Discussiones Mathematicae
Graph Theory 42 (2022) 321–332
https://doi.org/10.7151/dmgt.2259

ON THE OPTIMALITY OF 3-RESTRICTED ARC CONNECTIVITY FOR DIGRAPHS AND BIPARTITE DIGRAPHS

Yaoyao Zhang and Jixiang Meng[1]

College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046, P.R. China

e-mail: yoyoyame@126.com
mjxxju@sina.com

Abstract

Let D be a strong digraph. An arc subset S is a k-restricted arc cut of D if $D-S$ has a strong component D' with order at least k such that $D \backslash V(D')$ contains a connected subdigraph with order at least k. If such a k-restricted arc cut exists in D, then D is called λ^k-connected. For a λ^k-connected digraph D, the k-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all k-restricted arc cuts of D. It is known that for many digraphs $\lambda^k(D) \leq \xi^k(D)$, where $\xi^k(D)$ denotes the minimum k-degree of D. D is called λ^k-optimal if $\lambda^k(D) = \xi^k(D)$. In this paper, we will give some sufficient conditions for digraphs and bipartite digraphs to be λ^3-optimal.

Keywords: restricted arc-connectivity, bipartite digraph, optimality, digraph, network.

2010 Mathematics Subject Classification: 05C40, 68R10.

1. Introduction

It is well-known that the network can be modelled as a digraph D with vertices $V(D)$ representing sites and arcs $A(D)$ representing links between sites of the network. Let $v \in V(D)$, the *out-neighborhood* of v is the set $N^+(v) = \{x \in V(D) : vx \in A(D)\}$ and the *out-degree* of v is $d^+(v) = |N^+(v)|$. The *in-neighborhood* of v is the set $N^-(v) = \{x \in V(D) : xv \in A(D)\}$ and the *in-degree* of v is $d^-(v) = |N^-(v)|$. The *neighborhood* of v is $N(v) = N^+(v) \cup N^-(v)$.

---
[1]Corresponding author.

Let $\delta^+(D), \delta^-(D)$ and $\delta(D)$ denote, respectively, the *minimum out-degree*, the *minimum in-degree* and the *minimum degree* of D.

For a pair nonempty vertex sets X and Y of D, $[X,Y] = \{xy \in A(D) : x \in X, y \in Y\}$. Specially, if $Y = \overline{X}$, where $\overline{X} = V(D)\backslash X$, then we write $\partial^+(X)$ or $\partial^-(Y)$ instead of $[X,Y]$. For $X \subseteq V(D)$, the *subdigraph* of D induced by X is denoted by $D[X]$. The *underlying graph* $U(D)$ of D is the unique graph obtained from D by deleting the orientation of all arcs and keeping one edge of a pair of multiple edges. D is *connected* if $U(D)$ is connected and D is *strongly connected* (or, just, *strong*) if there exists a directed (x,y)-path and a directed (y,x)-path for any $x, y \in V(D)$. We define a digraph with one vertex to be strong. A *connected* (*strong*) *component* of D is a maximal induced subdigraph of D which is connected (strong). If D has p strong components, then these strong components can be labeled $D_1, \ldots, D_p$ such that there is no arc from D_j to D_i unless $j < i$. We call such an ordering an *acyclic ordering of the strong components* of D.

In a strong digraph D, we often use *arc connectivity* of D to measure the reliability. An arc set S is a *arc cut* of D if $D - S$ is not strong. The arc connectivity $\lambda(D)$ is the minimum cardinality over all arc cuts of D. The arc cut S of D with cardinality $\lambda(D)$ is called a λ-*cut*. Whitney's inequality shows $\lambda(D) \leq \delta(D)$. A strong digraph D with $\lambda(D) = \delta(D)$ is called λ-*optimal*. However, only using arc connectivity to measure the reliability is not enough. In [12], Volkmann introduced the concept of *restricted arc connectivity*. An arc subset S of D is a *resrtricted arc cut* if $D - S$ has a strong component D' with order at least 2 such that $D\backslash V(D')$ contains an arc. If such an arc cut exists in D, then D is called λ'-*connected*. For a λ'-connected digraph D, the restricted arc connectivity, denoted by $\lambda'(D)$, is the minimum cardinality over all restricted arc cuts of D. The restricted arc cut S of D with cardinality $\lambda'(D)$ is called a λ'-*cut*. In [13], Wang and Lin introduced the notion of *minimum arc degree*. Let $xy \in A(D)$. Then

$$\Omega(\{x,y\}) = \{\partial^+(\{x,y\}), \partial^-(\{x,y\}), \partial^+(\{x\}) \cup \partial^-(\{y\}), \partial^+(\{y\}) \cup \partial^-(\{x\})\}.$$

The *arc degree* of xy is $\xi'(xy) = \min\{|S| : S \in \Omega(\{x,y\})\}$ and the minimum arc degree of D is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$.

It was proved in [3, 13] that for many λ'-connected digraphs, $\xi'(D)$ is an upper bound of $\lambda'(D)$. In [13], Wang and Lin introduced the concept of λ'-optimality. A λ'-connected digraph D with $\xi'(D) = \lambda'(D)$ is called λ'-*optimal*. As a generalization of restricted arc connectivity, in [10], Lin *et al.* introduced the concept of *k-restricted arc connectivity*.

Definition [10]**.** Let D be a strong digraph. An arc subset S is a *k-restricted arc cut* of D if $D - S$ has a strong component D' with order at least k such that

$D \backslash V(D')$ contains a connected subdigraph with order at least k. If such a k-restricted arc cut exists in D, then D is called λ^k-*connected*. For a λ^k-connected digraph D, the k-restricted arc connectivity, denoted by $\lambda^k(D)$, is the minimum cardinality over all k-restricted arc cuts of D. The k-restricted arc cut S of D with cardinality $\lambda^k(D)$ is called a λ^k-*cut*.

Definition [10]**.** Let D be a strong digraph. For any $X \subseteq V(D)$, let $\Omega(X) = \{\partial^+(X_1) \cup \partial^-(X \backslash X_1) : X_1 \subseteq X\}$ and $\xi(X) = \min\{|S| : S \in \Omega(X)\}$. Define the *minimum k-degree* of D to be

$$\xi^k(D) = \min\{\xi(X) : X \subseteq V(D), |X| = k, D[X] \text{ is connected}\}.$$

Clearly, $\lambda^1(D) = \lambda(D)$, $\lambda^2(D) = \lambda'(D)$, $\xi^1(D) = \delta(D)$ and $\xi^2(D) = \xi'(D)$. Let D be a λ^k-connected digraph, where $k \geq 2$. Then D is λ^{k-1}-connected and $\lambda^{k-1}(D) \leq \lambda^k(D)$. It was shown in [10] that $\xi^k(D)$ is an upper bound of $\lambda^k(D)$ for many digraphs. And a λ^k-connected digraph D with $\lambda^k(D) = \xi^k(D)$ is called λ^k-*optimal*.

The research on the λ^k-optimality of digraph D is considered to be a hot issue. In [11], Hellwig and Volkmann concluded many sufficient conditions for digraphs to be λ-optimal. Besides, sufficient conditions for digraphs to be λ'-optimal were also given by several authors, for example by Balbuena *et al.* [1–4], Chen *et al.* [5,6], Grüter and Guo [7,8], Liu and Zhang [9], Volkmann [12] and Wang and Lin [13]. However, closely related conditions for λ^3-optimal digraphs have received little attention until recently. In [10], Lin *et al.* gave some sufficient conditions for digraphs to be λ^3-optimal. In this paper, we will give some sufficient conditions for digraphs to be λ^3-optimal. As corollaries, degree conditions or degree sum conditions for a digraph or a bipartite digraph to be λ^3-optimal are given. The main contributions in this paper are as following.

Theorem 1. *Let D be a digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then D is λ^3-optimal.*

Theorem 2. *Let $D = (X, Y, A(D))$ be a bipartite digraph with $|V(D)| \geq 6$. If $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite, then D is λ^3-optimal.*

2. Proof of Theorem 1

We first introduce three useful lemmas.

Lemma 3 (Theorem 1.4 in [10])**.** *Let D be a strong digraph with $\delta^+(D) \geq 2k - 1$ or $\delta^-(D) \geq 2k - 1$. Then D is λ^k-connected and $\lambda^k(D) \leq \xi^k(D)$.*

Lemma 4. *Let D be a strong digraph with $\delta^+(D) \ge 2k-1$ or $\delta^-(D) \ge 2k-1$, and let $S = \partial^+(X)$ be a λ^k-cut of D, where X is a subset of $V(D)$. If $D[X]$ contains a connected subdigraph B with order k such that $|N^+(x) \cap \overline{X}| \ge k$ for any $x \in X \backslash V(B)$ or $D[\overline{X}]$ contains a connected subdigraph C with order k such that $|N^-(y) \cap X| \ge k$ for any $y \in \overline{X} \backslash V(C)$, then D is λ^k-optimal.*

Proof. By Lemma 3, D is λ^k-connected and $\lambda^k(D) \le \xi^k(D)$. By symmetry, we only prove the case that $D[X]$ contains a connected subdigraph B with order k such that $|N^+(x) \cap \overline{X}| \ge k$ for any $x \in X \backslash V(B)$. The hypotheses imply that

$$\begin{aligned}\xi^k(D) &\le |\partial^+(V(B))| = |[V(B), X\backslash V(B)]| + |[V(B), \overline{X}]| \\ &\le k|X\backslash V(B)| + |[V(B), \overline{X}]| \le \sum_{x \in X\backslash V(B)} |N^+(x) \cap \overline{X}| + |[V(B), \overline{X}]| \\ &= |[X\backslash V(B), \overline{X}]| + |[V(B), \overline{X}]| = |[X, \overline{X}]| = |S| = \lambda^k(D).\end{aligned}$$

Thus $\lambda^k(D) = \xi^k(D)$ and D is λ^k-optimal. ■

Lemma 5 (Lemma 4.1 in [10]). *Let D be a strong digraph with $|V(D)| \ge 6$ and $\delta(D) \ge 4$, and let S be a λ^3-cut of D. If D is not λ^3-optimal, then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.*

Proof of Theorem 1. Clearly, D is a strong digraph with $\delta(D) \ge 5$. By Lemma 3, D is λ^3-connected and $\lambda^3(D) \le \xi^3(D)$. Suppose, on the contrary, that D is not λ^3-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let S be a λ^3-cut of D. By Lemma 5, there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ contain a connected subdigraph with order 3.

Let $Y = \overline{X}$, and let $X_i = \{x \in X : |N^+(x) \cap Y| = i\}$, $Y_i = \{y \in Y : |N^-(y) \cap X| = i\}$, $i = 0, 1, 2$, and let $X_3 = \{x \in X : |N^+(x) \cap Y| \ge 3\}$, $Y_3 = \{y \in Y : |N^-(y) \cap X| \ge 3\}$.

Claim 1. $\min\{|X|, |Y|\} \ge 4$.

Proof. Suppose that $|X| = 3$. Then $\lambda^3(D) = |S| = |\partial^+(X)| \ge \xi(X) \ge \xi^3(D)$, contrary to the assumption. Suppose that $|Y| = 3$. Then $\lambda^3(D) = |S| = |\partial^-(Y)| \ge \xi(Y) \ge \xi^3(D)$, contrary to the assumption. Claim 1 follows. □

Claim 2. $X_0 = Y_0 = \emptyset$.

Proof. By symmetry, we only prove that $X_0 = \emptyset$ by contradiction. Suppose $X_0 \neq \emptyset$ and let $x \in X_0$. Then for any $\overline{x} \in Y$, $x\overline{x} \notin A(D)$ and we have that $5 \le |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X| + |N^+(x) \cap N^-(\overline{x}) \cap Y| \le |N^-(\overline{x}) \cap X| + |N^+(x) \cap Y| = |N^-(\overline{x}) \cap X|$. It implies that $|N^-(\overline{x}) \cap X| \ge 5$. Therefore $Y \subseteq Y_3$. So D is λ^3-optimal by Lemma 4, a contradiction to our assumption. □

Combining Claim 2 with Lemma 4, we have that $Y_1 \cup Y_2 \neq \emptyset$ and $X_1 \cup X_2 \neq \emptyset$. Otherwise we will obtain that D is λ^3-optimal, which is a contradiction. Next, we consider two cases.

Case 1. $X_1 \neq \emptyset$. Let $x' \in X_1$ and suppose $N^+(x') \cap Y = \{y'\}$. Then for any $y \in Y \backslash \{y'\}$, $x'y \notin A(D)$, so we have that $5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 1$. So $|N^-(y) \cap X| \geq 4$ and $Y \backslash \{y'\} \subseteq Y_3$. On the other hand, since $Y_1 \cup Y_2 \neq \emptyset$, so $y' \in Y_1 \cup Y_2$. Besides, $5 \leq \delta(D) \leq \delta^-(y') = |N^-(y')| = |N^-(y') \cap Y| + |N^-(y') \cap X| \leq |N^-(y') \cap Y| + 2$, thus $|N^-(y') \cap Y| \geq 3$. Let $y_1, y_2 \in N^-(y') \cap Y$, then $D[y', y_1, y_2]$ is connected and $|N^-(y) \cap X| \geq 4$ for any $y \in Y \backslash \{y', y_1, y_2\}$. By Lemma 4, we have that D is λ^3-optimal, a contradiction.

Case 2. $X_2 \neq \emptyset$. Let $x' \in X_2$ and suppose $N^+(x') \cap Y = \{y', y''\}$. Then for any $y \in Y \backslash \{y', y''\}$, $x'y \notin A(D)$, thus $5 \leq |N^+(x') \cap N^-(y)| = |N^+(x') \cap N^-(y) \cap X| + |N^+(x') \cap N^-(y) \cap Y| \leq |N^-(y) \cap X| + |N^+(x') \cap Y| = |N^-(y) \cap X| + 2$. So $|N^-(y) \cap X| \geq 3$ and $Y \backslash \{y', y''\} \subseteq Y_3$. On the other hand, since $Y_1 \cup Y_2 \neq \emptyset$, $y' \in Y_1 \cup Y_2$ or $y'' \in Y_1 \cup Y_2$. If $|Y_1 \cup Y_2| = 1$, then we can prove that D is λ^3-optimal by a proof similar to Case 1, which is a contradiction. If $Y_1 \cup Y_2 = \{y', y''\}$, then we consider two subcases.

Subcase 2.1. $y'y'' \in A(D)$ or $y''y' \in A(D)$. Since $y'' \in Y_1 \cup Y_2$ and $\delta(D) \geq 5$, then there exists $y_1 \in N^-(y'') \cap Y$ such that $y_1 \neq y'$. Therefore $D[y', y'', y_1]$ is connected and $|N^-(y) \cap X| \geq 3$ for any $y \in Y \backslash \{y', y'', y_1\}$. By Lemma 4, we have that D is λ^3-optimal, a contradiction.

Subcase 2.2. $y'y'' \notin A(D)$ and $y''y' \notin A(D)$. Since $y'y'' \notin A(D)$ and $y''y' \notin A(D)$, then $5 \leq |N^+(y') \cap N^-(y'')| = |N^+(y') \cap N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq |N^-(y'') \cap X| + |N^+(y') \cap N^-(y'') \cap Y| \leq 2 + |N^+(y') \cap N^-(y'') \cap Y|$. Therefore $|N^+(y') \cap N^-(y'') \cap Y| \geq 3$. Let $y_1 \in N^+(y') \cap N^-(y'') \cap Y$. Then $D[y', y'', y_1]$ is connected and $|N^-(y) \cap X| \geq 3$ for any $y \in Y \backslash \{y', y'', y_1\}$. By Lemma 4, we have that D is λ^3-optimal, a contradiction.

The proof is complete. ∎

From Theorem 1, we have following corollaries.

Corollary 6. *Let D be a digraph with $|V(D)| \geq 6$. If $d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$, then D is λ^3-optimal.*

Corollary 7 (Theorem 1.7 in [10])**.** *Let D be a digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \frac{|V(D)|+3}{2}$, then D is λ^3-optimal.*

Remark 8. To show the condition that "$|N^+(u) \cap N^-(v)| \geq 5$ for any $u, v \in V(D)$ with $uv \notin A(D)$ "in Theorem 1 is sharp, we give a class of digraphs. Let m, k be positive integers with $m \geq 3$, and let D be a digraph with $|V(D)| = 4m + 4$.

Define the vertex set of D as $V(D) = B \cup C$, where $B = \{x_0, \ldots, x_m, w_0, \ldots, w_m\}$ and $C = \{y_0, \ldots, y_m, z_0, \ldots, z_m\}$. And define the arc set of D as $A(D) = A(D[B]) \cup A(D[C]) \cup M_1 \cup M_2 \cup M_3 \cup M_4$, where $A(D[B]) \cup A(D[C]) = \{uv :$ for any $u, v \in B$ or $C\}$, $M_1 = \{x_i y_{k(\mathrm{mod}\, m+1)} : 0 \leq i \leq m$ and $0 \leq k - i \leq 1\}$, $M_2 = \{w_i z_{k(\mathrm{mod}\, m+1)} : 0 \leq i \leq m$ and $0 \leq k - i \leq 2\}$, $M_3 = \{y_i x_{k(\mathrm{mod}\, m+1)} : 0 \leq i \leq m$ and $0 \leq k - i \leq 2\}$ and $M_4 = \{z_i w_{k(\mathrm{mod}\, m+1)} : 0 \leq i \leq m$ and $0 \leq k - i \leq 2\}$.

Clearly, D is strong and there exists $0 \leq i, j \leq m$ such that $|N^+(x_i) \cap N^-(y_j)| = 4$ and $x_i y_j \notin A(D)$. And $\partial^+(B)$ is a 3-restricted edge cut with $|\partial^+(B)| = (2+3) \cdot (m+1) = 5m+5$. On the other hand, $\xi^3(D) = \xi(\{x_l, x_p, x_q\}) = |\partial^+(\{x_l, x_p, x_q\})| = 3 \cdot (2m+3) - 6 = 6m+3$, where $0 \leq l, p, q \leq m$. So $\lambda^3(D) \leq |\partial^+(B)| = 5m+5 < 6m+3 = \xi^3(D)$ for $m \geq 3$. Thus D is not λ^3-optimal.

Besides, in D, there exists $0 \leq i, j \leq m$ such that $x_i y_j \notin A(D)$ and $d^+(x_i) + d^-(y_j) = 2 \cdot (2m+3) = |V(D)| + 2 < |V(D)| + 3$, and $\delta(D) = 2m+3 = \frac{|V(D)|}{2} + 1 < \frac{|V(D)|+3}{2}$. So this example also shows that the conditions that "$d^+(u) + d^-(v) \geq |V(D)| + 3$ for any $u, v \in V(D)$ with $uv \notin A(D)$" in Corollary 6 and "$\delta(D) \geq \frac{|V(D)|+3}{2}$" in Corollary 7 are sharp.

3. Proof of Theorem 2

We first introduce several useful lemmas.

Lemma 9 (Lemma 2.1 in [10])**.** *Let D be a strong digraph and X_1, Y_1 disjoint subsets of $V(D)$. If $D[X_1]$ contains a connected subdigraph with order at least k and $D[Y_1]$ contains a strong subdigraph with order at least k, then D is λ^k-connected and each arc set in $\{\partial^-(Y_1), \partial^+(Y_1)\} \cup \Omega(X_1)$ is a k-restricted arc cut of D.*

Lemma 10. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ^3-connected and $\lambda^3(D) \leq \xi^3(D)$.*

Proof. By symmetry, we only consider the case that $\delta^-(D) \geq 3$. Let X' be a subset of $V(D)$ with $|X'| = 3$ such that $D[X']$ is connected and $\xi^3(D) = \xi(X')$. Without loss of generality, assume that $|X' \cap X| = 1$ and $|X' \cap Y| = 2$. Let $X' \cap X = \{x\}$ and $X' \cap Y = \{y, z\}$. Let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D \backslash X'$.

First, we claim that $V(D_1) \cap Y \neq \emptyset$. Otherwise, we have that $V(D_1) \subseteq X$ and $|V(D_1)| = 1$. Let $V(D_1) = \{u\}$. Then $N^-(u) \subseteq \{y, z\}$. So $3 \leq \delta^-(D) \leq d^-(u) = |N^-(u)| \leq |\{y, z\}| = 2$, a contradiction. Next, we aim to prove $|V(D_1)| \geq 3$.

Since $N^-(v) \subseteq \{x\} \cup (V(D_1) \cap X)$ for any $v \in V(D_1) \cap Y$, we have $3 \leq \delta^-(D) \leq d^-(v) = |N^-(v)| \leq |\{x\} \cup (V(D_1) \cap X)| = |\{x\}| + |V(D_1) \cap X| = 1 + |V(D_1) \cap X|$. Thus $|V(D_1) \cap X| \geq 2$ and $|V(D_1)| = |V(D_1) \cap X| + |V(D_1) \cap Y| \geq 2 + 1 = 3$. It follows that $|V(D_1)| \geq 3$. Since $D[X']$ is connected and

$D[X'] \subseteq D \backslash V(D_1)$, by Lemma 9, each arc set in $\Omega(X')$ is a 3-restricted arc cut of D. Therefore, D is λ^3-connected and $\lambda^3(D) \leq \xi(X') = \xi^3(D)$. ■

Lemma 11. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$, and let $S = \partial^+(X')$ be a λ^3-cut of D, where X' is a subset of $V(D)$. If $D[X']$ contains a connected subdigraph B with order 3 such that $|N^+(x) \cap \overline{X'}| \geq 2$ for any $x \in X' \backslash V(B)$ or $D[\overline{X'}]$ contains a connected subdigraph C with order 3 such that $|N^-(y) \cap X'| \geq 2$ for any $y \in \overline{X'} \backslash V(C)$, then D is λ^3-optimal.*

Proof. By Lemma 10, D is λ^3-connected and $\lambda^3(D) \leq \xi^3(D)$. By symmetry, we only prove the case that $D[X']$ contains a connected subdigraph B with order 3 such that $|N^+(x) \cap \overline{X'}| \geq 2$ for any $x \in X' \backslash V(B)$. The hypotheses imply that

$$\begin{aligned} \xi^3(D) &\leq |\partial^+(V(B))| = \big|[V(B), X' \backslash V(B)]\big| + \big|[V(B), \overline{X'}]\big| \\ &\leq 2|X' \backslash V(B)| + \big|[V(B), \overline{X'}]\big| \leq \sum_{x \in X' \backslash V(B)} \big|N^+(x) \cap \overline{X'}\big| + \big|[V(B), \overline{X'}]\big| \\ &= \big|[X' \backslash V(B), \overline{X'}]\big| + \big|[V(B), \overline{X'}]\big| = \big|[X', \overline{X'}]\big| = |S| = \lambda^3(D). \end{aligned}$$

Thus $\lambda^3(D) = \xi^3(D)$ and D is λ^3-optimal. ■

By a proof similar to that of Lemma 4.1 shown in [10], we can get the following Lemma 12.

Lemma 12. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$, and let S be a λ^3-cut of D. If D is not λ^3-optimal, then there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X'}]$ contain a connected subdigraph with order 3.*

Proof of Theorem 2. Since $|V(D)| \geq 6$, for any $u, v \in V(D)$ in the same partite, $|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3$. Therefore D is strong and $\delta(D) \geq 3$. By Lemma 10, D is λ^3-connected and $\lambda^3(D) \leq \xi^3(D)$. Suppose, on the contrary, that D is not λ^3-optimal, that is, $\lambda^3(D) < \xi^3(D)$. Let S be a λ^3-cut of D. Then by Lemma 12, there exists a subset of vertices $X' \subset V(D)$ such that $S = \partial^+(X')$ and both induced subdigraphs $D[X']$ and $D[\overline{X'}]$ contain a connected subdigraph with order 3.

Let $\overline{X'} = X''$, and let $X'_X = X' \cap X$, $X'_Y = X' \cap Y$, $X''_X = X'' \cap X$ and $X''_Y = X'' \cap Y$. And let $X'_{Xi} = \{x \in X'_X : |N^+(x) \cap X''_Y| = i\}$, $X'_{Yi} = \{y \in X'_Y : |N^+(y) \cap X''_X| = i\}$, $X''_{Xi} = \{x \in X''_X : |N^-(x) \cap X'_Y| = i\}$, $X''_{Yi} = \{y \in X''_Y : |N^-(y) \cap X'_X| = i\}$, $i = 0, 1$, and $X'_{X2} = \{x \in X'_X : |N^+(x) \cap X''_Y| \geq 2\}$, $X'_{Y2} = \{y \in X'_Y : |N^+(y) \cap X''_X| \geq 2\}$, $X''_{X2} = \{x \in X''_X : |N^-(x) \cap X'_Y| \geq 2\}$, $X''_{Y2} = \{y \in X''_Y : |N^-(y) \cap X'_X| \geq 2\}$.

Claim 1. $\min\{|X'_X|, |X'_Y|, |X''_X|, |X''_Y|\} \geq 2$.

Proof. If, on the contrary $|X'_X| = 1$, let $X'_X = \{v\}$. Then $|N(v) \cap X'_Y| \geq 2$ for $D[X']$ contains a connected subdigraph with order 3. Let $y_1, y_2 \in N(v) \cap X'_Y$. Then $D[v, y_1, y_2]$ is connected, and for any $x' \in X' \backslash \{v, y_1, y_2\}$, $N^+(x') \subseteq \{v\} \cup (N^+(x') \cap X'')$, we have $3 \leq \delta(D) \leq d^+(x') = |N^+(x')| \leq |\{v\}| + |N^+(x') \cap X''| = 1 + |N^+(x') \cap X''|$. Therefore $|N^+(x') \cap X''| \geq 2$. By Lemma 11, D is λ^3-optimal, a contradiction to our assumption. Thus $|X'_X| \geq 2$. Similarly, we can prove that $\min\left\{|X'_Y|, |X''_X|, |X''_Y|\right\} \geq 2$. □

Claim 2. *Either* $X'_{X0} = \emptyset$ *or* $X''_{X0} = \emptyset$ *and either* $X'_{Y0} = \emptyset$ *or* $X''_{Y0} = \emptyset$.

Proof. If $X'_{X0} \neq \emptyset$ and $X''_{X0} \neq \emptyset$, then there exists $x \in X'_{X0} \subseteq X$ and $\overline{x} \in X''_{X0} \subseteq X$ such that $|N^+(x) \cap N^-(\overline{x})| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. On the other hand, since $x \in X'_{X0}$ and $\overline{x} \in X''_{X0}$, $N^+(x) \subseteq X'_Y$ and $N^-(\overline{x}) \subseteq X''_Y$, which implies that $N^+(x) \cap N^-(\overline{x}) = \emptyset$, a contradiction. Thus either $X'_{X0} = \emptyset$ or $X''_{X0} = \emptyset$. Similarly, we can obtain that either $X'_{Y0} = \emptyset$ or $X''_{Y0} = \emptyset$. □

We consider the following two cases.

Case 1. $X'_{X0} = X'_{Y0} = \emptyset$ or $X''_{X0} = X''_{Y0} = \emptyset$. By symmetry, we only prove the case that $X'_{X0} = X'_{Y0} = \emptyset$.

Claim 1.1. *Either* $X'_{X1} = \emptyset$ *and* $X'_{Y1} \neq \emptyset$ *or* $X'_{X1} \neq \emptyset$ *and* $X'_{Y1} = \emptyset$.

Proof. Since D is not λ^3-optimal, by Lemma 11, we have that $X'_{X1} \cup X'_{Y1} \neq \emptyset$. Suppose $X'_{X1} \neq \emptyset$ and $X'_{Y1} \neq \emptyset$. Take $x_1 \in X'_{X1}$. Then for any $\overline{x} \in X''_X$, we have that $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(\overline{x})| = |N^+(x_1) \cap N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap N^-(\overline{x}) \cap X''| \leq |N^-(\overline{x}) \cap X'| + |N^+(x_1) \cap X''| = |N^-(\overline{x}) \cap X'| + 1$. It implies that $|N^-(\overline{x}) \cap X'| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2$. So $X''_X \subseteq X''_{X2}$. By a similar proof, we can also prove that $X''_Y \subseteq X''_{Y2}$. Therefore D is λ^3-optimal by Lemma 11, a contradiction. The proof of Claim 1.1 is complete. □

Without loss of generality, let $X'_{X1} \neq \emptyset$ and $X'_{Y1} = \emptyset$.

Case 1.1. $|X'_{X1}| = 1$. Let $x_1 \in X'_{X1}$. Then $3 \leq \delta(D) \leq d^+(x_1) = |N^+(x_1)| = |N^+(x_1) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap X'_Y| + 1$, therefore $|N^+(x_1) \cap X'_Y| \geq 2$. Let $y_1, y_2 \in N^+(x_1) \cap X'_Y$. Then $D[x_1, y_1, y_2]$ is connected, and for any $v \in X' \backslash \{x_1, y_1, y_2\}$, $|N^+(v) \cap X''| \geq 2$. By Lemma 11, D is λ^3-optimal, a contradiction.

Case 1.2. $|X'_{X1}| \geq 2$. Let $x_1, x_2 \in X'_{X1}$. Then $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x_1) \cap N^-(x_2)| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + |N^+(x_1) \cap N^-(x_2) \cap X''_Y| \leq |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + |N^+(x_1) \cap X''_Y| = |N^+(x_1) \cap N^-(x_2) \cap X'_Y| + 1$. So $|N^+(x_1) \cap$

$N^-(x_2) \cap X'_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil \geq 2$. Let $y_1 \in N^+(x_1) \cap N^-(x_2) \cap X'_Y$. Then

$$\begin{aligned}
\xi^3(D) &\leq \xi(\{x_1, x_2, y_1\}) \leq |\partial^+(\{x_1, x_2, y_1\})| \\
&= \big|[\{x_1\}, X'_Y \backslash \{y_1\}]\big| + \big|[\{x_1\}, X''_Y]\big| + \big|[\{x_2\}, X'_Y \backslash \{y_1\}]\big| + \big|[\{x_2\}, X''_Y]\big| \\
&+ \big|[\{y_1\}, X'_X \backslash \{x_1, x_2\}]\big| + \big|[\{y_1\}, X''_X]\big| \\
&\leq 2 \cdot \left(|X'_Y| - 1\right) + 2 + |X'_X| - 2 + \big|[\{y_1\}, X''_X]\big| \leq |S| = \lambda^3(D).
\end{aligned}$$

Thus D is λ^3-optimal, a contradiction.

Case 2. $X'_{X0} = X''_{Y0} = \emptyset$ or $X''_{X0} = X'_{Y0} = \emptyset$. By symmetry, we only prove the case that $X'_{X0} = X''_{Y0} = \emptyset$. Without loss of generality, we may assume that $X'_{Y0} \neq \emptyset$ and $X''_{X0} \neq \emptyset$. Otherwise, by Case 1, D is λ^3-optimal, a contradiction. On the other hand, since for any $u \in X'_{Y0}$, $N^+(u) \subseteq X'_X$, we have $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq \delta(D) \leq d^+(u) = |N^+(u)| \leq |X'_X|$. Therefore $|X'_X| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. Similarly, we can also prove that $|X''_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$. Thus

$$\begin{aligned}
|X'_Y| + |X''_X| &= |V(D)| - |X'_X| - |X''_Y| \\
&\leq |V(D)| - 2 \cdot \left(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \right) \leq \frac{|V(D)|}{2} - 2.
\end{aligned} \tag{1}$$

Claim 2.1. $|X'_X| \geq |X'_Y| + 1$ *or* $|X''_Y| \geq |X''_X| + 1$.

Proof. Otherwise, we have that $|X'_Y| + |X''_X| \geq |X'_X| + |X''_Y| \geq 2 \cdot \left(\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \right)$ $\geq \frac{|V(D)|}{2} + 2$, a contradiction to (1). □

Without loss of generality, we assume that $|X'_X| \geq |X'_Y| + 1$ in the following discussion.

Claim 2.2. $|N^+(x) \cap X''_Y| \geq 3$ *and* $|N^-(y) \cap X'_X| \geq 3$ *for any* $x \in X'_X$ *and* $y \in X''_Y$.

Proof. By symmetry, we only prove that for any $x \in X'_X$, $|N^+(x) \cap X''_Y| \geq 3$. Since $X''_{X0} \neq \emptyset$, for any $x \in X'_X$ and $\overline{x} \in X''_{X0}$, $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \leq |N^+(x) \cap N^-(\overline{x})| = |N^+(x) \cap N^-(\overline{x}) \cap X'_Y| + |N^+(x) \cap N^-(\overline{x}) \cap X''_Y| \leq |N^-(\overline{x}) \cap X'_Y| + |N^+(x) \cap X''_Y| = |N^+(x) \cap X''_Y|$, so $|N^+(x) \cap X''_Y| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \geq 3$. □

Claim 2.3. $X'_{Y2} = X''_{X2} = \emptyset$.

Proof. Here, we only prove that $X'_{Y2} = \emptyset$. The proof of the statement that $X''_{X2} = \emptyset$ is similar. Suppose, by a contradiction, there exists $y \in X'_{Y2}$. Let

$x_1, x_2 \in N^+(y) \cap X''_X$. Then

$$\begin{aligned}\xi^3(D) &\le \xi(\{x_1,x_2,y\}) \le \left|\partial^+(\{y\}) \cup \partial^-(\{x_1,x_2\})\right| \\ &= \left|\partial^+(\{y\})\right| + \left|\partial^-(\{x_1,x_2\})\right| - 2 = \left|\left[\{y\}, X'_X\right]\right| + \left|\left[\{y\}, X''_X\right]\right| \\ &+ \left|\left[X'_Y, \{x_1\}\right]\right| + \left|\left[X''_Y, \{x_1\}\right]\right| + \left|\left[X'_Y, \{x_2\}\right]\right| + \left|\left[X''_Y, \{x_2\}\right]\right| - 2 \\ &\le |X'_X| + \left|\left[\{y\}, X''_X\right]\right| + \left|\left[X'_Y, \{x_1\}\right]\right| + 2|X''_Y| + \left|\left[X'_Y, \{x_2\}\right]\right| - 2 \\ &\le 3\max\left\{|X'_X|, |X''_Y|\right\} + |[\{y\}, X''_X]| + \left|\left[X'_Y, \{x_1\}\right]\right| \\ &+ \left|\left[X'_Y, \{x_2\}\right]\right| - 2 \le |S| = \lambda^3(D).\end{aligned}$$

So D is λ^3-optimal, a contradiction. □

Claim 2.4. *For any $x \in X'_X$, $|N(X) \cap X'_Y| \ge 2$.*

Proof. Let $X'_Y = \{y_1, y_2, \ldots, y_p\}$ and let $S^* = \{s^* : s^* \in N^+(y_i) \cap N^-(y_j) \cap X'_X$, where $i, j \in \{1, \ldots, p\}$ and $i \ne j\}$. Then $D[S^* \cup X'_Y]$ is strong. Besides, by Claim 2.3, we have that for any $i, j \in \{1, \ldots, p\}$ and $i \ne j$, $y_i, y_j \in X'_{Y0} \cup X'_{Y1}$. Therefore $\left\lceil \frac{|V(D)|}{4} \right\rceil + 1 \le \left|N^+(y_i) \cap N^-(y_j)\right| = \left|N^+(y_i) \cap N^-(y_j) \cap X'_X\right| + \left|N^+(y_i) \cap N^-(y_j) \cap X''_X\right| \le \left|N^+(y_i) \cap N^-(y_j) \cap X'_X\right| + \left|N^+(y_i) \cap X''_X\right| \le \left|N^+(y_i) \cap N^-(y_j) \cap X'_X\right| + 1$. So $|N^+(y_i) \cap N^-(y_j) \cap X'_X| \ge \left\lceil \frac{|V(D)|}{4} \right\rceil \ge 2$. Similarly, we can prove that $\left|N^+(y_j) \cap N^-(y_i) \cap X'_X\right| \ge 2$. On the other hand, since $\left|X'_Y\right| \ge 2$, we have $\left|S^* \cup X'_Y\right| \ge 3$. For any $x \in S^*$, clearly, $\left|N(x) \cap X'_Y\right| \ge 2$. Next, we claim that for any $x \in X'_X \backslash S^*$, $\left|N^+(x) \cap X''_Y\right| \le \left|\left[X'_Y, \{x\}\right]\right|$.

Suppose there exists $x^* \in X'_X \backslash S^*$ such that $|N^+(x^*) \cap X''_Y| > |[X'_Y, \{x^*\}]|$. Since $D[S^* \cup X'_Y]$ is strong and $\left|S^* \cup X'_Y\right| \ge 3$, we have $X' \backslash \{x^*\}$ is a 3-restricted edge cut. Therefore $|\partial^+(X' \backslash \{x^*\})| = |S| - \left|N^+(x^*) \cap X''_Y\right| + \left|\left[X'_Y, \{x^*\}\right]\right| < |S|$, a contradiction to the minimality of S. Thus $\left|\left[X'_Y, \{x\}\right]\right| \ge \left|N^+(x) \cap X''_Y\right|$. By Claim 2.2, we have that $\left|\left[X'_Y, \{x\}\right]\right| \ge 3$. The proof of Claim 2.4 is complete. □

Let $x_1 \in X'_X$ such that $\left|N^+(x_1) \cap X''_Y\right| \le \left|N^+(u) \cap X''_Y\right|$ for any $u \in X'_X$, and let $y_1, y_2 \in N(x_1) \cap X'_Y$. Then

$$\begin{aligned}\xi^3(D) &\le |\partial^+(\{x_1,y_1,y_2\})| = \left|\left[\{x_1,y_1,y_2\}, X' \backslash \{x_1,y_1,y_2\}\right]\right| + \left|\left[\{x_1,y_1,y_2\}, X''\right]\right| \\ &\le 2\big(|X'_X| - 1\big) + \left|X'_Y\right| - 2 + \left|\left[\{x_1\}, X''_Y\right]\right| + \left|\left[\{y_1\}, X''_X\right]\right| + \left|\left[\{y_2\}, X''_X\right]\right| \\ &\le 3\left|X'_X\right| - 5 + \left|\left[\{x_1\}, X''_Y\right]\right| + \left|\left[\{y_1\}, X''_X\right]\right| \\ &+ \left|\left[\{y_2\}, X''_X\right]\right| \big(|X'_X| \ge |X'_Y| + 1\big) \le \left|\left[\{x_1\}, X''_Y\right]\right| \times \left|X'_X\right| \\ &+ \left|\left[\{y_1\}, X''_X\right]\right| + \left|\left[\{y_2\}, X''_X\right]\right| \le |S| = \lambda^3(D).\end{aligned}$$

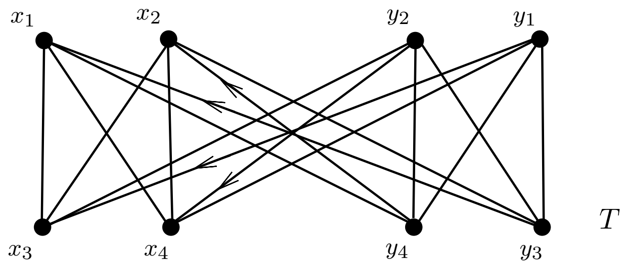
So D is λ_3-optimal, a contradiction.

The proof is complete. ■

From Theorem 2, we have following corollaries.

Corollary 13. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $\delta(D) \geq 3$. If for any $u, v \in V(D)$ in the same partite, $d^+(u) + d^-(v) \geq |V(D)| - 1$, then D is λ^3-optimal.*

Corollary 14. *Let $D = (X, Y, A(D))$ be a strong bipartite digraph with $|V(D)| \geq 6$. If $\delta(D) \geq \left\lfloor \frac{|V(D)|}{2} \right\rfloor$, then D is λ^3-optimal.*



(Unordered edges represent two arcs with the same end-vertices and opposite directions.)

Figure 1. The example from Remark 15.

Remark 15. To show that the condition "$|N^+(u) \cap N^-(v)| \geq \left\lceil \frac{|V(D)|}{4} \right\rceil + 1$ for any $u, v \in V(D)$ in the same partite" in Theorem 2 is sharp, we consider the digraph T shown in Figure 1. Clearly, $|V(D)| \geq 6$ and D is strong. There exists x_1, y_1 in the same partite such that $|N^+(x_1) \cap N^-(y_1)| = 2 < 3 = \left\lceil \frac{|V(T)|}{4} \right\rceil + 1$. Clearly, $\partial^+(\{x_1, x_2, x_3, x_4\})$ is a 3-restricted edge cut and $\xi^3(T) = |\partial^+(\{x_1, x_2, x_3\})| = 5$. Therefore, $\lambda^3(T) \leq |\partial^+(\{x_1, x_2, x_3, x_4\})| = 4 < 5 = \xi^3(T)$ and T is not λ^3-optimal.

Besides, since $d^+(x_3) + d^-(y_4) = 6 < 7 = |V(T)| - 1$ and $\delta(T) = 3 < 4 = \left\lfloor \frac{|V(D)|}{2} \right\rfloor$, this example also shows that the conditions "$d^+(u)+d^-(v) \geq |V(D)|-1$ for any $u, v \in V(D)$ in the same partite" in Corollary 13 and "$\delta(D) \geq \left\lfloor \frac{|V(D)|}{2} \right\rfloor$" in Corollary 14 are sharp.

Acknowledgements

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions. This work is supported by National Natural Science Foundation of China (Nos. 11531011).

References

[1] C. Balbuena and P. García-Vázquez, *On the restricted arc-connectivity of s-geodetic digraphs*, Acta Math. Sin. (Engl. Ser.) **26** (2010) 1865–1876. https://doi.org/10.1007/s10114-010-9313-y

[2] C. Balbuena, P. García-Vázquez, A. Hansberg and L.P. Montejano, *Restricted arc-connectivity of generalized p-cycles*, Discrete Appl. Math. **160** (2012) 1325–1332. https://doi.org/10.1016/j.dam.2012.02.006

[3] C. Balbuena, P. García-Vázquez, A. Hansberg and L.P. Montejano, *On the super-restricted arc-connectivity of s-geodetic digraphs*, Networks **61** (2013) 20–28. https://doi.org/10.1002/net.21462

[4] C. Balbuena, C. Cera, A. Diánez, P. García-Vázquez and X. Marcote, *Sufficient conditions for λ'-optimality of graphs with small conditional diameter*, Inform. Process. Lett. **95** (2005) 429–434. https://doi.org/10.1016/j.ipl.2005.05.006

[5] X. Chen, J. Liu and J. Meng, *The restricted arc connectivity of Cartesian product digraphs*, Inform. Process. Lett. **109** (2009) 1202–1205. https://doi.org/10.1016/j.ipl.2009.08.005

[6] X. Chen, J. Liu and J. Meng, *λ'-optimality of bipartite bigraphs*, Inform. Process. Lett. **112** (2012) 701–705. https://doi.org/10.1016/j.ipl.2012.05.003

[7] S. Grüter, Y. Guo and A. Holtkamp, *Restricted arc-connectivity of bipartite tournaments*, Discrete Appl. Math. **161** (2013) 2008–2013. https://doi.org/10.1016/j.dam.2013.01.028

[8] S. Grüter, Y. Guo and A. Holtkamp, *Restricted arc-connectivity in tournaments*, Discrete Appl. Math. **161** (2013) 1467–1471. https://doi.org/10.1016/j.dam.2013.02.010

[9] Z. Liu and Z. Zhang, *Restricted connectivity of total digraph*, Discrete Math. Algorithms Appl. **08** (2016) 1–9. https://doi.org/10.1142/S1793830916500221

[10] S. Lin, Y. Jin and C. Li, 3*-restricted arc connectivity of digraphs*, Discrete Math. **340** (2017) 2341–2348. https://doi.org/10.1016/j.disc.2017.05.004

[11] A. Hellwig and L. Volkmann, *Maximally edge-connected and vertex-connected graphs and digraphs: A survey*, Discrete Math. **308** (2008) 3265–3296. https://doi.org/10.1016/j.disc.2007.06.035

[12] L. Volkmann, *Restricted arc-connectivity of digraphs*, Inform. Process. Lett. **103** (2007) 234–239. https://doi.org/10.1016/j.ipl.2007.04.004

[13] S.Y. Wang and S.W. Lin, *λ'-optimal digraphs*, Inform. Process. Lett. **108** (2008) 386–389. https://doi.org/10.1016/j.ipl.2008.07.008

Received 8 October 2018
Revised 5 October 2019
Accepted 5 October 2019