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ON THE OPTIMALITY OF 3-RESTRICTED ARC CONNECTIVITY FOR DIGRAPHS AND BIPARTITE DIGRAPHS

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Abstract

Let D be a strong digraph. An arc subset S is a k-restricted arc cut of D if $D - S$ has a strong component D' with order at least k such that $D \backslash V(D')$ contains a connected subdigraph with order at least k. If such a k-restricted arc cut exists in D, then D is called λ^k-connected. For a λ^k-connected digraph D, the k-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all k-restricted arc cuts of D. It is known that for many digraphs $\lambda^k(D) \leq \xi^k(D)$, where $\xi^k(D)$ denotes the minimum k-degree of D. D is called λ^k-optimal if $\lambda^k(D) = \xi^k(D)$. In this paper, we will give some sufficient conditions for digraphs and bipartite digraphs to be λ^3-optimal.

Keywords: restricted arc-connectivity, bipartite digraph, optimality, digraph, network.

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1. Introduction

It is well-known that the network can be modelled as a digraph D with vertices $V(D)$ representing sites and arcs $A(D)$ representing links between sites of the network. Let $v \in V(D)$, the *out-neighborhood* of v is the set $N^+(v) = \{x \in V(D) : vx \in A(D)\}$ and the *out-degree* of v is $d^+(v) = |N^+(v)|$. The *in-neighborhood* of v is the set $N^-(v) = \{x \in V(D) : xv \in A(D)\}$ and the *in-degree* of v is $d^-(v) = |N^-(v)|$. The *neighborhood* of v is $N(v) = N^+(v) \cup N^-(v)$.

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Let $\delta^+(D), \delta^-(D)$ and $\delta(D)$ denote, respectively, the *minimum out-degree*, the *minimum in-degree* and the *minimum degree* of D.

For a pair nonempty vertex sets X and Y of D, $[X,Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Specially, if $Y = \overline{X}$, where $\overline{X} = V(D)\backslash X$, then we write $\partial^+(X)$ or $\partial^-(Y)$ instead of $[X,Y]$. For $X \subseteq V(D)$, the *subdigraph* of D induced by X is denoted by $D[X]$. The *underlying graph* $U(D)$ of D is the unique graph obtained from D by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. D is *connected* if $U(D)$ is connected and D is *strongly connected* (or, just, *strong*) if there exists a directed (x,y)-path and a directed (y,x)-path for any $x, y \in V(D)$. We define a digraph with one vertex to be strong. A *connected* (*strong*) *component* of D is a maximal induced subdigraph of D which is connected (strong). If D has p strong components, then these strong components can be labeled $D_1, \ldots, D_p$ such that there is no arc from D_j to D_i unless $j < i$. We call such an ordering an *acyclic ordering of the strong components* of D.

In a strong digraph D, we often use *arc connectivity* of D to measure the reliability. An arc set S is a *arc cut* of D if $D - S$ is not strong. The arc connectivity $\lambda(D)$ is the minimum cardinality over all arc cuts of D. The arc cut S of D with cardinality $\lambda(D)$ is called a λ-*cut*. Whitney's inequality shows $\lambda(D) \leq \delta(D)$. A strong digraph D with $\lambda(D) = \delta(D)$ is called λ-*optimal*. However, only using arc connectivity to measure the reliability is not enough. In [12], Volkmann introduced the concept of *restricted arc connectivity*. An arc subset S of D is a *resrtricted arc cut* if $D - S$ has a strong component D' with order at least 2 such that $D\backslash V(D')$ contains an arc. If such an arc cut exists in D, then D is called λ'-*connected*. For a λ'-connected digraph D, the restricted arc connectivity, denoted by $\lambda'(D)$, is the minimum cardinality over all restricted arc cuts of D. The restricted arc cut S of D with cardinality $\lambda'(D)$ is called a λ'-*cut*. In [13], Wang and Lin introduced the notion of *minimum arc degree*. Let $xy \in A(D)$. Then

$$\Omega(\{x,y\}) = \{\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^+(\{x\}) \cup \partial^-(\{y\}), \partial^+(\{y\}) \cup \partial^-(\{x\})\}.$$

The *arc degree* of xy is $\xi'(xy) = \min\{|S| : S \in \Omega(\{x,y\})\}$ and the minimum arc degree of D is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$.

It was proved in [3, 13] that for many λ'-connected digraphs, $\xi'(D)$ is an upper bound of $\lambda'(D)$. In [13], Wang and Lin introduced the concept of λ'-optimality. A λ'-connected digraph D with $\xi'(D) = \lambda'(D)$ is called λ'-*optimal*. As a generalization of restricted arc connectivity, in [10], Lin *et al.* introduced the concept of *k-restricted arc connectivity*.

Definition [10]**.** Let D be a strong digraph. An arc subset S is a *k-restricted arc cut* of D if $D - S$ has a strong component D' with order at least k such that

$D \backslash V(D')$ contains a connected subdigraph with order at least k. If such a k-restricted arc cut exists in D, then D is called λ^k-*connected*. For a λ^k-connected digraph D, the k-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all k-restricted arc cuts of D. The k-restricted arc cut S of D with cardinality $\lambda^k(D)$ is called a λ^k-*cut*.

Definition [10]**.** Let D be a strong digraph. For any $X \subseteq V(D)$, let $\Omega(X) = \{\partial^+(X_1) \cup \partial^-(X \backslash X_1) : X_1 \subseteq X\}$ and $\xi(X) = \min\{|S| : S \in \Omega(X)\}$. Define the *minimum k-degree* of D to be

$$\xi^k(D) = \min\{\xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected}\}.$$

Clearly, $\lambda^1(D) = \lambda(D)$, $\lambda^2(D) = \lambda'(D)$, $\xi^1(D) = \delta(D)$ and $\xi^2(D) = \xi'(D)$. Let D be a λ^k-connected digraph, where $k \geq 2$. Then D is λ^{k-1}-connected and $\lambda^{k-1}(D) \leq \lambda^k(D)$. It was shown in [10] that $\xi^k(D)$ is an upper bound of $\lambda^k(D)$ for many digraphs. And a λ^k-connected digraph D with $\lambda^k(D) = \xi^k(D)$ is called λ^k-*optimal*.

The research on the λ^k-optimality of digraph D is considered to be a hot issue. In [11], Hellwig and Volkmann concluded many sufficient conditions for digraphs to be λ-optimal. Besides, sufficient conditions for digraphs to be λ'-optimal were also given by several authors, for example by Balbuena *et al.* [1–4], Chen *et al.* [5,6], Grüter and Guo [7,8], Liu and Zhang [9], Volkmann [12] and Wang and Lin [13]. However, closely related conditions for λ^3-optimal digraphs have received little attention until recently. In [10], Lin *et al.* gave some sufficient conditions for digraphs to be λ^3-optimal. In this paper, we will give some sufficient conditions for digraphs to be λ^3-optimal. As corollaries, degree conditions or degree sum conditions for a digraph or a bipartite digraph to be λ^3-optimal are given. The main contributions in this paper are as following.

Theorem 1. *Let D be a digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then D is λ^3-optimal.*

Theorem 2. *Let $D = (X, Y, A(D))$ be a bipartite digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite, then D is λ^3-optimal.*

2. Proof of Theorem 1

We first introduce three useful lemmas.

Lemma 3 (Theorem 1.4 in [10])**.** *Let D be a strong digraph with $\delta^+(D) \geq 2k-1$ or $\delta^-(D) \geq 2k-1$. Then D is λ^k-connected and $\lambda^k(D) \leq \xi^k(D)$.*

Lemma 4. *Let D be a strong digraph with $\delta^+(D) \geq 2k-1$ or $\delta^-(D) \geq 2k-1$, and let $S = \partial^+(X)$ be a λ^k-cut of D, where X is a subset of $V(D)$. If $D[X]$ contains a connected subdigraph B with order k such that $|N^+(x) \cap \overline{X}| \geq k$ for any $x \in X \backslash V(B)$ or $D[\overline{X}]$ contains a connected subdigraph C with order k such that $|N^-(y) \cap X| \geq k$ for any $y \in \overline{X} \backslash V(C)$, then D is λ^k-optimal.*

Proof. By Lemma 3, D is λ^k-connected and $\lambda^k(D) \leq \xi^k(D)$. By symmetry, we only prove the case that $D[X]$ contains a connected subdigraph B with order k such that $|N^+(x) \cap \overline{X}| \geq k$ for any $x \in X \backslash V(B)$. The hypotheses imply that

$$\begin{aligned}\xi^k(D) &\leq |\partial^+(V(B))| = |[V(B), X\backslash V(B)]| + |[V(B), \overline{X}]| \\ &\leq k|X\backslash V(B)| + |[V(B), \overline{X}]| \leq \sum_{x \in X\backslash V(B)} |N^+(x) \cap \overline{X}| + |[V(B), \overline{X}]| \\ &= |[X\backslash V(B), \overline{X}]| + |[V(B), \overline{X}]| = |[X, \overline{X}]| = |S| = \lambda^k(D).\end{aligned}$$

Thus $\lambda^k(D) = \xi^k(D)$ and D is λ^k-optimal. ■

Lemma 5 (Lemma 4.1 in [10])**.** *Let D be a strong digraph with $|V(D)| \geq 6$ and $\delta(D) \geq 4$, and let S be a λ^3-cut of D. If D is not λ^3-optimal, then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.*

***Proof of Theorem* 1.** Clearly, D is a strong digraph with $\delta(D) \geq 5$. By Lemma 3, D is λ^3-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that D is not λ^3-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let S be a λ^3-cut of D. By Lemma 5, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $Y = \overline{X}$, and let $X_i = \{x \in X : |N^+(x) \cap Y| = i\}$, $Y_i = \{y \in Y : |N^-(y) \cap X| = i\}$, $i = 0, 1, 2$, and let $X_3 = \{x \in X : |N^+(x) \cap Y| \geq 3\}$, $Y_3 = \{y \in Y : |N^-(y) \cap X| \geq 3\}$.

Claim 1. $\min\{|X|, |Y|\} \geq 4$.

Proof. Suppose that $|X| = 3$. Then $\lambda^3(D) = |S| = |\partial^+(X)| \geq \xi(X) \geq \xi^3(D)$, contrary to the assumption. Suppose that $|Y| = 3$. Then $\lambda^3(D) = |S| = |\partial^-(Y)| \geq \xi(Y) \geq \xi^3(D)$, contrary to the assumption. Claim 1 follows. □

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. By symmetry, we only prove that $X_0 = \emptyset$ by contradiction. Suppose $X_0 \neq \emptyset$ and let $x \in X_0$. Then for any $\overline{x} \in Y$, $x\overline{x} \notin A(D)$ and we have that $5 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X| + |N^+(x) \cap N^-(\overline{x}) \cap Y| \leq |N^-(\overline{x}) \cap X| + |N^+(x) \cap Y| = |N^-(\overline{x}) \cap X|$. It implies that $|N^-(\overline{x}) \cap X| \geq 5$. Therefore $Y \subseteq Y_3$. So D is λ^3-optimal by Lemma 4, a contradiction to our assumption. □

Combining Claim 2 with Lemma 4, we have that $Y_1 \cup Y_2 \neq \emptyset$ and $X_1 \cup X_2 \neq \emptyset$. Otherwise we will obtain that D is λ^3-optimal, which is a contradiction. Next, we consider two cases.

Case 1. $X_1 \neq \emptyset$. Let $x' \in X_1$ and suppose $N^+(x') \cap Y = \{y'\}$. Then for any $y \in Y \backslash \{y'\}$, $x'y \notin A(D)$, so we have that $5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 1$. So $|N^-(y) \cap X| \geq 4$ and $Y \backslash \{y'\} \subseteq Y_3$. On the other hand, since $Y_1 \cup Y_2 \neq \emptyset$, so $y' \in Y_1 \cup Y_2$. Besides, $5 \leq \delta(D) \leq \delta^-(y') = |N^-(y')| = |N^-(y') \cap Y| + |N^-(y') \cap X| \leq |N^-(y') \cap Y| + 2$, thus $|N^-(y') \cap Y| \geq 3$. Let $y_1, y_2 \in N^-(y') \cap Y$, then $D[y', y_1, y_2]$ is connected and $|N^-(y) \cap X| \geq 4$ for any $y \in Y \backslash \{y', y_1, y_2\}$. By Lemma 4, we have that D is λ^3-optimal, a contradiction.

Case 2. $X_2 \neq \emptyset$. Let $x' \in X_2$ and suppose $N^+(x') \cap Y = \{y', y''\}$. Then for any $y \in Y \backslash \{y', y''\}$, $x'y \notin A(D)$, thus $5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 2$. So $|N^-(y) \cap X| \geq 3$ and $Y \backslash \{y', y''\} \subseteq Y_3$. On the other hand, since $Y_1 \cup Y_2 \neq \emptyset$, $y' \in Y_1 \cup Y_2$ or $y'' \in Y_1 \cup Y_2$. If $|Y_1 \cup Y_2| = 1$, then we can prove that D is λ^3-optimal by a proof similar to Case 1, which is a contradiction. If $Y_1 \cup Y_2 = \{y', y''\}$, then we consider two subcases.

Subcase 2.1. $y'y'' \in A(D)$ or $y''y' \in A(D)$. Since $y'' \in Y_1 \cup Y_2$ and $\delta(D) \geq 5$, then there exists $y_1 \in N^-(y'') \cap Y$ such that $y_1 \neq y'$. Therefore $D[y', y'', y_1]$ is connected and $|N^-(y) \cap X| \geq 3$ for any $y \in Y \backslash \{y', y'', y_1\}$. By Lemma 4, we have that D is λ^3-optimal, a contradiction.

Subcase 2.2. $y'y'' \notin A(D)$ and $y''y' \notin A(D)$. Since $y'y'' \notin A(D)$ and $y''y' \notin A(D)$, then $5 \leq |N^+(y') \cap N^-(y'')| = |N^+(y') \cap N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq |N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq 2 + |N^+(y') \cap N^-(y'') \cap Y|$. Therefore $|N^+(y') \cap N^-(y'') \cap Y| \geq 3$. Let $y_1 \in N^+(y') \cap N^-(y'') \cap Y$. Then $D[y', y'', y_1]$ is connected and $|N^-(y) \cap X| \geq 3$ for any $y \in Y \backslash \{y', y'', y_1\}$. By Lemma 4, we have that D is λ^3-optimal, a contradiction.

The proof is complete. ∎

From Theorem 1, we have following corollaries.

Corollary 6. *Let D be a digraph with $|V(D)| \geq 6$. If $d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then D is λ^3-optimal.*

Corollary 7 (Theorem 1.7 in [10])**.** *Let D be a digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \frac{|V(D)|+3}{2}$, then D is λ^3-optimal.*

Remark 8. To show the condition that "$|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$ "in Theorem 1 is sharp, we give a class of digraphs. Let m, k be positive integers with $m \geq 3$, and let D be a digraph with $|V(D)| = 4m + 4$.

Define the vertex set of D as $V(D) = B \cup C$, where $B = \{x_0, \ldots, x_m, w_0, \ldots, w_m\}$ and $C = \{y_0, \ldots, y_m, z_0, \ldots, z_m\}$. And define the arc set of D as $A(D) = A(D[B]) \cup A(D[C]) \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $A(D[B]) \cup A(D[C]) = \{uv :$ for any $u, v \in B$ or $C\}$, $M_1 = \{x_i y_{k(\text{mod } m+1)} : 0 \le i \le m$ and $0 \le k - i \le 1\}$, $M_2 = \{w_i z_{k(\text{mod } m+1)} : 0 \le i \le m$ and $0 \le k - i \le 2\}$, $M_3 = \{y_i x_{k(\text{mod } m+1)} : 0 \le i \le m$ and $0 \le k - i \le 2\}$ and $M_4 = \{z_i w_{k(\text{mod } m+1)} : 0 \le i \le m$ and $0 \le k - i \le 2\}$.

Clearly, D is strong and there exists $0 \le i, j \le m$ such that $|N^+(x_i) \cap N^-(y_j)| = 4$ and $x_i y_j \notin A(D)$. And $\partial^+(B)$ is a 3-restricted edge cut with $|\partial^+(B)| = (2+3) \cdot (m+1) = 5m+5$. On the other hand, $\xi^3(D) = \xi(\{x_l, x_p, x_q\}) = |\partial^+(\{x_l, x_p, x_q\})| = 3 \cdot (2m+3) - 6 = 6m+3$, where $0 \le l, p, q \le m$. So $\lambda^3(D) \le |\partial^+(B)| = 5m+5 < 6m+3 = \xi^3(D)$ for $m \ge 3$. Thus D is not λ^3-optimal.

Besides, in D, there exists $0 \le i, j \le m$ such that $x_i y_j \notin A(D)$ and $d^+(x_i) + d^-(y_j) = 2 \cdot (2m+3) = |V(D)| + 2 < |V(D)| + 3$, and $\delta(D) = 2m + 3 = \frac{|V(D)|}{2} + 1 < \frac{|V(D)|+3}{2}$. So this example also shows that the conditions that "$d^+(u) + d^-(v) \ge |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$" in Corollary 6 and "$\delta(D) \ge \frac{|V(D)|+3}{2}$" in Corollary 7 are sharp.

3. Proof of Theorem 2

We first introduce several useful lemmas.

Lemma 9 (Lemma 2.1 in [10]). *Let D be a strong digraph and X_1, Y_1 disjoint subsets of $V(D)$. If $D[X_1]$ contains a connected subdigraph with order at least k and $D[Y_1]$ contains a strong subdigraph with order at least k, then D is λ^k-connected and each arc set in $\{\partial^-(Y_1), \partial^+(Y_1)\} \cup \Omega(X_1)$ is a k-restricted arc cut of D.*

Lemma 10. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \ge 3$ or $\delta^-(D) \ge 3$. Then D is λ^3-connected and $\lambda^3(D) \le \xi^3(D)$.*

Proof. By symmetry, we only consider the case that $\delta^-(D) \ge 3$. Let X' be a subset of $V(D)$ with $|X'| = 3$ such that $D[X']$ is connected and $\xi^3(D) = \xi(X')$. Without loss of generality, assume that $|X' \cap X| = 1$ and $|X' \cap Y| = 2$. Let $X' \cap X = \{x\}$ and $X' \cap Y = \{y, z\}$. Let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D \backslash X'$.

First, we claim that $V(D_1) \cap Y \ne \emptyset$. Otherwise, we have that $V(D_1) \subseteq X$ and $|V(D_1)| = 1$. Let $V(D_1) = \{u\}$. Then $N^-(u) \subseteq \{y, z\}$. So $3 \le \delta^-(D) \le d^-(u) = |N^-(u)| \le |\{y, z\}| = 2$, a contradiction. Next, we aim to prove $|V(D_1)| \ge 3$.

Since $N^-(v) \subseteq \{x\} \cup (V(D_1) \cap X)$ for any $v \in V(D_1) \cap Y$, we have $3 \le \delta^-(D) \le d^-(v) = |N^-(v)| \le |\{x\} \cup (V(D_1) \cap X)| = |\{x\}| + |V(D_1) \cap X| = 1 + |V(D_1) \cap X|$. Thus $|V(D_1) \cap X| \ge 2$ and $|V(D_1)| = |V(D_1) \cap X| + |V(D_1) \cap Y| \ge 2 + 1 = 3$. It follows that $|V(D_1)| \ge 3$. Since $D[X']$ is connected and

$D[X'] \subseteq D\backslash V(D_1)$, by Lemma 9, each arc set in $\Omega(X')$ is a 3-restricted arc cut of D. Therefore, D is λ^3-connected and $\lambda^3(D) \le \xi(X') = \xi^3(D)$. ∎

Lemma 11. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \ge 3$ or $\delta^-(D) \ge 3$, and let $S = \partial^+(X')$ be a λ^3-cut of D, where X' is a subset of $V(D)$. If $D[X']$ contains a connected subdigraph B with order 3 such that $|N^+(x) \cap \overline{X'}| \ge 2$ for any $x \in X'\backslash V(B)$ or $D[\overline{X'}]$ contains a connected subdigraph C with order 3 such that $|N^-(y) \cap X'| \ge 2$ for any $y \in \overline{X'}\backslash V(C)$, then D is λ^3-optimal.*

Proof. By Lemma 10, D is λ^3-connected and $\lambda^3(D) \le \xi^3(D)$. By symmetry, we only prove the case that $D[X']$ contains a connected subdigraph B with order 3 such that $|N^+(x) \cap \overline{X'}| \ge 2$ for any $x \in X'\backslash V(B)$. The hypotheses imply that

$$\begin{aligned}\xi^3(D) &\le |\partial^+(V(B))| = \big|[V(B), X'\backslash V(B)]\big| + \big|[V(B), \overline{X'}]\big| \\ &\le 2|X'\backslash V(B)| + \big|[V(B), \overline{X'}]\big| \le \sum_{x \in X'\backslash V(B)} \big|N^+(x) \cap \overline{X'}\big| + \big|[V(B), \overline{X'}]\big| \\ &= \big|[X'\backslash V(B), \overline{X'}]\big| + \big|[V(B), \overline{X'}]\big| = \big|[X', \overline{X'}]\big| = |S| = \lambda^3(D).\end{aligned}$$

Thus $\lambda^3(D) = \xi^3(D)$ and D is λ^3-optimal. ∎

By a proof similar to that of Lemma 4.1 shown in [10], we can get the following Lemma 12.

Lemma 12. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \ge 3$, and let S be a λ^3-cut of D. If D is not λ^3-optimal, then there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X'}]$ contain a connected subdigraph with order 3.*

Proof of Theorem 2. Since $|V(D)| \ge 6$, for any $u, v \in V(D)$ in the same partite, $|N^+(u) \cap N^-(v)| \ge \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \ge 3$. Therefore D is strong and $\delta(D) \ge 3$. By Lemma 10, D is λ^3-connected and $\lambda^3(D) \le \xi^3(D)$. Suppose, on the contrary, that D is not λ^3-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let S be a λ^3-cut of D. Then by Lemma 12, there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X'}]$ contain a connected subdigraph with order 3.

Let $\overline{X'} = X''$, and let $X'_X = X' \cap X$, $X'_Y = X' \cap Y$, $X''_X = X'' \cap X$ and $X''_Y = X'' \cap Y$. And let $X'_{Xi} = \{x \in X'_X : |N^+(x) \cap X''_Y| = i\}$, $X'_{Yi} = \{y \in X'_Y : |N^+(y) \cap X''_X| = i\}$, $X''_{Xi} = \{x \in X''_X : |N^-(x) \cap X'_Y| = i\}$, $X''_{Yi} = \{y \in X''_Y : |N^-(y) \cap X'_X| = i\}$, $i = 0, 1$, and $X'_{X2} = \{x \in X'_X : |N^+(x) \cap X''_Y| \ge 2\}$, $X'_{Y2} = \{y \in X'_Y : |N^+(y) \cap X''_X| \ge 2\}$, $X''_{X2} = \{x \in X''_X : |N^-(x) \cap X'_Y| \ge 2\}$, $X''_{Y2} = \{y \in X''_Y : |N^-(y) \cap X'_X| \ge 2\}$.

Claim 1. $\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \ge 2$.

Proof. If, on the contrary $|X'_X| = 1$, let $X'_X = \{v\}$. Then $|N(v) \cap X'_Y| \geq 2$ for $D[X']$ contains a connected subdigraph with order 3. Let $y_1, y_2 \in N(v) \cap X'_Y$. Then $D[v, y_1, y_2]$ is connected, and for any $x' \in X' \backslash \{v, y_1, y_2\}$, $N^+(x') \subseteq \{v\} \cup (N^+(x') \cap X'')$, we have $3 \leq \delta(D) \leq d^+(x') = |N^+(x')| \leq |\{v\}| + |N^+(x') \cap X''| = 1 + |N^+(x') \cap X''|$. Therefore $|N^+(x') \cap X''| \geq 2$. By Lemma 11, D is λ^3-optimal, a contradiction to our assumption. Thus $|X'_X| \geq 2$. Similarly, we can prove that $\min\left\{|X'_Y|, |X''_X|, |X''_Y|\right\} \geq 2$. □

Claim 2. *Either $X'_{X0} = \emptyset$ or $X''_{X0} = \emptyset$ and either $X'_{Y0} = \emptyset$ or $X''_{Y0} = \emptyset$.*

Proof. If $X'_{X0} \neq \emptyset$ and $X''_{X0} \neq \emptyset$, then there exists $x \in X'_{X0} \subseteq X$ and $\overline{x} \in X''_{X0} \subseteq X$ such that $|N^+(x) \cap N^-(\overline{x})| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. On the other hand, since $x \in X'_{X0}$ and $\overline{x} \in X''_{X0}$, $N^+(x) \subseteq X'_Y$ and $N^-(\overline{x}) \subseteq X''_Y$, which implies that $N^+(x) \cap N^-(\overline{x}) = \emptyset$, a contradiction. Thus either $X'_{X0} = \emptyset$ or $X''_{X0} = \emptyset$. Similarly, we can obtain that either $X'_{Y0} = \emptyset$ or $X''_{Y0} = \emptyset$. □

We consider the following two cases.

Case 1. $X'_{X0} = X'_{Y0} = \emptyset$ or $X''_{X0} = X''_{Y0} = \emptyset$. By symmetry, we only prove the case that $X'_{X0} = X'_{Y0} = \emptyset$.

Claim 1.1. *Either $X'_{X1} = \emptyset$ and $X'_{Y1} \neq \emptyset$ or $X'_{X1} \neq \emptyset$ and $X'_{Y1} = \emptyset$.*

Proof. Since D is not λ^3-optimal, by Lemma 11, we have that $X'_{X1} \cup X'_{Y1} \neq \emptyset$. Suppose $X'_{X1} \neq \emptyset$ and $X'_{Y1} \neq \emptyset$. Take $x_1 \in X'_{X1}$. Then for any $\overline{x} \in X''_X$, we have that $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(\overline{x})| = |N^+(x_1) \cap N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap N^-(\overline{x}) \cap X''| \leq |N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap X''| = |N^-(\overline{x}) \cap X'| + 1$. It implies that $|N^-(\overline{x}) \cap X'| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2$. So $X''_X \subseteq X''_{X2}$. By a similar proof, we can also prove that $X''_Y \subseteq X''_{Y2}$. Therefore D is λ^3-optimal by Lemma 11, a contradiction. The proof of Claim 1.1 is complete. □

Without loss of generality, let $X'_{X1} \neq \emptyset$ and $X'_{Y1} = \emptyset$.

Case 1.1. $|X'_{X1}| = 1$. Let $x_1 \in X'_{X1}$. Then $3 \leq \delta(D) \leq d^+(x_1) = |N^+(x_1)| = |N^+(x_1) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap X'_Y| + 1$, therefore $|N^+(x_1) \cap X'_Y| \geq 2$. Let $y_1, y_2 \in N^+(x_1) \cap X'_Y$. Then $D[x_1, y_1, y_2]$ is connected, and for any $v \in X' \backslash \{x_1, y_1, y_2\}$, $|N^+(v) \cap X''| \geq 2$. By Lemma 11, D is λ^3-optimal, a contradiction.

Case 1.2. $|X'_{X1}| \geq 2$. Let $x_1, x_2 \in X'_{X1}$. Then $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(x_2)| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + |N^+(x_1) \cap N^-(x_2) \cap X''_Y| \leq |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + 1$. So $|N^+(x_1) \cap$

$N^-(x_2) \cap X'_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2$. Let $y_1 \in N^+(x_1) \cap N^-(x_2) \cap X'_Y$. Then

$$\begin{aligned}\xi^3(D) &\leq \xi(\{x_1, x_2, y_1\}) \leq |\partial^+(\{x_1, x_2, y_1\})| \\ &= \big|[\{x_1\}, X'_Y \backslash \{y_1\}]\big| + \big|[\{x_1\}, X''_Y]\big| + \big|[\{x_2\}, X'_Y \backslash \{y_1\}]\big| + \big|[\{x_2\}, X''_Y]\big| \\ &+ \big|[\{y_1\}, X'_X \backslash \{x_1, x_2\}]\big| + \big|[\{y_1\}, X''_X]\big| \\ &\leq 2 \cdot \left(|X'_Y| - 1\right) + 2 + |X'_X| - 2 + \big|[\{y_1\}, X''_X]\big| \leq |S| = \lambda^3(D).\end{aligned}$$

Thus D is λ^3-optimal, a contradiction.

Case 2. $X'_{X0} = X''_{Y0} = \emptyset$ or $X''_{X0} = X'_{Y0} = \emptyset$. By symmetry, we only prove the case that $X'_{X0} = X''_{Y0} = \emptyset$. Without loss of generality, we may assume that $X'_{Y0} \neq \emptyset$ and $X''_{X0} \neq \emptyset$. Otherwise, by Case 1, D is λ^3-optimal, a contradiction. On the other hand, since for any $u \in X'_{Y0}$, $N^+(u) \subseteq X'_X$, we have $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq \delta(D) \leq d^+(u) = |N^+(u)| \leq |X'_X|$. Therefore $|X'_X| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. Similarly, we can also prove that $|X''_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. Thus

$$\begin{aligned}|X'_Y| + |X''_X| &= |V(D)| - |X'_X| - |X''_Y| \\ &\leq |V(D)| - 2 \cdot \left(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \right) \leq \frac{|V(D)|}{2} - 2.\end{aligned} \tag{1}$$

Claim 2.1. $|X'_X| \geq |X'_Y| + 1$ *or* $|X''_Y| \geq |X''_X| + 1$.

Proof. Otherwise, we have that $|X'_Y| + |X''_X| \geq |X'_X| + |X''_Y| \geq 2 \cdot \left(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \right) \geq \frac{|V(D)|}{2} + 2$, a contradiction to (1). □

Without loss of generality, we assume that $|X'_X| \geq |X'_Y| + 1$ in the following discussion.

Claim 2.2. $|N^+(x) \cap X''_Y| \geq 3$ *and* $|N^-(y) \cap X'_X| \geq 3$ *for any* $x \in X'_X$ *and* $y \in X''_Y$.

Proof. By symmetry, we only prove that for any $x \in X'_X$, $|N^+(x) \cap X''_Y| \geq 3$. Since $X''_{X0} \neq \emptyset$, for any $x \in X'_X$ and $\overline{x} \in X''_{X0}$, $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X'_Y| + |N^+(x) \cap N^-(\overline{x}) \cap X''_Y| \leq |N^-(\overline{x}) \cap X'_Y| + |N^+(x) \cap X''_Y| = |N^+(x) \cap X''_Y|$, so $|N^+(x) \cap X''_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3$. □

Claim 2.3. $X'_{Y2} = X''_{X2} = \emptyset$.

Proof. Here, we only prove that $X'_{Y2} = \emptyset$. The proof of the statement that $X''_{X2} = \emptyset$ is similar. Suppose, by a contradiction, there exists $y \in X'_{Y2}$. Let

$x_1, x_2 \in N^+(y) \cap X''_X$. Then

$$\begin{aligned}\xi^3(D) &\le \xi(\{x_1, x_2, y\}) \le \left|\partial^+(\{y\}) \cup \partial^-(\{x_1, x_2\})\right| \\ &= \left|\partial^+(\{y\})\right| + \left|\partial^-(\{x_1, x_2\})\right| - 2 = \left|[\{y\}, X'_X]\right| + \left|[\{y\}, X''_X]\right| \\ &+ \left|[X'_Y, \{x_1\}]\right| + \left|[X''_Y, \{x_1\}]\right| + \left|[X'_Y, \{x_2\}]\right| + \left|[X''_Y, \{x_2\}]\right| - 2 \\ &\le |X'_X| + \left|[\{y\}, X''_X]\right| + \left|[X'_Y, \{x_1\}]\right| + 2|X''_Y| + \left|[X'_Y, \{x_2\}]\right| - 2 \\ &\le 3\max\left\{|X'_X|, |X''_Y|\right\} + |[\{y\}, X''_X]| + \left|[X'_Y, \{x_1\}]\right| \\ &+ \left|[X'_Y, \{x_2\}]\right| - 2 \le |S| = \lambda^3(D).\end{aligned}$$

So D is λ^3-optimal, a contradiction. □

Claim 2.4. *For any $x \in X'_X$, $|N(X) \cap X'_Y| \ge 2$.*

Proof. Let $X'_Y = \{y_1, y_2, \ldots, y_p\}$ and let $S^* = \{s^* : s^* \in N^+(y_i) \cap N^-(y_j) \cap X'_X$, where $i, j \in \{1, \ldots, p\}$ and $i \ne j\}$. Then $D[S^* \cup X'_Y]$ is strong. Besides, by Claim 2.3, we have that for any $i, j \in \{1, \ldots, p\}$ and $i \ne j$, $y_i, y_j \in X'_{Y0} \cup X'_{Y1}$. Therefore $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \le \left|N^+(y_i) \cap N^-(y_j)\right| = \left|N^+(y_i) \cap N^-(y_j) \cap X'_X\right| + \left|N^+(y_i) \cap N^-(y_j) \cap X''_X\right| \le \left|N^+(y_i) \cap N^-(y_j) \cap X'_X\right| + \left|N^+(y_i) \cap X''_X\right| \le \left|N^+(y_i) \cap N^-(y_j) \cap X'_X\right| + 1$. So $|N^+(y_i) \cap N^-(y_j) \cap X'_X| \ge \left\lceil \frac{|V(D)|}{4} \right\rceil \ge 2$. Similarly, we can prove that $\left|N^+(y_j) \cap N^-(y_i) \cap X'_X\right| \ge 2$. On the other hand, since $\left|X'_Y\right| \ge 2$, we have $\left|S^* \cup X'_Y\right| \ge 3$. For any $x \in S^*$, clearly, $\left|N(x) \cap X'_Y\right| \ge 2$. Next, we claim that for any $x \in X'_X \backslash S^*$, $\left|N^+(x) \cap X''_Y\right| \le \left|[X'_Y, \{x\}]\right|$.

Suppose there exists $x^* \in X'_X \backslash S^*$ such that $|N^+(x^*) \cap X''_Y| > |[X'_Y, \{x^*\}]|$. Since $D[S^* \cup X'_Y]$ is strong and $\left|S^* \cup X'_Y\right| \ge 3$, we have $X' \backslash \{x^*\}$ is a 3-restricted edge cut. Therefore $|\partial^+(X' \backslash \{x^*\})| = |S| - \left|N^+(x^*) \cap X''_Y\right| + \left|[X'_Y, \{x^*\}]\right| < |S|$, a contradiction to the minimality of S. Thus $\left|[X'_Y, \{x\}]\right| \ge \left|N^+(x) \cap X''_Y\right|$. By Claim 2.2, we have that $\left|[X'_Y, \{x\}]\right| \ge 3$. The proof of Claim 2.4 is complete. □

Let $x_1 \in X'_X$ such that $\left|N^+(x_1) \cap X''_Y\right| \le \left|N^+(u) \cap X''_Y\right|$ for any $u \in X'_X$, and let $y_1, y_2 \in N(x_1) \cap X'_Y$. Then

$$\begin{aligned}\xi^3(D) &\le |\partial^+(\{x_1, y_1, y_2\})| = \left|[\{x_1, y_1, y_2\}, X' \backslash \{x_1, y_1, y_2\}]\right| + \left|[\{x_1, y_1, y_2\}, X'']\right| \\ &\le 2\big(|X'_X| - 1\big) + \left|X'_Y\right| - 2 + \left|[\{x_1\}, X''_Y]\right| + \left|[\{y_1\}, X''_X]\right| + \left|[\{y_2\}, X''_X]\right| \\ &\le 3\left|X'_X\right| - 5 + \left|[\{x_1\}, X''_Y]\right| + \left|[\{y_1\}, X''_X]\right| \\ &+ \left|[\{y_2\}, X''_X]\right| \big(|X'_X| \ge |X'_Y| + 1\big) \le \left|[\{x_1\}, X''_Y]\right| \times \left|X'_X\right| \\ &+ \left|[\{y_1\}, X''_X]\right| + \left|[\{y_2\}, X''_X]\right| \le |S| = \lambda^3(D).\end{aligned}$$

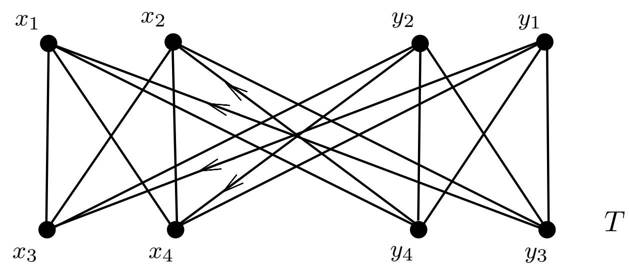
So D is λ_3-optimal, a contradiction.

The proof is complete. ■

From Theorem 2, we have following corollaries.

Corollary 13. *Let* $D = (X, Y, A(D))$ *be a strong bipartite digraph with* $\delta(D) \geq 3$. *If for any* $u, v \in V(D)$ *in the same partite,* $d^+(u) + d^-(v) \geq |V(D)| - 1$, *then* D *is* λ^3*-optimal.*

Corollary 14. *Let* $D = (X, Y, A(D))$ *be a strong bipartite digraph with* $|V(D)| \geq 6$. *If* $\delta(D) \geq \left\lfloor \frac{|V(D)|}{2} \right\rfloor$, *then* D *is* λ^3*-optimal.*



(Unordered edges represent two arcs with the same end-vertices and opposite directions.)

Figure 1. The example from Remark 15.

Remark 15. To show that the condition "$|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite" in Theorem 2 is sharp, we consider the digraph T shown in Figure 1. Clearly, $|V(D)| \geq 6$ and D is strong. There exists x_1, y_1 in the same partite such that $|N^+(x_1) \cap N^-(y_1)| = 2 < 3 = \left\lceil \frac{|V(T)|}{4} \right\rceil + 1$. Clearly, $\partial^+(\{x_1, x_2, x_3, x_4\})$ is a 3-restricted edge cut and $\xi^3(T) = |\partial^+(\{x_1, x_2, x_3\})| = 5$. Therefore, $\lambda^3(T) \leq |\partial^+(\{x_1, x_2, x_3, x_4\})| = 4 < 5 = \xi^3(T)$ and T is not λ^3-optimal.

Besides, since $d^+(x_3) + d^-(y_4) = 6 < 7 = |V(T)| - 1$ and $\delta(T) = 3 < 4 = \left\lfloor \frac{|V(D)|}{2} \right\rfloor$, this example also shows that the conditions "$d^+(u) + d^-(v) \geq |V(D)| - 1$ for any $u, v \in V(D)$ in the same partite" in Corollary 13 and "$\delta(D) \geq \left\lfloor \frac{|V(D)|}{2} \right\rfloor$" in Corollary 14 are sharp.

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