{a, b\} > \{d, e\}$, a contradiction. Hence, $1 \notin \{h, i\}$. By symmetry, $1 \notin \{d, e\}$.

Now, assume that $g = 1$. To avoid I_3, J_4 and Q_4, there are two choices for 2, namely, $i = 2$ or $a = 2$. If $i = 2$, then it can be checked that $h = 3$ and $c = 4$. To avoid I_3, we have $5 \notin \{b, d, f\}$. To avoid J_4 and Q_4, we have $5 \notin \{a, e\}$. This means that there is no position for 5. Hence, $i \neq 2$. Similarly, setting $a = 2$ makes us find no place for 3. It follows that $g \neq 1$. By symmetry, $c \neq 1$.

Assume that $f = 1$, then $h = 2$ or $d = 2$. Without loss of generality assume that $h = 2$. Then $i = 3$. Then, to avoid I_3, J_4 and Q_4, there is no position for 4. Hence, we deduce that $f \neq 1$.

Thus, there exists no good labeling for X, which completes the proof. ■

If a graph G does not contain an induced subgraph in F, then we say that G is *F-avoiding.* By Lemma 2.1, any 12-representable graph is F-avoiding.

Definition 2.2. A square S in a grid graph is called an *end-square* if it is incident with an edge ab such that neither a nor b is a corner point of a square different from S.

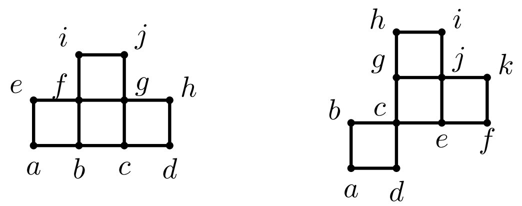

Figure 2.6. The graphs G_1 and G_2, respectively. Non-12-representability of these graphs follows from the fact that they contain X as an induced subgraph.

For example, in Figure 2.6, all squares in G_1 but $bcgf$ are end-squares, and all squares in G_2 but $cgje$ are end-squares. In Lemma 2.8 below we will show that each F-avoiding square grid graph contains an end-square.

Lemma 2.3. *Let $G = (V, E)$ be a labeled 12-representable square grid graph. Then 1 must be the label of a node of an end-square in G.*

Proof. To prove this lemma, we assume to the contrary that 1 is not a label of a node of an end-square of G. Since G is 12-representable, G must be F-avoiding by Lemma 2.1, so each node of G belongs to at most three squares. We consider the following three cases.

Case 1. The node labeled by 1 belongs to only one square. Possible situations in this case are given schematically in Figure 2.7, where the ovals indicate the rest of the respective graphs. For the first subcase, it is easy to see that the number of nodes in A or B are at least 2. Choosing $\{x, y\}$ as a cutset, by Lemma 1.5 we see that all labels of the nodes in A are larger than those of B. While, choosing $\{y, z\}$ as a cutset, we have all labels of the nodes in B are larger than those of A, a contradiction. Hence, this subcase is impossible. Similarly, by viewing $\{a, c\}$ and $\{b\}$ as a cutset, respectively, we prove that the second subcase is impossible. Viewing $\{d\}$ and $\{e\}$ as a cutset, respectively, we obtain that the third subcase is impossible as well.

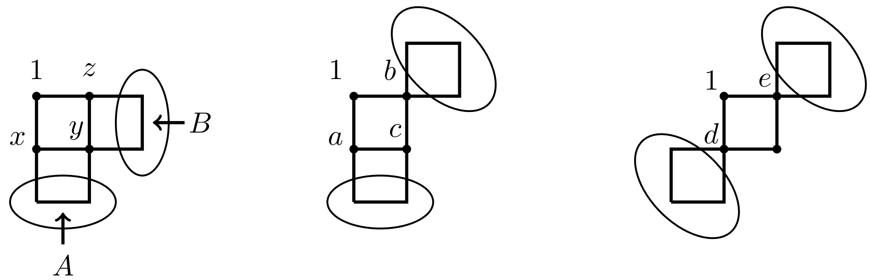

Figure 2.7. Three subcases when 1 belongs to only one square.

Case 2. The node labeled by 1 belongs to exactly two squares. Possible situations in this case are given in Figure 2.8. For the first subcase, the number of nodes in C is at least 5 and the number of nodes in D is at least 4, which follows from the fact that the two squares 1 belongs to are not end-squares. Choosing $\{x, y\}$ as a cutset, we see that all labels in C except x, y are larger than those in D. Choosing $\{y, z\}$ as a cutset, we see that all labels in C except y are smaller than those in D except z, a contradiction. Thus, this subcase is impossible. Similarly, by viewing $\{a, c\}$ and $\{b, d\}$ as a cutset, respectively, we can prove that the second subcase is impossible.

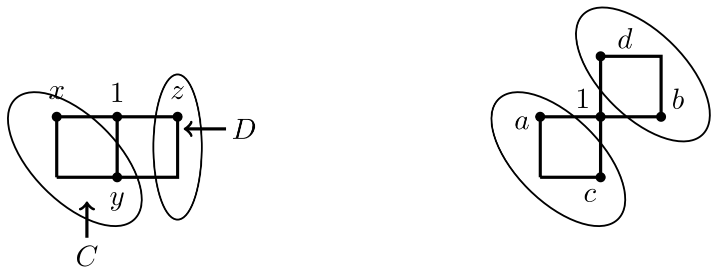

Figure 2.8. Two subcases when 1 belongs to exactly two squares.

Case 3. The node labeled 1 belongs to three squares. There is only one possibility here shown schematically in Figure 2.9. Since the three squares 1 belongs to are not end-squares, we see that the number of nodes in E, as well as that in F, is at least 4. Choosing $\{x, y\}$ as a cutset, all labels in E except x are larger than those in F. On the other hand, choosing $\{z, k\}$ as a cutset, all labels in E are larger than those in F except k, a contradiction. Hence, this case is impossible.

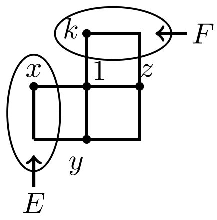

Figure 2.9. The situation when 1 belongs to three squares.

Summarizing the cases above, we see that 1 is never the label of a node of a non-end-square. ■

Lemma 2.4. *Given a labeled* 12*-representable square grid graph* $G = (V, E)$ *with an end-square* S *shown schematically in Figure* 2.10, *with neither* 2 *nor* 3 *being a node of another square and* 1 *being a node of another square, we can assume that a representant of* G *is* $w = 3w_1 2w_2 12w_3$ *with any letter in* w_1, w_2, w_3 *being at least* 4*.*

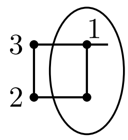

Figure 2.10. An end-square with 2 and 3 belonging to a single square.

Proof. Let w be a 12-representant of G. Firstly, note that erasing all 3s in w and placing a 3 at the beginning of the obtained word, we have a 12-representant of G since 3 is only connected to 1 and 2 and no other connections are changed.

Secondly, we claim that we can assume that 1 occurs only once in w. This can be verified by removing all but the leftmost 1 in w. Since 1 is the smallest letter, and the labels of the nodes connected to it must be to the left of the leftmost 1 in w, nothing will be changed after this operation.

Thirdly, we can assume that there are two copies of 2 in w and 1 is between the 2s. Indeed, by Theorem 1.2, we can assume that 2 occurs at most twice in w, and clearly since there is no edge between 1 and 2, after 1 there must be at least one 2. However, the other 2 must be before 1, since there exist nodes connected to 1 but not to 2.

Lastly, we claim that the only 1 and the 2 after it can be assumed to be next to each other in w. If not, we can move all the letters between 1 and the second 2 right after 2 keeping their relative order. This operation is allowable since all letters between 1 and 2 must be not connected to 2 because of the 2 before 1. Hence, moving the letters does not introduce any change. This completes the proof. ■

Proposition 2.5. *Let $G = (V, E)$ be a 12-representable square grid graph. Then there exists a labeled copy G' of G, which realizes 12-representability of G, with an end-square given in Figure 2.11, with neither 1 nor 3 belonging to another square.*

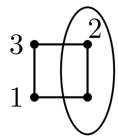

Figure 2.11. An end-square with neither 1 nor 3 belonging to another square.

Proof. By Lemma 2.3, 1 must be the label of a node in an end-square. Then for the labeling of the end-square, it is easy to check that there are only two cases up to symmetry, which are given in Figures 2.10 and 2.11, respectively. If it is the case given in Figure 2.11, we are done. If not, by Lemma 2.4 we assume that a 12-representant of G is $w = 3w_1 2w_2 12w_3$. Let G'' be the graph obtained from G' by exchanging the labels 1 and 2. We claim that G'' is also 12-representable and its representant is given by $\overline{w} = 3w_1 1w_2 2w_3$. This can be verified by the fact that the nodes connected to both 1 and 2 in G' remain connected to 1 and 2 in G'', while the nodes connected to 1 but not 2 in G' become connected to 2 but not 1 in G''. This completes the proof. ■

As a by-product of the above lemma, we obtain the following corollary, which will be frequently used in the rest of the paper.

Corollary 2.6. *Let $G = (V, E)$ be a 12-representable square grid graph with an end-square given in Figure 2.11. Then G can be represented by some $w = 3w_1 1w_2 2w_3$ with each letter in w_1, w_2, w_3 being at least 4.*

Lemma 2.7. *Given a 12-representable square grid graph G, we can obtain a new 12-representable square grid graph G' by extending G in the five ways presented schematically in Figure 2.12, where the extensions are applied to end-squares, which are indicated by dashed lines, and the assumption is that the dashed edges are the only new edges in the obtained graph.*

Proof. In view of Corollary 2.6, we may assume that a 12-representant of G is $w = 3w_1 1w_2 2w_3$, with each letter in w_1, w_2, w_3 being larger than 3. It is enough to find the corresponding 12-representant of G' obtained under each extension. Assume that π' is obtained from π by adding 2 to each letter of π, while π'' is obtained from π by adding 3 to each letter of π. Then it is not difficult to see that a representant of G' for Cases 1–5 is, respectively, $\overline{w} = 351w_1' 4w_2' 24w_3'$, $\overline{w} = 35w_1' 152w_2' 4w_3'$, $\overline{w} = 351w_1' 2w_2' 4w_3'$, $\overline{w} = 3612w_1'' 4w_2'' 5w_3''$, and $\overline{w} = 3416w_1'' 2w_2'' 5w_3''$. Hence, in each of the five cases, G' is 12-representable. ■

Lemma 2.8. *Each F-avoiding square grid graph G has an end-square.*

Proof. We consider two cases for a square grid graph G. If no two squares in G have a common edge, then G must be as in Figure 2.13. Clearly, since G is

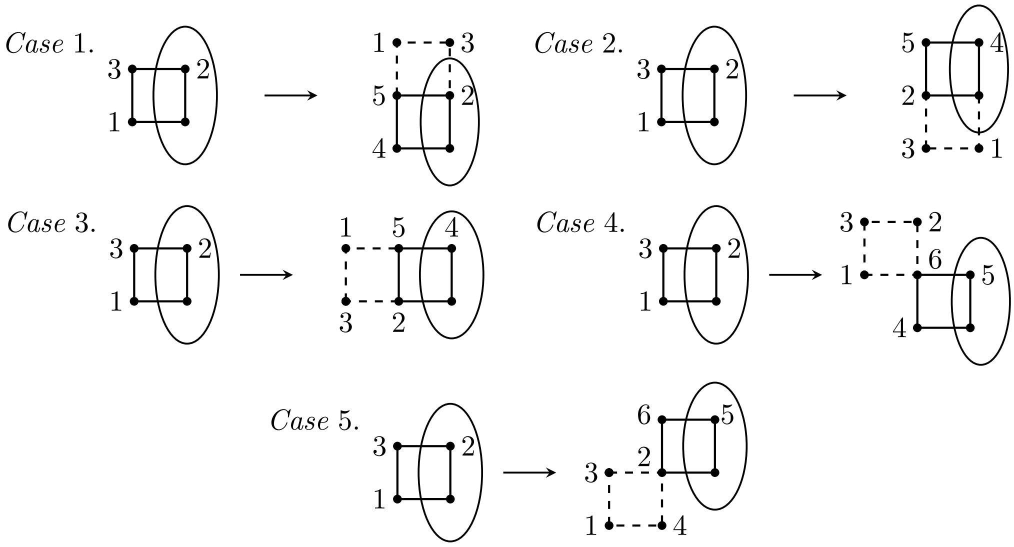

Figure 2.12. Five ways in extending a 12-representable square grid graph.

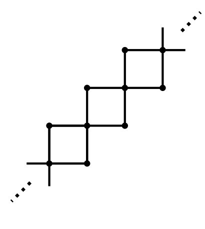

Figure 2.13. Grid graph with no common edges.

finite, it has an end-square. So, assume that at least two squares in G have a common edge. We start with one of such squares. If it is an end-square, we are done. If it is not, we go to the second square which has a common edge with it. If it is an end-square, we are done. Otherwise, we go to the third one which shares an edge with the second square but not the first one, if such a square exists. If there is no such square, we choose the third square to be the square that shares a node with the second square but not the first one. Thus, squares with common edges always come first, and then squares with common nodes. Continue in this fashion. Figure 2.14 illustrates possible cases in the $(i + 1)$-th step up to symmetry.

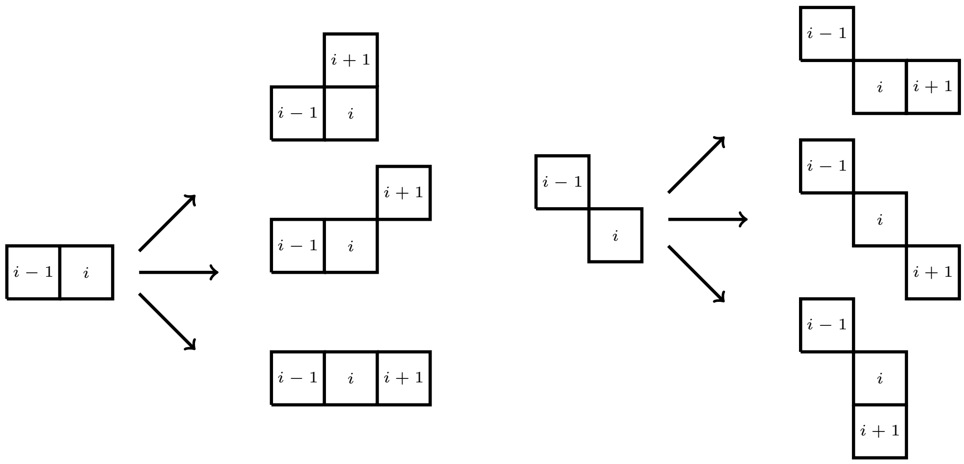

Figure 2.14. Possible cases in the $(i+1)$-th step of searching for an end-square. Squares are labeled according to the step on which they were added.

Since we avoid the cycle graphs C_{2n} for $n \geq 4$, the square labeled by $i+1$ will never be connected to that labeled by a smaller number. Because the graph is finite, sooner or latter we will find an end-square. ∎

Actually, we can prove the following stronger version of Lemma 2.8.

Lemma 2.9. *In any* 12*-representable square grid graph* G *with more than* 4 *nodes, there exist exactly two end-squares.*

Proof. Firstly, we show that there exist at least two end-squares in G. This can be done essentially by repeating the arguments in the proof of Lemma 2.8. Begin with considering the square labeled by 1. If it is an end-square, then there exists a square different from 1 connected to it since G has more than 4 nodes. Repeating the entire argument given in Lemma 2.8, we can find another end-square labeled by, say, t. If the square labeled by 1 is not an end-square, then there exists another square different from square 2 connected to it, which we label by $t+1$. Repeating the entire argument given in Lemma 2.8 again, we obtain another end-square.

Next, we will show that the number of end-squares is no more than two. Indeed, assume that G has at least three end-squares, which implies that the number of nodes in G is more than 12 as no two of end-squares can share a node to be F-avoiding; see Figure 2.15 for a schematic representation of G.

By Lemma 2.3, 1 must be the label of a node of an end-square. Without loss of generality assume that $1 \in \{a, b, x_1, x_2\}$. Viewing $\{x_3, x_4\}$ as a cutset, we deduce that $\{c, d\} > \{e, f\}$ by Lemma 1.5. On the other hand, if we choose $\{x_5, x_6\}$ as a cutset, we obtain that $\{c, d\} < \{e, f\}$, a contradiction. Thus, there

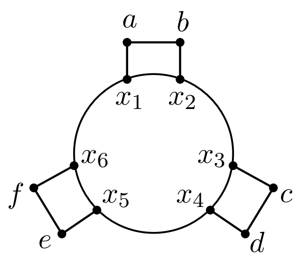

Figure 2.15. Square grid graph with at least three end-squares.

is no place for the label 1 in G assuming G has at least three end-squares, so G must have at most two squares. ■

Let $u = u_1 \cdots u_j \in \{1, \ldots, n\}^*$ and $\mathrm{red}(u) = u$. Let the *reverse* of u be the word $u^r = u_j u_{j-1} \cdots u_1$ and the *complement* of u be the word $u^c = (n+1-u_1) \cdots (n+1-u_j)$. Jones *et al.* [2] introduced the definition of the *supplement* of a graph. Given a graph $G = (\{1, \ldots, n\}, E)$, let the supplement of G be defined by $\overline{G} = (V, \overline{E})$ where for all $x, y \in V$, $xy \in E$ if and only if $n+1-x$ is adjacent to $n+1-y$ in $\overline{E}$. One can think of the supplement of $G = (V, E)$ as a relabeling of G by replacing each label x by the label $n+1-x$.

The following observation can be obtained by combining Observations 2 and 3 in [2].

Observation 2.10. *Let $G = (V, E)$ be a 12-representable graph and w be a 12-representant of G. Then $\overline{G}$ is also 12-representable with a 12-representant $(w^r)^c$.*

Corollary 2.11. *Let $G = (V, E)$ be a 12-representable square grid graph with $|V| = n$. Then there exists a labeled graph G', which realizes 12-representability of G, with one end-square given in Figure 2.11 and the other end-square given in Figure 2.16.*

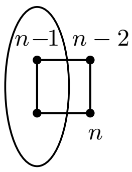

Figure 2.16. An end-square with neither n nor $n-2$ being a node of another square.

Proof. By Proposition 2.5, there exists a labeled copy G' of G, which realizes 12-representability of G with one end-square given in Figure 2.11. By Observation 2.10, we can relabel G so that it remains 12-representable with an end square being as in Figure 2.16. By Lemma 2.9, there is another end-square in G, and by Lemma 2.3, 1 must the label of one of its nodes. If this endpoint as in Figure 2.11, we are done. Otherwise, it must be like in Figure 2.10, and the arguments in the proof of Proposition 2.5 can be applied to relabel the square of the form in Figure 2.10 into that in Figure 2.11. We are done. ■

Our main result in this section is the following characterization theorem.

Theorem 2.12. *A square grid graph is F-avoiding if and only if it is* 12*-representable.*

Proof. Following from Oberservation 1.1 and Lemma 2.1, it is easy to check that 12-representable square grid graph is F-avoiding. In the following, we proceed to show that an F-avoiding square grid graph is 12-representable by induction on the number of squares.

Firstly, if a square grid graph G has at most two squares, which is certainly F-avoiding, then it can be easily checked that G is 12-representable as G is then either a square, or a two-square ladder graph, or two squares joint in a node. So, we assume that any F-avoiding square grid graph with n, $n > 2$, squares is 12-representable. We wish to show that this still holds for an F-avoiding square grid graph G with $n + 1$ squares.

Since G is F-avoiding, it must contain an end-square S_1 by Lemma 2.8. Let G' be a graph obtained form G by removing S_1. Clearly, G' is F-avoiding, which implies that G' is 12-representable by the induction hypothesis. We claim that removing S_1 from G produces a new end-suquare S_2 which is connected to S_1. Since $n > 2$, the number of nodes of G is not less than 8. We see that S_2 cannot be an end-square of G since G is connected. Then it is easy to check this claim through the following five cases up to symmetry. We omit the details here.

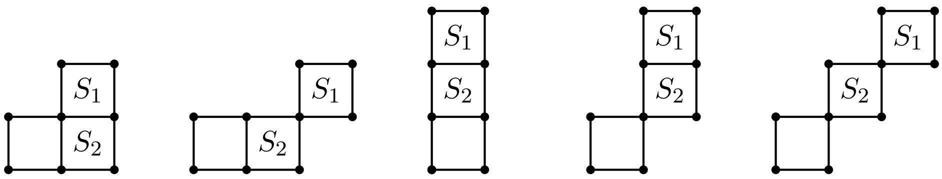

Figure 2.17. Five cases up to symmetry when removing an end-square of G.

By Corollary 2.11, there exists a labeling with square S_2 being labeled as given in Figure 2.11 or Figure 2.16. Viewing Obeservation 2.10, we can always assume that S_2 is labeled as given in Figure 2.11, otherwise, we will take its supplement to obtain a labeling of S_2 as in Figure 2.11. Thus, by Lemma 2.7, we can simply find a labeling of G, as well as its 12-representant. This shows that G is 12-representable and the proof is completed. ■

Remark 2.13. We note that Theorem 2.12 is still true if in F the graph X will be replaced by the square grid graphs G_1 and G_2 in Figure 2.6. However, such a change would not result in a set of *minimal* forbidden induced subgraphs.

By Theorem 2.12, if a square grid graph is F-avoidng, then it is 12-representable, and thus it has a good labeling. On the other hand, by Lemma 2.1 we know that if a square grid graph has a good labeling, then it is F-avoiding, and thus it is 12-representable by Theorem 2.12. This leads us to the following corollary, which is still an open question in the case of arbitrary (not necessarily grid) graphs.

Corollary 2.14. *A square grid graph has a good labeling if and only if it is* 12*-representable.*

In fact, we can prove a stronger version of Corollary 2.14, which is again an open question in the case of arbitrary graphs (no counter-example to this statement is known).

Theorem 2.15. *For any good labeling of a square grid graph G, there exists a word w* 12*-representing G.*

Proof. We prove, by induction on the number of squares, even a stronger statement that any such w begins with the only occurrence of the third smallest letter.

The base case is given in Figure 2.4, and its 12-representant is 3412. Now, assume that this theorem holds for square grid graphs with n squares. We wish to show that it still holds for $n+1$ squares. Notice that in a good labeling of G, 1 is always the label of a node in an end-square. Hence, we need to consider three cases in Figure 2.18, where the labels arrowed to each other can be swapped.

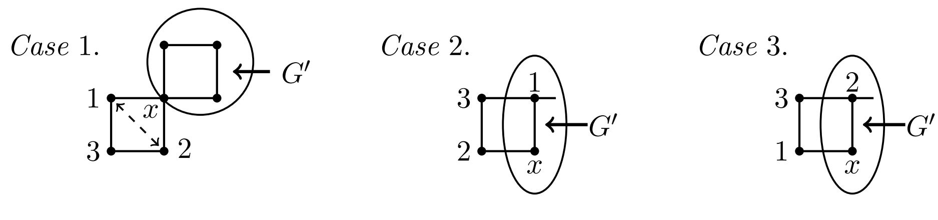

Figure 2.18. Three cases for a gooding labeling square grid graph.

For Case 1, let G be the graph in Figure 2.19 with exactly two squares. Then its 12-representant is 36712645, which is what we need to show.

Otherwise, we claim that x is always the third smallest label in G'. Indeed, consider four possible subcases in Figure 2.20. Remove the nodes labeled $1, 2$ and 3. Then, by induction hypothesis, G' can be represented by $w' = 6w''$. But then it can be checked that G is represented by $w = 3612w''$.

For Case 2, if x is the third smallest label in G', then $w' = xw''$ represents G' and $3x12w''$ represents G. Now, we suppose that x is not the third smallest in G'. Consider four possible subcases given in Figure 2.21.

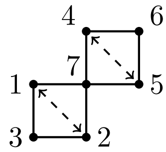

Figure 2.19. A special subcase in Case 1.

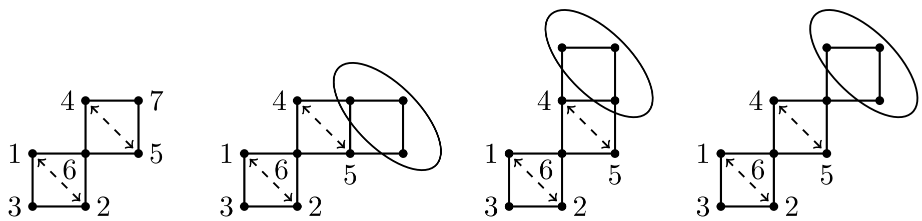

Figure 2.20. Four other subcases in Case 1.

For Subcase (1), remove the nodes labeled by $2, 3, 4$, and by induction hypothesis, the left graph G' is represented by some word w'. Let w be the word obtained from w' by appending 342 to w' to the left, and inserting 2 directly after the leftmost 1 in w'. Clearly, w ensures that 3 and 4 are connected with 1 and 2 only, and 2 is disconnected from any other node. Moreover, the third smallest letter 3 occurs once at the first place of w.

For Subcase (2), clearly, its 12-representant is $w = 35625124$.

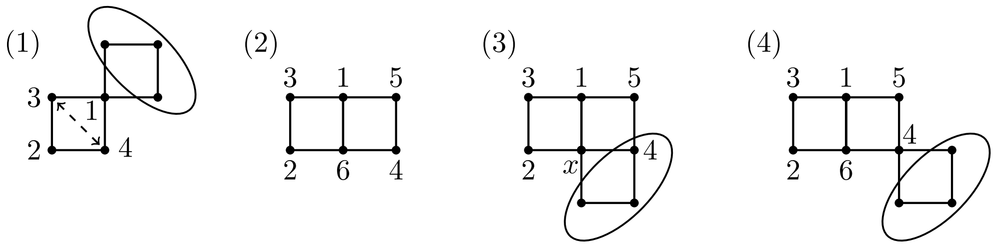

Figure 2.21. Four other subcases in Case 2.

For Subcase (3), remove the nodes labeled 2 and 3 and let the obtained graph be G'. By the induction hypothesis, assume that the 12-representant of G' is w'. Notice that the node labeled by x is the only node connected to both 1 and 4, except the node labeled by 5. Then, by Corollary 2.6, we may assume that the 12-representant of G' is $w' = 5x1w_2 4w_3$ with each letter in w_2 and w_3 being larger than 5. Let $w = 35x2512w_2 4w_3$. It can be checked readily that w 12-represents G given in Subcase (3). The same construction is still valid for Subcase (4), and hence, we omit the details.

For Case 3, we swap the labels 1 and 2 in G. Then we are led to a new graph G'', which is just the graph given in Case 2. It should be mentioned that the labeling after swaping is still good, because 1 and 2 are indistinguishable with respect to the other elements. Then, following from the result of Case 2, we see that there is alway a 12-representant for G'', which implies that G'' is 12-representable. By the proof of Proposition 2.5, we see that there is a 12-representant for G. ∎

Remark 2.16. We have proved that the existence of a good labeling for a square grid graph is equivalent to the square grid graph being 12-representable. Moreover, we have shown that any good labeling of a square grid graph can be used to find a word 12-representing the graph. It is an interesting open question if such a property holds for any other graph. Namely, is the existence of a good labeling in a graph equivalent to the graph being 12-representable? If so, then can any good labeling be turned into a word-representant?

Definition 2.17. A *corner node* in a 12-representable square grid graph is a node that belongs to exactly one square, which is a non-end square and shares edges with two other squares.

For example, in the leftmost graph in Figure 2.7, the node labeled by 1 is a corner node, but the nodes labeled by x, y and z are not.

Theorem 2.18. *Let G be a 12-representable square grid graph. Then there exists a word-representant w of G, such that in w,*

- *the label of each corner node is repeated twice (for no good labeling it can appear only once in w);*
- *between the two copies of the label of a corner node there are exactly two other letters, and*
- *the label of any other (non-corner) node appears exactly once.*

Proof. We wish to prove this by induction on the number of squares. The base case is given in Figure 2.22 and its 12-representant is $w = 351748246$.

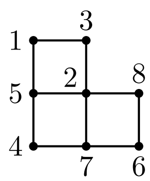

Figure 2.22. The base case in the proof of Theorem 2.18.

Now, assuming that the theorem holds for square grid graphs with n squares, we proceed to show that it still holds for $n + 1$ squares. This can be done by going through the cases given in Lemma 2.7. In the generation of square grid graphs, only Cases 1 and 2 will bring new corner nodes. In Case 1, 4 is labeled at the corner and repeats in the 12-representant $\overline{w} = 351w_1'4w_2'24w_3'$, with letters in w_1', w_2' and w_3' larger than 5. Notice that the elements in w_2' are connected with 2, but not connected with $1, 3, 4, 5$. Hence, actually, there is exactly one element in w_2'. This completes the proof of this case.

For the second case, 5 is labeled at the corner and repeats in the 12-representant $\overline{w} = 35w_1'152w_2'4w_3'$, with letters in w_1', w_2' and w_3' larger than 5. Notice that the elements in w_1' are connected with $1, 2, 4$, but not connected with $3, 5$. Hence, there is exactly one element in w_1'. ■

3. 12-Representability of Line Grid Graphs

In this section, we study the 12-representability of induced subgraphs of a grid graph with "lines" (i.e., edges not belonging to any square), which we call line grid graphs. Unfortunately, we cannot give a characterization in this case (we can only conjecture it; see Conjecture 3.6). Still, we give a number of results on 12-representability of line grid graphs that should be useful in achieving the desired characterization.

Definition 3.1. Given a line grid graph G, we call a node v in G *k-suitable* if there is a way to attach an induced path to v of length k that would result in a line grid graph. We refer to such induced paths as lines in what follows for brevity.

Proposition 3.2. *Let G be a 12-representable square grid graph, and G' be obtained from G by attaching a line of length k, $k \geq 2$, to a k-suitable node in G. If G' is 12-representable, then the line must be attached to a node of an end-square in G.*

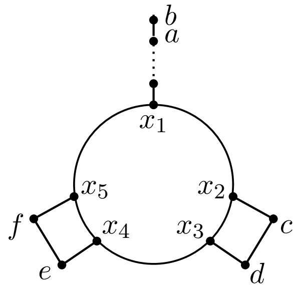

Figure 3.23. A line grid graph with a line of length k, $k \geq 2$, attached to a k-suitable node not belonging to an end-square.

Proof. We show that a line of length k, $k \geq 2$, cannot be attached to a node of a non-end-square. Indeed, assume to the contrary, that this was the case, as illustrated schematically in Figure 3.23. Clearly, x_i, $i = 1, \ldots, 5$, are distinct. If 1 is the label of a node on the line, then viewing $\{x_2, x_3\}$ as a cutset, we have $\{c, d\} > \{e, f\}$. On the other hand, viewing $\{x_4, x_5\}$ as a cutset, we have $\{c, d\} < \{e, f\}$, a contradiction. If $1 \in \{x_2, x_3, c, d\}$, then viewing $\{x_1\}$ as a cutset, we deduce that $\{a, b\} > \{e, f\}$. However, viewing $\{x_4, x_5\}$ as a cutset, we deduce that $\{a, b\} < \{e, f\}$, a contradiction. Similarly, we can prove that $1 \notin \{x_4, x_5, e, f\}$. Summarising the cases, and combining with Lemma 2.3, we see that there is no place for the label 1, as desired. ■

We believe that in Proposition 3.2 the "if then" statement can be replaced by an "if and only if" statement, but we were not able to prove it.

Proposition 3.3. *Let G be the line grid graph obtained by adding a line A to a node x on another line B as shown in Figure* 3.24. *If G is* 12*-representable, then the length of A is at most* 2*, or the smallest distance between x and an endpoint of B is at most* 2.

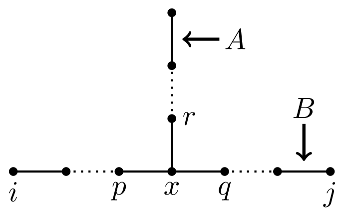

Figure 3.24. A line grid graph with a line of length k, $k \geq 2$, not attached to a node of an end-square.

Proof. Assume that the part of line B between i and x (respectively, x and j) is called C (respectively, D). We wish to prove that if all of the lengths of A, C and D are larger than 2, then G is not 12-representable. We prove this by showing that there is no good labeling for G in this case. If 1 is the label of a node on A, then choosing p as a cutset, we see that the nodes on C, except for p and x, are larger than those on D. Choosing q as a cutset, we see that the nodes in D, except for q and x, are larger than those on C, which is a contradiction. Hence, 1 cannot be the label of a node on A. By a similar analysis, we can also show that 1 cannot be the label of a node on C or D, as desired. ■

Note that the graph in Figure 3.24 is a subdivision of the claw $K_{1,3}$. Denote a graph of this form with three branches of length s, t, p by $B(s, t, p)$ and observe that Proposition 3.3 proves that $B(3, 3, 3)$, the 3-subdivision of $K_{1,3}$ presented in Figure 3.25, is not 12-representable. Once again, we believe that in Proposition 3.3 the "if then" statement can be replaced by an "if and only if" statement, but we were not able to prove it.

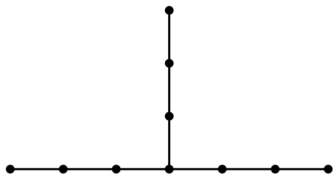

Figure 3.25. The non-12-representable graph $B(3,3,3)$.

Theorem 3.4. *Given a* 12*-representable line grid graph G, G' is obtained by gluing a line of length* 1 *to a node of G which is not a corner node and is* 1*-suitable. Then G' is* 12*-representable.*

Proof. First, we claim that each node in G is either a local maximum or a local minimum, and moreover, a local maximum must connect to a local minimum. Otherwise, there will occur an induced subgraph equal to I_3, which contradicts to the fact that G is 12-representable.

By Theorem 2.18, we may find a labeling of G with w being its 12-representant, where each non-corner label occurs only once. Assume that the node we glue to is i. Now, we consider two cases as follows.

Case 1. i is a local maximum. Labeling the node new added i and increasing each label in G not less than i by 1, we obtain a labeling of G'. Let w' be the word obtained from w by increasing each letter in w larger than i by 1, changing i to $(i+1)i$ and insering an i at the end of the word.

We claim that w' 12-represents G'. Since i is a local maximum of G, all letters larger than $i+1$ will occur after $(i+1)i$ in w'. This implies that i is disconnected with all letters larger than $i+1$. The i at the end of w' kills all smaller letters. Lastly, i and $i+1$ are connected. The claim is verified.

Case 2. i is a local minimum. Labeling the node new added $i+1$ and increasing each label in G larger than i by 1, we obtain a labeling of G'. Let w' be the word obtained from w by increasing each letter in w larger than i by 1. Then changing i to $(i+1)i$ and lastly inserting an $i+1$ at the begining of the word. By a similar analysis as given in Case 1, we may see that w' 12-represents G'.

Summarizing, in each case, we find a 12-representant for G'. Hence, G' is 12-representable and we complete the proof. ■

Proposition 3.5. *Assume that G' is obtained by gluing a line of length* 1 *to a corner node which is* 1*-suitable in G. Then G' is not* 12*-representable.*

Proof. The statement follows from non-12-representability of the graph X in Figure 2.5. ∎

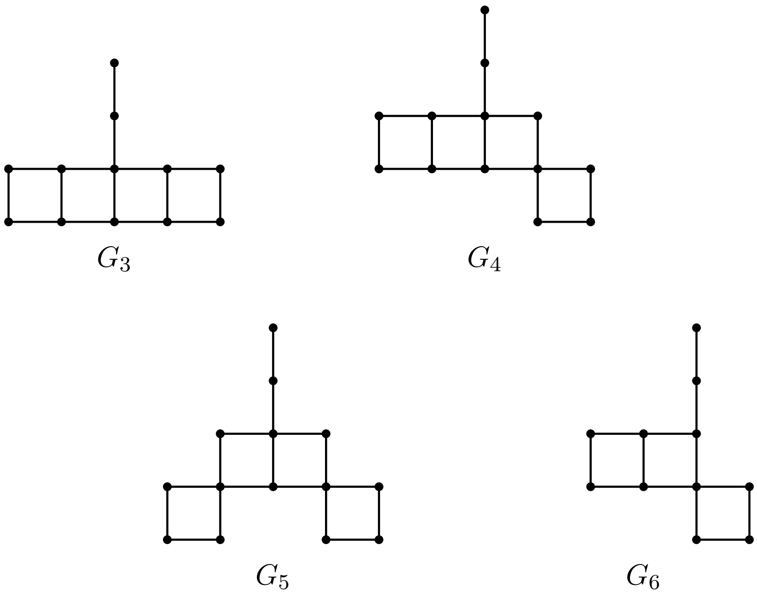

Figure 3.26. The non-12-representable graphs G_i, $i \in \{3,4,5,6\}$; non-12-representability of these graphs follows from Proposition 3.2.

We end up this paper with the following conjecture.

Conjecture 3.6. *A line grid graph is* 12*-representable if and only if it is* P*-avoiding, where* $P = F \cup \{B(3,3,3), G_3, G_4, G_5, G_6\}$ *with* $B(3,3,3)$ *and* $G_i, i \in \{3,4,5,6\}$*, given in Figures* 3.25 *and* 3.26*, respectively.*

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