

TOTAL DOMINATION IN GENERALIZED PRISMS AND A NEW DOMINATION INVARIANT

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Abstract

In this paper we complement recent studies on the total domination of prisms by considering generalized prisms, i.e., Cartesian products of an arbitrary graph and a complete graph. By introducing a new domination invariant on a graph G , called the k -rainbow total domination number and denoted by $\gamma_{krt}(G)$, it is shown that the problem of finding the total domination number of a generalized prism $G \square K_k$ is equivalent to an optimization problem of assigning subsets of $\{1, 2, \dots, k\}$ to vertices of G . Various properties of the new domination invariant are presented, including, inter alia, that $\gamma_{krt}(G) = n$ for a nontrivial graph G of order n as soon as $k \geq 2\Delta(G)$. To prove the mentioned result as well as the closed formulas for the k -rainbow total domination number of paths and cycles for every k , a new weight-redistribution method is introduced, which serves as an efficient tool for establishing a lower bound for a domination invariant.

Keywords: domination, k -rainbow total domination, total domination.

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1. INTRODUCTION AND PRELIMINARIES

Graphs considered in this paper are finite, simple and undirected. If G is a graph, then $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to some vertex in D . The *domination number* $\gamma(G)$ is the size of a smallest dominating set of G . If every vertex of G is adjacent to a vertex in D , then D is called a *total dominating set* of G . The *total domination number* $\gamma_t(G)$ is the size of a

smallest total dominating set of G . For graphs G and H , the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$. Vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

A conjecture made by V.G. Vizing in 1968, which asserts that the domination number of the Cartesian product of two graphs is at least as large as the product of their domination numbers, has been a motivating force for the study of domination in Cartesian products (see recent papers and a survey [5, 6, 7]). Inability of settling this conjecture led authors to pose different variations of the original problem.

A version of Vizing's conjecture for total domination, the most fundamental concept in domination theory besides classical domination, has been studied by Henning and Rall, [14]. They proved that the product of the total domination numbers of any nontrivial tree and any graph without isolated vertices is at most twice the total domination number of their Cartesian product. Their conjecture that the (sharp) upper bound of the product of the total domination numbers of arbitrary two graphs without isolated vertices is at most twice the total domination number of their Cartesian product, was proved by Ho, [16]. Choudhary *et al.* [11] extended this result to the n -product case. In the subsequent studies the total domination number of prisms, i.e., Cartesian products of an arbitrary graph and the complete graph on two vertices, was considered. Lu and Hou [18] characterized graphs H which satisfy $\gamma_t(H \square K_2) = \gamma_t(H)$ and $\gamma_t(C_n) \gamma_t(H) = 2\gamma_t(C_n \square H)$, respectively. Brešar *et al.* [4] extended their result by characterizing the pairs of graphs G and H for which $2\gamma_t(G \square H) = \gamma_t(G) \gamma_t(H)$, whenever $\gamma_t(H) = 2$. A recent result of Azarija *et al.* [3] that $\gamma_t(Q_{n+1}) = 2\gamma_t(Q_n)$ holds for all $n \geq 1$ follows from a more general result on bipartite prisms.

Theorem 1 [3]. *If G is a bipartite graph, then $\gamma_t(G \square K_2) = 2\gamma(G)$.*

Motivated by a conjecture from [3], Goddard and Henning [13] proved the following.

Theorem 2 [13]. *If G is a graph, then $\gamma_t(G \square K_2) \geq \frac{4}{3}\gamma(G)$, and this bound is tight.*

Another example of a well-studied domination invariant is the k -rainbow domination, introduced in [8]. Let $[k] = \{1, 2, \dots, k\}$. The elements of $[k]$ will be referred to as colors, and in the remainder of this paper we will assume the vertex set of the complete graph K_k is $[k]$. The *open neighborhood* (or just neighborhood for short) of a vertex v in G is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. Let G be a graph and let f be a function that assigns to each vertex a subset of integers chosen from the set $[k]$. The *weight*, $\|f\|$, of f is defined as $\|f\| = \sum_{v \in V(G)} |f(v)|$.

The function f is called a *k-rainbow dominating function* (*kRDF* for short) of G if for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ it is the case that

$$\bigcup_{u \in N_G(v)} f(u) = [k].$$

Given a graph G , the minimum weight of a *k-rainbow dominating function* is called the *k-rainbow domination number of G*, and denoted by $\gamma_{rk}(G)$. The authors of [8] observed that $\gamma_{rk}(G)$, for $k \geq 1$ and every graph G , equals the domination number of the generalized prism $G \square K_k$,

$$\gamma_{rk}(G) = \gamma(G \square K_k).$$

Similarly, the *k-rainbow independent domination number* of a graph G , $\gamma_{rik}(G)$, was introduced in [20] in such a way that it coincides with the ordinary independent domination number i of the generalized prism $G \square K_k$,

$$\gamma_{rik}(G) = i(G \square K_k).$$

In both cases the problem of computing an invariant on the generalized prism $G \square K_k$ was reduced to a problem of finding the value of the corresponding invariant on the factor G , which is sometimes easier to consider, see for example [2, 9, 10, 19].

As a tool in studying the total domination of $G \square K_k$ we introduce a new invariant. A function $f: V(G) \rightarrow 2^{[k]}$ is called a *k-rainbow total dominating function* (*kRTDF* for short) of G if the following two conditions hold.

1. If $v \in V(G)$ and $f(v) = \emptyset$, we have $\bigcup_{u \in N_G(v)} f(u) = [k]$.
2. For every $v \in V(G)$ such that $f(v) = \{i\}$ for some $i \in [k]$ there exists $u \in N_G(v)$ such that $i \in f(u)$.

In other words, a *k-rainbow total dominating function* f is a *k-rainbow dominating function* such that no vertex v with $f(v) = \{i\}$ is isolated in the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq \emptyset\}$, and moreover, the color i is contained in $f(u)$ where u is a neighbor of v in G . For a vertex $v \in V(G)$, $|f(v)|$ will be called the *weight of v*. As in the case of *k-rainbow domination*, the *weight of f* is defined as $\|f\| = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a *k-rainbow total dominating function* is called the *k-rainbow total domination number of G* and is denoted by $\gamma_{krt}(G)$. A *k-rainbow total dominating function* with minimum weight will be also called a *γ_{krt}-function*. Note that since every *kRTDF* f of a graph G is a *kRDF*, we have

$$\gamma_{rk}(G) \leq \gamma_{krt}(G).$$

We will show that for $k \geq 2$, $\gamma_{krt}(G)$ coincides with the total domination of $G \square K_k$, i.e.,

$$(1) \quad \gamma_{krt}(G) = \gamma_t(G \square K_k).$$

The above equality holds also in the case $k = 1$ if G does not contain isolated vertices, thus the new concept can be seen as a generalization of total domination.

It needs to be mentioned that recently two variations of the k -rainbow domination have been introduced in [1] and [17], both under the same name, the *total k -rainbow domination*, but from different perspectives, however, only our perspective relates to the original introduction of the k -rainbow domination. According to the equality (1), total k -rainbow domination would be justified choice of the name for our concept too, however, to distinguish it from already introduced ones, we use the term *k -rainbow total domination*. Ahangar *et al.* [1] defined a total k -rainbow dominating function f of a graph with no isolated vertex as a k -rainbow dominating function such that the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertices. On the other hand, Kumbargoudra *et al.* [17] for $k \geq 2$ defined a total k -rainbow dominating function f as a function $f: V(G) \rightarrow 2^{[k]} \setminus \emptyset$ such that for every $v \in V(G)$ we have $\bigcup_{u \in N_G[v]} f(u) = [k]$. In both papers the notation $\gamma_{trk}(G)$ was used for the minimum weight $\|f\| = \sum_{v \in V(G)} |f(v)|$ of a total k -rainbow dominating function on G . A simple example of a 4-cycle shows, that our concept differs from the total k -rainbow dominating function in the sense of Ahangar *et al.*, as we have $\gamma_{2rt}(C_4) = 4$ and $\gamma_{tr2}(C_4) = 3$. It also differs from the concept as defined by Kumbargoudra *et al.*, we have for instance $\gamma_{2rt}(C_5) = 4$ and $\gamma_{tr2}(C_5) = 5$.

We proceed as follows. In Section 2 various properties and bounds for the k -rainbow total domination number are presented. The main theorem of this section shows that for a nontrivial graph G of order n we have $\gamma_{krt}(G) = n$ as soon as $k \geq 2\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G . In Section 3 we complement a result by Azarija *et al.* [3], who considered the total domination of the Cartesian products $C_n \square K_2$ when $n = 6\ell + 1$ for $\ell \geq 1$, or n is an even number, by establishing a complete formula for $\gamma_{krt}(C_n) = \gamma_t(C_n \square K_k)$ for arbitrary k . In addition, exact values of the k -rainbow total domination number of paths for every k are given. Along the way it is demonstrated how a new weight-redistribution method can be used to find a lower bound for a domination invariant.

2. BASIC PROPERTIES AND BOUNDS

It follows from the definition that 1-rainbow total domination in a graph G without isolated vertices coincides with the total domination on G . We next prove

that for $k \geq 2$, the k -rainbow total domination in a graph G is equivalent to the total domination of the Cartesian product $G \square K_k$.

Observation 3. *If $k \geq 2$ is a positive integer and G a graph, then*

$$\gamma_{\text{krt}}(G) = \gamma_t(G \square K_k).$$

Proof. Let D be a smallest total dominating set of $G \square K_k$. We define a function $f : V(G) \rightarrow 2^{[k]}$ as follows: if $(v, i) \in V(G \square K_k)$ is contained in D , then $i \in f(v)$. The set $\{(v, h) \in V(G \square K_k) \mid h \in V(K_k)\}$ will be called a vK_k -layer. If none of the vertices in vK_k -layer belongs to D , we set $f(v) = \emptyset$. We will show that the function f is a k RTDF on G . Indeed, if $f(v) = \emptyset$, then every vertex (v, i) in vK_k -layer must have a neighbor in D , and since ${}^vK_k \cap D = \emptyset$, such a neighbor must lie within the G^i -layer. The structure of the Cartesian product thus implies that we find every color from $[k]$ in the open neighborhood of v . If $f(v) = \{i\}$, then in vK_k there is only one vertex from D , namely (v, i) . Since D is a total dominating set, there must exist $(u, i) \in D$, a neighbor of (v, i) . Hence $i \in f(u)$ for $u \in N_G(v)$. Since f is a k RTDF, we have

$$\gamma_t(G \square K_k) = |D| = \sum_{v \in V(G)} |f(v)| \geq \gamma_{\text{krt}}(G).$$

To prove the opposite inequality, $\gamma_{\text{krt}}(G) \geq \gamma_t(G \square K_k)$, let f be a $\gamma_{\text{krt}}(G)$ -function. We define a subset S of $V(G \square K_k)$ as follows: if for $v \in V(G)$ we have $f(v) = A$ and $A \neq \emptyset$, then (v, i) for every $i \in A$ belongs to S . To show that S is a total dominating set in $V(G \square K_k)$ we need to distinguish three possibilities. Every vertex in a vK_k -layer where $|f(v)| \geq 2$ is dominated within this layer, moreover, a vertex in ${}^vK_k \cap S$ has a neighbor which also belongs to ${}^vK_k \cap S$. If $|f(v)| = 1$, i.e., $f(v) = \{i\}$ for some $i \in [k]$, then again all vertices in vK_k -layer except v are dominated by the vertex $(v, i) \in S$. Also, since v has a neighbor u in G such that $i \in f(u)$, (v, i) is adjacent to $(u, i) \in S$. In the last case let $f(v) = \emptyset$. Then in the open neighborhood of v in G all colors appear, i.e., $\bigcup_{u \in N(v)} f(u) = [k]$. Thus for every $i \in [k]$, (v, i) is dominated by a neighbor (u, i) where $i \in f(u)$ (meaning that $(u, i) \in S$), and as we have seen before, (u, i) must have a neighbor in S . So S is a total dominating set in $V(G \square K_k)$, thus $|S| \geq \gamma_t(G \square K_k)$. To end the proof, observe that $|S|$ equals the weight of the $\gamma_{\text{krt}}(G)$ -function f . ■

For a k -rainbow total dominating function f of G we let $V_\emptyset = \{x \in V(G) \mid f(x) = \emptyset\}$. For a trivial graph K_1 we obviously have $\gamma_{\text{krt}}(K_1) = 2$ for every $k \geq 2$. If a graph G does not contain isolated vertices, then we clearly have $\gamma_{\text{krt}}(G) \leq n$ as $f : V(G) \rightarrow 2^{[k]}$ defined with $f(v) = \{1\}$ for every $v \in V(G)$ is a k RTDF. Moreover, if $k \geq n$ and f is a k RTDF with $V_\emptyset = \emptyset$, then clearly $\gamma_{\text{krt}}(G) \geq n$. But

the same holds also if there is a vertex v with $f(v) = \emptyset$, as every color from $[k]$ must appear in the open neighborhood of v . Thus $\gamma_{\text{krt}}(G) \geq k \geq n$, and we have the following observation.

Observation 4. *If G is a graph without isolated vertices and $k \geq n$, then $\gamma_{\text{krt}}(G) = n$.*

If G has n vertices and $n > k$, then clearly $\gamma_{\text{krt}}(G) \geq k$. Also, if f is a γ_{krt} -function of G and $S = \{v \in V(G) \mid f(v) \neq \emptyset\}$, then S is clearly a dominating set and we have $\gamma_{\text{krt}}(G) \geq |S| \geq \gamma(G)$. If D is a minimum dominating set in G , i.e., $\gamma(G) = |D|$, then $f : V(G) \rightarrow 2^{[k]}$ defined with $f(v) = [k]$ if $v \in D$, and $f(v) = \emptyset$ otherwise, is clearly a k RTDF. We can summarize the above observations in the following.

Observation 5. *If G is a graph of order n and $n > k > 1$, then*

$$\max\{k, \gamma(G)\} \leq \gamma_{\text{krt}}(G) \leq k\gamma(G).$$

Both bounds in the above observation are sharp, as can easily be seen in the case of a star graph S_n with $n > k$. In next proposition all graphs for which $\gamma_{\text{krt}}(G) = k$ are described.

Proposition 6. *Let k and n be positive integers such that $n > k > 1$. For a connected graph G of order n we have $\gamma_{\text{krt}}(G) = k$ if and only if G contains a spanning subgraph isomorphic to a complete bipartite graph $K_{s, n-s}$ where $s \leq \lfloor \frac{k}{2} \rfloor$.*

Proof. Suppose that $\gamma_{\text{krt}}(G) = k$ and let f be a γ_{krt} -function of G . Since $n > k$, there exists $x \in V_\emptyset$ such that $\bigcup_{v \in N_G(x)} f(v) = [k]$. Since $\gamma_{\text{krt}}(G) = k$ we have $f(u) \cap f(v) = \emptyset$ for every $u, v \in N_G(x)$ such that $f(u) \neq \emptyset$ and $f(v) \neq \emptyset$. Note that it must also hold that $|f(v)| \geq 2$ for every $v \in N_G(x)$ such that $f(v) \neq \emptyset$, and every vertex from V_\emptyset is adjacent to every vertex v from $N_G(x)$ such that $f(v) \neq \emptyset$. This implies that G contains a spanning complete bipartite subgraph $K_{s, n-s}$ where $s = |V(G) \setminus V_\emptyset| \leq \lfloor \frac{k}{2} \rfloor$ and $n - s = |V_\emptyset|$.

Now suppose that G contains a spanning complete bipartite subgraph $K_{s, n-s}$ with bipartite sets A and B , and $|A| = s \leq \lfloor \frac{k}{2} \rfloor$. Since $n > k$, we already know that $\gamma_{\text{krt}}(G) \geq k$. To prove the opposite inequality one can define a function $f : V(G) \rightarrow 2^{[k]}$ such that $f(u) = \emptyset$ for every $u \in B$, $|f(v)| \geq 2$ for every $v \in A$, and $\bigcup_{v \in A} f(v) = [k]$. As f with these properties is clearly a k RTDF of G we have $\gamma_{\text{krt}}(G) \leq k$. Thus $\gamma_{\text{krt}}(G) = k$. ■

Besides the obvious condition from Observation 4 we present another non-trivial and useful condition which assures that $\gamma_{\text{krt}}(G)$ equals the order of a graph.

Theorem 7. *Let G be a graph without isolated vertices of order n and $k \geq 2\Delta(G)$. Then $\gamma_{\text{krt}}(G) = n$.*

Proof. Let f be a γ_{krt} -function, i.e., $\|f\| = \sum_{v \in V(G)} |f(v)| = \gamma_{\text{krt}}(G)$. We already know that $\|f\| \leq n$. In order to show $\|f\| \geq n$ we will redistribute weights $|f(v)|$ of vertices between the neighboring vertices, more precisely, we will assign new weights to vertices in such a way that their sum will be equal to the sum of original weights. Let $n(v)$ denote the number of neighbors of v that belong to V_\emptyset . We redistribute weights as follows: each vertex v with $|f(v)| \geq 1$ and $n(v) \geq 1$ contributes $\frac{|f(v)|-1}{n(v)}$ to the weight of each vertex in $N(v) \cap V_\emptyset$, and keeps the rest of the original weight for itself. The new weight of a vertex $v \in V(G)$ will be denoted by $c(v)$. (Figure 1 shows an example where vertices x, y, z have original weights $|f(x)| = 0, |f(y)| = 3$ and $|f(z)| = 0$, and the above redistribution rule implies that their new weights are $c(x) = c(y) = c(z) = 1$.) We claim that after redistributing weights in this way $c(v) \geq 1$ for every $v \in V(G)$.

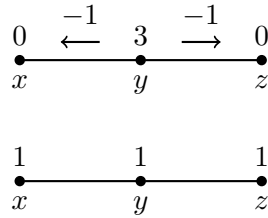


Figure 1. Redistribution of weights.

First, suppose $v \in V(G) \setminus V_\emptyset$. If $n(v) = 0$, then $c(v) = |f(v)| \geq 1$. If $n(v) \geq 1$, then $c(v) = |f(v)| - \frac{|f(v)|-1}{n(v)}n(v) = 1$.

If $v \in V_\emptyset$, and $|N_G(v)| - n(v) = \ell$, then v receives parts of weights from its neighbors $v_1, v_2, \dots, v_\ell \in V(G) \setminus V_\emptyset$. In estimating the value of $c(v)$ we will use the assumption that $k \geq 2\Delta(G)$ and obvious inequalities $\ell \leq \Delta(G)$ and $|f(v_1)| + |f(v_2)| + \dots + |f(v_\ell)| \geq k$ (the latter holds by the definition of a γ_{krt} -function). We derive

$$\begin{aligned} c(v) &= \frac{|f(v_1)|-1}{n(v_1)} + \frac{|f(v_2)|-1}{n(v_2)} + \dots + \frac{|f(v_\ell)|-1}{n(v_\ell)} \\ &\geq \frac{|f(v_1)|-1}{\Delta(G)} + \frac{|f(v_2)|-1}{\Delta(G)} + \dots + \frac{|f(v_\ell)|-1}{\Delta(G)} \\ &= \frac{|f(v_1)| + |f(v_2)| + \dots + |f(v_\ell)| - \ell}{\Delta(G)} \geq \frac{k - \ell}{\Delta(G)} \geq \frac{2\Delta(G) - \Delta(G)}{\Delta(G)} = 1. \end{aligned}$$

Thus $c(v) \geq 1$ for every $v \in V(G)$ and since $\sum_{v \in V(G)} c(v) = \sum_{v \in V(G)} |f(v)| = \|f\|$ we conclude that $\|f\| \geq n$. Therefore $\gamma_{\text{krt}}(G) = n$. ■

For $k = 3$ and $G = C_6$ we have $k = 2\Delta(G) - 1$, $\gamma_{\text{3rt}}(G) = 5$ (see Theorem 12 in the next section) and $n = 6$. This example shows that the lower bound for k

in terms of $\Delta(G)$ in the above theorem cannot be improved. The following result is a direct corollary of Theorem 7.

Corollary 8. *If $k \geq 4$, then $\gamma_{krt}(P_n) = \gamma_{krt}(C_n) = n$.*

3. SOME EXACT VALUES

As a corollary of Observation 3 and Theorems 1 and 2, respectively, we obtain the following.

Corollary 9. *If G is a bipartite graph, then $\gamma_{2rt}(G) = 2\gamma(G)$.*

Corollary 10. *If G is a graph, then $\gamma_{2rt}(G) \geq \frac{4}{3}\gamma(G)$, and this bound is tight.*

By Corollary 8 the remaining interesting cases for cycles and paths are for $k = 2$ and $k = 3$. It is well known and easy to see that $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$. Thus, if $k = 2$ and G is either a path or an even cycle, by Observation 3 and Corollary 9 we immediately obtain $\gamma_{2rt}(G) = 2\gamma(G) = 2\lceil \frac{n}{3} \rceil$. Azarija *et al.* [3] also considered cycles C_n where $n = 6\ell + 1$ and $\ell \geq 1$, and obtained that in this case $\gamma_{2rt}(C_n) = 2\lceil \frac{n}{3} \rceil - 1$. To settle the remaining cases, i.e., $n = 6\ell + 3$ and $n = 6\ell + 5$, recall the Theorem 2.11 from [15], which states that for a graph G of order n with no isolated vertex it holds $\gamma_t(G) \geq \frac{n}{\Delta(G)}$. Hence $\gamma_{2rt}(C_n) = \gamma_t(C_n \square K_2) \geq \frac{2n}{3}$. On the other hand, by Observation 5, we have $\gamma_{2rt}(C_n) \leq 2\lceil \frac{n}{3} \rceil$. We derive

$$\gamma_{2rt}(C_{6\ell+3}) \leq 2\left\lceil \frac{6\ell+3}{3} \right\rceil = 4\ell + 2 = \frac{2(6\ell+3)}{3} \leq \gamma_{2rt}(C_{6\ell+3}).$$

Similarly,

$$\gamma_{2rt}(C_{6\ell+5}) \leq 2\left\lceil \frac{6\ell+5}{3} \right\rceil = 4\ell + 4 \quad \text{and} \quad \frac{2(6\ell+5)}{3} \leq \gamma_{2rt}(C_{6\ell+5}).$$

However, since $\gamma_{2rt}(C_{6\ell+5})$ is an integer number, it must hold $\gamma_{2rt}(C_{6\ell+5}) \geq 4\ell + 4$. To summarize, we have the following complete formula for the 2-rainbow total domination number of cycles.

Theorem 11. *For $n \geq 3$ it holds*

$$\gamma_{2rt}(C_n) = \gamma_t(C_n \square K_2) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 2\ell, \ell \geq 2, \text{ or } n \in \{6\ell + 3, 6\ell + 5\}, \ell \geq 0, \\ 2\left\lceil \frac{n}{3} \right\rceil - 1, & \text{if } n = 6\ell + 1, \ell \geq 1. \end{cases}$$

The case $k = 3$ for cycles and paths is more interesting. We begin with the observation that for every positive integer k we have

$$(2) \quad \gamma_{krt}(C_n) \leq \gamma_{krt}(P_n).$$

Namely, if f is a γ_{krt} -function of P_n , then it is clearly also a k RTDF of C_n . Thus, by finding a lower bound for $\gamma_{3rt}(C_n)$, also a lower bound for $\gamma_{3rt}(P_n)$ will be established. By the aforementioned result from [15], $\gamma_{3rt}(C_n) = \gamma_t(C_n \square K_3) \geq \frac{3n}{4}$. However, this bound can be improved by using the weight-redistribution method.

Theorem 12. *For $n \geq 3$ it holds*

$$\gamma_{3rt}(C_n) = \left\lceil \frac{4n}{5} \right\rceil.$$

Proof. First we prove that $\gamma_{3rt}(C_n) \geq \lceil \frac{4n}{5} \rceil$. This holds for $n = 3$, as we have $\gamma_{3rt}(C_3) = 3$ by Observation 4. It is also straightforward to check that $\gamma_{3rt}(C_4) = 4$. Thus assume that $n \geq 5$ and let f be a γ_{3rt} -function of C_n , i.e., $\|f\| = \sum_{v \in V(C_n)} |f(v)| = \gamma_{3rt}(C_n)$. Similarly as in previous proofs, we will redistribute weights of vertices of C_n by defining the function $c : V(G) \rightarrow \mathbb{Q}$ according to the following:

- each vertex v with the initial weight 3, i.e., with the property $|f(v)| = 3$, contributes $\frac{4}{5}$ to the new weight of each vertex from V_\emptyset ,
- if $|f(v)| = 2$, then v contributes $\frac{3}{5}$ to each vertex from V_\emptyset ,
- if $|f(v)| = 1$, then v contributes $\frac{1}{5}$ to each vertex from V_\emptyset .

To prove that $\sum_{v \in V(C_n)} c(v) \geq \frac{4n}{5}$ we will show that $c(v) \geq \frac{4}{5}$ for every $v \in V(C_n)$. If $|f(v)| = 3$, then $c(v) \geq 3 - 2\frac{4}{5} = \frac{7}{5} > \frac{4}{5}$. If $|f(v)| = 2$, then $c(v) \geq 2 - 2\frac{3}{5} = \frac{4}{5}$. If $|f(v)| = 1$, then by the definition of f , v has at most one neighbor in V_\emptyset , thus $c(v) \geq 1 - \frac{1}{5} = \frac{4}{5}$. Lastly, let $f(v) = \emptyset$. Then by the definition of f , v is either adjacent to at least one neighbor u with $|f(u)| = 3$ (namely to a vertex u such that $f(u) = \{1, 2, 3\}$) or v has two neighbors u, w such that $|f(u)| = |f(w)| = 2$, or v has one neighbor u with $|f(u)| = 2$ and one neighbor w with $|f(w)| = 1$. In either of the three cases, we obtain that $c(v) \geq \frac{4}{5}$. Now we can derive

$$\gamma_{3rt}(C_n) = \sum_{v \in V(C_n)} |f(v)| = \sum_{v \in V(C_n)} c(v) \geq \frac{4n}{5},$$

which concludes the proof of the first inequality since $\gamma_{3rt}(C_n)$ is an integer number. The opposite inequality, $\gamma_{3rt}(C_n) \leq \lceil \frac{4n}{5} \rceil$, will follow from the following constructions of a 3RTDF f of $C_n = v_1 v_2 \cdots v_n v_1$.

If $n = 5k$, $n = 5k + 3$ or $n = 5k + 4$ we define the function $f : V(G) \rightarrow 2^{[k]}$ as follows:

- if $i \equiv 1 \pmod{5}$, then $f(v_i) = \{1, 2\}$,
- if $i \equiv 0 \pmod{5}$ or $i \equiv 2 \pmod{5}$, then $f(v_i) = \emptyset$,
- if $i \equiv 3 \pmod{5}$ or $i \equiv 4 \pmod{5}$, then $f(v_i) = \{3\}$.

(Small cases are presented in Figure 2, where for the sake of simplicity we write 12, 3 and 0 instead of $\{1, 2\}$, $\{3\}$ and \emptyset , respectively, and vertices are numbered in a clockwise manner.) It can easily be seen that f is a 3RTDF, thus $\gamma_{3\text{rt}}(C_n) \leq \|f\|$. If $n = 5k$, we obtain $\|f\| = 4k = \frac{4n}{5}$, if $n = 5k + 3$, we have $\|f\| = 4k + 3 = \frac{4n}{5} + \frac{3}{5}$, and if $n = 5k + 4$, we obtain $\|f\| = 4k + 4 = \frac{4n}{5} + \frac{4}{5}$. So in every case it holds $\|f\| \leq \lceil \frac{4n}{5} \rceil$.

For the cases $n = 5k + 1$ and $n = 5k + 2$ we slightly change the above definition of f : if $n = 5k + 1$ we set $f(v_{5k}) = \{3\}$ and $f(v_{5k+1}) = \emptyset$, and if $n = 5k + 2$, we set $f(v_{5k}) = f(v_{5k+1}) = \{3\}$ and $f(v_{5k+2}) = \emptyset$. For every other i , $f(v_i)$ is defined as above. In both cases f is a 3RTDF. If $n = 5k + 1$ we derive $\|f\| = 4k + 1 = \frac{4n}{5} + \frac{1}{5} = \lceil \frac{4n}{5} \rceil$. Similarly, for $n = 5k + 2$, we obtain $\|f\| = 4k + 2 = \frac{4n}{5} + \frac{2}{5} = \lceil \frac{4n}{5} \rceil$, which completes the proof. ■

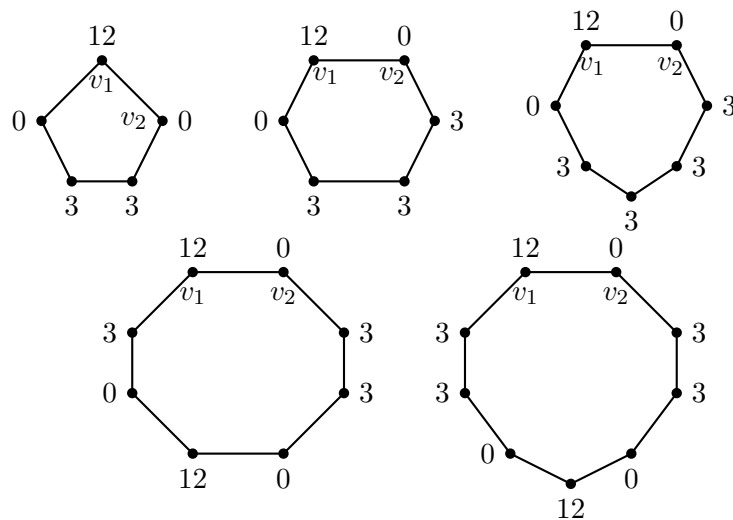


Figure 2. $\gamma_{3\text{rt}}$ -functions for small cycles.

For a trivial path it clearly holds $\gamma_{3\text{rt}}(P_1) = 2$, and for $n \geq 2$ we have the following.

Theorem 13. For $n \geq 2$ it holds

$$\gamma_{3\text{rt}}(P_n) = \begin{cases} \lceil \frac{4n}{5} \rceil, & \text{if } n \in \{5k + 2, 5k + 3, 5k + 4\}, \\ \lceil \frac{4n}{5} \rceil + 1, & \text{if } n \in \{5k, 5k + 1\}. \end{cases}$$

Proof. For $n = 2$ or $n = 3$, theorem holds by Observation 4. It is also straightforward to check that $\gamma_{3\text{rt}}(P_4) = 4$. Thus assume $n \geq 5$. It follows from Theorem 12 and inequality (2) that $\gamma_{3\text{rt}}(P_n) \geq \lceil \frac{4n}{5} \rceil$. We distinguish the following cases.

If n is congruent to 2 or 4 modulo 5, let $f : V(G) \rightarrow 2^{[k]}$ be the function defined as follows:

- if $i \equiv 1 \pmod{5}$ or $i \equiv 2 \pmod{5}$, then $f(v_i) = \{3\}$,
- if $i \equiv 0 \pmod{5}$ or $i \equiv 3 \pmod{5}$, then $f(v_i) = \emptyset$,
- if $i \equiv 4 \pmod{5}$, then $f(v_i) = \{1, 2\}$.

(Small cases are depicted in Figure 3.) Observe that f is a 3RTDF in both cases. For $n = 5k + 4$ we have $\|f\| = 4k + 4 = \frac{4n}{5} + \frac{4}{5}$, and for $n = 5k + 2$ it holds $\|f\| = 4k + 2 = \frac{4n}{5} + \frac{2}{5}$.

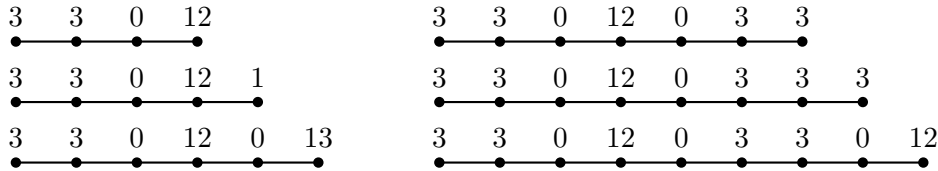


Figure 3. $\gamma_{3\text{rt}}$ -functions for small paths.

If $n = 5k + 3$, we define the function f as above with one exception: we set $f(v_n) = \{3\}$. Then f is a 3RTDF and $\|f\| = 4k + 3 = \frac{4n}{5} + \frac{3}{5}$. Thus in all of the above cases we have $\gamma_{3\text{rt}}(P_n) \leq \lceil \frac{4n}{5} \rceil$.

The upper bound in the case $n = 5k$ will be proved by defining f as in the first case with the exception $f(v_n) = \{1\}$. It follows from the construction of f that it is a 3RTDF and $\|f\| = 4k + 1$. Thus $\gamma_{3\text{rt}}(P_{5k}) \leq 4k + 1$. To prove the opposite inequality let g be a $\gamma_{3\text{rt}}$ -function of P_{5k} , i.e., $\gamma_{3\text{rt}}(P_{5k}) = \|g\|$. We will redistribute weights of g by the same rules as presented in the proof of Proposition 12. From this proof we also know that the new weight $c(v)$ is at least $\frac{4}{5}$ for every $v \in V(P_n)$. Moreover, if $g(v_1) = \emptyset$, then $g(v_2) = \{1, 2, 3\}$, thus $c(v_2) \geq \frac{7}{5} > \frac{4}{5}$. We derive that $\|g\| = \sum_{v \in V(P_n)} |g(v)| = \sum_{v \in V(P_n)} c(v) > n \cdot \frac{4}{5} = 4k$, i.e., $\|g\| \geq 4k + 1$. If $|g(v_1)| = 1$, then it must hold $|g(v_2)| \geq 1$, hence $c(v_1) = 1$ and thus $\sum_{v \in V(P_n)} c(v) > n \cdot \frac{4}{5}$. If $|g(v_1)| = 2$, we have $c(v_1) \geq 2 - \frac{3}{5} = \frac{7}{5}$ and if $|g(v_1)| = 3$, we have $c(v_1) \geq 3 - \frac{4}{5} = \frac{11}{5}$. Thus in every case we have $\|g\| = \sum_{v \in V(P_n)} c(v) \geq 4k + 1 = \frac{4n}{5} + 1$.

We deal with the last case $n = 5k + 1$ similarly. For the upper bound $\gamma_{3\text{rt}}(P_{5k+1}) \leq 4k + 2$ we construct a function f as in the first case, with the only exception $f(v_n) = \{1, 3\}$. As f is a 3RTDF we have $\gamma_{3\text{rt}}(P_{5k+1}) \leq \|f\| = 4k + 2$. To prove the opposite inequality, we let g be a $\gamma_{3\text{rt}}$ -function of P_{5k+1} and consider the following cases. If $g(v_1) = \emptyset$, then $c(v_1) = \frac{4}{5}$ and $c(v_2) \geq \frac{7}{5}$, thus $c(v_1) + c(v_2) \geq \frac{11}{5}$. If $|g(v_1)| = 1$, then $|g(v_2)| \geq 1$, hence $c(v_1) + c(v_2) \geq 1 + \frac{4}{5} = \frac{9}{5}$. If $|g(v_1)| = 2$, then $c(v_1) = 2 - \frac{3}{5} = \frac{7}{5}$ and $c(v_1) + c(v_2) \geq \frac{11}{5}$. In the last case, when $|g(v_1)| = 3$, we have $c(v_1) \geq \frac{11}{5}$. Thus in every case we have

$c(v_1) + c(v_2) \geq \frac{9}{5}$, and by symmetry we also have $c(v_{n-1}) + c(v_n) \geq \frac{9}{5}$. This implies $\|g\| = \sum_{v \in V(P_n)} c(v) \geq 2 \cdot \frac{9}{5} + (n-4) \frac{4}{5} = \frac{4n}{5} + \frac{2}{5} = 4k + \frac{6}{5} > 4k + 1$, i.e., $\|g\| \geq 4k + 2$. Since $4k + 2 = \frac{4n}{5} + 1 + \frac{1}{5} = \lceil \frac{4n}{5} \rceil + 1$, and the proof is complete. ■

4. CONCLUDING REMARKS

In this paper we have introduced a new domination invariant which serves as a tool for easier study of the total domination of generalized prisms, and presented the weight-redistribution technique which could be an efficient way of establishing lower bounds also for other domination invariants. The newly introduced concept could be interesting on its own. One possible direction for further studies may be motivated by the question of characterizing graphs which attain bounds in Observation 5, i.e., graphs for which it holds $\gamma_{krt}(G) = \gamma(G)$ and $\gamma_{krt}(G) = k\gamma(G)$, respectively, for given k . The characterization of graphs satisfying $\gamma_{krt}(G) = k\gamma(G)$ could be a challenging problem though, see Concluding remarks in [3], where the case $k = 2$ is discussed. It would be interesting to explore relationships of the new concept with already introduced domination invariants, paired domination [12] for instance, since by Theorem 1, Observation 3 and a result from [13] (see Theorem 2.1) it follows that the paired domination number and the 2-rainbow total domination number of a bipartite graph coincide.

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