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ON THE ρ-EDGE STABILITY NUMBER OF GRAPHS

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Abstract

For an arbitrary invariant $\rho(G)$ of a graph G the ρ-edge stability number $es_\rho(G)$ is the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$ or with $E(H) = \emptyset$.

In the first part of this paper we give some general lower and upper bounds for the ρ-edge stability number. In the second part we study the χ'-edge stability number of graphs, where $\chi' = \chi'(G)$ is the chromatic index of G. We prove some general results for the so-called chromatic edge stability index $es_{\chi'}(G)$ and determine $es_{\chi'}G)$ exactly for specific classes of graphs.

Keywords: edge stability number, line stability, invariant, chromatic edge stability index, chromatic index, edge coloring.

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1. Introduction

We consider in this paper finite simple graphs $G = (V(G), E(G))$. A graph is empty if $E(G) = \emptyset$.

Definition. A (*graph*) *invariant* $\rho(G)$ is a function $\rho : \mathcal{I} \to \mathbb{R}_0^+ \cup \{\infty\}$, where $\mathcal{I}$ is the class of finite simple graphs. An invariant $\rho(G)$ is *integer valued* if its image set consists of non-negative integers, that is, $\rho(\mathcal{I}) \subseteq \mathbb{N}_0$.

An invariant $\rho(G)$ is *monotone increasing* if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$, and *monotone decreasing* if $H \subseteq G$ implies $\rho(H) \geq \rho(G)$; $\rho(G)$ is *monotone* if it is monotone increasing or monotone decreasing.

If H_1 and H_2 are disjoint graphs, then an invariant is called *additive* if $\rho(H_1 \cup H_2) = \rho(H_1) + \rho(H_2)$ and *maxing* if $\rho(H_1 \cup H_2) = \max\{\rho(H_1), \rho(H_2)\}$.

Example 1. The chromatic number $\chi(G)$ is integer valued, monotone increasing, and maxing, and $\chi(G) - 1 \leq \chi(G - e) \leq \chi(G)$ holds for any edge e of G.

Example 2. The domination number $\gamma(G)$ is integer valued, not monotone, and additive, and $\gamma(G) \leq \gamma(G - e) \leq \gamma(G) + 1$ holds for any edge e of G.

Definition. Let $\rho(G)$ be an arbitrary invariant of a graph G. We define the *ρ-edge stability number $es_\rho(G)$* of G as the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$ or with $E(H) = \emptyset$.

In [2] the ρ-edge stability number is also defined and called *ρ-line-stability*.

The ρ-edge stability number $es_\rho(G)$ is an integer valued invariant. Some easy observations follow directly from the definition. For example, $es_\rho(G) = 0$ if and only if G is empty. If G is not empty, then $1 \leq es_\rho(G) \leq |E(G)|$. If $\rho(G)$ does not change by any edge removal, for example, the order of the graph G, then $es_\rho(G) = |E(G)|$.

If $\rho(G)$ is monotone increasing, then the subgraph H in the definition on the previous page fulfills $\rho(H) < \rho(G)$ or H is empty. Conversely, if $\rho(G)$ is monotone decreasing, then this subgraph H fulfills $\rho(H) > \rho(G)$ or H is empty.

For some specific invariants $\rho(G)$ the problem of determining the ρ-edge stability number was already considered, for example for the chromatic number $\chi(G)$ and particularly for the domination number $\gamma(G)$. In this paper the attention is drawn to the χ'-edge stability number where $\chi' = \chi'(G)$ is the chromatic index of G.

The χ-edge stability number or *chromatic edge stability number $es_\chi(G)$* was introduced in [1, 5] and also studied in [3].

The increase of the domination number $\gamma(G)$ with respect to edge removal was extensively studied (see e.g. [2] or [6] for a survey). The so-called *bondage number* $b(G)$ is equal to the γ-edge stability number $es_\gamma(G)$ if G is not empty, and $b(G) = \infty$ if G is empty.

In this paper we first consider the general case and give bounds for arbitrary ρ-edge stability numbers of graphs. Section 3 contains some examples of invariants $\rho(G)$ for which $es_\rho(G)$ can easily be determined. In Sections 4 and 5 we study the χ'-edge stability number of graphs.

2. General Results

An easy observation for the bondage number $b(G)$ and some implications (see [6]) can be transferred to arbitrary edge stability numbers $es_\rho(G)$.

Proposition 3. *Let H be a spanning subgraph of G obtained from G by removing k edges. Then $es_\rho(G) \leq es_\rho(H)+k$. Moreover, if $\rho(G) \neq \rho(H)$, then $es_\rho(G) \leq k$.*

Proof. Let $H = G - E'$ where $E' \subset E(G)$ with $|E'| = k$. If $\rho(G) \neq \rho(H) = \rho(G - E')$, then $es_\rho(G) \leq |E'| = k \leq es_\rho(H) + k$.

Therefore, assume in the following that $\rho(G) = \rho(H)$. If $\rho(H)$ cannot be changed by edge removal, then $es_\rho(H) = |E(H)|$, and $es_\rho(G) \leq |E(G)| = |E(H)| + |E'| = es_\rho(H) + k$ follows.

Otherwise, let E'' be a set of edges of H such that $|E''| = es_\rho(H)$ and $\rho(H - E'') \neq \rho(H)$. Set $E''' = E' \cup E''$ with $|E'''| = |E'| + |E''| = k + es_\rho(H)$. It follows that $\rho(G) = \rho(H) \neq \rho(H - E'') = \rho(G - E''')$, which implies $es_\rho(G) \leq |E'''| = es_\rho(H) + k$. ■

Upper bounds for $es_\rho(G)$ can be obtained by carefully selecting spanning subgraphs H with a fixed ρ-edge stability number. The next result considers as an example the case $es_\rho(H) = 1$.

Corollary 4. *If H is a spanning subgraph of G with $es_\rho(H) = 1$, then $es_\rho(G) \leq 1 + |E(G)| - |E(H)|$.*

Proof. The result immediately follows from Proposition 3, since H is obtained from G by removing $k = |E(G)| - |E(H)|$ edges. ■

In [2] it was stated that if there is at least one vertex $v \in V(G)$ such that $\gamma(G - v) \geq \gamma(G)$, then $b(G) \leq d(v) \leq \Delta(G)$, where $d(v)$ is the degree of v and $\Delta(G)$ the maximum degree of the graph G. This result can be generalized as follows.

Proposition 5. *Let $\rho(G)$ be additive. If there is a vertex $v \in V(G)$ such that $\rho(G - v) > \rho(G)$, or $\rho(G - v) = \rho(G)$ and $\rho(K_1) > 0$, or $\rho(G - v) < \rho(G)$ and $\rho(K_1) = 0$, then $es_\rho(G) \leq d(v) \leq \Delta(G)$.*

Proof. The given conditions imply $\rho(G) < \rho(G - v) \leq \rho(G - v) + \rho(K_1) = \rho(G - E_v)$, or $\rho(G) = \rho(G - v) < \rho(G - v) + \rho(K_1) = \rho(G - E_v)$, or $\rho(G) > \rho(G - v) = \rho(G - v) + \rho(K_1) = \rho(G - E_v)$, where E_v is the set of edges incident to v. Therefore, $\rho(G) \neq \rho(G - E_v)$ and thus $es_\rho(G) \leq |E_v| = d(v) \leq \Delta(G)$ by Proposition 3. ■

Alternatively, the condition $\gamma(G - v) \geq \gamma(G)$ implies that there is a minimal dominating set of $G - v$ which contains a neighbor w of v, that is, there is an induced subgraph $H = G - E_v + vw$, obtained from G by removing all edges incident to v except vw, with $b(H) = 1$, and the conclusion $b(G) \leq d(v)$ follows by Corollary 4 (see [6]). This second proof method leads to the following result.

Corollary 6. *If there is an edge set $E' \subseteq E_v$ such that $\rho(G) \neq \rho(G - E')$ or $\rho(G - E_v) \neq \rho(G - E')$, where E_v is the set of edges incident to v, then $es_\rho(G) \leq d(v) \leq \Delta(G)$.*

Proof. If $\rho(G) \neq \rho(G - E')$, then $es_\rho(G) \leq |E'| \leq d(v) \leq \Delta(G)$. If $\rho(G - E_v) \neq \rho(G - E')$, then $\rho(G) \neq \rho(G - E_v)$ or $\rho(G) \neq \rho(G - E')$ and the result follows by Proposition 3. ■

If removing a pending edge always changes $\rho(G)$, then Corollary 6 implies that $es_\rho(G) \leq d(v)$ for each non-isolated vertex $v \in V(G)$. Thus the following holds.

Corollary 7. *If G is a graph without isolated vertices and if removing a pending edge always changes the invariant ρ, then $es_\rho(G) \leq \delta(G)$.*

This holds for example for the number $k(G)$ of components of G, since a pending edge is a bridge (see also Section 3).

Another result from [2] can be generalized by requesting appropriate conditions for the considered invariant.

Proposition 8. *If $\rho(G)$ is additive and $\rho(K_2) \neq \rho(2K_1)$, then $es_\rho(G) \leq \min\{d(u) + d(v) - 1 : uv \in E(G)\}$.*

Proof. For an arbitrary edge $uv \in E(G)$ set $H = G - E_u - E_v + uv \cong G - \{u, v\} \cup K_2$, which is obtained from G by removing $k = d(u) + d(v) - 2$ edges. Since $\rho(H) = \rho(G - \{u, v\}) + \rho(K_2) \neq \rho(G - \{u, v\}) + \rho(2K_1) = \rho(H - uv)$ implies $es_\rho(H) = 1$, the result follows from Corollary 4. ■

This result can be generalized by considering an arbitrary subgraph S of G instead of K_2. The additivity of $\rho(G)$ gives an upper bound on $es_\rho(G)$ which only depends on S and the number of removed edges.

Theorem 9. *Let $\rho(G)$ be additive and $S \subseteq G$ a subgraph for which $\rho(S)$ can be changed by edge deletions. Then $es_\rho(G) \leq es_\rho(S) + |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$, where $E(U, W)$ is the set of edges between vertex sets U and W.*

Proof. Consider the spanning subgraph $H = G - V(S) \cup S$ of G which is obtained from G by removing $k = |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$ edges, namely all edges between $V(S)$ and $V(G) \setminus V(S)$ as well all edges in $G[V(S)]$ not contained in S. By Proposition 3, $es_\rho(G) \leq es_\rho(H) + k$.

Let $E' \subseteq E(S)$ be an edge set such that $|E'| = es_\rho(S)$ and $\rho(S) \neq \rho(S - E')$. Then by the additivity of the invariant, $\rho(H) = \rho(G - V(S)) + \rho(S) \neq \rho(G - V(S)) + \rho(S - E') = \rho(H - E')$ which implies by Proposition 3 that $es_\rho(H) \leq |E'| = es_\rho(S)$. Thus, $es_\rho(G) \leq es_\rho(S) + k$. ■

If S is a spanning subgraph, then $V(S) = V(G)$, thus Theorem 9 gives the bound $es_\rho(G) \leq es_\rho(S) + |E(G)| - |E(S)|$, which follows directly by Proposition 3. If S is an induced subgraph, then the bound of Theorem 9 simplifies to $es_\rho(G) \leq es_\rho(S) + |E(V(S), V(G) \setminus V(S))|$.

An additional condition on the invariant ρ is necessary to prove the corresponding result for maxing invariants.

Theorem 10. *Let $\rho(G)$ be maxing and $S \subseteq G$ a subgraph for which $\rho(S)$ can be changed by edge deletions and $\rho(S) > \rho(G - V(S))$. Then $es_\rho(G) \leq es_\rho(S) + |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$.*

Proof. As in the proof of Theorem 9, consider the spanning subgraph $H = G - V(S) \cup S$ of G which is obtained by removing $k = |E(V(S), V(G) \setminus V(S))| + |E(G[V(S)])| - |E(S)|$ edges from G. By Proposition 3, $es_\rho(G) \leq es_\rho(H) + k$.

Let $E' \subseteq E(S)$ be an edge set such that $|E'| = es_\rho(S)$ and $\rho(S) \neq \rho(S - E')$. Since the invariant is maxing and $\rho(S) > \rho(G - V(S))$ by assumption and $\rho(S) \neq \rho(S - E')$, it holds that $\rho(H) = \max\{\rho(G - V(S)), \rho(S)\} = \rho(S) \neq \max\{\rho(G - V(S)), \rho(S - E')\} = \rho(H - E')$. By Proposition 3, $es_\rho(H) \leq |E'| = es_\rho(S)$ and therefore $es_\rho(G) \leq es_\rho(S) + k$. ■

In the proofs of Theorems 9 and 10, the disjoint union of two graphs was considered. The proof idea can be transferred to the disjoint union of arbitrarily many graphs.

Theorem 11. *Let $\rho(G)$ be additive, let $G = H_1 \cup \cdots \cup H_k$ be a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 0$ are defined such that $\rho(H_i)$ can be changed by edge deletion if and only if $1 \leq i \leq s$. Then $es_\rho(G) = |E(G)|$ if $s = 0$ and $es_\rho(G) = \min\{es_\rho(H_i) : 1 \leq i \leq s\}$ if $s \neq 0$.*

Proof. If $s = 0$, then $\rho(H_i)$ cannot be changed by edge deletion for every subgraph H_i, which implies by the additivity that also $\rho(G) = \rho(H_1) + \cdots + \rho(H_k)$ cannot be changed by edge deletions, that is, $es_\rho(G) = |E(G)|$.

If $s \neq 0$, then let H_j be a subgraph with $es_\rho(H_j) = \min\{es_\rho(H_i) : 1 \leq i \leq s\}$ and $E' \subseteq E(H_j)$ be an edge set with $|E'| = es_\rho(H_j)$ and $\rho(H_j - E') \neq \rho(H_j)$. By the additivity, $\rho(G - E') = \rho(H_1) + \cdots + \rho(H_{j-1}) + \rho(H_j - E') + \rho(H_{j+1}) + \cdots + \rho(H_k) \neq \rho(H_1) + \cdots + \rho(H_{j-1}) + \rho(H_j) + \rho(H_{j+1}) + \cdots + \rho(H_k) = \rho(G)$, which implies $es_\rho(G) \leq |E'| = es_\rho(H_j)$.

Let $E'' \subseteq E(G)$ be an edge set with $|E''| < es_\rho(H_j)$. By the minimality of $es_\rho(H_j)$, $\rho(H_i - E'') = \rho(H_i)$ for $i = 1, \ldots, k$, which implies $\rho(G - E'') = \rho(G)$ since $\rho(G)$ is additive. Therefore, $es_\rho(G) = es_\rho(H_j)$. ■

For maxing invariants we can prove the following result.

Theorem 12. *Let $\rho(G)$ be maxing and monotone increasing, let $G = H_1 \cup \cdots \cup H_k$ be a graph whose subgraphs $H_1, \ldots, H_k$ and the integer $s \geq 1$ are defined* *such that* $\rho(H_i) = \rho(G)$ *if and only if* $1 \leq i \leq s$*. Then* $es_\rho(G) = |E(G)|$ *if there is a subgraph* H_j*,* $1 \leq j \leq s$*, such that* $\rho(H_j)$ *cannot be changed by edge deletions, and* $es_\rho(G) = \sum_{i=1}^{s} es_\rho(H_i)$ *otherwise.*

Proof. If there is a subgraph H_j, $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge deletions, then $\rho(G) = \rho(H_j) = \rho(G - E')$ for every $E' \subseteq E(G)$, since the invariant is maxing and monotone increasing (that is, removing edges does not increase the invariant). Therefore, $es_\rho(G) = |E(G)|$.

Otherwise, let $E' = E'_1 \cup \cdots \cup E'_s$ with $E'_i \subseteq E(H_i)$, $|E'_i| = es_\rho(H_i)$, and $\rho(H_i - E'_i) \neq \rho(H_i)$ for $i = 1, \ldots, s$. Since the invariant is maxing, $\rho(G - E') = \max\{\rho(H_i - E'_i) : 1 \leq i \leq s\} \cup \{\rho(H_i) : s+1 \leq i \leq k\} \neq \rho(G)$ which implies $es_{\rho(G)} \leq |E'| = \sum_{i=1}^{s} es_\rho(H_i)$. If an edge set E'' with less than $|E'|$ edges is removed from G, then there is a subgraph H_j, $1 \leq j \leq s$, from which less than $es_\rho(H_j)$ edges are removed, which implies $\rho(H_j - E'') = \rho(H_j)$ and thus, since the invariant is maxing and monotone increasing, $\rho(G - E'') = \rho(H_j) = \rho(G)$. Therefore, $es_{\rho(G)} = |E'| = \sum_{i=1}^{s} es_\rho(H_i)$. ■

Theorems 11 and 12 imply that $\rho(G)$ can be computed by the ρ-edge stability numbers of the components of G if the invariant is additive or if it is maxing and monotone increasing. Therefore, it is sufficient to consider connected graphs G in these cases.

A lower bound for $es_\chi(G)$ given in [3] can be generalized as follows.

Theorem 13. *Let* $\rho(G)$ *be monotone and let* G *be a nonempty graph with* $\rho(G) = k$*. If* G *contains* s *nonempty subgraphs* $G_1, \ldots, G_s$ *with* $\rho(G_1) = \cdots = \rho(G_s) = k$ *such that* $a \geq 0$ *is the number of edges that occur in at least two of these subgraphs and* $q \geq 1$ *is the maximum number of these subgraphs with a common edge, then both* $es_\rho(G) \geq \frac{1}{q}\sum_{i=1}^{s} es_\rho(G_i) \geq s/q$ *and* $es_\rho(G) \geq \sum_{i=1}^{s} es_\rho(G_i) - a(q-1)$ *hold.*

Proof. Let $\rho(G)$ be monotone increasing. Let E' be a set of edges of G with $|E'| = es_\rho(G)$ such that $\rho(G - E') < k$ or $G - E'$ is empty. If $\rho(G - E') < k$, then the set E' must contain at least $es_\rho(G_i)$ edges of each graph G_i, $1 \leq i \leq s$, since otherwise $k > \rho(G - E') \geq \rho(G_j - E' \cap E(G_j)) = k$ for some j, $1 \leq j \leq s$, a contradiction. If $G - E'$ is empty, then $E' = E(G)$ contains all edges of G_i, $1 \leq i \leq s$. Therefore, $b = \sum_{i=1}^{s} |E' \cap E(G_i)| \geq \sum_{i=1}^{s} es_\rho(G_i) \geq s$.

On the other hand, at most $\bar{a} = \min\{a, |E'|\}$ edges of E' are counted at most q times in b, every other edge of E' is counted at most once, so $b \leq \bar{a} \cdot q + (|E'| - \bar{a}) \cdot 1 = |E'| + \bar{a}(q-1)$.

Since $\bar{a} \leq |E'|$, $b \leq q\,|E'|$ and therefore $es_\rho(G) = |E'| \geq b/q \geq \frac{1}{q}\sum_{i=1}^{s} es_\rho(G_i) \geq s/q$. On the other hand, $\bar{a} \leq a$ implies $es_\rho(G) = |E'| \geq b - a(q-1) \geq \sum_{i=1}^{s} es_\rho(G_i) - a(q-1)$.

The proof for monotone decreasing $\rho(G)$ runs analogously. ■

Note that we do not require that the graphs G_i are distinct in Theorem 13. The lower bound of the first inequality can be improved by considering additional subgraphs G_i with $\rho(G_i) = k$ that do not increase the number q. A refinement of the latter inequality can be achieved if the number of occurrences of fixed edges in the subgraphs is taken into account.

Corollary 14. *Let $\rho(G)$ be monotone and let G be a nonempty graph with $\rho(G) = k$. If G contains s nonempty subgraphs $G_1, \ldots, G_s$ with $\rho(G_1) = \cdots = \rho(G_s) = k$ and pairwise disjoint edge sets, then $es_\rho(G) \geq \sum_{i=1}^{s} es_\rho(G_i) \geq s$.*

Proof. Each edge of G is contained in at most $q = 1$ of the given subgraphs since they are pairwise edge disjoint. The result follows from Theorem 13. ∎

Corollary 15. *Let $\rho(G)$ be monotone. If $H \subseteq G$ and $\rho(H) = \rho(G)$, then $es_\rho(H) \leq es_\rho(G)$.*

Proof. If H is empty, then $es_\rho(H) = 0 \leq es_\rho(G)$; otherwise Corollary 14 with $s = 1$ implies the result. ∎

Note that in general $es_\rho(G)$ must not be monotone even if $\rho(G)$ is monotone.

3. Examples for Edge Stability Numbers

In this section the edge stability numbers for some well-known invariants are considered, beginning with the minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$.

Proposition 16. *$es_\delta(G) = |E(G)|$ if $\delta(G) = 0$ and $es_\delta(G) = 1$ if $\delta(G) \neq 0$.*

Proof. If $\delta(G) = 0$, that is, G has isolated vertices, then the minimum degree cannot be decreased by edge removal, hence $es_\delta(G) = |E(G)|$ by definition. If $\delta(G) \neq 0$, then it suffices to remove one edge incident to a vertex of degree $\delta(G)$ in order to decrease the minimum degree, hence $es_\delta(G) = 1$. ∎

Proposition 17. *$es_\Delta(G) = 0$ if G is empty and $es_\Delta(G) = |V_\Delta| - \alpha'(G[V_\Delta])$ if G is not empty, where V_Δ is the set of vertices of G of degree $\Delta(G)$ and $\alpha'(G)$ is the edge independence number or matching number of G.*

Proof. If G is empty, then $es_\Delta(G) = 0$ by definition. If G is not empty, then $\Delta(G) \geq 1$. Let E' be an edge set of G with $\Delta(G - E') = \Delta(G) - 1 \geq 0$. Each vertex from V_Δ is incident with at least one edge from E'. At most $\alpha'(G[V_\Delta])$ edges from E' connect two vertices each from V_Δ such that all these vertices are distinct. The remaining vertices of V_Δ need one additional incident edge from E' each. Therefore, $es_\Delta(G) \geq \alpha'(G[V_\Delta]) + |V_\Delta| - 2\alpha'(G[V_\Delta]) = |V_\Delta| - \alpha'(G[V_\Delta])$.

Equality holds by selecting an appropriate maximum matching in $G[V_\Delta]$ and an incident edge for each not matched vertex from V_Δ. ■

If G is regular and not empty, then $es_\Delta(G) = |V(G)| - \alpha'(G)$. For example, $es_\Delta(K_n) = \frac{1}{2}n$ if n is even and $es_\Delta(K_n) = \frac{1}{2}(n+1)$ if $n \geq 3$ is odd.

Let $k(G)$ be the number of *components* of a graph G and $\lambda(G)$ the *edge connectivity* of G, that is, the minimum number of edges whose removal gives a disconnected graph or the singleton K_1. By the definitions it follows that if G is connected, then $es_k(G) = \lambda(G)$. A direct implication of Theorem 11 is the following general result which also covers disconnected graphs.

Proposition 18. *Let G be a graph with $k(G)$ components $H_1, \ldots, H_{k(G)}$. Then $es_k(G) = 0$ if G is empty and $es_k(G) = \min\{\lambda(H_i) : 1 \leq i \leq k(G), H_i \ncong K_1\}$ if G is not empty.*

Proof. The number of components $k(H)$ is additive and can be increased by edge deletions for nonempty graphs. Let $H_1, \ldots, H_s$ be the nonempty components of G and $H_{s+1}, \ldots, H_{k(G)}$ be singletons K_1, $0 \leq s \leq k(G)$. Then Theorem 11 gives $es_k(G) = 0$ if $s = 0$, that is, if G is empty, and $es_k(G) = \min\{es_k(H_i) : 1 \leq i \leq s\} = \min\{\lambda(H_i) : 1 \leq i \leq s\}$ otherwise. ■

Proposition 19. $es_\lambda(G) = 1$ *if G is connected and not a singleton, and* $es_\lambda(G) = |E(G)|$ *otherwise.*

Proof. If G is connected and not a singleton, then let E' be an edge set with $|E'| = \lambda(G) \geq 1$ such that $G - E'$ is disconnected. For any edge $e \in E'$, $\lambda(G-e) = \lambda(G) - 1$, hence $es_\lambda(G) = 1$. If G is disconnected or a singleton, then $\lambda(G) = 0$ and the invariant cannot be changed by edge removal, hence $es_\lambda(G) = |E(G)|$ by definition. ■

4. General Results for the Chromatic Edge Stability Index

If $G = (V(G), E(G))$ is a graph, a function $c : E(G) \to \{1, \ldots, k\}$ such that $c(e_1) \neq c(e_2)$ for any two adjacent edges e_1 and e_2 is called a *k-edge-coloring* of G, and G is called *k-edge-colorable*. The minimum k for which G is k-edge-colorable is the *chromatic index* $\chi'(G)$ of G. By Vizing's Theorem, the chromatic index can only attain one of two values, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. Graphs with $\chi'(G) = \Delta(G)$ are called *class* 1 *graphs* and graphs with $\chi'(G) = \Delta(G) + 1$ are called *class* 2 *graphs*. We define the invariant $\text{class}(G) = \chi'(G) - \Delta(G) + 1 \in \{1, 2\}$. A graph G is called *overfull* if its order n is odd and if it contains more than $\Delta(G)(n-1)/2$ edges. Obviously, an overfull graph must be a class 2 graph.

Note that $\chi'(G)$ is an invariant which is monotone increasing, integer valued, and maxing, and it holds that $\chi'(G-e) \le \chi'(G) \le \chi'(G-e)+1$ for any edge e of G.

In this section we consider the χ'-edge stability number $es_{\chi'}(G)$ which we also call *chromatic edge stability index* of G. Using Theorem 12 we can compute $es_{\chi'}(G)$ by the chromatic edge stability indexes of its components. Let $G = H_1 \cup \cdots \cup H_{k(G)}$ such that $\chi'(G) = \chi'(H_i)$ if and only if $1 \le i \le s$ for $s \le k(G)$. Then $es_{\chi'}(G) = \sum_{i=1}^{s} es_{\chi'}(H_i)$. Therefore, we can assume without loss of generality in the following that G is connected.

Proposition 20. *$es_{\chi'}(G) \le \lfloor |E(G)| / \chi'(G) \rfloor \le \alpha'(G)$ if G is nonempty, and $es_{\chi'}(G) = \alpha'(G) = 0$ if G is empty.*

Proof. Let $t'(G)$ be the minimum number of edges in a color class of the graph G where the minimum is taken over all edge colorings of G with $\chi'(G)$ colors.

If G is nonempty, then removing any color class from G reduces the chromatic index, thus $es_{\chi'}(G) \le t'(G)$ follows. By the pigeonhole principle, any edge coloring of G with $\chi'(G)$ colors has a color class with at most $\lfloor |E(G)| / \chi'(G) \rfloor$ edges, which implies $t'(G) \le \lfloor |E(G)| / \chi'(G) \rfloor$. On the other hand, the lower bound $\chi'(G) \ge |E(G)| / \alpha'(G)$ implies the second inequality.

If G is empty, then the result is obvious. ∎

Lemma 21. *If G is a class 1 graph, then $es_{\chi'}(G) \ge es_{\Delta}(G)$.*

Proof. If G is empty, then $es_{\chi'}(G) = es_{\Delta}(G) = 0$. If G is nonempty, then there is a set E' of edges of G such that $|E'| = es_{\chi'}(G)$ and $\Delta(G-E') \le \chi'(G-E') < \chi'(G) = \Delta(G)$. It follows that $\Delta(G-E') < \Delta(G)$ which implies $es_{\chi'}(G) = |E'| \ge es_{\Delta}(G)$. ∎

The following proposition gives a class of graphs for which equality always holds.

Proposition 22. *If G is a regular class 1 graph, then $es_{\chi'}(G) = es_{\Delta}(G) = \alpha'(G)$.*

Proof. If G is empty, then $es_{\chi'}(G) = es_{\Delta}(G) = \alpha'(G) = 0$. If G is nonempty, then $es_{\Delta}(G) \le es_{\chi'}(G) \le \alpha'(G) = \frac{1}{2}|V(G)|$ by Lemma 21 and Proposition 20. Since $es_{\Delta}(G) = |V(G)| - \alpha'(G) = \frac{1}{2}|V(G)|$ by Proposition 17, $es_{\chi'}(G) = es_{\Delta}(G) = \alpha'(G) = \frac{1}{2}|V(G)|$ follows. ∎

More generally, we can characterize in a certain way all class 1 graphs with $es_{\chi'}(G) = es_{\Delta}(G)$.

Proposition 23. *If G is a class 1 graph, then $es_{\chi'}(G) = es_{\Delta}(G)$ if and only if G is empty or if there is an edge set E' such that $|E'| = es_{\Delta}(G)$, $\Delta(G-E') < \Delta(G)$, and $G-E'$ is in class 1.*

Proof. Let G be a non-empty class 1 graph.

If $es_{\chi'}(G) = es_{\Delta}(G)$, then let E' be an arbitrary edge set with $|E'| = es_{\chi'}(G)$ and $\chi'(G-E') < \chi'(G)$, which implies $\chi'(G-E') = \chi'(G)-1$. Since $\Delta(G-E') \leq \chi'(G-E')$ and $\chi'(G) = \Delta(G)$, $\Delta(G-E') < \Delta(G)$. Moreover, $|E'| = es_{\Delta(G)}$, which implies that $\Delta(G-E') = \Delta(G) - 1$. Therefore, $\chi'(G-E') = \Delta(G-E')$, that is, $G - E'$ is in class 1.

The second assertion follows from Lemma 21, which states $es_{\chi'}(G) \geq es_{\Delta}(G)$, and from the properties of the given set E', since $\chi'(G-E') = \Delta(G-E') < \Delta(G) = \chi'(G)$ which implies $|E'| = es_{\Delta}(G) \geq es_{\chi'}(G)$. ∎

Proposition 22 follows from this characterization, since a regular class 1 graph is 1-factorable, and removing a 1-factor E' leaves a class 1 graph.

Proposition 23 and Lemma 21 imply that if G is in class 1 but $G - E'$ is in class 2 for all sets E' with $|E'| = es_{\Delta}(G)$ and $\Delta(G-E') < \Delta(G)$, then $es_{\chi'}(G) > es_{\Delta}(G)$. An example for such graphs is given in Theorem 31.

Theorem 24. *If G is a class 2 graph, then* $es_{\chi'}(G) = \min\{es_{\Delta}(G), es_{\text{class}}(G)\}$.

Proof. Since G is in class 2, the graph G is not empty and the invariants $\Delta(G)$, $\text{class}(G) = 2$, and $\chi'(G) = \Delta(G) + 1$ can be reduced by edge removal.

By removing $es_{\Delta}(G)$ edges E' such that $\Delta(G-E') < \Delta(G)$ we obtain $\chi'(G-E') \leq \Delta(G-E') + 1 < \Delta(G) + 1 = \chi'(G)$ which implies $|E'| \geq es_{\chi'}(G)$. By removing $es_{\text{class}}(G)$ edges E'' such that $\text{class}(G-E'') = 1$ we obtain $\chi'(G-E'') = \Delta(G-E'') \leq \Delta(G) < \Delta(G)+1 = \chi'(G)$ which implies $|E''| \geq es_{\chi'}(G)$. It follows that $\min\{es_{\Delta}(G), es_{\text{class}}(G)\} \geq es_{\chi'}(G)$.

Consider now a set of edges E''' such that $\chi'(G-E''') < \chi'(G) = \Delta(G) + 1$, that is, $\chi'(G-E''') \leq \Delta(G)$. Then $G - E'''$ cannot both be in class 2 and have the same maximum degree as G since this would imply $\chi'(G-E''') = \Delta(G)+1$. Therefore, $|E'''| \geq es_{\Delta}(G)$ or $|E'''| \geq es_{\text{class}}(G)$ which implies $es_{\chi'}(G) \geq \min\{es_{\Delta}(G), es_{\text{class}}(G)\}$. ∎

For overfull graphs we can give a lower bound.

Corollary 25. *If G is an overfull graph, then* $es_{\chi'}(G) \geq |E(G)| - \Delta(G)(|V(G)|-1)/2$.

Proof. Since G is overfull, G is in class 2, $|E(G)| > \Delta(G)(n-1)/2$, and the invariants $\Delta(G)$ and $\text{class}(G)$ can be reduced by edge deletions.

Let E' be an edge set such that $|E'| = es_{\Delta}(G)$ and $\Delta(G-E') < \Delta(G)$. By the handshake lemma, $G - E'$ may contain at most $\Delta(G-E')n/2 \leq (\Delta(G)-1)n/2$ edges which implies $es_{\Delta}(G) = |E'| \geq |E(G)| - (\Delta(G)-1)n/2 > |E(G)| - \Delta(G)(n-1)/2$, since $n > \Delta(G)$.

Let E'' be an edge set such that $|E''| = es_{\text{class}}(G)$ and $\text{class}(G-E'') = 1$. Then $G - E''$ may contain at most $\Delta(G-E'')(n-1)/2 \leq \Delta(G)(n-1)/2$ edges (otherwise $G - E''$ would be still overfull) which implies $es_{\text{class}}(G) = |E''| \geq |E(G)| - \Delta(G)(n-1)/2$.

By Theorem 24, $es_{\chi'}(G) = \min\{es_{\Delta}(G), es_{\text{class}}(G)\} \geq |E(G)| - \Delta(G)(n-1)/2$. ■

5. Chromatic Edge Stability Index for Specific Graph Classes

In this section we use general results of the previous section to determine the chromatic edge stability index of some well-known graph classes.

Theorem 26. *If G is bipartite, then $es_{\chi'}(G) = es_{\Delta}(G)$.*

Proof. The result follows from Proposition 23, since every subgraph $G - E'$ of G is bipartite and thus in class 1. ■

Theorem 26 and Proposition 17 imply the following results for complete bipartite graphs and paths.

Corollary 27. $es_{\chi'}(K_{m,n}) = es_{\Delta}(K_{m,n}) = \min\{m, n\}$.

Corollary 28. $es_{\chi'}(P_n) = es_{\Delta}(P_n) = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ \lceil (n-2)/2 \rceil & \text{if } n \geq 3. \end{cases}$

Proposition 29. *$es_{\chi'}(C_n) = n/2$ if n is even, $es_{\chi'}(C_n) = 1$ if n is odd, and $es_{\Delta}(C_n) = \lceil n/2 \rceil$.*

Proof. By Proposition 17, $es_{\Delta}(C_n) = \lceil n/2 \rceil$. If n is even, then C_n is bipartite, and the result follows from Theorem 26. If n is odd, then $\chi'(C_n) = 3$ and removing one edge from the cycle gives a 2-edge-colorable path P_n, which implies $es_{\chi'}(C_n) = 1$. ■

This proposition shows that the difference between the two invariants $es_{\Delta}(G)$ and $es_{\chi'}(G)$ may be arbitrarily large, since $es_{\Delta}(C_{2s+1}) - es_{\chi'}(C_{2s+1}) = s$. Moreover, Lemma 21 does not necessarily hold for class 2 graphs.

Next we consider complete graphs and complete graphs with an additional vertex.

Proposition 30. $es_{\chi'}(K_n) = \lfloor n/2 \rfloor = \begin{cases} n/2 & \text{if } n \text{ is even}, \\ (n-1)/2 & \text{if } n \text{ is odd}, \end{cases}$ *and $es_{\Delta}(K_n) = \lceil n/2 \rceil$ if $n \geq 2$.*

Proof. If $n \geq 2$, then $es_\Delta(K_n) = \lceil n/2 \rceil$ follows from Propositions 17.

If $n = 1$ or if n is even, then K_n is a regular class 1 graph, and the result is an implication of Proposition 22.

If $n \geq 3$ is odd, then K_n is overfull and Corollary 25 implies $es_{\chi'}(K_n) \geq \binom{n}{2} - (n-1)^2/2 = (n-1)/2$. On the other hand, $es_{\chi'}(K_n) \leq \alpha'(K_n) = (n-1)/2$ follows from Proposition 20, that is, equality holds. ■

Theorem 31. *Let G be a graph which consists of a complete graph K_n, $n \geq 2$, and an additional vertex w connected to $d = d(w)$ vertices of K_n. Then we have the following.*

1. $es_\Delta(G) = \lceil n/2 \rceil$ *if* $d = 0$, $es_\Delta(G) = \lceil d/2 \rceil$ *if* $1 \leq d \leq n-1$, $es_\Delta(G) = \lceil (n+1)/2 \rceil$ *if* $d = n$.
2. *If* $n \geq 3$ *odd, then* $es_{\chi'}(G) = \lfloor n/2 \rfloor$ *if* $0 \leq d \leq n-1$, $es_{\chi'}(G) = (n+1)/2$ *if* $d = n$.
3. *If* n *even, then* $es_{\chi'}(G) = \lfloor n/2 \rfloor$ *if* $d = 0$, $es_{\chi'}(G) = d$ *if* $1 \leq d \leq n/2$, $es_{\chi'}(G) = d - n/2$ *if* $n/2 < d \leq n$.

Proof. If $d = 0$, then $G \cong K_n \cup K_1$ and if $d = n$, then $G \cong K_{n+1}$, therefore the results follow from Propositions 17 and 30. Let $1 \leq d \leq n-1$ in the following and denote the vertices of K_n by $v_1, \ldots, v_n$ such that w is adjacent to $v_1, \ldots, v_d$.

1. If $1 \leq d \leq n-1$, then G has d vertices of maximum degree $\Delta(G) = n$, namely the neighbors of w. Then, by Proposition 17, $es_\Delta(G) = d - \alpha'(K_d) = d - \lfloor d/2 \rfloor = \lceil d/2 \rceil$.

2. Since $n \geq 3$, n is odd, and $K_n \subseteq G \subseteq K_{n+1}$, $\chi'(K_n) = \chi'(G) = \chi'(K_{n+1}) = n$. By Corollary 15 and Proposition 30, $es_{\chi'}(G) \geq es_{\chi'}(K_n) = \lfloor n/2 \rfloor$. Since $d < n$, there is a color class with $\lfloor n/2 \rfloor$ edges in every proper n-edge-coloring of G, whose removal reduces the chromatic index. Therefore, $es_{\chi'}(G) = \lfloor n/2 \rfloor$.

3. If n even, then we consider two cases.

Case 3(a): If $1 \leq d \leq n/2$, then G is in class 1. Consider the natural edge coloring of K_n with $n-1$ colors where the vertices are in order v_1, v_{d+1}, v_2, $v_{d+2}, \ldots, v_{d-1}$, v_{2d-1}, v_d, v_{2d}, $v_{2d+1}, \ldots, v_n$. Then the edges $v_1v_{d+1}, \ldots, v_{d-1}v_{2d-1}$ are colored pairwise differently. Color these edges as well as edge wv_d with the new color n and then color wv_i with the old color of v_iv_{d+i}, $i = 1, \ldots, d-1$. This implies $\chi'(G) = \Delta(G) = n$.

Let E' be a set of edges of G with $|E'| = es_{\chi'}(G)$ and $\chi'(G - E') < \chi'(G) = \Delta(G) = n$. Then $\Delta(G - E') \leq n-1$. If $\Delta(G - E') \leq n-2$, then the degree of the d vertices of maximum degree must be reduced by 2, which implies $|E'| \geq d$. If $\Delta(G - E') = n-1$, then $G - E'$ cannot be overfull since otherwise $\chi'(G - E') = \Delta(G - E') + 1 = \Delta(G) = \chi'(G)$, a contradiction. Hence $|E(G - E')| =$

$\binom{n}{2} + d - |E'| \leq \Delta(G - E')(|V(G - E')| - 1)/2 = n(n-1)/2 = \binom{n}{2}$ which again implies $|E'| \geq d$. Therefore, $es_{\chi'}(G) \geq d$.

On the other hand, removing all d edges incident to w gives a graph $G - E' \cong K_n \cup K_1$ with $\chi'(G - E') = n - 1$, that is, $es_{\chi'}(G) \leq d$, and equality follows.

Case 3(b): If $n/2 < d \leq n - 1$, then G is overfull since $|E(G)| = \binom{n}{2} + d > n(n-1)/2 + n/2 = n^2/2 = \Delta(G)(|V(G)| - 1)/2$. Therefore, $es_{\chi'}(G) \geq d - n/2$ by Corollary 25. On the other hand, removing $d - n/2$ edges incident to w gives a class 1 graph (see Case 3(a)), which implies $es_{\chi'}(G) \leq d - n/2$ and therefore $es_{\chi'}(G) = d - n/2$. ■

Parts of Proposition 30 and Theorem 31 follow also from a result by Plantholt [4], which states the following. If G is a graph of odd order $n \geq 3$ with a spanning star, then G is in class 1 if and only if it has at most $(n-1)^2/2$ edges. This implies for example that if K_n is a complete graph of odd order $n \geq 3$ and $E' \subseteq E(K_n)$, then $\chi'(K_n - E') = n$ if and only if $|E'| \leq (n-3)/2$.

The result of Theorem 31 implies that the difference between $es_{\chi'}(G)$ and $es_{\Delta}(G)$ may be arbitrarily large for class 1 graphs.

Theorem 32. *For every pair of positive integers a, b there is a graph G with $es_{\Delta}(G) = a$ and $es_{\chi'}(G) = b$.*

Proof. If $a \leq b$, then for $d = 2a$ and $n = 2b+1$ it holds that $n \geq 3$, n is odd, and $1 \leq d \leq n-1$, and the class 1 graph G of Theorem 31 fulfills $es_{\Delta}(G) = \lceil d/2 \rceil = a$ and $es_{\chi'}(G) = (n-1)/2 = b$.

If $a > b$, then for $d = 2a$ and $n = 2d - 2b$ (n even) it holds that $n/2 = d - b < d < 2d - 2b = n$, and the class 2 graph G of Theorem 31 fulfills again $es_{\Delta}(G) = \lceil d/2 \rceil = a$ and $es_{\chi'}(G) = d - n/2 = b$. ■

Note that a graph G with $es_{\Delta}(G) = a$, $es_{\chi'}(G) = b$, and $a > b$ is a class 2 graph by Lemma 21.

A wheel W_n with $n \geq 3$ is the join of a cycle C_n, say with consecutive vertices $v_1, \ldots, v_n$, and a single vertex w. Wheels are class 1 graphs.

Proposition 33. $es_{\Delta}(W_3) = 2$*,* $es_{\Delta}(W_n) = 1$ *for* $n \geq 4$*,* $es_{\chi'}(W_n) = 2$ *for* $n \in \{3, 4\}$*,* $es_{\chi'}(W_n) = 1$ *for* $n \geq 5$*.*

Proof. If $n = 3$, then $W_3 \cong K_4$ and $es_{\Delta}(W_3) = es_{\chi'}(W_3) = 2$ follows from Propositions 17 and 30.

The wheel W_n has only one vertex of maximum degree for $n \geq 4$, hence $es_{\Delta}(W_n) = 1$ for $n \geq 4$.

If $n = 4$, then $G \cong W_4 - wv_i$ ($i \in \{1, \ldots, 4\}$) has 5 vertices, 7 edges, and maximum degree 3. Since $7 = |E(G)| > \Delta(G)(|V(G)| - 1)/2 = 6$, the graph G is overfull and thus $\chi'(G) = \Delta(G) + 1 = 4 = \chi'(W_4)$, which implies $es_{\chi'}(W_4) \geq 2$. On the other hand, $es_{\chi'}(W_4) \leq \alpha'(W_4) = 2$, hence equality follows.

Let $n \geq 5$ and consider the n-edge-coloring of W_n which assigns color $i \in \{1, \ldots, n\}$ to edges wv_i and $v_{i+1}v_{i+2}$ (indices modulo n), and recolor edge v_1v_2 with color 3. Removing color class n with only one edge wv_n reduces the chromatic index, which implies $es_{\chi'}(W_n) = 1$ if $n \geq 5$. ■

It would be an interesting task to determine the chromatic edge stability index for some other classes of graphs. For example, $es_{\chi'}(P) = 2$ and $es_{\Delta}(P) = 5$ hold for the Petersen graph P.

References

[1] S. Arumugam, I. Sahul Hamid and A. Muthukamatchi, *Independent domination and graph colorings*, Ramanujan Math. Soc. Lect. Notes Ser. **7** (2008) 195–203.

[2] D. Bauer, F. Harary, J. Nieminen and C.L. Suffel, *Domination alteration sets in graphs*, Discrete Math. **47** (1983) 153–161.
https://doi.org/10.1016/0012-365X(83)90085-7

[3] A. Kemnitz, M. Marangio and N. Movarraei, *On the chromatic edge stability number of graphs*, Graphs Combin. **34** (2018) 1539–1551.
https://doi.org/10.1007/s00373-018-1972-y

[4] M. Plantholt, *The chromatic index of graphs with a spanning star*, J. Graph Theory **5** (1981) 45–53.
https://doi.org/10.1002/jgt.3190050103

[5] W. Staton, *Edge deletions and the chromatic number*, Ars Combin. **10** (1980) 103–106.

[6] J.-M. Xu, *On bondage numbers of graphs: A survey with some comments*, Int. J. Comb. **2013** (2013) ID 595210.
https://doi.org/10.1155/2013/595210

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