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SPANNING TREES WITH DISJOINT DOMINATING AND 2-DOMINATING SETS

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Abstract

In this paper, we provide a structural characterization of graphs having a spanning tree with disjoint dominating and 2-dominating sets.

Keywords: domination, 2-domination, spanning tree.

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1. Introduction

Let $\mathcal{P}$ and $\mathcal{R}$ be two graph-theoretic properties. We say that a pair (P, R) of disjoint subsets of vertices of a graph $G = (V_G, E_G)$ is a $\mathcal{PR}$*-pair in* G if the set P satisfies the property $\mathcal{P}$ in G while the set R satisfies the property $\mathcal{R}$ (in G either). A graph that has a $\mathcal{PR}$-pair is called a $\mathcal{PR}$*-graph*. The concept of $\mathcal{PR}$-graphs naturally generalize several graph classes, where existence of a bipartition of the vertex set with some properties is required. There are considered properties not only those well-known as, for example, bipartite graphs (as $(\mathcal{I}, \mathcal{I})$-graphs, where the property $\mathcal{I}$ is "the induced subgraph is an empty graph", in addition imposing $R = V_G \backslash P$) or split graphs [10] (as $(\mathcal{I}, \mathcal{K})$-graphs, where the property $\mathcal{K}$ is "the induced subgraph is a clique", again restricted to $R = V_G \backslash P$), but also those very specific, for example, the class of trees [11] with the smallest possible size of two disjoint dominating sets (as $(\mathcal{I}_2, \mathcal{M})$-trees, where the property $\mathcal{I}_2$ is "all vertices are of degree two and the induced subgraph is an empty graph" while the property $\mathcal{M}$ is "the induced subgraph is a perfect matching", again imposing $R = V_G \backslash P$). Herein, following a bunch of papers, see [1, 8, 9, 12, 16, 18, 19] to mention just a few most recent, we focus on two graph domination properties: (standard) domination and 2-domination.

We say that a set $D \subseteq V_G$ is a *dominating set* of a graph $G = (V_G, E_G)$ if each vertex in the set $V_G \backslash D$ has a neighbor in D; this property, denoted by $\mathcal{D}$, plays a role of the property $\mathcal{P}$. Analogously, a subset $D_2 \subseteq V_G$ of G is a 2-*dominating set* of G if each vertex in the set $V_G \backslash D_2$ has at least two neighbors in D_2; this property, denoted by $\mathcal{D}_2$, serves as the property $\mathcal{R}$. The study on $\mathcal{DD}_2$-graphs has been initiated in 2013 by Henning and Rall [12]. In particular, they showed that every graph with minimum degree at least two is a $\mathcal{DD}_2$-graph and provided a constructive characterization of trees that have a $\mathcal{DD}_2$-pair. The complete structural characterization of all $\mathcal{DD}_2$-graphs was given only in 2019 by Miotk *et al.* [16]. In addition, the authors provided a characterization of minimal $\mathcal{DD}_2$-graphs and also proved several algorithmic and hardness results concerning spanning minimal $\mathcal{DD}_2$-graphs and $\mathcal{DD}_2$-supergraphs of non-$\mathcal{DD}_2$-graphs.

In this paper, taking into account an analogous problem studied in [18], we continue our study on $\mathcal{DD}_2$-graphs and characterize the class of these $\mathcal{DD}_2$-graphs that have a spanning tree being a $\mathcal{DD}_2$-graph itself; we shall refer to such a tree as a *spanning* $\mathcal{DD}_2$*-tree*. Observe that not every $\mathcal{DD}_2$-graph has a spanning $\mathcal{DD}_2$-tree — as an example can serve the subdivision graph of any corona graph having a cycle (see formal definitions below). Therefore, our problem perfectly fits into the bunch of problems related to the concept of spanning trees having the same or approximately the same properties as the input graph, i.e., center-preserving spanning trees [3], diameter-preserving spanning trees [4, 5, 15], degree-preserving spanning trees [2, 14], and t-spanners [6, 17].

2. Preliminaries

For usual notation and graph theory terminology we generally follow [7]. Specifically, for a vertex v of a graph $G = (V_G, E_G)$, its *neighborhood*, denoted by $N_G(v)$, is the set of all vertices adjacent to v, and the cardinality of $N_G(v)$, denoted by $d_G(v)$, is called the *degree* of v. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. In general, for a subset $X \subseteq V_G$ of vertices, the *neighborhood* of X, denoted by $N_G(X)$, is defined to be $\bigcup_{v \in X} N_G(v)$, and the *closed neighborhood* of X, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. The minimum degree of a vertex in G is denoted by $\delta(G)$. A vertex of degree one is called a *leaf*, and the only neighbor of a leaf is called its *support vertex* (or simply, its *support*). If a support vertex has at least two leaves as neighbors, we call it a *strong* support, otherwise it is a *weak* support. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of G is denoted by L_G, S'_G, S''_G, and S_G, respectively.

Next, a graph G is said to be a *minimal* $\mathcal{DD}_2$*-graph* if G is a $\mathcal{DD}_2$-graph and no proper spanning subgraph of G is a $\mathcal{DD}_2$-graph. A multigraph H is called a *corona graph* if every vertex of H is a leaf or it is adjacent to a leaf of H. The *subdivision graph* $S(H)$ of a multigraph H is the graph obtained from H by inserting a new vertex onto each edge of H. For the purpose of characterizing the class of graphs possessing a spanning $\mathcal{DD}_2$-tree we shall employ the following two results.

Theorem 1 [16]. *Let G be a graph with no isolated vertex. Then G is a $\mathcal{DD}_2$-graph if and only if $N_G(s)\backslash(L_G \cup S_G) \neq \emptyset$ for every weak support s of G.*

Theorem 2 [16]. *A connected graph G is a minimal $\mathcal{DD}_2$-graph if and only if G is a star $K_{1,n}$ $(n \geq 2)$, a cycle C_4, or G is the subdivision graph of a corona graph, that is, $G = S(H)$ for some connected corona multigraph H.*

Finally, let $\Pi = v_1v_2, \ldots, v_k$ be a path in a graph G, with $k \geq 2$. We say that Π is a *pendant P_k-path* in G if $d_G(v_1) = 1$ and $d_G(v_i) = 2$ for $2 \leq i \leq k$, and the vertex $v_{k+1} \in N_G(v_k)\backslash\{v_{k-1}\}$, called the *$P_k$-support*, is of degree at least three in G, see Figure 1 for an illustration. Similarly, we say that the path Π forms a *P_k-hammock*, if $d_G(v_i) = 2$ for $1 \leq i \leq k$, vertices $v_0 \in N_G(v_1)\backslash\{v_2\}$ and $v_{k+1} \in N_G(v_k)\backslash\{v_{k-1}\}$, called *$P_k$-junctions*, are of degree at least three in G, and $v_0 \neq v_{k+1}$; if under the same assumptions we have $v_0 = v_{k+1}$, then we refer to Π as a *P_k-loop* (with the vertex v_0 being the only its P_k-junction), again see Figure 1. The set of all P_k-supports in G is denoted by S_G^k.

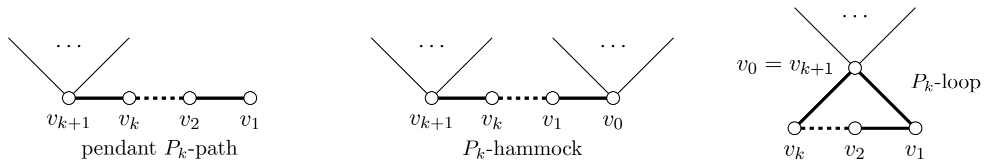

Figure 1. A pendant P_k-path, a P_k-hammock, and a P_k-loop, respectively.

3. Graphs with Spanning $\mathcal{DD}_2$-Tree

Let $G = (V_G, E_G)$ be a $\mathcal{DD}_2$-graph and let F be a spanning subgraph of G. We say that F is a *spanning $\mathcal{DD}_2$-forest* (of G) if each connected component of F is an acyclic $\mathcal{DD}_2$-graph. As observed in [16], a connected $\mathcal{DD}_2$-graph G has a spanning $\mathcal{DD}_2$-forest if and only if G has a spanning $\mathcal{DD}_2$-tree. Therefore, from now on, we shall focus on spanning $\mathcal{DD}_2$-forests rather then explicitly on spanning $\mathcal{DD}_2$-trees.

Due to the fact that being a $\mathcal{DD}_2$-graph strictly depends on the neighborhoods of weak supports (see Theorem 1), the neighborhoods of these and only these vertices will naturally play a key role in our characterization. Namely, let G be a connected $\mathcal{DD}_2$-graph of order at least five. Taking into account Theorem 2, observe that none of the edges connecting any two supports or any two P_2-supports in G belongs to any spanning minimal $\mathcal{DD}_2$-graph of G. Therefore, the graph H_G resulting from G by deleting all such edges (if they exist) is also $\mathcal{DD}_2$-graph, and moreover, we have the following observation.

Observation 3. *A connected graph G has a spanning $\mathcal{DD}_2$-forest if and only if the graph H_G has a spanning $\mathcal{DD}_2$-forest.*

Next, let us observe that if v_1v_2 is a P_2-hammock in H_G such that its both P_2-junctions are supports in H_G, then the edge v_1v_2 does not belong to any spanning minimal $\mathcal{DD}_2$-graph of H_G. Therefore, the graph B_G, called the *$\mathcal{DD}_2$-backbone of G*, resulting from H_G by deleting all such edges of P_2-hammocks (if they exist) is also $\mathcal{DD}_2$-graph, and we have the following observation.

Observation 4. *The graph H_G has a spanning $\mathcal{DD}_2$-forest if and only if the graph B_G has a spanning $\mathcal{DD}_2$-forest. Consequently, a connected graph G has a spanning $\mathcal{DD}_2$-forest if and only if its $\mathcal{DD}_2$-backbone has a spanning $\mathcal{DD}_2$-forest.*

We point out that the $\mathcal{DD}_2$-backbone B_G of G is unique (of course, $B_G = G$ may hold) which follows from the fact that the above operation of deleting edges of P_2-hammocks never creates a (new) edge connecting any two supports or any two P_2-supports. Furthermore, notice that the number of weak supports in the backbone B_G is at most the number of weak supports in the graph G itself.

Taking into account Observations 3 and 4, in the following, for simplicity of the presentation, we shall assume that G is a connected *$\mathcal{DD}_2$-backbone*, that is, G is a $\mathcal{DD}_2$-graph being the $\mathcal{DD}_2$-backbone of itself (recall that the order of G is at least five). For the same reason, starting from Lemma 5 establishing the necessity condition, we shall split our discussion into several pieces. But before we proceed further, let us define the property, which we shall refer to as the *$\mathcal{X}$-property*, that plays a crucial role in our characterization of $\mathcal{DD}_2$-graphs having a spanning $\mathcal{DD}_2$-forest.

Let $G = (V_G, E_G)$ be a connected graph of order at least five, and let X be a subset of $V_G \setminus L_G$ such that $S'_G \subseteq N_G(X)$. We say that X (and so the graph G itself) has the *$\mathcal{X}$-property* if for each non-empty independent set I satisfying $N_G(I) \subseteq X$, we have $|I| < |N_G(I)|$. Notice that if there is no weak support in G, then the empty set has the $\mathcal{X}$-property in G by the definition. Also, if G has only one weak support, say s, then any 1-element set $X = \{x\}$, where $x \in N_G(s) \setminus L_G$, has the $\mathcal{X}$-property in G (since in this case, the only independent set I in G satisfying $N_G(I) \subseteq X$ is the empty set).

Lemma 5 (Necessity)**.** *Let $G = (V_G, E_G)$ be a connected $\mathcal{DD}_2$-backbone of order at least five. If G has a spanning $\mathcal{DD}_2$-forest, then G has the $\mathcal{X}$-property.*

Proof. Let F be a minimal spanning $\mathcal{DD}_2$-forest of G and suppose contrary to our claim that there is no subset $X \subseteq V_G \backslash L_G$ satisfying the $\mathcal{X}$-property in G; observe that G must have at least two weak supports (by our discussion above).

Let (D, D_2) be a $\mathcal{DD}_2$-pair in F. Since F is a $\mathcal{DD}_2$-graph, each weak support $s \in S'_G$ has a neighbor in $D_2 \backslash L_G$ (by Theorem 1). Now, let $X \subseteq D_2 \backslash L_G$ be a minimal set such that $S'_G \subseteq N_G(X)$ and let I be a non-empty independent set in G such that $N_G(I) \subseteq X$ and $|I| \geq |N_G(I)| \geq 1$ (such a set I exists by the assumption). Since F is a minimal $\mathcal{DD}_2$-graph and $N_G(I) \subseteq X$, it follows from Theorem 2 that $d_F(v) = 2$ for each $v \in I$. Consequently, since $|I| \geq |N_G(I)|$, we obtain (see the next paragraph) that the induced subgraph $F[I \cup N_G(I)]$ has a cycle, a contradiction.

To argue non-acyclity of $F[I \cup N_G(I)]$, consider the graph $H = (V_H, E_H)$, where $V_H = N_G(I)$ and $E_H = \{xy : x, y \in V_H$ share a (degree two) neighbor in $F\}$. Observe that H has $|I| \geq |V_H|$ edges, and so it has a cycle whose subdivision is a cycle in $F[I \cup N_G(I)]$. ■

The remaining part of the section is devoted to the sufficient condition related with the $\mathcal{X}$-property. In particular, we will show that any connected $\mathcal{DD}_2$-graph of order at least five with at most one weak support has a spanning $\mathcal{DD}_2$-forest.

Lemma 6 (Sufficiency). *Let G be a connected $\mathcal{DD}_2$-backbone of order at least five. If G satisfies the $\mathcal{X}$-property, then G has a spanning $\mathcal{DD}_2$-forest.*

Proof. Suppose contrarily that $G = (V_G, E_G)$ is a smallest (with respect to the order primarily and the size secondarily) connected $\mathcal{DD}_2$-backbone that has the $\mathcal{X}$-property but does not have a spanning $\mathcal{DD}_2$-forest. We split our discussion into four cases, depending on the minimum degree of G and the number of its weak supports.

Case 1. $\delta(G) \geq 2$. Observe that G is not a cycle, and so there exists $v_0 \in V_G$ of degree at least three. Let $e = v_0v_1$ be an edge in G. If $d_G(v_1) \geq 3$, then the graph $G' = G - e$ also has no support, and hence it is a $\mathcal{DD}_2$-graph by Theorem 1. Moreover, G' is $\mathcal{DD}_2$-backbone and so the minimality of G implies that G' has a spanning $\mathcal{DD}_2$-forest F (notice that G' may be disconnected, but this is not a problem at all). Since F is also a spanning $\mathcal{DD}_2$-forest of G, we obtain a contradiction. If $d_G(v_1) = 2$, then G has a P_{k+1}-loop $L = v_1 \cdots v_{k+1}$ or a P_k-hammock $H = v_1 \cdots v_k$, for some $k \geq 1$. Consider again the graph $G' = G - e$. Now, the vertex v_2 is the only weak support in G'. It follows again from Theorem 1 that G' is a $\mathcal{DD}_2$-graph, moreover, it is the $\mathcal{DD}_2$-backbone of itself. Since the set $X = \{x\}$, where $x \in N_G(v_2) \backslash \{v_1\}$, has the $\mathcal{X}$-property, it follows from the minimality of G that G' has a spanning $\mathcal{DD}_2$-forest. But such a forest is also a spanning $\mathcal{DD}_2$-forest of G, a contradiction.

Case 2. $\delta(G) = 1$ and G has only strong supports. Notice that G is not a star, since otherwise G would be a $\mathcal{DD}_2$-spanning tree of itself, a contradiction. Consider the strong support v_0 and delete an edge $e = v_0v_1 \in E_G$ from G, with $v_1 \notin L_G$. It follows from Theorem 1 that the resulting graph $G' = G - e$ is a $\mathcal{DD}_2$-graph; notice that G' has at most one weak support. Since any graph with at most one weak support satisfies the $\mathcal{X}$-property and the number of weak supports in the $\mathcal{DD}_2$-backbone of a $\mathcal{DD}_2$-graph is at most the number of weak supports in that graph, G' or some of its spanning subgraphs is a $\mathcal{DD}_2$-backbone that has the $\mathcal{X}$-property. But then, the minimality of G implies a contradiction.

Case 3. G has at least one weak support. Recall — our supposition is that G is a smallest $\mathcal{DD}_2$-backbone that satisfies the $\mathcal{X}$-property but does not have a spanning $\mathcal{DD}_2$-forest. Now, for the purpose of a contradiction, we shall prove the following, more restricted, implication (and so our supposition involves now the relevant set D as well).

If D is a minimal set satisfying the $\mathcal{X}$-property in a given $\mathcal{DD}_2$-backbone G, then G has a spanning $\mathcal{DD}_2$-forest F such that for each $d \in D$, the edge ds_d belongs to F, where $s_d \in S'_G \cap (N(d)\backslash N[D\backslash\{d\}])$, that is, s_d is any weak support being an (*external*) *private neighbor of d in G.*

We note in passing that if $S'_G = \emptyset$, then the only minimal set satisfying the $\mathcal{X}$-property in G is the empty set, and so in this case the above stronger implication is also satisfied (by Case 1 and Case 2, respectively).

Assume first that G has exactly one weak support, that is $|S'_G| = 1$. Let $s \in S'_G$ and consider a vertex $d_s \in N_G(s)\backslash L_G$ (such a vertex d_s exists since G is a $\mathcal{DD}_2$-graph and it is of order at least five). Notice that the set $\{d_s\}$ makes G satisfying the $\mathcal{X}$-property. Now, one can observe, by arguments similar to those in Cases 1 and 2, that to avoid a contradiction, the only possible adjacent vertices of degree at least three in G are d_s and s; we omit the details. Furthermore, an analogous argument also implies a contradiction when $d_G(d_s) = 2$ and d_s has a neighbor $v \notin S'_G$ such that $d_G(v) \geq 3$; we again omit the details. Next, we consider three subcases.

Subcase 3.1. Assume that all neighbors of d_s that do not belong to S'_G are of degree two, and all neighbors of s that do not belong to D are also of degree two. If some of those neighbors of d_s, say $v_1, \ldots, v_j$, $j \geq 2$, share a neighbor $v \neq s$, then by deleting from G all edges d_sv_i, for each $i \in \{1, \ldots, j\}$, we do not create any new weak support in the resulting graph G', and the vertex v becomes — if not has already been — a strong support. Again, similarly as in Cases 1 and 2 (we omit the details), the minimality of G implies the existence of a spanning $\mathcal{DD}_2$-forest $F = (V_F, E_F)$ of G' with $ds_d \in E_F$, and so of G as well, a contradiction.

Otherwise, if the only such shared neighbor of d_s is $v = s$, then we shall handle this subcase as Subcase 3.3, assuming that the following Subcase 3.2 has not occurred.

Subcase 3.2. Assume that none two of the neighbors of d_s share a neighbor $v \neq s$ and $\deg_G(v) = 2$ for each $v \in N_G(d_s)\backslash\{s\}$. Let $v_1 \in N_G(d_s)\backslash S'_G$ and let $v_2 \in N_G(v_1)\backslash\{d_s\}$ (notice $v_2 \notin L_G$ as $|S'_G| = 1$). If $d_G(v_2) \geq 3$ and v_2 is a strong support, then by deleting the edge d_sv_1 we do not create any new support in the resulting graph, and thus we can apply a reasoning similar to that in Case 1; we omit the formal proof.

Otherwise, if $d_G(v_2) \geq 3$ and v_2 is not a strong support, then deleting the edge d_sv_1 creates one new weak support v_2 in the graph $G' = G - d_sv_1$, and observe that in that case $d_G(v) = 2$ must hold for each $v \in N_G(v_2)$ (by so far handled cases), and so $v \neq s$. Consequently, although G' has now two weak supports, any vertex $v \in N_G(v_2)\backslash\{v_1\}$ shares at most one (degree two) neighbor with d_s. Therefore G' or some of its spanning graphs is the $\mathcal{DD}_2$-backbone of itself that satisfies the $\mathcal{X}$-property with $D' = \{d_s, v\}$ if $d_G(d_s) \geq 3$, and with $D' = \{v\}$ if $d_G(d_s) = 2$, for any $v \in N_G(v_2)\backslash\{v_1\}$. In both cases, the minimality of G implies a contradiction. Notice that if $d_G(d_s) = 2$, then d_s becomes a leaf in G' while s becomes a strong support, and so the edge d_ss always belongs to the resulting $\mathcal{DD}_2$-forest, regardless of the case.

Otherwise, if $d_G(v_2) = 2$, then consider $v_3 \in N_G(v_2)\backslash\{v_1\}$ (notice that $v_3 \in \{s, d_s\}$ may hold). If $d_G(v_3) \geq 3$, then deleting the edge v_2v_3 results in the graph G' such that v_2 is the only new leaf while v_1 is the only new weak support in G', and v_1 can be dominated only by d_s in G'. Since G' is a $\mathcal{DD}_2$-backbone that satisfies the $\mathcal{X}$-property with $D = \{d_s\}$ (and only such D), it has a spanning $\mathcal{DD}_2$-forest $F = (V_F, E_F)$ with $ds_d \in E_F$ by the minimality of G, being a spanning $\mathcal{DD}_2$-forest of G as well, a contradiction.

Otherwise, if also $d_G(v_3) = 2$ (observe that $v_3 \notin \{s, d_s\}$), then the graph $G' = G - d_sv_1$ is a $\mathcal{DD}_2$-graph having exactly two weak supports s and v_2 if $d_G(d_s) \geq 3$, and respectively, having only one weak support v_2 if $d_G(d_s) = 2$ (in this case, s becomes a strong support in G'). Since v_3 is the only non-leaf vertex adjacent to v_2 in G' and it is of degree two, the graph G' or some of its spanning graphs is a $\mathcal{DD}_2$-backbone that satisfies the $\mathcal{X}$-property with $D' = \{d_s, v_3\}$ if $d_G(d_s) \geq 3$, and with $D' = \{v_3\}$ if $d_G(d_s) = 2$. And again, in both cases, the minimality of G implies a contradiction. Notice that if $d_G(d_s) = 2$ and so $d_s \in L_{G'}$, then the edge d_ss always belongs to the resulting $\mathcal{DD}_2$-forest, regardless of the case.

Subcase 3.3. Assume that $d_G(d_s) \geq 3$, all but vertex s neighbors of d_s are of degree two, and $N_G(v) = \{s, d_s\}$ for any non-leaf vertex $v \neq s, d_s$. Then deleting all edges d_sv, where $v \neq s$, results in a star graph with the centre at s, and so in a spanning $\mathcal{DD}_2$-forest of G with ds_d as one of its edges, a final contradiction for $|S'_G| = 1$.

Case 4. Assume that G has at least two weak supports. Let $D \subset V_G \backslash L_G$ be a minimal subset such that $S'_G \subseteq N_G(D)$, and let I be a largest independent set such that $N_G(I) \subseteq D$ (recall that G satisfies the $\mathcal{X}$-property, and so $|I'| < |N_G(I')|$ holds for any non-empty subset $I' \subseteq I$; also notice that I may be the empty set). Let G' be the (simple) graph resulting from deleting all vertices in I from G, then identifying all vertices in D with one vertex $d_D \notin V_G$, identifying all weak supports in S'_G with one weak support $s_D \notin V_G$, and finally identifying all leaves incident to weak supports in G with one leaf $l_D \notin V_G$ (being a $\mathcal{DD}_2$-backbone by G implies that no loops can be created, and any multiedge, if appears, is reduced). Observe that G' is a connected $\mathcal{DD}_2$-backbone and $S'_{G'} = \{s_D\}$. Consider now the set $D' = \{d_D\}$. It follows from Case 3 and the minimality of G that G' has a spanning $\mathcal{DD}_2$-forest $F' = (V_{F'}, E_{F'})$ such that $d_D s_D \in E_{F'}$ (and the edge $s_D l_D \in E_{F'}$ as well). Next, let $F'' = (V_G \backslash I, E_{F''})$ be the subgraph of G, where $E_{F''}$ contains these and only these edges:

- all edges of F' with both endpoints in $V_G \backslash (I \cup D \cup S'_G \cup (N_G(S'_G) \cap L_G))$;
- all edges ds_d and $s_d l_{s_d}$, where $d \in D$, $s_d \in S'_G$ is a private neighbor of d, and l_{s_d} is the unique leaf adjacent to s_d in G (observe that such a vertex d may have many private neighbors in S'_G — then, for each such weak support, we add the appropriate set of two edges);
- for each edge $e' = xy \in E_{F'}$ such that $x \in \{d_D, s_D\}$ and $y \notin \{d_D, s_D, L_G\}$ (and so $y \in V_G \backslash (I \cup D \cup S'_G \cup (N_G(S'_G) \cap L_G))$), we add to $E_{F''}$ the edge $e \in E_G$ that corresponds to e'.

Observe that acyclicity of F' implies acyclicity of F'', and F'' has at least $|D|$ connected components each of which is a $\mathcal{DD}_2$-graph. Moreover, none of two vertices in D belongs to the same component of F'' and, by the construction, for each $d \in D$, the edge ds_d belongs to F'', where $s_d \in S'_G \cap (N(d) \backslash N[D \backslash \{d\}])$.

Now, to reach the final contradiction, we have to handle vertices in I, if any. Namely, if $I = \emptyset$, then F'' is just a spanning $\mathcal{DD}_2$-forest of G, a contradiction. So assume $I \neq \emptyset$. First, we add all vertices in I to F''. Next, let $B = (I \cup N_G(I), E_B)$ be the maximal bipartite subgraph of G whose edges have one endpoint in I and the other — in $N_G(I) \subseteq D$, respectively. Since $|I'| < |N_G(I')|$ for any subset $I' \subseteq I$ and $d_B(v) \geq 2$ for each $v \in I$, the graph B has a spanning tree T such that $d_T(v) \geq 2$ for each $v \in I$ [13, Theorem 1]. Consequently, by adding to F'' also all edges of T, we obtain a spanning $\mathcal{DD}_2$-forest F of G. Clearly, F is also a $\mathcal{DD}_2$-forest of G such that for each $d \in D$, the edge ds_d belongs to F, where $s_d \in S'_G \cap (N_G(d) \backslash N_G[D \backslash \{d\}])$, the final contradiction. ■

Summarizing, for $\mathcal{DD}_2$-graphs of order at least five, the necessity condition is established by Lemma 5, while the sufficiency condition is established by Lemma 6. Therefore, supported by a simple enumeration of graphs of order at most four, we may conclude with the following theorem.

Theorem 7. *Let $G \neq C_4$ be a connected $\mathcal{DD}_2$-graph.*

(a) *If G has at most one weak support, then G has a spanning $\mathcal{DD}_2$-tree.*

(b) *Otherwise, if $|S'_G| \geq 2$, then G has a spanning $\mathcal{DD}_2$-tree if and only if in the $\mathcal{DD}_2$-backbone B_G of G, there exists a subset $X \subset V_{B_G} \backslash L_{B_G}$ such that $S'_{B_G} \subseteq N_{B_G}(X)$ and for each non-empty independent set I in B_G satisfying $N_{B_G}(I) \subseteq X$, we have $|I| < |N_{B_G}(I)|$.*

Notice that the above characterization of $\mathcal{DD}_2$-graphs having spanning $\mathcal{DD}_2$-trees is non-practical from the algorithmic point of view. Therefore, we close our paper with the open question concerning the computational complexity status of the following problem.

The spanning $\mathcal{DD}_2$-tree problem.

For a given graph G, determine whether G has a spanning $\mathcal{DD}_2$-tree.

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