# SPANNING TREES WITH DISJOINT DOMINATING AND 2-DOMINATING SETS 

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#### Abstract

In this paper, we provide a structural characterization of graphs having a spanning tree with disjoint dominating and 2-dominating sets.


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## 1. Introduction

Let $\mathcal{P}$ and $\mathcal{R}$ be two graph-theoretic properties. We say that a pair $(P, R)$ of disjoint subsets of vertices of a graph $G=\left(V_{G}, E_{G}\right)$ is a $\mathcal{P R}$-pair in $G$ if the set $P$ satisfies the property $\mathcal{P}$ in $G$ while the set $R$ satisfies the property $\mathcal{R}$ (in $G$ either). A graph that has a $\mathcal{P R}$-pair is called a $\mathcal{P R}$-graph. The concept of $\mathcal{P R}$-graphs naturally generalize several graph classes, where existence of a bipartition of the vertex set with some properties is required. There are considered properties not only those well-known as, for example, bipartite graphs (as ( $\mathcal{I}, \mathcal{I})$-graphs, where the property $\mathcal{I}$ is "the induced subgraph is an empty graph", in addition imposing $R=V_{G} \backslash P$ ) or split graphs [10] (as ( $\left.\mathcal{I}, \mathcal{K}\right)$-graphs, where the property $\mathcal{K}$ is "the induced subgraph is a clique", again restricted to $R=V_{G} \backslash P$ ), but also those very specific, for example, the class of trees [11] with the smallest possible size of two disjoint dominating sets (as $\left(\mathcal{I}_{2}, \mathcal{M}\right)$-trees, where the property $\mathcal{I}_{2}$ is "all vertices are of degree two and the induced subgraph is an empty graph" while the property $\mathcal{M}$ is "the induced subgraph is a perfect matching", again imposing $\left.R=V_{G} \backslash P\right)$. Herein, following a bunch of papers, see $[1,8,9,12,16,18,19]$ to mention just a few most recent, we focus on two graph domination properties: (standard) domination and 2-domination.

We say that a set $D \subseteq V_{G}$ is a dominating set of a graph $G=\left(V_{G}, E_{G}\right)$ if each vertex in the set $V_{G} \backslash D$ has a neighbor in $D$; this property, denoted by $\mathcal{D}$, plays a role of the property $\mathcal{P}$. Analogously, a subset $D_{2} \subseteq V_{G}$ of $G$ is a 2dominating set of $G$ if each vertex in the set $V_{G} \backslash D_{2}$ has at least two neighbors in $D_{2}$; this property, denoted by $\mathcal{D}_{2}$, serves as the property $\mathcal{R}$. The study on $\mathcal{D D}_{2}$-graphs has been initiated in 2013 by Henning and Rall [12]. In particular, they showed that every graph with minimum degree at least two is a $\mathcal{D D}_{2}$-graph and provided a constructive characterization of trees that have a $\mathcal{D D}_{2}$-pair. The complete structural characterization of all $\mathcal{D D}_{2}$-graphs was given only in 2019 by Miotk et al. [16]. In addition, the authors provided a characterization of minimal $\mathcal{D D}_{2}$-graphs and also proved several algorithmic and hardness results concerning spanning minimal $\mathcal{D D}_{2}$-graphs and $\mathcal{D D}_{2}$-supergraphs of non- $\mathcal{D D}_{2}$-graphs.

In this paper, taking into account an analogous problem studied in [18], we continue our study on $\mathcal{D D}_{2}$-graphs and characterize the class of these $\mathcal{D D}_{2}$-graphs that have a spanning tree being a $\mathcal{D D}_{2}$-graph itself; we shall refer to such a tree as a spanning $\mathcal{D D}_{2}$-tree. Observe that not every $\mathcal{D D}_{2}$-graph has a spanning $\mathcal{D D}_{2}$-tree - as an example can serve the subdivision graph of any corona graph having a cycle (see formal definitions below). Therefore, our problem perfectly fits into the bunch of problems related to the concept of spanning trees having the same or approximately the same properties as the input graph, i.e., center-preserving spanning trees [3], diameter-preserving spanning trees [4, 5, 15], degree-preserving spanning trees $[2,14]$, and $t$-spanners $[6,17]$.

## 2. Preliminaries

For usual notation and graph theory terminology we generally follow [7]. Specifically, for a vertex $v$ of a graph $G=\left(V_{G}, E_{G}\right)$, its neighborhood, denoted by $N_{G}(v)$, is the set of all vertices adjacent to $v$, and the cardinality of $N_{G}(v)$, denoted by $d_{G}(v)$, is called the degree of $v$. The closed neighborhood of $v$, denoted by $N_{G}[v]$, is the set $N_{G}(v) \cup\{v\}$. In general, for a subset $X \subseteq V_{G}$ of vertices, the neighborhood of $X$, denoted by $N_{G}(X)$, is defined to be $\bigcup_{v \in X} N_{G}(v)$, and the closed neighborhood of $X$, denoted by $N_{G}[X]$, is the set $N_{G}(X) \cup X$. The minimum degree of a vertex in $G$ is denoted by $\delta(G)$. A vertex of degree one is called a leaf, and the only neighbor of a leaf is called its support vertex (or simply, its support). If a support vertex has at least two leaves as neighbors, we call it a strong support, otherwise it is a weak support. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of $G$ is denoted by $L_{G}, S_{G}^{\prime}, S_{G}^{\prime \prime}$, and $S_{G}$, respectively.

Next, a graph $G$ is said to be a minimal $\mathcal{D D}_{2}$-graph if $G$ is a $\mathcal{D D}_{2}$-graph and


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a corona graph if every vertex of $H$ is a leaf or it is adjacent to a leaf of $H$. The subdivision graph $S(H)$ of a multigraph $H$ is the graph obtained from $H$ by inserting a new vertex onto each edge of $H$. For the purpose of characterizing the class of graphs possessing a spanning $\mathcal{D D}_{2}$-tree we shall employ the following two results.

Theorem 1 [16]. Let $G$ be a graph with no isolated vertex. Then $G$ is a $\mathcal{D D}_{2}$ graph if and only if $N_{G}(s) \backslash\left(L_{G} \cup S_{G}\right) \neq \emptyset$ for every weak support $s$ of $G$.

Theorem 2 [16]. A connected graph $G$ is a minimal $\mathcal{D D}_{2}$-graph if and only if $G$ is a star $K_{1, n}(n \geq 2)$, a cycle $C_{4}$, or $G$ is the subdivision graph of a corona graph, that is, $G=S(H)$ for some connected corona multigraph $H$.

Finally, let $\Pi=v_{1} v_{2}, \ldots, v_{k}$ be a path in a graph $G$, with $k \geq 2$. We say that $\Pi$ is a pendant $P_{k}$-path in $G$ if $d_{G}\left(v_{1}\right)=1$ and $d_{G}\left(v_{i}\right)=2$ for $2 \leq i \leq k$, and the vertex $v_{k+1} \in N_{G}\left(v_{k}\right) \backslash\left\{v_{k-1}\right\}$, called the $P_{k}$-support, is of degree at least three in $G$, see Figure 1 for an illustration. Similarly, we say that the path $\Pi$ forms a $P_{k}$-hammock, if $d_{G}\left(v_{i}\right)=2$ for $1 \leq i \leq k$, vertices $v_{0} \in N_{G}\left(v_{1}\right) \backslash\left\{v_{2}\right\}$ and $v_{k+1} \in N_{G}\left(v_{k}\right) \backslash\left\{v_{k-1}\right\}$, called $P_{k}$-junctions, are of degree at least three in $G$, and $v_{0} \neq v_{k+1}$; if under the same assumptions we have $v_{0}=v_{k+1}$, then we refer to $\Pi$ as a $P_{k}$-loop (with the vertex $v_{0}$ being the only its $P_{k}$-junction), again see Figure 1. The set of all $P_{k}$-supports in $G$ is denoted by $S_{G}^{k}$.


Figure 1. A pendant $P_{k}$-path, a $P_{k}$-hammock, and a $P_{k}$-loop, respectively.

## 3. Graphs with Spanning $\mathcal{D D}_{2}$-Tree

Let $G=\left(V_{G}, E_{G}\right)$ be a $\mathcal{D D}_{2}$-graph and let $F$ be a spanning subgraph of $G$. We say that $F$ is a spanning $\mathcal{D D}_{2}$-forest (of $G$ ) if each connected component of $F$ is an acyclic $\mathcal{D D}_{2}$-graph. As observed in [16], a connected $\mathcal{D D}_{2}$-graph $G$ has a spanning $\mathcal{D D}_{2}$-forest if and only if $G$ has a spanning $\mathcal{D D}_{2}$-tree. Therefore, from now on, we shall focus on spanning $\mathcal{D D}_{2}$-forests rather then explicitly on spanning $\mathcal{D D}_{2}$-trees.

Due to the fact that being a $\mathcal{D D}_{2}$-graph strictly depends on the neighborhoods of weak supports (see Theorem 1), the neighborhoods of these and only these vertices will naturally play a key role in our characterization. Namely, let $G$ be
a connected $\mathcal{D D}_{2}$-graph of order at least five. Taking into account Theorem 2, observe that none of the edges connecting any two supports or any two $P_{2^{-}}$ supports in $G$ belongs to any spanning minimal $\mathcal{D D}_{2}$-graph of $G$. Therefore, the graph $H_{G}$ resulting from $G$ by deleting all such edges (if they exist) is also $\mathcal{D D}_{2}$-graph, and moreover, we have the following observation.

Observation 3. A connected graph $G$ has a spanning $\mathcal{D D}_{2}$-forest if and only if the graph $H_{G}$ has a spanning $\mathcal{D D}_{2}$-forest.

Next, let us observe that if $v_{1} v_{2}$ is a $P_{2}$-hammock in $H_{G}$ such that its both $P_{2^{-}}$ junctions are supports in $H_{G}$, then the edge $v_{1} v_{2}$ does not belong to any spanning minimal $\mathcal{D D}_{2}$-graph of $H_{G}$. Therefore, the graph $B_{G}$, called the $\mathcal{D D}_{2}$-backbone of $G$, resulting from $H_{G}$ by deleting all such edges of $P_{2}$-hammocks (if they exist) is also $\mathcal{D D}_{2}$-graph, and we have the following observation.

Observation 4. The graph $H_{G}$ has a spanning $\mathcal{D D}_{2}$-forest if and only if the graph $B_{G}$ has a spanning $\mathcal{D D}_{2}$-forest. Consequently, a connected graph $G$ has a spanning $\mathcal{D D}_{2}$-forest if and only if its $\mathcal{D D}_{2}$-backbone has a spanning $\mathcal{D D}_{2}$-forest.

We point out that the $\mathcal{D D}_{2}$-backbone $B_{G}$ of $G$ is unique (of course, $B_{G}=G$ may hold) which follows from the fact that the above operation of deleting edges of $P_{2}$-hammocks never creates a (new) edge connecting any two supports or any two $P_{2}$-supports. Furthermore, notice that the number of weak supports in the backbone $B_{G}$ is at most the number of weak supports in the graph $G$ itself.

Taking into account Observations 3 and 4, in the following, for simplicity of the presentation, we shall assume that $G$ is a connected $\mathcal{D D}_{2}$-backbone, that is, $G$ is a $\mathcal{D D}_{2}$-graph being the $\mathcal{D D}_{2}$-backbone of itself (recall that the order of $G$ is at least five). For the same reason, starting from Lemma 5 establishing the necessity condition, we shall split our discussion into several pieces. But before we proceed further, let us define the property, which we shall refer to as the $\mathcal{X}$ property, that plays a crucial role in our characterization of $\mathcal{D D}_{2}$-graphs having a spanning $\mathcal{D D}_{2}$-forest.

Let $G=\left(V_{G}, E_{G}\right)$ be a connected graph of order at least five, and let $X$ be a subset of $V_{G} \backslash L_{G}$ such that $S_{G}^{\prime} \subseteq N_{G}(X)$. We say that $X$ (and so the graph $G$ itself) has the $\mathcal{X}$-property if for each non-empty independent set $I$ satisfying $N_{G}(I) \subseteq X$, we have $|I|<\left|N_{G}(I)\right|$. Notice that if there is no weak support in $G$, then the empty set has the $\mathcal{X}$-property in $G$ by the definition. Also, if $G$ has only one weak support, say $s$, then any 1-element set $X=\{x\}$, where $x \in N_{G}(s) \backslash L_{G}$, has the $\mathcal{X}$-property in $G$ (since in this case, the only independent set $I$ in $G$ satisfying $N_{G}(I) \subseteq X$ is the empty set).

Lemma 5 (Necessity). Let $G=\left(V_{G}, E_{G}\right)$ be a connected $\mathcal{D D}_{2}$-backbone of order at least five. If $G$ has a spanning $\mathcal{D D}_{2}$-forest, then $G$ has the $\mathcal{X}$-property.

Proof. Let $F$ be a minimal spanning $\mathcal{D D}_{2}$-forest of $G$ and suppose contrary to our claim that there is no subset $X \subseteq V_{G} \backslash L_{G}$ satisfying the $\mathcal{X}$-property in $G$; observe that $G$ must have at least two weak supports (by our discussion above).

Let $\left(D, D_{2}\right)$ be a $\mathcal{D D}_{2}$-pair in $F$. Since $F$ is a $\mathcal{D D}_{2}$-graph, each weak support $s \in S_{G}^{\prime}$ has a neighbor in $D_{2} \backslash L_{G}$ (by Theorem 1). Now, let $X \subseteq D_{2} \backslash L_{G}$ be a minimal set such that $S_{G}^{\prime} \subseteq N_{G}(X)$ and let $I$ be a non-empty independent set in $G$ such that $N_{G}(I) \subseteq X$ and $|I| \geq\left|N_{G}(I)\right| \geq 1$ (such a set $I$ exists by the assumption). Since $F$ is a minimal $\mathcal{D D}_{2}$-graph and $N_{G}(I) \subseteq X$, it follows from Theorem 2 that $d_{F}(v)=2$ for each $v \in I$. Consequently, since $|I| \geq\left|N_{G}(I)\right|$, we obtain (see the next paragraph) that the induced subgraph $F\left[I \cup N_{G}(I)\right]$ has a cycle, a contradiction.

To argue non-acyclity of $F\left[I \cup N_{G}(I)\right]$, consider the graph $H=\left(V_{H}, E_{H}\right)$, where $V_{H}=N_{G}(I)$ and $E_{H}=\left\{x y: x, y \in V_{H}\right.$ share a (degree two) neighbor in $F\}$. Observe that $H$ has $|I| \geq\left|V_{H}\right|$ edges, and so it has a cycle whose subdivision is a cycle in $F\left[I \cup N_{G}(I)\right]$.

The remaining part of the section is devoted to the sufficient condition related with the $\mathcal{X}$-property. In particular, we will show that any connected $\mathcal{D D}_{2}$-graph of order at least five with at most one weak support has a spanning $\mathcal{D D _ { 2 }}$-forest.

Lemma 6 (Sufficiency). Let $G$ be a connected $\mathcal{D D}_{2}$-backbone of order at least five. If $G$ satisfies the $\mathcal{X}$-property, then $G$ has a spanning $\mathcal{D D}_{2}$-forest.

Proof. Suppose contrarily that $G=\left(V_{G}, E_{G}\right)$ is a smallest (with respect to the order primarily and the size secondarily) connected $\mathcal{D D}_{2}$-backbone that has the $\mathcal{X}$-property but does not have a spanning $\mathcal{D D}_{2}$-forest. We split our discussion into four cases, depending on the minimum degree of $G$ and the number of its weak supports.

Case 1. $\delta(G) \geq 2$. Observe that $G$ is not a cycle, and so there exists $v_{0} \in V_{G}$ of degree at least three. Let $e=v_{0} v_{1}$ be an edge in $G$. If $d_{G}\left(v_{1}\right) \geq 3$, then the graph $G^{\prime}=G-e$ also has no support, and hence it is a $\mathcal{D D}_{2}$-graph by Theorem 1. Moreover, $G^{\prime}$ is $\mathcal{D D}_{2}$-backbone and so the minimality of $G$ implies that $G^{\prime}$ has a spanning $\mathcal{D D}_{2}$-forest $F$ (notice that $G^{\prime}$ may be disconnected, but this is not a problem at all). Since $F$ is also a spanning $\mathcal{D D}_{2}$-forest of $G$, we obtain a contradiction. If $d_{G}\left(v_{1}\right)=2$, then $G$ has a $P_{k+1}$-loop $L=v_{1} \cdots v_{k+1}$ or a $P_{k^{-}}$ hammock $H=v_{1} \cdots v_{k}$, for some $k \geq 1$. Consider again the graph $G^{\prime}=G-e$. Now, the vertex $v_{2}$ is the only weak support in $G^{\prime}$. It follows again from Theorem 1 that $G^{\prime}$ is a $\mathcal{D D _ { 2 } \text { -graph, moreover, it is the } \mathcal { D D } _ { 2 } \text { -backbone of itself. Since the }}$ set $X=\{x\}$, where $x \in N_{G}\left(v_{2}\right) \backslash\left\{v_{1}\right\}$, has the $\mathcal{X}$-property, it follows from the minimality of $G$ that $G^{\prime}$ has a spanning $\mathcal{D D}_{2}$-forest. But such a forest is also a spanning $\mathcal{D D}_{2}$-forest of $G$, a contradiction.

Case 2. $\delta(G)=1$ and $G$ has only strong supports. Notice that $G$ is not a star, since otherwise $G$ would be a $\mathcal{D D}_{2}$-spanning tree of itself, a contradiction. Consider the strong support $v_{0}$ and delete an edge $e=v_{0} v_{1} \in E_{G}$ from $G$, with $v_{1} \notin L_{G}$. It follows from Theorem 1 that the resulting graph $G^{\prime}=G-e$ is
 with at most one weak support satisfies the $\mathcal{X}$-property and the number of weak supports in the $\mathcal{D D}_{2}$-backbone of a $\mathcal{D D}_{2}$-graph is at most the number of weak supports in that graph, $G^{\prime}$ or some of its spanning subgraphs is a $\mathcal{D D}_{2}$-backbone that has the $\mathcal{X}$-property. But then, the minimality of $G$ implies a contradiction.

Case 3. $G$ has at least one weak support. Recall - our supposition is that $G$ is a smallest $\mathcal{D D}_{2}$-backbone that satisfies the $\mathcal{X}$-property but does not have a spanning $\mathcal{D D}_{2}$-forest. Now, for the purpose of a contradiction, we shall prove the following, more restricted, implication (and so our supposition involves now the relevant set $D$ as well).

If $D$ is a minimal set satisfying the $\mathcal{X}$-property in a given $\mathcal{D D}_{2}$-backbone $G$, then $G$ has a spanning $\mathcal{D D}_{2}$-forest $F$ such that for each $d \in D$, the edge ds $s_{d}$ belongs to $F$, where $s_{d} \in S_{G}^{\prime} \cap(N(d) \backslash N[D \backslash\{d\}])$, that is, $s_{d}$ is any weak support being an (external) private neighbor of $d$ in $G$.

We note in passing that if $S_{G}^{\prime}=\emptyset$, then the only minimal set satisfying the $\mathcal{X}$ property in $G$ is the empty set, and so in this case the above stronger implication is also satisfied (by Case 1 and Case 2, respectively).

Assume first that $G$ has exactly one weak support, that is $\left|S_{G}^{\prime}\right|=1$. Let $s \in S_{G}^{\prime}$ and consider a vertex $d_{s} \in N_{G}(s) \backslash L_{G}$ (such a vertex $d_{s}$ exists since $G$ is a $\mathcal{D D}_{2}$-graph and it is of order at least five). Notice that the set $\left\{d_{s}\right\}$ makes $G$ satisfying the $\mathcal{X}$-property. Now, one can observe, by arguments similar to those in Cases 1 and 2, that to avoid a contradiction, the only possible adjacent vertices of degree at least three in $G$ are $d_{s}$ and $s$; we omit the details. Furthermore, an analogous argument also implies a contradiction when $d_{G}\left(d_{s}\right)=2$ and $d_{s}$ has a neighbor $v \notin S_{G}^{\prime}$ such that $d_{G}(v) \geq 3$; we again omit the details. Next, we consider three subcases.

Subcase 3.1. Assume that all neighbors of $d_{s}$ that do not belong to $S_{G}^{\prime}$ are of degree two, and all neighbors of $s$ that do not belong to $D$ are also of degree two. If some of those neighbors of $d_{s}$, say $v_{1}, \ldots, v_{j}, j \geq 2$, share a neighbor $v \neq s$, then by deleting from $G$ all edges $d_{s} v_{i}$, for each $i \in\{1, \ldots, j\}$, we do not create any new weak support in the resulting graph $G^{\prime}$, and the vertex $v$ becomes - if not has already been - a strong support. Again, similarly as in Cases 1 and 2 (we omit the details), the minimality of $G$ implies the existence of a spanning $\mathcal{D D}_{2}$-forest $F=\left(V_{F}, E_{F}\right)$ of $G^{\prime}$ with $d s_{d} \in E_{F}$, and so of $G$ as well, a contradiction.

Otherwise, if the only such shared neighbor of $d_{s}$ is $v=s$, then we shall handle this subcase as Subcase 3.3, assuming that the following Subcase 3.2 has not occurred.

Subcase 3.2. Assume that none two of the neighbors of $d_{s}$ share a neighbor $v \neq s$ and $\operatorname{deg}_{G}(v)=2$ for each $v \in N_{G}\left(d_{s}\right) \backslash\{s\}$. Let $v_{1} \in N_{G}\left(d_{s}\right) \backslash S_{G}^{\prime}$ and let $v_{2} \in N_{G}\left(v_{1}\right) \backslash\left\{d_{s}\right\}$ (notice $v_{2} \notin L_{G}$ as $\left|S_{G}^{\prime}\right|=1$ ). If $d_{G}\left(v_{2}\right) \geq 3$ and $v_{2}$ is a strong support, then by deleting the edge $d_{s} v_{1}$ we do not create any new support in the resulting graph, and thus we can apply a reasoning similar to that in Case 1; we omit the formal proof.

Otherwise, if $d_{G}\left(v_{2}\right) \geq 3$ and $v_{2}$ is not a strong support, then deleting the edge $d_{s} v_{1}$ creates one new weak support $v_{2}$ in the graph $G^{\prime}=G-d_{s} v_{1}$, and observe that in that case $d_{G}(v)=2$ must hold for each $v \in N_{G}\left(v_{2}\right)$ (by so far handled cases), and so $v \neq s$. Consequently, although $G^{\prime}$ has now two weak supports, any vertex $v \in N_{G}\left(v_{2}\right) \backslash\left\{v_{1}\right\}$ shares at most one (degree two) neighbor with $d_{s}$. Therefore $G^{\prime}$ or some of its spanning graphs is the $\mathcal{D D}_{2}$-backbone of itself that satisfies the $\mathcal{X}$-property with $D^{\prime}=\left\{d_{s}, v\right\}$ if $d_{G}\left(d_{s}\right) \geq 3$, and with $D^{\prime}=\{v\}$ if $d_{G}\left(d_{s}\right)=2$, for any $v \in N_{G}\left(v_{2}\right) \backslash\left\{v_{1}\right\}$. In both cases, the minimality of $G$ implies a contradiction. Notice that if $d_{G}\left(d_{s}\right)=2$, then $d_{s}$ becomes a leaf in $G^{\prime}$ while $s$ becomes a strong support, and so the edge $d_{s} s$ always belongs to the resulting $\mathcal{D D}_{2}$-forest, regardless of the case.

Otherwise, if $d_{G}\left(v_{2}\right)=2$, then consider $v_{3} \in N_{G}\left(v_{2}\right) \backslash\left\{v_{1}\right\}$ (notice that $v_{3} \in$ $\left\{s, d_{s}\right\}$ may hold). If $d_{G}\left(v_{3}\right) \geq 3$, then deleting the edge $v_{2} v_{3}$ results in the graph $G^{\prime}$ such that $v_{2}$ is the only new leaf while $v_{1}$ is the only new weak support in $G^{\prime}$, and $v_{1}$ can be dominated only by $d_{s}$ in $G^{\prime}$. Since $G^{\prime}$ is a $\mathcal{D D}_{2}$-backbone that satisfies the $\mathcal{X}$-property with $D=\left\{d_{s}\right\}$ (and only such $D$ ), it has a spanning $\mathcal{D D}_{2}{ }^{-}$ forest $F=\left(V_{F}, E_{F}\right)$ with $d s_{d} \in E_{F}$ by the minimality of $G$, being a spanning $\mathcal{D D}_{2}$-forest of $G$ as well, a contradiction.

Otherwise, if also $d_{G}\left(v_{3}\right)=2$ (observe that $v_{3} \notin\left\{s, d_{s}\right\}$ ), then the graph $G^{\prime}=$ $G-d_{s} v_{1}$ is a $\mathcal{D D}_{2}$-graph having exactly two weak supports $s$ and $v_{2}$ if $d_{G}\left(d_{s}\right) \geq 3$, and respectively, having only one weak support $v_{2}$ if $d_{G}\left(d_{s}\right)=2$ (in this case, $s$ becomes a strong support in $G^{\prime}$ ). Since $v_{3}$ is the only non-leaf vertex adjacent to $v_{2}$ in $G^{\prime}$ and it is of degree two, the graph $G^{\prime}$ or some of its spanning graphs is a $\mathcal{D D}_{2}$-backbone that satisfies the $\mathcal{X}$-property with $D^{\prime}=\left\{d_{s}, v_{3}\right\}$ if $d_{G}\left(d_{s}\right) \geq 3$, and with $D^{\prime}=\left\{v_{3}\right\}$ if $d_{G}\left(d_{s}\right)=2$. And again, in both cases, the minimality of $G$ implies a contradiction. Notice that if $d_{G}\left(d_{s}\right)=2$ and so $d_{s} \in L_{G^{\prime}}$, then the edge $d_{s} s$ always belongs to the resulting $\mathcal{D D}_{2}$-forest, regardless of the case.

Subcase 3.3. Assume that $d_{G}\left(d_{s}\right) \geq 3$, all but vertex $s$ neighbors of $d_{s}$ are of degree two, and $N_{G}(v)=\left\{s, d_{s}\right\}$ for any non-leaf vertex $v \neq s, d_{s}$. Then deleting all edges $d_{s} v$, where $v \neq s$, results in a star graph with the centre at $s$, and so in a spanning $\mathcal{D D}_{2}$-forest of $G$ with $d s_{d}$ as one of its edges, a final contradiction for $\left|S_{G}^{\prime}\right|=1$.

Case 4. Assume that $G$ has at least two weak supports. Let $D \subset V_{G} \backslash L_{G}$ be a minimal subset such that $S_{G}^{\prime} \subseteq N_{G}(D)$, and let $I$ be a largest independent set such that $N_{G}(I) \subseteq D$ (recall that $G$ satisfies the $\mathcal{X}$-property, and so $\left|I^{\prime}\right|<\left|N_{G}\left(I^{\prime}\right)\right|$ holds for any non-empty subset $I^{\prime} \subseteq I$; also notice that $I$ may be the empty set). Let $G^{\prime}$ be the (simple) graph resulting from deleting all vertices in $I$ from $G$, then identifying all vertices in $D$ with one vertex $d_{D} \notin V_{G}$, identifying all weak supports in $S_{G}^{\prime}$ with one weak support $s_{D} \notin V_{G}$, and finally identifying all leaves incident to weak supports in $G$ with one leaf $l_{D} \notin V_{G}$ (being a $\mathcal{D D}_{2}$-backbone by $G$ implies that no loops can be created, and any multiedge, if appears, is reduced). Observe that $G^{\prime}$ is a connected $\mathcal{D D}_{2}$-backbone and $S_{G^{\prime}}^{\prime}=\left\{s_{D}\right\}$. Consider now the set $D^{\prime}=\left\{d_{D}\right\}$. It follows from Case 3 and the minimality of $G$ that $G^{\prime}$ has a spanning $\mathcal{D D}_{2}$-forest $F^{\prime}=\left(V_{F^{\prime}}, E_{F^{\prime}}\right)$ such that $d_{D} s_{D} \in E_{F^{\prime}}$ (and the edge $s_{D} l_{D} \in E_{F^{\prime}}$ as well). Next, let $F^{\prime \prime}=\left(V_{G} \backslash I, E_{F^{\prime \prime}}\right)$ be the subgraph of $G$, where $E_{F^{\prime \prime}}$ contains these and only these edges:

- all edges of $F^{\prime}$ with both endpoints in $V_{G} \backslash\left(I \cup D \cup S_{G}^{\prime} \cup\left(N_{G}\left(S_{G}^{\prime}\right) \cap L_{G}\right)\right)$;
- all edges $d s_{d}$ and $s_{d} l_{s_{d}}$, where $d \in D, s_{d} \in S_{G}^{\prime}$ is a private neighbor of $d$, and $l_{s_{d}}$ is the unique leaf adjacent to $s_{d}$ in $G$ (observe that such a vertex $d$ may have many private neighbors in $S_{G}^{\prime}$ - then, for each such weak support, we add the appropriate set of two edges);
- for each edge $e^{\prime}=x y \in E_{F^{\prime}}$ such that $x \in\left\{d_{D}, s_{D}\right\}$ and $y \notin\left\{d_{D}, s_{D}, L_{G}\right\}$ (and so $y \in V_{G} \backslash\left(I \cup D \cup S_{G}^{\prime} \cup\left(N_{G}\left(S_{G}^{\prime}\right) \cap L_{G}\right)\right)$ ), we add to $E_{F^{\prime \prime}}$ the edge $e \in E_{G}$ that corresponds to $e^{\prime}$.
Observe that acyclicity of $F^{\prime}$ implies acyclicity of $F^{\prime \prime}$, and $F^{\prime \prime}$ has at least $|D|$ connected components each of which is a $\mathcal{D D}_{2}$-graph. Moreover, none of two vertices in $D$ belongs to the same component of $F^{\prime \prime}$ and, by the construction, for each $d \in D$, the edge $d s_{d}$ belongs to $F^{\prime \prime}$, where $s_{d} \in S_{G}^{\prime} \cap(N(d) \backslash N[D \backslash\{d\}])$.

Now, to reach the final contradiction, we have to handle vertices in $I$, if any. Namely, if $I=\emptyset$, then $F^{\prime \prime}$ is just a spanning $\mathcal{D D}_{2}$-forest of $G$, a contradiction. So assume $I \neq \emptyset$. First, we add all vertices in $I$ to $F^{\prime \prime}$. Next, let $B=\left(I \cup N_{G}(I), E_{B}\right)$ be the maximal bipartite subgraph of $G$ whose edges have one endpoint in $I$ and the other - in $N_{G}(I) \subseteq D$, respectively. Since $\left|I^{\prime}\right|<\left|N_{G}\left(I^{\prime}\right)\right|$ for any subset $I^{\prime} \subseteq I$ and $d_{B}(v) \geq 2$ for each $v \in I$, the graph $B$ has a spanning tree $T$ such that $d_{T}(v) \geq 2$ for each $v \in I\left[13\right.$, Theorem 1]. Consequently, by adding to $F^{\prime \prime}$ also all edges of $T$, we obtain a spanning $\mathcal{D D}_{2}$-forest $F$ of $G$. Clearly, $F$ is also a $\mathcal{D D}_{2}$-forest of $G$ such that for each $d \in D$, the edge $d s_{d}$ belongs to $F$, where $s_{d} \in S_{G}^{\prime} \cap\left(N_{G}(d) \backslash N_{G}[D \backslash\{d\}]\right)$, the final contradiction.

Summarizing, for $\mathcal{D D}_{2}$-graphs of order at least five, the necessity condition is established by Lemma 5, while the sufficiency condition is established by Lemma 6 . Therefore, supported by a simple enumeration of graphs of order at most four, we may conclude with the following theorem.

Theorem 7. Let $G \neq C_{4}$ be a connected $\mathcal{D D}_{2}$-graph.
(a) If $G$ has at most one weak support, then $G$ has a spanning $\mathcal{D D}_{2}$-tree.
(b) Otherwise, if $\left|S_{G}^{\prime}\right| \geq 2$, then $G$ has a spanning $\mathcal{D D}_{2}$-tree if and only if in the $\mathcal{D D}_{2}$-backbone $B_{G}$ of $G$, there exists a subset $X \subset V_{B_{G}} \backslash L_{B_{G}}$ such that $S_{B_{G}}^{\prime} \subseteq N_{B_{G}}(X)$ and for each non-empty independent set $I$ in $B_{G}$ satisfying $N_{B_{G}}(I) \subseteq X$, we have $|I|<\left|N_{B_{G}}(I)\right|$.

Notice that the above characterization of $\mathcal{D D _ { 2 }}$-graphs having spanning $\mathcal{D D}_{2^{-}}$ trees is non-practical from the algorithmic point of view. Therefore, we close our paper with the open question concerning the computational complexity status of the following problem.

## The spanning $\mathcal{D D}_{\mathbf{2}}$-tree problem.

For a given graph $G$, determine whether $G$ has a spanning $\mathcal{D D}_{2}$-tree.

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