Discussiones Mathematicae
Graph Theory 42 (2022) 277–298
https://doi.org/10.7151/dmgt.2261

THE STAR DICHROMATIC NUMBER

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Abstract

We introduce a new notion of circular colourings for digraphs. The idea of this quantity, called *star dichromatic number* $\vec{\chi}^*(D)$ of a digraph D, is to allow a finer subdivision of digraphs with the same dichromatic number into such which are "easier" or "harder" to colour by allowing fractional values. This is related to a coherent notion for the vertex arboricity of graphs introduced in [G. Wang, S. Zhou, G. Liu and J. Wu, *Circular vertex arboricity*, J. Discrete Appl. Math. 159 (2011) 1231–1238] and resembles the concept of the *star chromatic number* of graphs introduced by Vince in [15] in the framework of digraph colouring. After presenting basic properties of the new quantity, including range, simple classes of digraphs, general inequalities and its relation to integer counterparts as well as other concepts of fractional colouring, we compare our notion with the notion of circular colourings for digraphs introduced in [D. Bokal, G. Fijavz, M. Juvan, P.M. Kayll and B. Mohar, *The circular chromatic number of a digraph*, J. Graph Theory 46 (2004) 227–224] and point out similarities as well as differences in certain situations. As it turns out, the star dichromatic number shares all positive characteristics with the circular dichromatic number of Bokal *et al.*, but has the advantage that it depends on the strong components of the digraph only, while the addition of a dominating source raises the circular dichromatic number to the ceiling. We conclude with a discussion of the case of planar digraphs and point out some open problems.

Keywords: dichromatic number, star chromatic number, circular dichromatic number.

2010 Mathematics Subject Classification: 05C15, 05C20.

1. Introduction

Digraphs and graphs in this paper are considered loopless, but are allowed to have multiple parallel and anti-parallel arcs between vertices. Digraphs without parallel or anti-parallel edges are referred to as *simple*. We will refer to edges e in graphs by uw where u, w are the end vertices of e, if this does not lead to confusion with parallel edges. Given an arc (or directed edge) e of a digraph, we use $e = u \to w$ or equivalently $e = (u, w)$ to express that e has tail u and head w. This is not to be understood as a proper equality but as a statement on the edge e. Cycles and paths in graphs and directed cycles and paths in digraphs are always considered without repeated vertices.

Vince in [15] introduced the concept of the *star chromatic number* of a graph, nowadays also known as the *circular chromatic number*. The original definition of Vince is based on so-called (k, d)-colourings, where colours at adjacent vertices are not only required to be distinct as usual but moreover 'far apart' in the following sense. For every $k \in \mathbb{N}$ and elements $x, y \in \{0, \dots, k-1\}$, let $\text{dist}_k(x, y) = |(x - y) \text{mod } k|_k$, where $|a|_k = \min\{|a|, |k - a|\}$, for all $a = 0, \dots, k-1$, denote the *circular k-distance between x and y*. Then we define the following.

Definition (cf. [15])**.** Let G be a graph and $(k, d) \in \mathbb{N}^2, k \geq d$. A *$(k, d)$-colouring of G* is an assignment $c : V(G) \to \{0, \dots, k-1\} \simeq \mathbb{Z}_k$ of colours to the vertices so that $\text{dist}_k(c(u), c(w)) \geq d$ whenever u, w are adjacent.

Fixing a graph G, Vince furthermore considered the smallest possible value of $\frac{k}{d}$ where (k, d) allows a legal colouring of G as a fractional measure of the "colourability" of G.

Definition. Let G be a graph. The quantity

$$\chi^*(G) = \inf \left\{ \frac{k}{d} \,\middle|\, \exists\, (k, d)\text{-colouring of } G \right\} \in \mathbb{R}_+$$

is called the *star chromatic number*, respectively *circular chromatic number* of G.

The following theorem captures the most important elementary properties of the star chromatic number.

Theorem 1 (cf. [15])**.** *Let G be a graph. Then the following holds.*

(i) *$\chi^*(G)$ is a positive rational number, and $\chi^*(G) \geq 2$ whenever $E(G) \neq \emptyset$ (otherwise $\chi^*(G) = 1$).*

(ii) *$\lceil \chi^*(G) \rceil = \chi(G)$, i.e., $\chi^*(G) \in (\chi(G) - 1, \chi(G)]$.*

(iii) *For each rational number $q \in \mathbb{Q}, q = \frac{m}{n} \geq 2$, there is a graph $G_{m,n}$ with $\chi^*(G_{m,n}) = \frac{m}{n} = q$.*

(iv) *For every $k, d \in \mathbb{N}$, there is a (k, d)-colouring of G if and only if $\frac{k}{d} \geq \chi^*(G)$.*

(v) *If $\chi^*(G) = \frac{m}{n}$, then there exists a (k, d)-colouring of G with $\frac{k}{d} = \frac{m}{n}$ and $k \leq |V(G)|$.*

For further details concerning circular chromatic numbers of graphs we refer to the survey article [18].

A first definition of circular colourings for digraphs was given by Bokal *et al.* in [1], leading to the notion of the circular dichromatic number of digraphs and graphs. Instead of (k, d)-pairs as in the case of Vince, they, equivalently, use real numbers for their definition. Given a $p \geq 1$, consider a plane-circle S_p of perimeter p and define a *strong circular p-colouring* of D to be an assignment $c : V(D) \to S_p$ of (colouring) points on S_p to the vertices, in such a way that for every edge $e = (u, w)$ in D, the one-sided distance of $c(u), c(w)$ (i.e., the length of a clockwise arc connecting $c(u)$ to $c(w)$ in S_p) is at least 1. More formally, we can identify S_p with the set $\mathbb{R}/p\mathbb{Z}$ and require that the unique representative of $c(w) - c(u) \in \mathbb{R}/p\mathbb{Z}$ in the interval $[0, p)$, denoted by $(c(w) - c(u)) \bmod p$ is at least one. In this representation, the clockwise direction on S_p is identified with the positive direction in $\mathbb{R}/p\mathbb{Z}$. Since the notion of a strong circular p-colouring turns out to be much less flexible, the authors also define so-called *weak circular p-colourings* of D, $p \in [1, \infty)$, as maps $c : V(D) \to S_p$, such that equal colours at both ends of an edge, i.e., $c(u) = c(w)$ where $e = (u, w) \in E(D)$, are allowed, but at the same time, the one-sided distance of $c(u), c(w)$ on S_p is at least 1 whenever they are distinct. Moreover, each so-called colour class, i.e., $c^{-1}(t)$, $t \in S_p$ has to induce an acyclic subdigraph of D. This seems much more intuitive and closer to the definition of legal digraph colourings.

The *circular dichromatic number* $\vec{\chi}_c(D)$ now is defined as the infimum over all real values $p \geq 1$ for which D admits a strong circular p-colouring, or, equivalently (as shown in their paper), as the infimum over all values $p \geq 1$ providing weak circular p-colourings of D. Moreover, in the case of weak circular p-colourings the infimum is always attained.

Proposition 2 [1]**.** *Let D be a digraph. The real value*

$$\begin{aligned}\vec{\chi}_c(D) &= \inf\{p \geq 1 \mid \exists \text{ weak circular } p\text{-colouring of } D\} \\ &= \inf\{p \geq 1 \mid \exists \text{ strong circular } p\text{-colouring of } D\}\end{aligned}$$

is called the circular dichromatic number of D. Furthermore, every digraph admits a weak circular $\vec{\chi}_c(D)$-colouring. If G is a graph and $\mathcal{O}(G)$ the set of its orientations, then we define the maximum

$$\vec{\chi}_c(G) = \max_{D \in \mathcal{O}(G)} \vec{\chi}_c(D)$$

to be the circular dichromatic number *of the graph G.*

The following sums up the most basic properties of this quantity.

Theorem 3 [1]**.** *Let D be a digraph. Then the following holds.*

(i) $\vec{\chi}_c(D) \geq 1$ *is a rational number with numerator at most $|V(D)|$.*

(ii) $\lceil \vec{\chi}_c(D) \rceil = \vec{\chi}(D)$, *i.e.,* $\vec{\chi}_c(D) \in (\vec{\chi}(D) - 1,\, \vec{\chi}(D)]$.

(iii) $\vec{\chi}_c(\cdot)$ *attains exactly the rational numbers $q \in \mathbb{Q}$, $q \geq 1$.*

It was furthermore pointed out in [14] that the following discrete notion of *circular (k, d)-colourings* corresponds to the above notion of weak circular p-colourings.

Definition. Let D be a digraph and $k \geq d$ natural numbers. A *circular (k, d)-colouring* is a vertex-colouring $c : V(D) \to \{0, \ldots, k-1\} \simeq \mathbb{Z}_k$ such that for all $e = (u, w) \in E(D)$ we have $(c(w) - c(u)) \bmod k \geq d$ or $c(u) = c(w)$ and each colour class $c^{-1}(i), i \in \mathbb{Z}_k$ induces an acyclic subdigraph of D.

Proposition 4 [14]**.** *For every digraph D and every $p \geq 1$, there exists a weak circular p-colouring of D if and only if there is a circular (k, d)-colouring of D for every pair $(k, d) \in \mathbb{N}^2$ with $\frac{k}{d} \geq p$. Thus,*

$$\vec{\chi}_c(D) = \inf \left\{ \frac{k}{d} \,\middle|\, \exists \text{ circular } (k, d)\text{-colouring of } D \right\}.$$

2. The Star Dichromatic Number, General Properties

In this section we introduce a new concept of fractional digraph colouring.

Definition. Let D be a digraph, $(k, d) \in \mathbb{N}^2$, $k \geq d$. An *acyclic (k, d)-colouring* of D is an assignment $c : V(D) \to \mathbb{Z}_k$ of colours to the vertices such that for every $i \in \mathbb{Z}_k$, the pre-image of the cyclic interval $A_i = \{i, i+1, \ldots, i+d-1\} \subseteq \mathbb{Z}_k$ of colours, $c^{-1}(A_i) \subseteq V(D)$, induces an acyclic subdigraph of D.

It will be handy to also have an equivalent formulation allowing real numbers ready, which deals with the circles $S_p, p \geq 1$.

Definition. Let $p \in \mathbb{R}, p \geq 1$. For $a, b \in [0, p)$, we denote by $(a, b)_p$ the open "interval" $(a, b)_p = \{y \in [0, p) \mid 0 < (y - a) \bmod p < (b - a) \bmod p\}$. Analogous definitions apply for $[a, b]_p, [a, b)_p, (a, b]_p$. In each case, we call $(b - a) \bmod p$ the *length* of the respective interval. For each $x \in S_p$, denote by $|x|_p = \min\{x, p - x\}$ its two-sided distance to 0.

Definition. Let D be a digraph and $p \geq 1$. An *acyclic p-colouring* of D is an assignment $c : V(D) \to [0, p) \simeq \mathbb{R}/p\mathbb{Z}$ of "colours" to the vertices, such that for every open interval $I = (a, b)_p$ of length 1 within $[0, p) \simeq \mathbb{R}/p\mathbb{Z}$, the subdigraph induced by the vertices in $c^{-1}(I)$ is acyclic. The *star dichromatic number of D* now is defined as the infimum over the numbers p for which D admits an acyclic p-colouring

$$\vec{\chi}^*(D) = \inf\{p \geq 1 \mid \exists \text{ acyclic } p\text{-colouring of } D\}.$$

The following ensures that there always exists a $\vec{\chi}^*(D)$-colouring of D.

Proposition 5. *Let $P = \{p \geq 1 \mid \exists$ acyclicp-colouring of $D\} \subseteq [1, \infty)$. Then P is closed. Furthermore, D admits an acyclic $\vec{\chi}^*(D)$-colouring.*

Proof. Since P is bounded from below, the latter claim is a consequence of the former. Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of elements of P convergent to some $p \geq 1$. We have to show that $p \in P$. Clearly, we may assume $p_n > p$ for all $n \in \mathbb{N}$. For given n let $c'_n : V(D) \to [0, p_n)$ denote a feasible p_n-colouring of D. Scaling by $\frac{p}{p_n}$ we derive maps $c_n : V(D) \to [0, p), x \mapsto \frac{p}{p_n} c'_n(x)$ with the property that for every open interval $I \subseteq \mathbb{R}/p\mathbb{Z}$ of length at most $\frac{p}{p_n}$ there is no directed cycle in the digraph induced by $c_n^{-1}(I)$. We may consider $(c_n)_{n \in \mathbb{N}}$ as a sequence of vectors in $S_p^{|V(D)|}$. Applying the Theorem of Heine-Borel to $(c_n)_{n \in \mathbb{N}}$ yields a convergent subsequence $(c_{n_l})_{l \in \mathbb{N}}$. Let $c = \lim_{l \to \infty} c_{n_l}$. Then $c : V(D) \to [0, p)$. We claim that c defines an acyclic p-colouring of D.

Assume to the contrary there was a directed cycle C in D such that $c(V(C))$ is contained in an open interval $I = (a, b)_p \subseteq S_p \simeq [0, p)$ of length 1. Since $c(V(C))$ is finite, there exists $0 < \varepsilon < \frac{1}{2}$ such that $c(V(C)) \subseteq (a + \varepsilon, b - \varepsilon)_p \subseteq (a, b)_p$. Since D is finite, $(c_{n_l})_{l \in \mathbb{N}}$ is a sequence convergent in $S_p^{|V(D)|}$ and $\lim_{n \to \infty} p_n = p$, we may choose $N \in \mathbb{N}$ such that $|c_N(x) - c(x)| < \frac{\varepsilon}{2}$ for all $x \in V(D)$ and $p_N < \frac{p}{1 - \varepsilon}$. Now, $c_N(V(C)) \subseteq \left(a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right) \bmod p$. Hence, we have found a directed cycle in the inverse image of an open interval of length $1 - \varepsilon < \frac{p}{p_N}$, contradicting the properties of c_N.

As D is considered loopless, $P \neq \emptyset$ and thus P is closed and bounded from below, which implies that it admits a minimum. ∎

The following equivalence now makes the relation between the discrete notion and the real-number-notion of acyclic colourings of digraphs precise:

Proposition 6. *Let D be a digraph. Then for every real number $p \geq 1$, D admits an acyclic p-colouring if and only if it admits an acyclic (k,d)-colouring for every $(k,d) \in \mathbb{N}^2$ fulfilling $\frac{k}{d} \geq p$. Consequently,*

$$\vec{\chi}^*(D) = \inf \left\{ \frac{k}{d} \,\middle|\, \exists \text{ acyclic } (k,d)\text{-colouring of } D \right\}.$$

Proof. For the first implication let $c : V(D) \to [0,p)$ be an acyclic p-colouring of D and let $(k,d) \in \mathbb{N}^2$ with $\frac{k}{d} \geq p$ be arbitrary. Define a colouring $c_{k,d}$ of the vertices by

$$\forall x \in V(D) : c_{k,d}(x) = \left\lfloor \frac{k}{p} c(x) \right\rfloor \in \{0, \ldots, k-1\}.$$

We claim that this defines an acyclic (k,d)-colouring of D. Assume to the contrary there was a directed cycle C within $c_{k,d}^{-1}(A_i)$ for some $i \in \{0, \ldots, k-1\} \simeq \mathbb{Z}_k$. Then for all $x \in V(C)$,

$$\left(\left\lfloor \frac{k}{p} c(x) \right\rfloor - i \right) \bmod k \leq d - 1 \Rightarrow \left(\frac{k}{p} c(x) - i \right) \bmod k < d.$$

Consequently, $\left(c(x) - \frac{ip}{k}\right) \bmod p = \frac{p}{k}\left(\left(\frac{k}{p}c(x) - i\right) \bmod k\right) < \frac{p}{k/d} \leq 1$. Hence, $c(V(C)) \subseteq \left(\frac{ip}{k}, \frac{ip}{k} + 1\right)_p$, contradicting the definition of an acyclic colouring.

For the reverse implication, assume that $p \geq 1$ such that for every $(k,d) \in \mathbb{N}^2$ with $\frac{k}{d} \geq p$, there is an acyclic (k,d)-colouring $c_{(k,d)} : V(D) \to \{0, \ldots, k-1\}$ of D. Let $((k_n, d_n))_{n \in \mathbb{N}}$ be some sequence in $\mathbb{N}^2$ such that $p_n = \frac{k_n}{d_n} \geq p$ for all $n \in \mathbb{N}$, and $\lim_{n \to \infty} \frac{k_n}{d_n} = p$. Let $c_n = c_{(k_n, d_n)} : V(D) \to \{0, \ldots, k_n - 1\}$ denote corresponding acyclic (k_n, d_n)-colourings of D. We define $c_{p_n} : V(D) \to [0, p_n)$ by

$$x \mapsto \frac{p_n}{k_n} c_n(x) \in [0, p_n).$$

We claim that for every n this defines an acyclic p_n-colouring. Assume to the contrary there was a cyclic open subinterval $(a,b)_{p_n} \subseteq [0, p_n)$ of length 1 containing the colours of a directed cycle C in D, then for every $x \in V(C)$, we would have

$$0 < \left(\frac{p_n}{k_n} c_n(x) - a \right) \bmod p_n < 1 \Leftrightarrow 0 < (c_n(x) - d_n a) \bmod k_n < \frac{k_n}{p_n} = d_n$$

and thus, with $i = \lceil d_n a \rceil \bmod k_n$, we get $0 \leq (c_n(x) - i) \bmod k_n \leq d_n - 1$ for all $x \in V(C)$, implying $c_n(V(C)) \subseteq A_i$. This contradicts c_n being an acyclic (k_n, d_n)-colouring and shows that indeed, $p_n \in P, n \geq 1$, where again, P denotes the set of $p \geq 1$ allowing an acyclic colouring of D. Since P is closed (Proposition 5), we finally deduce that $p = \lim_{n \to \infty} p_n \in P$, and thus the claimed equivalence follows. ∎

Although theoretically, the definition of $\vec{\chi}^*(D)$ as the infimum of the set P of real numbers might include irrational values of $\vec{\chi}^*(D)$, the following statement shows that due to the conditions on acyclic p-colourings which are given in terms of a finite object, namely D, $\vec{\chi}^*(D)$ only attains rational numbers with a certain bound on the numerator. Analogous statements hold for other notions of circular colourings.

Theorem 7. *Let D be a digraph, $n = |V(D)|$. Then $\vec{\chi}^*(D)$ is a rational number of the form $\frac{k}{d}$ with $1 \le d \le k \le n$.*

Proof. Our proof follows the lines of the one given for the same result for $\vec{\chi}_c(D)$ in [1], respectively [11].

Let in the following $p = \vec{\chi}^*(D)$. We may assume $p > 1$. For a given acyclic p-colouring $c : V(D) \to S_p \simeq [0, p)$ of D we consider the digraph $D_1(c)$, defined over the vertex set $V(D)$ where $(u, w) \in E(D_1(c))$ if $(c(w) - c(u)) \bmod p = 1$. Let $v_0 \in V(D)$ be a fixed reference vertex, we may assume that $c(v_0) = 0$. We will show that we can choose c such that for every vertex $v \in V(D)$, there is a directed path from v_0 to v in $D_1(c)$. For this purpose, let c be an acyclic p-colouring maximal with respect to the cardinality of the set $S(c)$ of vertices reachable from v_0 via directed paths in $D_1(c)$. Assume for a contradiction that $S(c) \neq V(D)$. For $s \in [0, p)$, we define

$$c_s(v) = \begin{cases} c(v), & \text{if } v \in S(c), \\ (c(v) - s) \bmod p, & \text{if } v \notin S(c). \end{cases}$$

Note that for each $s \in [0, p)$ so that c_s is an acyclic p-colouring, we have $S(c_s) \supseteq S(c)$, and due to the maximality of c, $S(c_s) = S(c)$. Now, choose s^* maximal with the property, that for all $s < s^*$ c_s is an acyclic p-colouring. The assumption $S(c) \neq V(D)$ now implies that $0 < s^* < p$ and $c_{s^*+\varepsilon}$ is not an acyclic p-colouring for arbitrarily small values of $\varepsilon > 0$. Therefore, there must exist a closed interval $[a, b]_p \subseteq S_p$ of length 1 such that $c_{s^*}^{-1}([a, b]_p)$ contains the vertices of a directed cycle C and such that there are $u, w \in V(C)$ with $c_{s^*}(u) = a, c_{s^*}(w) = b$ and $u \in S(c)$, $w \notin S(c)$. But this implies that $S(c) \cup \{w\} \subseteq S(c_{s^*})$ contradicting the choice of c.

We now consider the case that there exists a vertex $v \in V(D) \setminus \{v_0\}$ and two directed v_0-v-walks P_1 and P_2 of lengths $\ell(P_1) > \ell(P_2)$ that visit at most one vertex (possibly v) twice. This includes the case, that there exists a directed cycle in $D_1(c)$. Since $c(v) = \ell(P_1) \bmod p = \ell(P_2) \bmod p$, there exists some $m \in \mathbb{N}$ such that $mp = \ell(P_1) - \ell(P_2)$. But clearly $0 < \ell(P_1) - \ell(P_2) < n$ and hence $p = \frac{\ell(P_1) - \ell(P_2)}{m}$ as required.

Thus we may assume, that for all vertices in $v \in V(D)$ all directed v_0-v paths have the same length, defining a map $f : V \to \mathbb{N}, v \mapsto \ell(P_v)$. We have $f(v) \bmod p = c(v)$ for all $v \in V(D)$. We will show that this contradicts the minimality of p. For that purpose choose $\delta > 0$ such that $p - \delta > 1$ and for each pair $u, w \in V(D)$ of vertices with $(f(w) - f(u)) \bmod p > 1$, we have $(f(w) - f(u)) \bmod (p - \delta) > 1$. We claim that $x \mapsto c_{-\delta}(x) = f(x) \bmod (p - \delta)$ defines an acyclic $(p - \delta)$-colouring of D. Assume to the contrary there was a directed cycle C in D such that its image under $c_{-\delta}$ is contained in a closed interval $[c_{-\delta}(u), c_{-\delta}(w)]_{p-\delta} \supseteq c_{-\delta}(V(C))$ of length < 1, where $u, w \in V(C)$. Let in the following $x \in V(C)$ be arbitrary. Then

$$(c_{-\delta}(x) - c_{-\delta}(u)) \bmod (p - \delta) = (f(x) - f(u)) \bmod (p - \delta) < 1$$

and thus $(u, x) \notin E(D_1(c))$, $(f(x) - f(u)) \bmod p = (c(x) - c(u)) \mathrm{mod}\, p \leq 1$. We conclude $(c(x) - c(u)) \bmod p < 1$ and that there exists $\varepsilon > 0$ such that $(c(x) - c(u)) \bmod p < 1 - \varepsilon$ for all $x \in V(C)$, contradicting c being an acyclic p-colouring. The claim follows. ■

Corollary 8. *For a digraph D we have $\vec{\chi}^*(D) \geq 1$ with equality if and only if D is acyclic.*

Proof. The inequality holds by definition. $\vec{\chi}^*(D) = 1$ implies the existence of an acyclic 1-colouring of D, and thus, since $V(D)$ is finite, that D is acyclic. ■

The following describes the relationship of $\vec{\chi}^*(D)$ with its integer counterpart.

Theorem 9. *Let D be a digraph. Then*

$$\lceil \vec{\chi}^*(D) \rceil = \vec{\chi}(D), \text{ i.e., } \vec{\chi}(D) - 1 < \vec{\chi}^*(D) \leq \vec{\chi}(D).$$

Proof. The latter inequality is an immediate consequence from Proposition 6 and the fact that the acyclic $(k, 1)$-colourings of D correspond exactly to legal k-digraph colourings of D in the usual sense, for $k \in \mathbb{N}$. On the other hand let $p = \vec{\chi}^*(D), k = \lceil p \rceil \in \mathbb{N}$ and let $c : V(D) \to S_p$ denote an acyclic p-colouring of D. Since $p \leq k$ and $V(D)$ is finite, we find k pairwise disjoint cyclic subintervals $I_1, \ldots, I_k$, each of length less than 1 such that all $v \in V(D)$ are mapped to the interior of one of these. Thus $c^{-1}(I_i)$, $i = 1, \ldots, k$ induces an acyclic subdigraph of D, this way defining a k-digraph colouring of D, proving $k \geq \vec{\chi}(D)$. ■

3. Relations to Other Fractional Digraph Colouring Parameters

We briefly review the notions of the fractional chromatic numbers of graphs and digraphs in order to draw a comparison with our new fractional colouring number. The fractional dichromatic number will be a main tool for deriving lower bounds on star dichromatic numbers.

Definition (cf. [13] and [12]).
(A) Let G be a graph. Denote by $\mathcal{I}(G)$ the collection of independent vertex subsets of G, and for each $v \in V(D)$, let $\mathcal{I}(G,v) \subseteq \mathcal{I}(G)$ be the subset containing only those sets including v. The *fractional chromatic number* $\chi_f(G)$ of G is now defined as the value of the following linear program

$$\begin{aligned} & \min \sum_{I \in \mathcal{I}(G)} x_I \\ \text{subject to} & \sum_{I \in \mathcal{I}(G,v)} x_I \geq 1, \text{ for all } v \in V(G) \\ & x \geq 0. \end{aligned} \tag{1}$$

(B) Let D be a digraph. Denote by $\mathcal{A}(D)$ the collection of vertex subsets of D inducing an acyclic subdigraph, and for each $v \in V(D)$, let $\mathcal{A}(D,v) \subseteq \mathcal{A}(D)$ be the subset containing only those sets including v. The *fractional dichromatic number* $\vec{\chi}_f(D)$ of D is now defined as the value of

$$\begin{aligned} & \min \sum_{A \in \mathcal{A}(D)} x_A \\ \text{subject to} & \sum_{A \in \mathcal{A}(D,v)} x_A \geq 1, \text{ for all } v \in V(D) \\ & x \geq 0. \end{aligned} \tag{2}$$

For a graph G, we define $\vec{\chi}_f(G) = \max_{D \in \mathcal{O}(G)} \vec{\chi}_f(D)$ to be its *fractional dichromatic number.*

The following inequality chain finally describes the behaviour of the three notions of fractional digraph colouring numbers introduced so far in general and shows that the star dichromatic number separates the fractional from the circular chromatic number.

Theorem 10. *Let D be a digraph. Then $\vec{\chi}_f(D) \leq \vec{\chi}^*(D) \leq \vec{\chi}_c(D)$.*

Proof. Let $\vec{\chi}^*(D) = \frac{k}{d}$ and $c_k : V(D) \to \mathbb{Z}_k$ be an acyclic (k,d)-colouring for two integers $0 < d \leq k$. Given $A \in \mathcal{A}(D)$ let

$$i_A = \left|\left\{i \in \mathbb{Z}_k \mid A = c_k^{-1}(\{i, \ldots, i+d-1\})\right\}\right|$$

and define $x_A = \frac{i_A}{d}$. Then for every vertex $v \in V(D)$, we have

$$\sum_{A \in \mathcal{A}(D,v)} x_A = \sum_{i \in \mathbb{Z}_k : c_k(v) \in \{i, \ldots, i+d-1\}} \frac{1}{d} = 1.$$

Hence, x is feasible for the program 2 implying $\vec{\chi}_f(D) \leq \sum_{i\in\mathbb{Z}_k} \frac{1}{d} = \frac{k}{d} = \vec{\chi}^*(D)$.

For the second inequality, it suffices to show that for every $p \geq 1$, any weak circular p-colouring $c : V(D) \to [0,p)$ in the sense of Bokal *et al.* is also an acyclic p-colouring of D. Assume to the contrary there was a directed cycle C in D such that $c(V(C))$ is contained in an open subinterval of length 1 in $S_p \simeq [0,p)$. We may assume $c(V(C)) \subseteq (0,1)_p$. Then obviously, $0 < (c(w) - c(u)) \bmod p < 1$ for every edge $(u,w) \in E(C)$ with $c(w) > c(u)$, contradicting the definition of weak circular colourings. Thus $c(C)$ consists of a single point $\{t\} \subseteq S_p$, which means that $c^{-1}(t)$ is not acyclic, a contradiction. Hence c is a weak colouring and the claim follows. ∎

It is well-known that for symmetric orientations of graphs the chromatic number of the original graph equals their dichromatic number. Similar relations hold for fractional, star and circular dichromatic number.

Remark 11. Let G be an undirected graph, and denote by $S(G)$ its symmetric orientation where every undirected egde in $E(G)$ is replaced by an anti-parallel pair of arcs. Then

$$\vec{\chi}_f(S(G)) = \chi_f(G), \vec{\chi}^*(S(G)) = \vec{\chi}_c(S(G)) = \chi^*(G).$$

Proof. The first equality follows from the fact that the vertex subsets in $S(G)$ inducing acyclic subdigraphs are exactly the independent vertex sets in G. Furthermore, since every parallel replacement pair of arcs gives rise to a directed 2-cycle, weak circular p-colourings as well as p-colourings according to our definition of $S(G)$, for every $p \geq 1$, are exactly those maps $c : V(G) \to [0,p)$ with $\mathrm{dist}_p(c(u), c(w)) \geq 1$ for every adjacent pair of vertices u, w, implying the latter two equalities. ∎

As we will see in the next section, when dealing with planar digraphs, finding digraphs without large acyclic vertex subsets yields good lower bounds for the fractional and thus also the star dichromatic number. This is made precise by the following inequality.

Lemma 12. *Let D be a digraph and denote by $\vec{\alpha}(D)$ the maximum size of a vertex subset of D inducing an acyclic subdigraph. Then $\vec{\chi}_f(D) \geq \frac{|V(D)|}{\vec{\alpha}(D)}$.*

Proof. Consider the dual of the linear program (2),

$$\begin{aligned} &\max \sum_{v\in V} y_v && (3)\\ &\text{subject to } \sum_{v\in A} y_v \leq 1, && \text{for all } A \in \mathcal{A}(D)\\ &y \geq 0. \end{aligned}$$

Define $y_v = \frac{1}{\vec{\alpha}(D)}$ for each vertex $v \in V$. The y clearly is feasible for (3) and the result follows by linear programming duality. ■

We now finally present a construction of digraphs (which are part of the class of so-called *circulant digraphs*) whose star dichromatic numbers attain every rational number $q \geq 1$. The same digraphs were used in [1].

Theorem 13. *Let $(k, d) \in \mathbb{N}^2$ with $k \geq d$. Denote by $\vec{C}(k, d)$ the digraph defined over the vertex set $V(\vec{C}(k, d)) = \{0, \ldots, k-1\} \simeq \mathbb{Z}_k$ so that each vertex $i \in \mathbb{Z}_k$ has exactly $k - d$ outgoing arcs, namely $(i, j), j = i + d, i + d + 1, \ldots, i + k - 1$. Then*

$$\vec{\chi}_f(D) = \vec{\chi}^*(D) = \vec{\chi}_c(D) = \frac{k}{d}.$$

Therefore, $\vec{\chi}^*(D)$ attains every rational number $q \geq 1$.

Proof. According to Theorem 10 it suffices to show that $\frac{k}{d} \leq \vec{\chi}_f(\vec{C}(k, d))$ and $\vec{\chi}_c(\vec{C}(k, d)) \leq \frac{k}{d}$.

Let $A \in \mathcal{A}(\vec{C}(k, d))$, then $\vec{C}(k, d)[A]$, being acyclic contains a sink $a \in A \subseteq \mathbb{Z}_k$ and therefore $A \cap \{a+d, \ldots, a+k-1\} = \emptyset$, proving $|A| \leq d$ and the first inequality follows using Lemma 12.

For the other inequality, note that $c_{k,d}(i) = \frac{i}{d} \in \left[0, \frac{k}{d}\right)$ for all $i \in V(\vec{C}(k, d))$ defines a strong $\frac{k}{d}$-colouring $c_{k,d}$ of D. ■

Putting $k = n$, $d = n - 1$ in the above, we immediately get the following.

Corollary 14. *For every $n \in \mathbb{N}$,*

$$\vec{\chi}_f(\vec{C}_n) = \vec{\chi}^*(\vec{C}_n) = \vec{\chi}_c(\vec{C}_n) = \frac{n}{n-1}.$$

While the above provides examples for digraphs where the three different concepts of fractional digraph colouring coincide, we now focus on constructing examples of digraphs where the numbers vary significantly in order to point out differences of the approaches.

First of all, it is well-known that contrary to the star chromatic number, the fractional chromatic number of a graph does not fulfil a ceiling-property, but can be arbitrarily far apart from the chromatic number of the graph. As a consequence we conclude that circular and star dichromatic number can be arbitrarily far apart of the fractional dichromatic number.

Theorem 15. *For every $C \in \mathbb{R}_+$, there is a digraph D with $\vec{\chi}^*(D) - \vec{\chi}_f(D) = \vec{\chi}_c(D) - \vec{\chi}_f(D) \geq C$.*

Proof. By Remark 11 the result follows from the same observation for undirected graphs. As is well known for the Kneser graphs $G_n = K(n, 2)$, $n \geq 4$, we have $\chi(G_n) - \chi_f(G_n) = (n-2) - \frac{n}{2} = \frac{n}{2} - 2 \to \infty$ (cf. [13], page 32). ■

Now we compare $\vec{\chi}^*$ and $\vec{\chi}_c$ in more detail. We see the main advantage of our new parameter in the fact, that it is sufficient to consider only the strong components of a digraph D in order to compute $\vec{\chi}^*(D)$.

Observation 16. *Let D be a digraph and $S = D(X, \overline{X})$ a directed cut. Let $D_1 = D[X], D_2 = D[\overline{X}]$. Then*

$$\vec{\chi}(D) = \max\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}, \vec{\chi}^*(D) = \max\{\vec{\chi}^*(D_1), \vec{\chi}^*(D_2)\}.$$

On the other hand, for the circular dichromatic number the existence of a dominating source completely destroys any extra information we hope to gain compared to the dichromatic number.

Proposition 17. *Let D be a digraph. We denote by D^s the digraph arising from D by adding an extra vertex s, which is a source adjacent to every vertex in $V(D)$. Then $\vec{\chi}_c(D^s) = \vec{\chi}(D)$.*

Proof. By Observation 16 we have $\vec{\chi}_c(D^s) \leq \vec{\chi}(D^s) = \vec{\chi}(D)$. Assume contrary to the assertion that there was a strong p-colouring c of D^s with $p < \vec{\chi}(D) =: k$. We may assume $c(s) = 0$. According to the definition of a strong colouring, the interval $[0, 1)_p$ does not contain any other vertices, hence $c(V(D)) \subseteq [1, p)_p$. Since $p - 1 < k - 1$, we can decompose the interval $[1, p)_p$ into $k - 1$ pairwise disjoint cyclic subintervals $I_1, \ldots, I_{k-1}$ of S_p, each of length less than one and covering all the finitely many colouring points. If (u, w) is an edge such that $c(u), c(w) \in I_l$ are contained in the same interval, then, since c is a strong colouring, we must have $c(u) > c(w)$. Hence, each $c^{-1}(I_l)$ induces an acyclic subdigraph of D for each l, all together defining a $(k - 1)$-digraph colouring of D, a contradiction. This proves the claim. ■

Example 18. $\vec{\chi}^*(\vec{C}_n^s) = \frac{n}{n-1} < 2 = \vec{\chi}_c(\vec{C}_n^s)$ for all $n \geq 3$.

Proof. According to Observation 16 and Theorem 14 we have $\vec{\chi}^*(\vec{C}_n^s) = \vec{\chi}^*(\vec{C}_n) = \frac{n}{n-1}$. The remaining equality follows immediately from Proposition 17. ■

Corollary 19. *For every $\varepsilon > 0$ there is a digraph D with $\vec{\chi}_c(D) - \vec{\chi}^*(D) \geq 1 - \varepsilon$.*

4. The Star Dichromatic Number of Simple Planar Digraphs and Circular Vertex Arboricity

Let G be a given (unoriented) graph. If we want to estimate $\vec{\chi}^*(G)$, we need to find (k, d)-digraph colourings for every possible orientation of G. The simplest way of doing this is to find a single colouring of $V(G)$ yielding a legal (k, d)-colouring on all the possible orientations at the same time. This leads to the following definition which is introduced in [16].

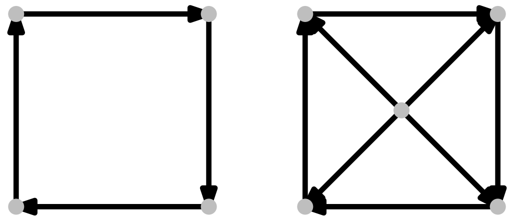

Figure 1. Stacking a source into a directed 4-cycle. While the star dichromatic number remains unchanged with value $\frac{4}{3}$, the circular dichromatic number jumps from $\vec{\chi}_c(\vec{C}_4) = \frac{4}{3}$ to $\vec{\chi}_c(\vec{C}_4^s) = 2$.

Definition [16]**.** Let G be a graph and $(k, d) \in \mathbb{N}^2, k \geq d$. A (k, d)-tree-colouring of G is a colouring $c : V(G) \to \mathbb{Z}_k \simeq \{0, \dots, k-1\}$ of the vertices so that with $A_i = \{i, i+1, \dots, i+d-1\} \subseteq \mathbb{Z}_k$, $c^{-1}(A_i)$ induces an acyclic subgraph of G for all $i \in \mathbb{Z}_k$.

The authors of [16] now define the *circular vertex arboricity* of a graph G as the minimal value

$$\mathrm{va}^*(G) = \inf\left\{\frac{k}{d} \,\middle|\, \exists\ (k, d)\text{-tree-colouring of } G\right\}.$$

The above now immediately implies

Remark 20. For every graph G, $\vec{\chi}^*(G) \leq \mathrm{va}^*(G)$.

As in the previous chapter, they also proved an alternative representation of this fractional quantity in terms of real numbers.

Definition [16]**.** Let G be a graph and $p \geq 1$. Then a *p-circular tree colouring* of G is defined as an assignment $c : V(G) \to S_p \simeq [0, p)$ so that for every open interval $I = (a, b)_p \subseteq [0, p)$ of length 1, $c^{-1}(I)$ induces an acyclic subgraph of G.

Theorem 21 [16]**.** *For every graph G we have*

$$va^*(G) = \inf\{p \mid \exists\ p\text{-circular tree colouring of } G\}.$$

An important conjecture related to colourings of digraphs is the 2-colour-conjecture by Victor-Neumann-Lara.

Conjecture 22 (Neumann-Lara, 1985)**.** $\vec{\chi}(D) \leq 2$ *for every simple planar digraph D.*

According to the above, this is equivalent to $\vec{\chi}^*(D) \leq 2$ for simple planar digraphs. While the conjecture still remains unproven and since the best known general result so far only guarantees the existence of 3-colourings of simple planar digraphs (via vertex arboricity, [2]), the following can be seen as an improvement of the upper bound 3 for the star dichromatic number of planar digraphs.

Theorem 23. *Let D be a simple planar digraph. Then $\vec{\chi}^*(D) \leq 2.5$.*

Proof. In [16] it is proved that $\mathrm{va}^*(G) \leq 2.5$ for simple planar graphs. The claim now follows from Remark 20. ■

While the star dichromatic number can be considered an oriented version of the circular vertex arboricity there does not seem to be an unoriented counterpart to the circular colourings introduced by Bokal *et al.*. Note that any map $c : V(G) \to S_p \simeq [0, p)$ that is a simultaneous weak circular p-colouring of each possible orientation of a graph G which is no forest necessarily must have $p \geq 2$.

The bound $\vec{\chi}^*(D) \leq 2$ for planar digraphs as a consequence of the 2-colour-conjecture is best-possible as there exist simple planar digraphs with star dichromatic number arbitrarily close to 2. This is a consequence of the case $g = 3$ of the following theorem.

Theorem 24. *For every $g \geq 3$ and every $\varepsilon > 0$, there exists a planar digraph D of digirth g with $\vec{\chi}^*(D) \in \left[\frac{g-1}{g-2} - \varepsilon, \frac{g-1}{g-2}\right]$.*

Proof. Knauer *et al.* [7] constructed a sequence $\left(D_f^g\right)_{f \geq 1}$ of planar digraphs of digirth g with $\left|V\left(D_f^g\right)\right| = f(g-1)+1$ and so that for the maximum order $\vec{\alpha}\left(D_f^g\right)$ of an induced acyclic subdigraph of D_f^g, we have $\vec{\alpha}\left(D_f^g\right) \leq \frac{\left|V\left(D_f^g\right)\right|(g-2)+1}{g-1}$ for all $f \geq 1$. Applying Lemma 12 this yields

$$\vec{\chi}^*\left(D_f^g\right) \geq \vec{\chi}_f\left(D_f^g\right) \geq \frac{\left|V\left(D_f^g\right)\right|}{\vec{\alpha}\left(D_f^g\right)} \geq \frac{\left|V\left(D_f^g\right)\right|(g-1)}{\left|V\left(D_f^g\right)\right|(g-2)+1}.$$

Since the latter expression is convergent to $\frac{g-1}{g-2}$ for $f \to \infty$, it remains to show that all the D_f^g, $f \geq 1$ admit acyclic $(g-1, g-2)$-colourings. This is easily seen using the inductive construction described in [7]. For $f \geq 2$, D_f^g arises from D_{f-1}^g by adding an extra directed path $P = s_1, \ldots, s_{g-1}$ with $g-1$ new vertices whose only connections to $V\left(D_f^g\right)$ consist of two vertices $x \neq y \in V\left(D_{f-1}^g\right)$ that both are adjacent to x_1 and x_{g-1} via edges that are oriented in such a way that x, P as well as y, P induce directed cycles. Now we inductively find an acyclic $(g-1, g-2)$-colouring by colouring the vertices of P with the $g-1$ pairwise distinct colours. Clearly, this cannot create any new directed cycle using at most $g-2$ colours. ■

There is some evidence that the construction given in [7] is asymptotically best-possible. Thus, we are tempted to generalize the 2-colour-conjecture as follows.

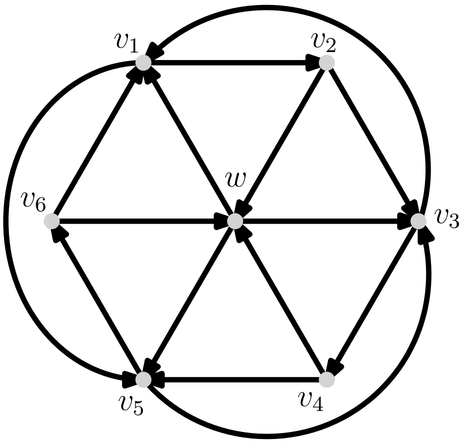

Figure 2. The simple planar digraph D_7 with star dichromatic number 2.

Conjecture 25. *For every planar digraph D of digirth at least $g \geq 3$, we have $\vec{\chi}^*(D) \leq \frac{g-1}{g-2}$. In other words, D admits a colouring with $g-1$ colours so that each directed cycle in D contains each colour at least once.*

The above implies that this bound for a given g, if true, is best-possible. We furthermore note that these upper bounds for $g \geq 4$ do not apply for the circular dichromatic number $\vec{\chi}_c(D)$ (take e.g. $\vec{C}_g^s$ from Example 18). We are not aware of an example for which the bound in the above inequality is attained with equality for $g \geq 4$. For the case $g = 3$, we have the following (minimal) example of a simple planar digraph with star dichromatic number exactly 2.

Proposition 26. *The digraph D_7 depicted in Figure 2 has $\vec{\chi}^*(D_7) = 2$, while $\vec{\chi}^*(D) \leq \frac{5}{3}$ for any simple planar digraph D on at most 6 vertices.*

Proof. The labels $w, v_1, \ldots, v_6$ of the vertices in D_7 refer to Figure 2. To see that $\vec{\chi}^*(D_7) \leq 2$, notice that $c : V(D_7) \to \{0, 1\}$, $c(w) = c(v_1) = c(v_3) = 0, c(v_2) = c(v_4) = c(v_5) = c(v_6) = 1$ defines a valid 2-colouring of D_7. Assume now for a proof by contradiction that we had $\vec{\chi}^*(D_7) < 2$ and thus (according to Theorem 7 and Proposition 6) that there was an acyclic (k, d)-colouring of D_7 where $1 \leq d \leq k \leq 7$ are integers such that $\frac{k}{d} < 2$. The latter implies $\frac{k}{d} \leq \frac{7}{4}$ and consequently the existence of an acyclic $(7, 4)$-colouring $c_{7,4} : V(D_7) \to \mathbb{Z}_7 \simeq \{0, \ldots, 6\}$ of D_7. Without loss of generality, we may assume that $c(w) = 0$. Because any pair of elements is contained in a cyclic subinterval of length 4 in $\mathbb{Z}_7$, the vertices of any directed triangle in D_7 must receive pairwise distinct colours. For any vertex v_i contained in a directed triangle together with w, we must have $c(v_i) \neq 0, i = 1, \ldots, 6$. Considering the directed triangle $v_1 v_5 v_3$, we find that at least one of the vertices v_1, v_3, v_5 must have colour 1 or 6. Because of the symmetry of D_7 we may assume that $c(v_1) \in \{6, 1\}$. Possibly after replacing c with $-c$ we can even assume $c(v_1) = 1$. Looking at the triangle v_1, v_2, w, this forces $c(v_2) = 4$. As $v_1v_2v_3$ forms a directed triangle, it follows that $c(v_3) \in \{5, 6\}$. Assume first that $c(v_3) = 6$. Looking at the triangle wv_3v_4, this forces $c(v_4) = 3$, and because $v_3v_4v_5$ is a directed triangle as well, this implies $c(v_5) \in \{1, 2\}$. This now is a contradiction, because it means that the colour set of the directed triangle $v_1v_3v_5$ is contained in the cyclic subinterval $\{6, 0, 1, 2\}$ of $\mathbb{Z}_7$. Consequently, we may assume that we are in the case of $c(v_3) = 5$. The directed triangle wv_3v_4 now forces $c(v_4) \in \{2, 3\}$, and as the colour set of $v_3v_4v_5$ must not be contained in $\{2, 3, 4, 5\}$, $c(v_5)$ has to be either 0, 1 or 6. This now leads to the contradiction that $\{c(v_1), c(v_3), c(v_5)\} \subseteq \{5, 6, 0, 1\}$.

The second part of the claim can be verified by checking the existence of $(5, 3)$-colourings of all simple planar digraphs on up to 6 vertices. This is a simple but lengthy case distinction by hand but can be easily checked using a brute force program run on a standard personal computer. ∎

Naturally, one might expect that the K_4 is an extremal example for colourings of simple planar digraphs. Surprisingly, this is not the case. Considering more generally odd and even wheels we find.

Example 27. For $k \geq 3$ denote by W_k the wheel with $k+1$ vertices.

(A) If k is odd, then

$$\vec{\chi}^*(W_k) = \vec{\chi}_f(W_k) = \frac{3}{2}.$$

(B) If k is even, then

$$\vec{\chi}^*(W_k) = \frac{5}{3} \quad \text{but} \quad \vec{\chi}_f(W_k) = \frac{3k-2}{2k-2}.$$

Proof. In the following, whenever we refer to a vertex w, it is to be understood as the dominating vertex of the respective wheel we deal with. In the following, wheels are considered to be canonically embedded in the plane such that w is the only inner vertex.

(A) As a wheel contains a triangle, using Corollary 14, we find $\frac{3}{2} = \vec{\chi}_f(C_3) \leq \vec{\chi}_f(W_k) \leq \vec{\chi}^*(W_k)$. Next, we construct an acyclic $(3, 2)$-coloring. Since k is odd, along the circular ordering of incoming and outgoing edges around w, there has to be a consecutive pair of edges with the same orientation, i.e., both incoming or both outgoing. Denote by x_1, x_2 their end vertices on the rim. We now color W_k by assigning 0 to w and colouring the outer cycle using alternatingly 1 and 2 except for x_1, x_2, which both receive 1. Clearly, this is an acyclic $(3, 2)$-coloring, unless the outer cycle is directed. In that case, not only the triangle wx_1x_2, but also one of its neighbouring triangles is not directed. Hence, we may assume that, say, x_1 is not a vertex of a directed triangle. Now recoloring x_1 with color

0 yields an acyclic $(3,2)$-coloring. Hence for every orientation D of W_k we find $\vec{\chi}^*(D) \leq \frac{3}{2}$.

(B) We first prove $\vec{\chi}^*(D) \leq \frac{5}{3}$ for all orientations D of W_k. If the outer cycle is undirected similar to case (A) we find an acyclic $(3,2)$-colouring of D by assigning 0 to the central vertex and alternatingly $1,2$ to the outer vertices. Hence we may assume that the outer cycle is directed in D. If there exists a pair of consecutive vertices x_1, x_2 on the outer cycle where wx_1x_2 is not a directed triangle, recoloring either x_1 or x_2 by 0 yields an acyclic $(3,2)$-coloring as in the odd case. So, we may assume that the outer cycle is directed and all edges incident to w are alternatingly incoming and outgoing in the cyclic order of $E_D(w)$. Hence, in the cyclic order, every second triangle is directed and the others are not. We define an acyclic $(5,3)$-colouring of D by starting with a $0,2,3$-colouring of the vertices, where w receives colour 0 and the outer vertices alternating colours 2 and 3 such that the directed triangles have its vertices coloured by $0,3,2$ in cyclic order. Now choose one directed edge whose tail is coloured by 2 and recolour its head with 4 and its tail with 1. It is now easily seen that the vertices of no directed triangle nor of the outer cycle are contained in the union of three consecutive colour classes of colours of $\mathbb{Z}_5 = \{0,1,2,3,4\}$, which proves $\vec{\chi}^*(D) \leq \frac{5}{3}$ also in this case.

Next we show that $\vec{\chi}^*(W_k) \geq \frac{5}{3}$. Clearly, this can be true only for the orientation considered the last for the upper bound. Let p be any real number admitting an acyclic colouring. As D contains a directed triangle, $p \geq \frac{3}{2}$. Assume for a contradiction $p < \frac{5}{3}$ and let $c : V(D) \to [0,p)$ be an acyclic p-colouring of D. We may assume $c(w) = 0$. We will show that $|c(v)|_p \geq 2-p$ for all $v \in V(D) \backslash \{w\}$.

Assume this was wrong. Possibly replacing c by $\tilde{c} = p - c \mod p$ this yields the existence of a vertex $v \in V(D)\backslash\{w\}$ such that $0 \leq c(v) < 2-p$. Let u be the other vertex in the unique directed triangle containing w and v. Let $m = \frac{c(v)}{2} \in S_p$, then $S_p \subseteq [0, (\frac{p}{2}+m) \bmod p]_p \cup [(\frac{p}{2}+m) \bmod p, c(v)]_p$ and $|\frac{p}{2}+m-c(v)|_p = \frac{p}{2}+m < \frac{p+2-p}{2} = 1$. Hence in any case $= \{c(w), c(u), c(x)\}$ is contained in an interval of length strictly smaller than 1 contradicting c being an acyclic p-colouring.

Thus, indeed $|c(v)|_p \geq 2-p > \frac{1}{3}$ for all outer vertices. Hence the image of the outer directed cycle under c is contained in an open cyclic subinterval of length $p - \frac{2}{3} < 1$, again contradicting the definition of an acyclic p-colouring. We conclude $\vec{\chi}^*(W_k) \geq \vec{\chi}^*(D) \geq \frac{5}{3}$ for this special orientation, proving the claims for the star dichromatic number.

We now turn to proving $\vec{\chi}_f(W_k) \leq \frac{3k-2}{2k-2}$. Denote by V^+, V^- a bipartition of the outer cycle of W_k. Note that $V^+ \cup \{w\}, V^- \cup \{w\}$ and all subsets of $V(W_k)\backslash\{w\}$ of size $k-1$ induce forests in W_k, hence also acyclic sets for any orientation of W_k. We construct an instance of (2) by putting a weight of $\frac{1}{2}$ on each of $V^+ \cup \{w\}, V^- \cup \{w\}$, and a weight of $\frac{1}{2(k-1)}$ on each of the k subsets of

$V(W_k)\backslash\{w\}$ of size $k-1$, all other acyclic vertex sets receive a weight of 0. We compute $\frac{1}{2}+\frac{1}{2}\geq 1$ for w and $(k-1)\cdot\frac{1}{2(k-1)}+\frac{1}{2}\geq 1$ for each outer vertex. Hence, we have a feasible instance (2), verifying $\vec{\chi}_f(W_k)\leq\frac{1}{2}+\frac{1}{2}+\frac{k}{2(k-1)}=\frac{3k-2}{2k-2}$ as claimed.

Finally, we prove $\vec{\chi}^*(W_k)\geq\frac{3k-2}{2k-2}$ using the same special orientation D of W_k where the outer cycle is directed and the orientations of edges incident to w are alternating in cyclic order. We construct a suitable instance of the dual program (3), defining $y_w=\frac{k-2}{2k-2}$ and $y_v=\frac{1}{k-1}$ for every outer vertex. Let A be a maximal acyclic set. If $w\notin A$, then, since the outer cycle is directed, $A=V(W_k)\backslash\{w,x\}$ for some outer vertex x. In this case, we verify

$$\sum_{v\in A}y_v=(k-1)\cdot\frac{1}{k-1}=1.$$

If $w\in A$, clearly A contains at most one vertex of each directed triangle. Therefore $|A\setminus\{w\}|\leq\frac{k}{2}$ and again we verify the restriction

$$\sum_{v\in A}y_v\leq\frac{k}{2}\cdot\frac{1}{k-1}+\frac{k-2}{2k-2}=1.$$

Using linear programming duality we find that $\vec{\chi}_f(D)\geq\sum_{v\in V(W_k)}y_v=\frac{k}{k-1}+\frac{k-2}{2k-2}=\frac{3k-2}{2k-2}$. ■

Concerning fractional dichromatic numbers, Conjecture 25 would imply a tight upper bound of $\vec{\chi}_f(D)\leq\frac{g-1}{g-2}$ for planar digraphs of digirth g. In the following, we want to approach this upper bound by showing that indeed, $\vec{\chi}_f(D)$ tends to 1 for planar digraphs of large digirth. This is not at all obvious, as it is known that when dropping the restriction of planarity, directed graphs with arbitrarily large digirth may have arbitrarily large dichromatic number at the same time (cf. [5]). In order to do so, we recall the following terminologies as well as a related famous max-min-principle, known as Lucchesi-Younger-Theorem.

Definition.

- A *clutter* is a set family with no members containing each other.
- A subset of arcs in a digraph is called *dijoin* if it intersects every directed cut.
- A subset of arcs in a digraph is called *feedback arc set* if it intersects every directed cycle.

Theorem 28 (Lucchesi-Younger, cf. [10]). *Let D be a digraph and $w:E(D)\to\mathbb{N}_0$ a weighting of the edges with non-negative integers. Then the minimal weight of a dijoin in D equals the maximum number of (minimal) dicuts in D so that every arc a is contained in at most $w(a)$ of them.*

The terminology used in the following refers to [3], especially Chapter 1.1. According to Definition 1.5 and Theorem 1.25 in [3], the Lucchesi-Younger-Theorem means that the clutter of minimal directed cuts in any digraph admits the Max-Flow-Min-Cut-Property (MFMC). Consecutive application of Theorem 1.8 and Theorem 1.17 in [3], where the latter is a theorem of Lehman (cf. [9]), yields that the blocker of the clutter of minimal directed cuts, namely the clutter of minimal dijoins, is ideal. This means the following statement which was also pointed out in the article [17] on Woodall's Conjecture in the Open Problem Garden.

Theorem 29. *Let D be a digraph and let g denote the minimal size of a directed cut in D. Then there is $m \in \mathbb{N}$ and a collection of dijoins $J_1, \dots, J_m$ equipped with a weighting $x_1, \dots, x_m \in \mathbb{R}_+$ such that $x_1 + \cdots + x_m = g$ and for every arc $e \in E(D)$, we have $\sum_{i:e \in J_i} x_i \leq 1$.*

By considering planar digraphs and their directed duals, the dualities between minimal directed cuts and directed cycles as well as of dijoins and feedback arc sets yield.

Corollary 30. *If D is a planar digraph of digirth g, then there are m feedback arc sets $F_1, \dots, F_m \subseteq E(D)$ equipped with a weighting $x_1, \dots, x_m \in \mathbb{R}_+$ so that $x_1 + \cdots + x_m = g$ and for each edge $e \in E(D)$, $\sum_{i:e \in F_i} x_i \leq 1$.*

From this we may now conclude an upper bound for the fractional dichromatic number which approaches 1 for planar digraphs of large digirth.

Theorem 31. *Let $g \geq 6$. Then for every planar digraph D of digirth g we have $\vec{\chi}_f(D) \leq \frac{g}{g-5}$.*

Proof. Without loss of generality assume D to be simple. Let $F_1, \dots, F_m$, $x_1, \dots, x_m$ be as given by Corollary 30. Use the 5-degeneracy of the underlying graph $U(D)$ of D to derive an ordering $v_1, \dots, v_n$, $n = |V(D)|$ of the vertices so that for each $i \in \{1, \dots, n\}$, v_i has degree at most 5 in $G_i = U(D)[v_1, \dots, v_i]$. For each v_i, let $c(v_i)$ denote the set of $j \in \{1, \dots, m\}$ so that v_i has an incident edge in $F_j \cap E(G_i)$. Then clearly

$$\sum_{j \in c(v_i)} x_j \leq \sum_{e \in E_{G_i}(v)} \sum_{j:e \in F_j} x_j \leq \deg_{G_i}(v_i) \leq 5$$

for each v_i. Furthermore, the vertex set $X_j = \{x \in V(D) \,|\, j \notin c(x)\}$ is acyclic in D for all $j = 1, \dots, m$. In any directed cycle C in D we find an arc contained in F_j, and thus, j is contained in at least one of the c-sets of its end vertices.

We now define an instance of the linear optimization program 2 defining $\vec{\chi}_f(D)$ according to $x_A = \frac{i_A}{g-5}$, where $i_A = \sum_{j \in \{1,\dots,m\}: A = X_j} x_j$ for each $A \in$

$\mathcal{A}(D)$. Then those variables are non-negative and for each vertex v, we have

$$\sum_{A\in\mathcal{A}(D,v)} i_A = \sum_{j\in\{1,\dots,m\}:v\in X_j} x_j = \sum_{j\notin c(v)} x_j = \sum_{j=1}^{m} x_j - \sum_{j\in c(v)} x_j \geq g-5.$$

Hence this is a legal instance proving $\vec{\chi}_f(D) \leq \sum_{A\in\mathcal{A}} \frac{i_A}{g-5} = \frac{\sum_{j=1}^{m} x_j}{g-5} = \frac{g}{g-5}$. ■

5. Conclusion and Some Open Problems

The star dichromatic number of a digraph introduced and analysed in this paper seems to share all desirable attributes of the competing parameter from [1], the circular chromatic number. But, while the star dichromatic number is always a lower bound for the circular dichromatic number, it has the additional advantage that it is immune to the existence of directed cuts, while the addition of a dominating source makes the circular dichromatic number hit the ceiling. We therefore believe that the parameter introduced in the present paper yields a preferable generalization of the star chromatic number of Vince to the directed case. This is also supported by the fact that it can be seen as oriented version of the circular vertex arboricity.

In the planar case it might be true that the star chromatic number approaches 1 when the digirth increases (Conjecture 25). Note that this is impossible for $\vec{\chi}_c(D)$. It might be rewarding to study in particular the case $g = 4$ of Conjecture 25, e.g., orientations of planar triangulations without directed triangles, as recently there has been substantial progress towards digraph colourings of this class [8].

Also, it would be interesting to determine the computational complexity of decision problems of the form.

Instance: A digraph D (possibly from a certain class) and a real number $p > 1$.

Decide whether $\vec{\chi}^*(D) \leq p$.

In [4] it was shown that corresponding decision problem for the circular dichromatic number is NP-complete, even if restricted to planar digraphs. We conjecture that the same should be true for the star dichromatic number. This is true at least for all integers $p \in \mathbb{N}, p \geq 2$, since in that case $\vec{\chi}^*(D) \leq p \Leftrightarrow \vec{\chi}_c(D) \leq p$ for all digraphs D.

We want to conclude with an incomplete list of other natural questions that remain unanswered in this paper.

- For any $g \geq 4$, is there a planar digraph of digirth g with $\vec{\chi}^*(D) = \frac{g-1}{g-2}$?
- Is there a meaningful characterization of digraphs with $\vec{\chi}^*(D) = \vec{\chi}_f(D)$?
- Which digraphs satisfy $\vec{\chi}^*(D) = \vec{\chi}(D)$ or, more generally, $\vec{\chi}^*(D) = \vec{\chi}_c(D)$?

- What about the star dichromatic number of tournaments?
- Does the following statement hold true: For any $\varepsilon > 0$, there is a $\delta > 0$ such that any planar digraph D with $\vec{\chi}_f(D) \leq 1 + \delta$ has $\vec{\chi}^*(D) \leq 1 + \varepsilon$?

Acknowledgements

Part of this research was carried out during a stay of the second author at the Workshop Cycles and Colourings 2018, High Tatras, Slovakia. The collaborative, inspiring and friendly atmosphere during the workshop is gratefully acknowledged. The second author was supported by DFG-GRK 2434.

Note added in proof

Since the submission of this paper, in [6] we could solve the complexity related questions for the star dichromatic number posed Section 5.

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Received 29 January 2019
Revised 3 October 2019
Accepted 3 October 2019