# THE SEMITOTAL DOMINATION PROBLEM IN BLOCK GRAPHS 

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#### Abstract

A set $D$ of vertices in a graph $G$ is a dominating set of $G$ if every vertex outside $D$ is adjacent in $G$ to some vertex in $D$. A set $D$ of vertices in $G$ is a semitotal dominating set of $G$ if $D$ is a dominating set of $G$ and every vertex in $D$ is within distance 2 from another vertex of $D$. Given a graph $G$ and a positive integer $k$, the semitotal domination problem is to decide whether $G$ has a semitotal dominating set of cardinality at most $k$. The semitotal domination problem is known to be NP-complete for chordal graphs and bipartite graphs as shown in [M.A. Henning and A. Pandey, Algorithmic aspects of semitotal domination in graphs, Theoret. Comput. Sci. 766 (2019) 46-57]. In this paper, we present a linear time algorithm to compute a minimum semitotal dominating set in block graphs. On the other hand, we show that the semitotal domination problem remains NP-complete for undirected path graphs.


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## 1. INTRODUCTION

A dominating set in a graph $G$ is a set $D$ of vertices of $G$ such that every vertex in $V(G) \backslash D$ is adjacent to at least one vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The concept of domination and its variations have been widely studied in theoretical, algorithmic and application aspects; a rough estimate says that it occurs in more than 6,000 papers to date. A thorough treatment of the fundamentals of domination theory in graphs can be found in the books $[4,5]$.

A total dominating set, abbreviated a TD-set, of a graph $G$ with no isolated vertex is a set $D$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $D$. The total domination number of $G$, denoted by $\gamma_{t}(G)$, is the minimum cardinality of a TD-set of $G$. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book [13].

A relaxed form of total domination called semitotal domination was introduced by Goddard, Henning and McPillan [3], and studied further in [6, 7, 8, $9,10,11,12$ ] and elsewhere. A set $D$ of vertices in a graph $G$ with no isolated vertices is a semitotal dominating set, abbreviated a semi-TD-set, of $G$ if $D$ is a dominating set of $G$ and every vertex in $D$ is within distance 2 of another vertex of $D$. The semitotal domination number of $G$, denoted by $\gamma_{t 2}(G)$, is the minimum cardinality of a semi-TD-set of $G$. Since every TD-set is a semi-TD-set, and since every semi-TD-set is a dominating set, we have the following observation.

Observation 1 [3]. For every isolate-free graph $G, \gamma(G) \leq \gamma_{t 2}(G) \leq \gamma_{t}(G)$.
As remarked in [3], by Observation 1 the semitotal domination number is squeezed between arguably the two most important domination parameters, namely the domination number and the total domination number. Goddard et al. [3] established tight upper bounds on the semitotal domination number of a connected graph in terms of its order. Henning [7] established tight upper bounds on the upper semitotal domination number of a regular graphs using edge weighting functions. Henning and Marcon [8] explored a relationship between the semitotal domination number and the matching number of a graph, and showed that the semitotal domination number of a connected graph is bounded above by the matching number plus one. Zhuang and Hao [15] established a lower bound on
the semitotal domination number of trees and characterized the extremal trees. Semitotal domination in claw-free cubic graphs has been studied in [10].

Given a graph $G$ and a positive integer $k$, the semitotal domination problem is to decide whether $G$ has a semitotal dominating set of cardinality at most $k$. The semitotal domination problem is known to be NP-complete for general graphs [3]. Henning and Pandey [12] showed that the semitotal domination problem remains NP-complete for chordal bipartite graphs, planar graphs and split graphs. On the positive side, linear time algorithms exist to find a minimum semi-TD-set in trees [3, 11]. A polynomial time algorithm to compute a minimum cardinality semi-TD-set in interval graphs, a subclass of chordal graphs, is presented in [12].

In this paper, we design in Section 3 a linear time algorithm for computing a minimum semitotal dominating set in block graphs, a superclass of trees. On the other hand, we show in Section 4 that the semitotal domination problem remains NP-complete for undirected path graphs, a subclass of chordal graphs.

## 2. Terminology and Notation

For notation and graph theory terminology, we in general follow [13]. Specifically, let $G=(V, E)$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is the set $N_{G}(v)=\{u \in$ $V \mid u v \in E\}$ and the closed neighborhood of $v$ is $N_{G}[v]=\{v\} \cup N_{G}(v)$. A vertex $v$ is said to dominate a vertex $u$ in $G$ if $u \in N_{G}[v]$. The open neighborhood of a set $S$ of vertices in $G$ is the set $N_{G}(S)=\bigcup_{v \in S} N_{G}(v)$ and its closed neighborhood is the set $N_{G}[S]=N_{G}(S) \cup S$. The degree of a vertex $v$ is $\left|N_{G}(v)\right|$ and is denoted by $d_{G}(v)$. For a set $S$ of vertices in $G$, the subgraph induced by $S$ in $G$ is denoted by $G[S]$. Thus, the edge set of $G[S]$ consists of those edges of $G$ with both ends in the set $S$. The set $S$ is a clique of $G$, if $G[S]$ is a complete subgraph of $G$.

The distance between two vertices $u$ and $v$ in a connected graph $G$, denoted by $d_{G}(u, v)$, is the length of a shortest $(u, v)$-path in $G$. For a vertex $v$ in $G$, the 2-distance neighborhood of $v$ is the set $N_{G}^{2}(v)=\left\{u \mid 1 \leq d_{G}(u, v) \leq 2\right\}$ of all vertices at distance 1 or 2 from $v$ in $G$, while the closed 2-distance neighborhood of $v$ is $N_{G}^{2}[v]=N_{G}^{2}(v) \cup\{v\}$. A vertex in $N_{G}^{2}(v)$ is called a 2-distance neighbor of the vertex $v$ in $G$.

A rooted tree is a tree $T$ in which there is a designated vertex $r$ named as root. For each vertex $v \neq r$ of $T$, the parent of $v$ is the neighbor of $v$ on the unique $(r, v)$-path, while a child of $v$ is any other neighbor of $v$.

For a vertex $v$ of $G$, the graph $G-v$ is the graph obtained from $G$ by deleting $v$ and deleting all edges of $G$ incident with $v$. A vertex $v$ is a cut-vertex of $G$ if the number of components increases in $G-v$. A block of a graph $G$ is a maximal connected subgraph of $G$ has no cut-vertex of its own. Thus, a block is a maximal

2-connected subgraph of $G$. Any two blocks of a graph have at most one vertex in common, namely a cut-vertex. If a connected graph contains a single block, we call the graph itself a block. A block graph is a connected graph in which every block is a clique. A block containing exactly one cut-vertex is called an end block. A non-complete block graph has at least two end blocks.

We use the standard notation $[k]=\{1,2, \ldots, k\}$. Let $G=(V, E)$ be a block graph, and let $\left\{B_{1}, B_{2}, \ldots B_{r}\right\}$ and $\left\{c_{1}, c_{2}, \ldots, c_{s}\right\}$ be the set of blocks and the set of cut-vertices of $G$, respectively. The cut-tree of $G$ is the tree $T_{G}$ defined by $V\left(T_{G}\right)=\left\{B_{1}, \ldots, B_{r}, c_{1}, \ldots, c_{s}\right\}$ and $E\left(T_{G}\right)=\left\{B_{i} c_{j} \mid c_{j} \in V\left(B_{i}\right), i \in[r], j \in\right.$ $[s]\}$. A block graph $G$ and its associated cut-tree $T_{G}$ is illustrated in Figure 1. The computation of blocks in a graph $G$ and the construction of the cut-tree $T_{G}$ can be done in $O(|V|+|E|)$ time by using depth-first search [1].


Figure 1. A block graph $G$ and its corresponding cut-tree $T_{G}$.

## 3. Semitotal Domination in Block Graphs

In this section, we present a linear algorithm to compute a minimum semi-TD-set of a block graph $G$ on at least two vertices. If $G$ itself is a block, then the graph $G$ is a complete graph. In this case, any two vertices in $G$ form a semi-TD-set of $G$, implying that $\gamma_{t 2}(G)=2$. Hence it is only of interest for us to consider non-complete block graphs; that is, block graphs containing at least two blocks.

Let $G=(V, E)$ be a non-complete block graph. The algorithm we present to compute a minimum semi-TD-set in $G$ runs in $O(|V|+|E|)$ time, and follows a certain ordering of the blocks. Let $\left\{B_{1}, B_{2}, \ldots, B_{r}\right\}$ and $\left\{c_{1}, c_{2}, \ldots, c_{s}\right\}$ be the set of blocks and the set of cut-vertices of $G$, respectively. Let $T_{G}$ be the cuttree associated with the graph $G$. Without loss of generality, we assume that
$T_{G}$ is rooted at the cut-vertex $c_{s}$ of $G$. Let $\sigma=\left(B_{1}, B_{2}, \ldots, B_{r}\right)$ be an ordering of blocks of $G$, where $\sigma^{-1}=\left(B_{r}, B_{r-1}, \ldots, B_{1}\right)$ is an ordering of blocks of $G$ obtained by applying a breadth-first search starting at the root $c_{s}$ of $T_{G}$. We call such an ordering of blocks of $G$ as a RBFS-Block-Ordering of the blocks of $G$. For every $i \in[r]$, we define $F\left(B_{i}\right)$ as the parent of the block $B_{i}$ in $T_{G}$. Further for every $i \in[r]$, we define

$$
G_{i}=G\left[\bigcup_{\ell=i}^{r} V\left(B_{\ell}\right)\right] .
$$

We note that for every $i \in[r-1]$, the block $B_{i}$ is an end block in the graph $G_{i}$ with $F\left(B_{i}\right)$ as the unique cut-vertex in $G_{i}$ that belongs to the block $B_{i}$. Since the $G_{r}$ is the block $B_{r}$, we treat any vertex of the block $B_{r}$ as $F\left(B_{r}\right)$. For the sake of simplicity, we denote the vertex $F\left(B_{i}\right)$ simply by $F_{i}$ for $i \in[r]$. The following observation follows immediately from the fact that any two blocks of $G$ have at most one vertex in common, namely a cut-vertex.

Observation 2. For every $i \in[r-1]$ and every $k>i$, we have $V\left(B_{i}\right) \cap V\left(B_{k}\right) \subseteq$ $\left\{F_{i}\right\}$.

Before formally presenting our algorithm MSTDS-BLock $(G)$, we discuss the main ideas of the algorithm. The algorithm constructs a set $D$ which upon termination of the algorithm is a semi-TD-set of the non-complete block graph $G$. We assign to each vertex $v$ of $G$ a label $L(v)=\left(L_{1}(v), L_{2}(v)\right)$ which we call its $L$-label. We call the labels $L_{1}(v)$ and $L_{2}(v)$ the $L_{1}$-label and $L_{2}$-label of $v$, respectively. The label $L_{1}(v)$ is used to determine whether the vertex $v$ is already dominated or has yet to be dominated. Initially, $L_{1}(v)=L_{2}(v)=0$ for every vertex $v$ of $G$. As the algorithm progresses, the label of the vertex $v$ changes. If the vertex $v$ is not dominated by the current set $D$, then the label $L_{1}(v)=0$ is unchanged; otherwise, $L_{1}(v)=1$. The label $L_{2}(v)$ is used to determine whether the vertex $v$ belongs to the current set $D$ or not. If the vertex $v$ does not belong to the current set $D$, then the label $L_{2}(v)=0$ is unchanged. If the vertex $v$ belongs to the current set $D$ but has no 2-distance neighbor in $D$, then $L_{2}(v)=1$. If the vertex $v$ belongs to the current set $D$ and has a 2 -distance neighbor in $D$, then $L_{2}(v)=2$.

At the $i$-th iteration, the algorithm systematically considers the vertices of the block $B_{i}$ with respect to the RBFS-BLock-Ordering $\sigma=\left(B_{1}, B_{2}, \ldots, B_{r}\right)$ of $G$ and takes some action (either the algorithm selects new vertices or updates some of the vertices of the graph) based on the values of $L_{1}$ and $L_{2}$ assigned to the vertices that belong to $V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$. If a vertex $u$ is selected by the algorithm and added to the set $D$, then $L_{1}(u)$ is updated to $1, L_{2}(u)$ is updated to 1 or 2 , and $L(y)$ is made $(1,0)$ for every neighbor $y$ of $u$ in $G$ such that $L(y)=(0,0)$. Upon
termination of the algorithm, the set $D$ consists precisely of the ( 1,2 -labeled vertices and forms a semi-TD-set of $G$. We now formally describe our algorithm to construct a semi-TD-set in a non-complete block graph.

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Algorithm 1: MSTDS-BLOCK \((G)\)
    Input: A non-complete connected block graph \(G=(V, E)\);
    Output: A semi-TD-set \(D\) of \(G\);
    Initialize \(D=\emptyset\);
    Initialize \(L(u)=(0,0)\) for each vertex \(u \in V\);
    Compute a RBFS-Block-Ordering \(\sigma=\left(B_{1}, B_{2}, \ldots, B_{r}\right)\) of the blocks of \(G\);
    \(i=1\);
    while \((i<r)\) do
        Let \(F_{i}\) be the unique cut-vertex of \(G_{i}\) present in \(B_{i}\) and \(\mathcal{C}\left(B_{i}\right)=V\left(B_{i}\right) \backslash\left\{F_{i}\right\}\);
        while \(\left(\mathcal{C}\left(B_{i}\right) \neq \emptyset\right)\) do
            Choose a vertex \(v \in \mathcal{C}\left(B_{i}\right)\);
            if \((L(v)=(0,0))\) then
                if (there exists a vertex \(u \in N_{G}\left[F_{i}\right]\) with \(L_{1}(u)=1\) ) then /* Case 1 */
                \(L\left(F_{i}\right)=(1,2)\) and \(L_{2}(x)=2\) for every vertex \(x \in N_{G}\left(F_{i}\right)\) such that
                    \(L_{2}(x)=1 ;\)
                    else
                            /* Case 2 */
                        \(L\left(F_{i}\right)=(1,1) ;\)
                \(L_{1}(x)=1\) for every vertex \(x \in N_{G}\left(F_{i}\right) ;\)
            else if \((L(v)=(1,0))\) then
                Let \(A(v)=\left\{y \in N_{G}(v) \mid L_{2}(y) \neq 0\right\}\);
                if \((|A(v)|>1)\) then /* Case 3 */
                    \(L_{2}(x)=2\) for every \(x \in N_{G}(v)\) such that \(L_{2}(x)=1\);
                else if \(\left(A(v)=\{u\}\right.\) such that \(L_{2}(u)=1\) and \(\left.u \notin V\left(B_{i}\right)\right)\) then /* Case 4 */
                    \(L\left(F_{i}\right)=(1,2)\) and \(L_{1}(x)=1\) for every vertex \(x \in N_{G}\left(F_{i}\right)\);
                    \(L_{2}(x)=2\) for every vertex \(x \in N_{G}\left(F_{i}\right) \cup\{u\}\) such that \(L_{2}(x)=1\);
                \(\mathcal{C}\left(B_{i}\right)=\mathcal{C}\left(B_{i}\right) \backslash\{v\} ;\)
        \(i=i+1 ;\)
    \(\mathcal{C}\left(B_{r}\right)=V\left(B_{r}\right) ;\)
    while \(\left(\mathcal{C}\left(B_{r}\right) \neq \emptyset\right)\) do
        Choose a vertex \(v \in \mathcal{C}\left(B_{r}\right)\);
        if \((L(v)=(0,0))\) then \(\quad\) /* Case 5 */
            \(L\left(c_{j}\right)=(1,2)\) for some cut-vertex \(c_{j}\) of \(G\) such that \(c_{j} \in V\left(B_{r}\right)\);
            \(L_{1}(x)=1\) for every \(x \in N_{G}\left(c_{j}\right)\);
            \(L_{2}(x)=2\) for every vertex \(x \in N_{G}\left(c_{j}\right)\) such that \(L_{2}(x)=1\);
        else if \(\left(L(u)=(1,1)\right.\) for some \(u \in N_{G}(v)\), where \(\left.v \in V\left(B_{r}\right)\right)\) then
            Let \(B(v)=\left\{y \in N_{G}(v) \mid L_{2}(y) \neq 0\right\}\);
            if \((|B(v)|>1)\) then /* Case 6 */
            \(L_{2}(x)=2\) for every \(x \in N_{G}(v)\) such that \(L_{2}(x)=1 ;\)
            else
                /* Case 7 */
            \(L(w)=(1,2)\) for some \(w \in V\left(B_{r}\right) \backslash\{u\}\) and \(L(u)=(1,2) ;\)
                \(L_{1}(x)=1\) for every vertex \(x \in N_{G}(w)\);
                \(L_{2}(x)=2\) for every vertex \(x \in N_{G}(w)\) such that \(L_{2}(x)=1\);
        \(\mathcal{C}\left(B_{r}\right)=\mathcal{C}\left(B_{r}\right) \backslash\{v\} ;\)
    return \(D=\{u \in V \mid L(u)=(1,2)\}\);
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In Table 1, we illustrate the different iterations of the algorithm MSTDS$\operatorname{Block}(G)$ on the graph $G$ shown in Figure 1, where we only show the iterations of the algorithm in which some update has been done. Moreover, in the column "Considered vertex $v \in V\left(B_{i}\right)$ with $L(v)$ " of Table 1, we have only shown those vertices of the block for which some update has been done. Upon termination of the algorithm, the resulting set $D=\left\{v_{1}, v_{6}, v_{9}, v_{12}, v_{15}, v_{18}, v_{19}\right\}$ a minimum semi-TD-set of the graph $G$ shown in Figure 1.

| Iteration $i$ | Considered block | Considered vertex $v \in V\left(B_{i}\right)$ with $L(v)$ | $\boldsymbol{F}_{\boldsymbol{i}}$ | $A(v)$ or $B(v)$ | Applied Case | Update |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $B_{1}$ | $L\left(v_{13}\right)=(0,0)$ | $v_{12}$ | Not computed | Case 2 | $\begin{aligned} & L\left(v_{12}\right)=(1,1) \\ & L\left(v_{11}\right)=(1,0) \\ & L\left(v_{13}\right)=(1,0) \end{aligned}$ |
| 2 | $B_{2}$ | $L\left(v_{20}\right)=(0,0)$ | $v_{18}$ | Not computed | Case 2 | $\begin{aligned} & L\left(v_{18}\right)=(1,1) \\ & L\left(v_{16}\right)=(1,0) \\ & L\left(v_{20}\right)=(1,0) \\ & \hline \end{aligned}$ |
| 3 | $B_{3}$ | $L\left(v_{21}\right)=(0,0)$ | $v_{19}$ | Not computed | Case 2 | $\begin{aligned} & L\left(v_{19}\right)=(1,1) \\ & L\left(v_{21}\right)=(1,0) \\ & L\left(v_{17}\right)=(1,0) \end{aligned}$ |
| 7 | $B_{7}$ | $L\left(v_{7}\right)=(0,0)$ | $v_{6}$ | Not computed | Case 2 | $\begin{aligned} & L\left(v_{6}\right)=(1,1) \\ & L\left(v_{7}\right)=(1,0) \\ & L\left(v_{5}\right)=(1,0) \end{aligned}$ |
| 8 | $B_{8}$ | (i) $L\left(v_{10}\right)=(0,0)$ | $v_{9}$ | (i) Not computed | (i) Case 1 | $\text { (i) } \begin{aligned} & L\left(v_{9}\right)=(1,2) \\ & L\left(v_{10}\right)=(1,0) \\ & L\left(v_{8}\right)=(1,0) \\ & L\left(v_{3}\right)=(1,0) \end{aligned}$ |
|  |  | (ii) $L\left(v_{11}\right)=(1,0)$ |  | (ii) $A(v)=\left\{v_{9}, v_{12}\right\}$ | (ii) Case 3 | (ii) $L\left(v_{12}\right)=(1,2)$ |
| 9 | $B 9$ | (i) $L\left(v_{16}\right)=(1,0)$ | $v_{15}$ | (i) $A(v)=\left\{v_{18}\right\}$; $v_{18} \notin V\left(B_{9}\right)$ | (i) Case 4 | $\text { (i) } \begin{aligned} & L\left(v_{15}\right)=(1,2) \\ & L\left(v_{1}\right)=(1,0) \\ & L\left(v_{14}\right)=(1,0) \\ & L\left(v_{18}\right)=(1,2) \end{aligned}$ |
|  |  | (ii) $L\left(v_{17}\right)=(1,0)$ |  | (ii) $A(v)=\left\{v_{15}, v_{19}\right\}$ | (ii) Case 3 | (ii) $L\left(v_{19}\right)=(1,2)$ |
| 13 | $B_{13}$ | $L\left(v_{2}\right)=(0,0)$ | $v_{1}$ | Not computed | Case 1 | $\begin{aligned} & L\left(v_{1}\right)=(1,2) \\ & L\left(v_{4}\right)=(1,0) \\ & L\left(v_{2}\right)=(1,0) \\ & \hline \end{aligned}$ |
| 14 | $B_{14}$ | $L\left(v_{5}\right)=(1,0)$ | $v_{1}$ | $B(v)=\left\{v_{1}, v_{6}\right\}$ | Case 6 | $L\left(v_{6}\right)=(1,2)$ |

Table 1. Illustration of the algorithm on the graph $G$ shown in Figure 1.
Recall that in the $i$-th iteration of the algorithm $\operatorname{MSTDS}-\operatorname{Block}(G)$, the labels of all vertices in $B_{i}$ are systematically considered. Furthermore, at the start of the $i$-th iteration, the labels $L(v)$ of all vertices $v$ in $B_{j}$ where $j<i$ are $(1,0),(1,1)$ or $(1,2)$. We state this formally as follows.
Observation 3. At the beginning of the $i$-th iteration of the algorithm MSTDS$\operatorname{BLOCK}(G)$ where $i \geq 2$, we have $L(v) \in\{(1,0),(1,1),(1,2)\}$ for all $v \in V\left(B_{j}\right) \backslash$ $\left\{F_{j}\right\}$ and $j \in[i-1]$.

Let $B_{i}$ be the block considered at the $i$-th iteration. If $L(v)=(1,0)$ for some $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, then the algorithm updates the $L$-labels of the neighbors
of $v$. In particular, upon completion of the $i$-th iteration, there is no neighbor $u \in N_{G}(v) \backslash V\left(B_{i}\right)$ of $v$ such that $L_{2}(u)=1$. We state this observation formally as follows.

Observation 4. Let $B_{i}$ be the block considered at the $i$-th iteration and let $L(y)=$ $(1,0)$ for all $y \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$. If $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$ and there exists a vertex $u \in N_{G}(v) \backslash\left\{F_{i}\right\}$ with $L_{2}(u) \neq 0$, then $L(u)=(1,2)$ upon completion of the $i$-th iteration of the algorithm.

We note that the algorithm $\operatorname{MSTDS}-\operatorname{Block}(G)$ has $r$ iterations where $r$ is the number of blocks in $G$. For $i \in[r] \cup\{0\}$, let $D_{i}$ denote the set $\{u \in V(G) \mid$ $\left.L_{2}(u) \neq 0\right\}$ after the $i$-th iteration of the algorithm $\operatorname{MSTDS}-\operatorname{Block}(G)$. We first prove that the set $D_{r}$ is a semi-TD-set of $G$.

Lemma 5. The set $D_{r}$ is a semi-TD-set of $G$.
Proof. Upon completion of the $i$-th iteration of the algorithm MSTDS$\operatorname{Block}(G)$, by Observation 3, $L_{1}(x)=1$ for all $x \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, where $i \in[r]$. This implies that $D_{r}$ is a dominating set of $G$. To prove that $D_{r}$ is a semi-TD-set of $G$, we show that for every $v \in D_{r}$, there exists a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$. Let $v \in D_{r}$ be arbitrary. Since $G$ is a block graph, $v \in V\left(B_{i}\right)$ for some $i \in[r]$. We consider two cases.

Case 1. $i<r$. We first prove that if $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, then there exists a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$. Let $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$. Since $i<r$, there exists a block $B_{j}$ with $j>i$ such that $F_{i} \in V\left(B_{j}\right)$. Since $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, the vertex $v \in N_{G}\left(F_{i}\right)$. If $F_{i} \in D_{r}$, then taking $q=F_{i}$ the desired result holds. Hence we may assume that $F_{i} \notin D_{r}$. If $z \in D_{r}$ for some $z \in N_{G}\left(F_{i}\right)$, then $d_{G}(v, z)=d_{G}\left(v, F_{i}\right)+d_{G}\left(F_{i}, z\right)=2$ and the desired result follows. Hence we may further assume that $z \notin D_{r}$ for every $z \in N_{G}\left(F_{i}\right)$. Thus, the set $\left\{y \in N_{G}\left(F_{i}\right) \mid\right.$ $\left.y \in D_{r}\right\}=\{v\}$.

If $j=r$, then $B\left(F_{i}\right)=\{v\}$, where $B(u)=\left\{y \in N_{G}(u) \mid L_{2}(y) \neq 0\right\}$. In this case, the algorithm selects a vertex $w \in V\left(B_{r}\right) \backslash\{v\}$ (see Line 36 of the algorithm) at the $r$-th iteration. Notice that $d_{G}(v, w) \leq 2$.

If $j<r$, then $F_{i} \in V\left(B_{j}\right) \backslash\left\{F_{j}\right\}$ and $F_{i} \in N_{G}(v)$. Recall that $z \notin D_{r}$ for every $z \in N_{G}\left(F_{i}\right)$. In this case, $A\left(F_{i}\right)=\{v\}$, where $A(u)=\left\{y \in N_{G}(u) \mid L_{2}(y) \neq 0\right\}$. This implies that $A\left(F_{i}\right)=\{v\}$ at the beginning of the $j$-th iteration of the algorithm noting that $D_{j} \subseteq D_{r}$. In this case since $j<r$, the algorithm selects $F_{j}$ (see Line 20 of the algorithm) at the $j$-th iteration. We note that $d_{G}\left(F_{j}, v\right) \leq 2$. In all the above cases, we have shown that if $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, then there exists a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$.

Now let $v=F_{i}$. Since $i<r$, we note that $v \in V\left(B_{j}\right)$ where $j>i$. If $j=r$, then the algorithm selects a vertex $w \in V\left(B_{r}\right) \backslash\{v\}$ (see Line 36 of the algorithm) at the $r$-th iteration. Since $d_{G}(v, w) \leq 2$, the desired result follows. If $j<r$, then
$v=F_{i} \in V\left(B_{j}\right) \backslash\left\{F_{j}\right\}$ where $j>i$. Thus by our earlier observations, there exists a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$. Therefore, $D_{r}$ is a semi-TD-set of $G$.

Case 2. $i=r$. Suppose that there does not exist a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$. Since the algorithm does not select any vertex with $L_{2}$-label 1 at the $r$-th iteration, $v \in D_{r}$ implies that $v \in D_{r-1}$. Since $G$ is a connected graph, $\left|V\left(B_{r}\right)\right| \geq 2$. Moreover, since there is no vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$, at the beginning of the $r$-th iteration, we note that $L_{2}(v)=1$. Thus in this case the algorithm selects a vertex $w \in V\left(B_{r}\right) \backslash\{v\}$ (see Line 36 of the algorithm) such that $d_{G}(v, w) \leq 2$. This is a contradiction to the fact that there does not exist a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$. Therefore there exists a vertex $q \in D_{r} \backslash\{v\}$ such that $d_{G}(v, q) \leq 2$, implying that $D_{r}$ is a semi-TD-set of $G$. This completes the proof of Lemma 5 .

We are now in a position to prove the following theorem.
Theorem 6. The set $D_{r}$ is a minimum semi-TD-set of $G$.
Proof. Recall that for $i \in[r] \cup\{0\}$, the set $D_{i}$ is the set $\left\{u \in V(G) \mid L_{2}(u) \in\right.$ $\{1,2\}\}$ after the $i$-th iteration of the algorithm MSTDS-Block $(G)$. By Lemma 5, the set $D_{r}$ is a semi-TD-set of $G$. We prove next that $D_{r}$ is a minimum semi-TD-set of $G$. For this purpose, we prove by induction on $i \geq 0$ that the set $D_{i}$ is contained in some minimum semi-TD-set of $G$. If $i=0$, then $D_{0}=\emptyset$ and hence the set $D_{i}$ is contained in every minimum semi-TD-set of $G$. This establishes the base case. Assume that $i \geq 1$ and that the set $D_{i-1}$ is contained in some minimum semi-TD-set $D^{\prime}$ of $G$. We now show that $D_{i}$ is contained in some minimum semi-TD-set of $G$. Recall that by our earlier assumptions, the graph $G$ is a non-complete block graph. We proceed further with a series of claims. In each claim, we construct a minimum semi-TD-set of $G$ containing $D_{i}$ from the minimum semi-TD-set $D^{\prime}$ of $G$.

Claim 7. If $i<r$ and $L(v)=(0,0)$ for some vertex $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, then there is a minimum semi-TD-set of $G$ containing $D_{i}=D_{i-1} \cup\left\{F_{i}\right\}$.

Proof. By our induction hypothesis, the set $D_{i-1}$ is contained in some minimum semi-TD-set $D^{\prime}$ of $G$. If $F_{i} \in D^{\prime}$, then we are done. So we may assume that $F_{i} \notin D^{\prime}$. Let $u$ be a vertex in $D^{\prime}$ that dominates the vertex $v$. Since $D^{\prime}$ is a semi-TD-set of $G$, there is a vertex $u^{\prime} \in D^{\prime}$ such that $d_{G}\left(u, u^{\prime}\right) \in\{1,2\}$. Since $L(v)=(0,0)$, we note that $u \notin D_{i-1}$. If $u \in V\left(B_{k}\right) \backslash\left\{F_{k}\right\}$ where $k>i$, then by Observation 2, the vertex $u=F_{i}$ noting that $u v \in E(G)$. This is a contradiction since $F_{i} \notin D^{\prime}$. Hence, $u \in V\left(B_{k}\right) \backslash\left\{F_{k}\right\}$ where $k \leq i$.

By Observation 3, $L(x) \in\{(1,0),(1,1),(1,2)\}$ for every vertex $x \in V\left(B_{j}\right) \backslash$ $\left\{F_{j}\right\}$ and all $j \in[i-1]$. Since $u v \in E(G)$, the vertex $v \in V\left(B_{k}\right)$. If $k<i$, then by

Observation 2, the vertex $v$ is the vertex $F_{k}$. Notice that $\left\{z \in N_{G}[u] \mid L_{1}(z)=\right.$ $0\} \subseteq N_{G}\left[F_{i}\right] \cup N_{G}\left[D_{i-1}\right]$, i.e., all the undominated vertices of $N_{G}[u]$ are dominated by $D_{i-1} \cup\left\{F_{i}\right\}$. Let $N_{2}\left(D^{\prime}, u\right)=\left\{x \mid x \in D^{\prime} \cap N_{G}^{2}(u)\right\}$. If $d_{G}\left(F_{i}, x\right) \leq 2$ for every $x \in N_{2}\left(D^{\prime}, u\right)$, then $D^{\prime \prime}=\left(D^{\prime} \backslash\{u\}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$, as desired. Hence, we may assume that $d_{G}\left(F_{i}, x\right)>2$ for some vertex $x \in N_{2}\left(D^{\prime}, u\right)$, for otherwise the desired result follows.

Let $p \in N_{2}\left(D^{\prime}, u\right)$ be an arbitrary vertex such that $d_{G}\left(F_{i}, p\right)>2$. Thus, $p \in V\left(B_{q}\right)$ for some $q<i$ and $F_{q} \notin V\left(B_{i}\right)$. By Observation 4, either $L_{2}(p)=0$ (hence $p \notin D_{i-1}$ ) or $L(p)=(1,2)$. Let $S=\left\{x \in N_{2}\left(D^{\prime}, u\right) \mid d_{G}\left(F_{i}, x\right)>2\right.$ and $\left.L_{2}(x)=0\right\}$ and $S^{\prime}=\left\{x \in S \mid N_{G}^{2}(x) \backslash\{u\}=\emptyset\right\}$. Notice that each element of $S$ does not belong to $D_{i-1}$ and belongs to the blocks that appear before $i$. Moreover, $N_{G}\left[S^{\prime}\right] \subseteq N_{G}\left[D_{i-1}\right] \cup N_{G}\left[F_{i}\right]$. If $\left|S^{\prime}\right| \geq 2$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\left\{F_{i}\right\}$ is a semi-TD-set of $G$ of cardinality less than $\left|D^{\prime}\right|$, contradicting the minimality of $D^{\prime}$. Hence, $\left|S^{\prime}\right| \leq 1$. If $\left|S^{\prime}\right|=1$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$, as desired. Hence we may assume that $S^{\prime}=\emptyset$.

If there is a vertex $q \in N_{G}\left[F_{i}\right]$ such that $L_{1}(q)=1$, then $q \in D_{i-1}$ or $q^{\prime} \in D_{i-1}$ where $q q^{\prime} \in E(G)$. In this case, $\left(D^{\prime} \backslash\{u\}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$ since $d_{G}\left(F_{i}, q\right) \leq 2$ or $d_{G}\left(F_{i}, q^{\prime}\right) \leq 2$. Hence, we may assume that $L_{1}(q)=0$ for every vertex $q \in N_{G}\left[F_{i}\right]$, for otherwise the desired result follows. We now let $b \in N_{G}\left[F_{i}\right]$, and let $b^{\prime}$ be a vertex in $D^{\prime}$ that dominates the vertex $b$. Since $i<r$, we note that vertices $b$ and $b^{\prime}$ exists. Further since $F_{i} \notin D^{\prime}$, we note that $b^{\prime} \neq F_{i}$. Since $L_{1}(q)=0$ for all $q \in N_{G}\left[F_{i}\right]$, the vertex $b^{\prime} \notin D_{i-1}$. Thus since $d_{G}\left(F_{i}, b^{\prime}\right) \leq 2$, the set $\left(D^{\prime} \backslash\{u\}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$. This completes the proof of Claim 7.

Recall that for each vertex $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$ with $L(v)=(1,0)$, the set $A(v)=\left\{y \in N_{G}(v) \mid L_{2}(y) \in\{1,2\}\right\}$. If $|A(v)| \geq 2$, then for every $x \in A(v)$, there exists a vertex $y \in A(v)$ different from $x$ such that $d_{G}(x, y) \leq 2$. So for every neighbor of $v$ with $L_{2}$-label 1 (if such a neighbor of $v$ exists), there is another neighbor of $v$ with $L_{2}$-label 1 or 2 . The following claim shows that if $A(v)=\{u\}$, $L_{2}(u)=1$, and $u \notin V\left(B_{i}\right)$, then we can find a neighbor of $v$ within distance 2 from $u$. Recall that $D^{\prime}$ is a minimum semi-TD-set of $G$ and $D_{i-1} \subseteq D^{\prime}$.
Claim 8. Suppose that $i<r$ and $L(v)=(1,0)$ for some vertex $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$. If $A(v)=\{u\}$, where $L_{2}(u)=1$ and $u \notin V\left(B_{i}\right)$, then there is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$.

Proof. If $F_{i} \in D^{\prime}$, then we are done. So we may assume that $F_{i} \notin D^{\prime}$. By the choice of $u$ and $v$, we note that $u \in V\left(B_{k}\right) \backslash\left\{F_{k}\right\}$ where $k<i$ as $u \notin V\left(B_{i}\right)$. Since $L(v)=(1,0)$ and $L_{2}(u)=1$, we have $u \in D_{i-1}$. Since $D^{\prime}$ is a semi-TD-set of $G$, there is a vertex $u^{\prime} \in D^{\prime}$ such that $d_{G}\left(u, u^{\prime}\right) \leq 2$. The fact that $L_{2}(u)=1$ implies that $u^{\prime} \notin D_{i-1}$. Let $u^{\prime} \in V\left(B_{\ell}\right) \backslash\left\{F_{\ell}\right\}$ for some integer $\ell \geq 1$. If $\ell>i$, then since
$u \notin V\left(B_{i}\right)$ and $d_{G}\left(u, u^{\prime}\right) \leq 2$, Observation 2 implies that $u^{\prime}=F_{i}$, contradicting the fact that $F_{i} \notin D^{\prime}$. Hence, $\ell \leq i$.

We note that $\left\{z \in N_{G}\left[u^{\prime}\right] \mid L_{1}(z)=0\right\} \subseteq N_{G}\left[F_{i}\right] \cup N_{G}\left[D_{i-1}\right]$, i.e., all the undominated vertices of $N_{G}\left[u^{\prime}\right]$ are dominated by $D_{i-1} \cup\left\{F_{i}\right\}$. Let $N_{2}\left(D^{\prime}, u^{\prime}\right)=$ $\left\{x \mid x \in D^{\prime} \cap N_{G}^{2}\left(u^{\prime}\right)\right\}$. If $d_{G}\left(F_{i}, x\right) \leq 2$ or $d_{G}(u, x) \leq 2$ for every $x \in N_{2}\left(D^{\prime}, u^{\prime}\right)$, then $D^{\prime \prime}=\left(D^{\prime} \backslash\left\{u^{\prime}\right\}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup$ $\left\{F_{i}\right\}$. Hence, we may assume that $d_{G}\left(F_{i}, x\right)>2$ and $d_{G}(u, x)>2$ for some vertex $x \in N_{2}\left(D^{\prime}, u\right)$, for otherwise the desired result follows.

Let $p \in N_{2}\left(D^{\prime}, u\right)$ be an arbitrary vertex such that $d_{G}\left(F_{i}, p\right)>2$ and $d_{G}(u, p)>2$. Thus, $p \in V\left(B_{q}\right)$ for some $q<i$ and $F_{q} \notin V\left(B_{i}\right)$. By Observation 4, either $L_{2}(p)=0$ (hence $p \notin D_{i-1}$ ) or $L(p)=(1,2)$. Let $S=\left\{x \in N_{2}\left(D^{\prime}, u^{\prime}\right) \mid\right.$ $d_{G}\left(F_{i}, x\right)>2, d_{G}(u, x)>2$ and $\left.L_{2}(x)=0\right\}$ and $S^{\prime}=\left\{x \in S \mid N_{G}^{2}(x) \backslash\left\{u^{\prime}\right\}=\emptyset\right\}$. Notice that each element of $S^{\prime}$ does not belong to $D_{i-1}$ and belongs to the blocks that appear before $i$. Moreover, $N_{G}\left[S^{\prime}\right] \subseteq N_{G}\left[D_{i-1}\right] \cup N_{G}\left[F_{i}\right]$. If $\left|S^{\prime}\right| \geq 2$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\left\{F_{i}\right\}$ is a semi-TD-set of $G$ of cardinality less than $\left|D^{\prime}\right|$, contradicting the minimality of $D^{\prime}$. Hence, $\left|S^{\prime}\right| \leq 1$. If $\left|S^{\prime}\right|=1$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$, as desired. Hence we may assume that $S^{\prime}=\emptyset$. Since $d_{G}\left(u, F_{i}\right) \leq 2$, the set $\left(D^{\prime} \backslash\left\{u^{\prime}\right\}\right) \cup\left\{F_{i}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{i-1} \cup\left\{F_{i}\right\}$. This completes the proof of Claim 8 .

Claim 9. If $i=r$ and $L(v)=(0,0)$ for some vertex $v \in V\left(B_{r}\right)$, then there is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\left\{c_{j}\right\}$, where $c_{j} \in V\left(B_{r}\right)$ is a cut-vertex of $G$.

Proof. We once again consider the minimum semi-TD-set $D^{\prime}$ of $G$. Recall that $D_{r-1} \subseteq D^{\prime}$. If $c_{j} \in D^{\prime}$, then we are done. Hence we may assume that $c_{j} \notin D^{\prime}$. Let $u$ be a vertex in $D^{\prime}$ that dominates the vertex $v$. Since $D^{\prime}$ is a semi-TD-set of $G$, there is a vertex $u^{\prime} \in D^{\prime}$ such that $d_{G}\left(u, u^{\prime}\right) \leq 2$. Since $L(v)=(0,0)$, we note that $u \notin D_{r-1}$. Further, we note that $L_{2}(x)=0$ for all $x \in V\left(B_{r}\right)$. Moreover, since $c_{j} \in V\left(B_{r}\right)$ is an arbitrary cut-vertex of $G$, if $u \in V\left(B_{r}\right)$, then the vertex $u$ is not a cut-vertex of $G$.

By Observation $3, L_{1}(x)=1$ for all $x \in V\left(B_{j}\right) \backslash\left\{F_{j}\right\}$, where $j \in[r-1]$. This implies that every vertex of $V(G) \backslash V\left(B_{r}\right)$ is dominated by $D_{r-1}$. We note that $\left\{z \in N_{G}[u] \cap V\left(B_{r}\right) \mid L_{1}(z)=0\right\} \subseteq N_{G}\left[c_{j}\right] \cup N_{G}\left[D_{r-1}\right]$, i.e., the undominated vertices of $N_{G}[u]$ present in $V\left(B_{r}\right)$ are dominated by $D_{r-1} \cup\left\{c_{j}\right\}$. If $u \in V\left(B_{r}\right)$, then $\left(D^{\prime} \backslash\{u\}\right) \cup\left\{c_{j}\right\}$ is a minimum semi-TD-set of $G$ since $u$ is not a cut-vertex of $G$ and $d_{G}\left(x, c_{j}\right) \leq 2$ for every $x$ such that $d_{G}(x, u) \leq 2$. Hence we may assume that $u \notin V\left(B_{r}\right)$, for otherwise the desired result follows.

Let $N_{2}\left(D^{\prime}, u\right)=\left\{x \mid x \in D^{\prime} \cap N_{G}^{2}(u)\right\}$. If $d_{G}\left(c_{j}, x\right) \leq 2$ for every $x \in$ $N_{2}\left(D^{\prime}, u\right)$, then $D^{\prime \prime}=\left(D^{\prime} \backslash\{u\}\right) \cup\left\{c_{j}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\left\{c_{j}\right\}$. Hence, we may assume that $d_{G}\left(c_{j}, p\right)>2$ for some vertex $p \in$ $N_{2}\left(D^{\prime}, u\right)$, for otherwise the desired result follows. Thus, $p \in V\left(B_{q}\right)$ for some
$q<r$ and $F_{q} \notin V\left(B_{r}\right)$. By Observation 4, either $L_{2}(p)=0$ (hence $p \notin D_{r-1}$ ) or $L(p)=(1,2)$.

Let $S=\left\{x \in N_{2}\left(D^{\prime}, u\right) \mid d_{G}\left(c_{j}, x\right)>2\right.$ and $\left.L_{2}(x)=0\right\}$ and $S^{\prime}=\{x \in$ $\left.S \mid N_{G}^{2}(x) \backslash\{u\}=\emptyset\right\}$. We note that each element of $S^{\prime}$ does not belong to $D_{r-1}$ and belongs to the blocks that appear before $r$. Moreover, $N_{G}\left[S^{\prime}\right] \subseteq N_{G}\left[D_{r-1}\right] \cup$ $N_{G}\left[c_{j}\right]$. If $\left|S^{\prime}\right| \geq 2$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\left\{c_{j}\right\}$ is a semi-TD-set of $G$ of cardinality less than $\left|D^{\prime}\right|$, contradicting the minimality of $D^{\prime}$. Hence, $\left|S^{\prime}\right| \leq 1$. If $\left|S^{\prime}\right|=1$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\left\{c_{j}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\left\{c_{j}\right\}$, as desired. Hence we may assume that $S^{\prime}=\emptyset$. Since $c_{j}$ is a cut-vertex of $G$ and $G$ is not complete, there must be a block $B_{k}$, where $k<r$, of $G$ such that $V\left(B_{k}\right) \cap V\left(B_{r}\right)=\left\{c_{j}\right\}$. By Observation $3, L_{1}(y)=1$ for all $y \in V\left(B_{k}\right) \backslash\left\{F_{k}\right\}$. This implies that there is a vertex $y^{\prime} \in N_{G}[y]$ such that $y^{\prime} \in D_{r-1}$. We note that $d_{G}\left(c_{j}, y^{\prime}\right) \leq 2$, implying that $\left(D^{\prime} \backslash\{u\}\right) \cup\left\{c_{j}\right\}$ is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\left\{c_{j}\right\}$. This completes the proof of Claim 9.

By Claim 9, if $L(v)=(0,0)$ for some vertex $v \in V\left(B_{r}\right)$, then the algorithm selects any cut-vertex $c_{j} \in V\left(B_{r}\right)$. Let $B_{k}$ where $k<r$ be the block such that $V\left(B_{r}\right) \cap V\left(B_{k}\right)=\left\{c_{j}\right\}$. By Observation $3, L(x) \in\{(1,0),(1,1),(1,2)\}$ for every $x \in V\left(B_{k}\right) \backslash\left\{c_{j}\right\}$. Thus there exists a vertex $y \in N_{G}(x)$ such that $L_{2}(y) \neq 0$. We note that $d_{G}\left(y, c_{j}\right) \leq 2$. The algorithm therefore assigns to $c_{j}$ the label $L\left(c_{j}\right)=(1,2)$. If there exists a vertex $z \in N_{G}(u)$ for some $u \in V\left(B_{r}\right) \backslash\{v\}$ such that $L(z)=(1,1)$, then $d_{G}\left(c_{j}, z\right)=2$. Let $L(v)=(1,0)$ for some $v \in V\left(B_{r}\right)$ and $B(v)=\left\{y \in N_{G}(v) \mid L_{2}(y) \neq 0\right\}$. If $|B(v)|>1$, then for every $x \in B(v)$, there exists a vertex $y \in B(v)$ different from $x$ such that $d_{G}(x, y) \leq 2$. Hence for every neighbor of $v$ with $L_{2}$-label 1 (if such a neighbor of $v$ exists), we can associate a vertex with $L_{2}$-label 1 or 2 . If $|B(v)|=1$ for any vertex $v \in V\left(B_{r}\right)$ with $L(v)=(1,0)$ that has a neighbor with label $(1,1)$, then the algorithm finds its 2-distance neighbor vertex by the following claim.

Claim 10. Suppose that $L(v)=(1,0)$ for some vertex $v \in V\left(B_{r}\right), L(u)=(1,1)$ for some $u \in N_{G}(v)$, and $B(v)=\left\{y \in N_{G}(v) \mid L_{2}(y) \neq 0\right\}$. If $|B(v)|=1$, then there is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\{w\}$, where $w \in$ $V\left(B_{r}\right) \backslash\{u\}$.

Proof. We once again consider the minimum semi-TD-set $D^{\prime}$ of $G$. Since $L(u)=$ $(1,1)$ for some $v \in V\left(B_{r}\right)$, the vertex $u \in D_{r-1}$. Since $D^{\prime}$ is a semi-TD-set of $G$, there is a vertex $u^{\prime} \in D^{\prime}$ such that $d_{G}\left(u, u^{\prime}\right) \leq 2$. If $u^{\prime}=w$, then we are done. Hence we may assume that $u^{\prime} \neq w$. Since $L(u)=(1,1)$, we note that $u^{\prime} \notin D_{r-1}$, and so $L_{2}\left(u^{\prime}\right)=0$. Since $|B(v)|=1$, there is no vertex $y \in N_{G}(v) \backslash\{u\}$ such that $L_{2}(y) \neq 0$. By Observation $3, L_{1}(x)=1$ for all $x \in V\left(B_{j}\right) \backslash\left\{F_{j}\right\}$, where $j \in[r-1]$. This implies that every vertex of $V(G) \backslash V\left(B_{r}\right)$ is dominated by $D_{r-1}$. We note that $\left\{z \in N_{G}\left[u^{\prime}\right] \cap V\left(B_{r}\right) \mid L_{1}(z)=0\right\} \subseteq N_{G}[w] \cup N_{G}\left[D_{r-1}\right]$, i.e., the undominated vertices of $N_{G}\left[u^{\prime}\right]$ present in $V\left(B_{r}\right)$ are dominated by $D_{r-1} \cup\{w\}$.

Let $N_{2}\left(D^{\prime}, u^{\prime}\right)=\left\{x \mid x \in D^{\prime} \cap N_{G}^{2}\left(u^{\prime}\right)\right\}$ and $w \in V\left(B_{r}\right) \backslash\{u\}$. Let $p$ be an arbitrary vertex in $N_{2}\left(D^{\prime}, u^{\prime}\right)$. If $d_{G}(p, u) \leq 2$ or $d_{G}(p, w) \leq 2$, then $\left(D^{\prime} \backslash\left\{u^{\prime}\right\}\right) \cup\{w\}$ is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\{w\}$, as desired. Hence we may assume that $d_{G}(p, u)>2$ and $d_{G}(p, w)>2$. In this case, $p \in V\left(B_{q}\right)$ for some $q<k$, where $u^{\prime} \in V\left(B_{k}\right) \backslash\left\{F_{k}\right\}$. We note that $L(x)=(1,0)$ for every $x \in V\left(B_{k}\right) \backslash\left\{F_{k}\right\}$ since $L(u)=(1,1)$. Thus by Observation 4, either $L_{2}(p)=0$ (hence $p \notin D_{r-1}$ ) or $L(p)=(1,2)$.

Let $S=\left\{x \in N_{2}\left(D^{\prime}, u^{\prime}\right) \mid d_{G}(x, w)>2, d_{G}(x, u)>2\right.$, and $\left.L_{2}(x)=0\right\}$ and $S^{\prime}=\left\{x \in S \mid N_{G}^{2}(x) \cap D^{\prime}=\left\{u^{\prime}\right\}\right\}$. We note that each element of $S^{\prime}$ does not belong to $D_{r-1}$ and belongs to the blocks that appear before $r$. Moreover, $N_{G}\left[S^{\prime}\right] \subseteq N_{G}\left[D_{r-1}\right] \cup N_{G}[w]$. If $\left|S^{\prime}\right| \geq 2$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\{w\}$ is a semi-TD-set of $G$ of cardinality less than $\left|D^{\prime}\right|$, contradicting the minimality of $D^{\prime}$. Hence, $\left|S^{\prime}\right| \leq 1$. If $\left|S^{\prime}\right|=1$, then $\left(D^{\prime} \backslash S^{\prime}\right) \cup\{w\}$ is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\{w\}$, as desired. Hence we may assume that $S^{\prime}=\emptyset$. Since $d_{G}(u, w) \leq 2$, the set $\left(D^{\prime} \backslash\left\{u^{\prime}\right\}\right) \cup\{w\}$ is a minimum semi-TD-set of $G$ containing $D_{r-1} \cup\{w\}$. This completes the proof of Claim 9.

We now return to the proof of Theorem 6. Recall that by the induction hypothesis, the set $D_{i-1}$ is contained in some minimum semi-TD-set $D^{\prime}$ of $G$. Now assume that the algorithm is at the $i$-th iteration and let $B_{i}$ be the block of $G$ considered at the $i$-th iteration. If $L(v)=(0,0)$ for some $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$ and $i<r$, then the algorithm selects the vertex $F_{i}$ (see Lines 11-13 of the algorithm $\operatorname{MSTDS}-\operatorname{Block}(G)$ and notice that in the algorithm $L\left(F_{i}\right)$ is made $(1,2)$ or $(1,1))$. By Claim 7, $D_{i}=D_{i-1} \cup\left\{F_{i}\right\}$ is contained in some minimum semi-TD-set of $G$. If $L(v)=(1,0)$ for some $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$ and $i<r$, then the algorithm checks the set $A(v)$. If $|A(v)|>1$, then the algorithm does not select any new vertex; rather it makes $L_{2}(x)=2$ for the neighbor $x$ of $v$ if $L_{2}(x)=1$. Hence, $D_{i}=D_{i-1}$ and therefore the set $D_{i}$ is contained in the minimum semi-TD-set $D^{\prime}$ of $G$. If $|A(v)|=1$, then the algorithm selects $F_{i}$ (see Line 20 of the algorithm MSTDS-Block $(G)$ and notice that $L\left(F_{i}\right)$ is made ( 1,2 )). By Claim 8, $D_{i}=D_{i-1} \cup\left\{F_{i}\right\}$ is contained in some minimum semi-TD-set of $G$. If $i=r$, then by Claim 9 and 10, the set $D_{i}$ is contained in some minimum semi-TD-set of $G$. Therefore, by induction, $D_{r}$ is a minimum semi-TD-set of $G$. This completes the proof of Theorem 6.

By Theorem 6, the algorithm MSTDS-BLOck $(G)$ produces a minimum semi-TD-set of $G$. This establishes the correctness of the algorithm. We discuss next how a minimum semi-TD-set of a given block graph $G$ can be computed in linear time. If $G$ is complete, then as observed earlier, any two vertices in $G$ form a semi-TD-set of $G$, implying that $\gamma_{t 2}(G)=2$. If $G$ is not complete, then the algorithm $\operatorname{MSTDS}-\operatorname{Block}(G)$ is used to compute a minimum semi-TD-set of $G$. We now show that the implementation of MSTDS-BLock $(G)$ can be done in linear time.

Suppose that $G$ has blocks $B_{1}, B_{2}, \ldots, B_{r}$ and cut-vertices $c_{1}, c_{2}, \ldots, c_{s}$. A cut-tree $T_{G}$ of $G$ can be constructed in linear time [1]. Once a cut-tree is constructed, a RBFS-Block-Ordering of the blocks for $G$ can be obtained in $O(r+s)$ time. The algorithm uses two dimensional array $L$ on each vertex $v$ of $G$. This two dimensional array can be seen as two arrays $L_{1}$ and $L_{2}$. Here, we use the array notation (.) instead of [.] for $L_{1}$ and $L_{2}$ to avoid confusion as we mean the same labels $L_{1}$ and $L_{2}$ used in the algorithm. Initially, $L_{1}(v)=0=L_{2}(v)$ for every vertex $v$ of $G$. We also maintain an array $F$ on each block of $G$, where $F$ is defined with respect to the RBFS-Block-Ordering $\sigma$ of the blocks for $G$. In particular, for $i \in[r-1], F[i]=t$ if $c_{t}$ is the cut-vertex common to the blocks $B_{i}$ and $B_{i+1}$. At the $i$-th iteration, the algorithm considers the block $B_{i}$.

- If $L_{1}(v)=0=L_{2}(v)$ for some vertex $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, then $L_{1}\left(F_{i}\right)$ is made 1 and $L_{2}\left(F_{i}\right)$ is made 1 or 2 . This takes at most $O\left(V\left(B_{i}\right)+d_{G}\left(F_{i}\right)\right)=O\left(d_{G}\left(F_{i}\right)\right)$ time. Thus, $L_{1}(x)$ is made 1 for every vertex $x \in N_{G}\left[F_{i}\right]$, which takes $O\left(d_{G}\left(F_{i}\right)\right)$ time.
- If $L_{1}(v)=1$, and $L_{2}(v)=0$ for some vertex $v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}$, then $A(v)$ is computed which can be done in $O\left(d_{G}(v)\right)$ time. If $|A(v)|>1$, then $L_{2}(x)$ is made 2 for every $x \in N_{G}(v)$ such that $L_{2}(x)=1$. This update can be done in $O\left(d_{G}(v)\right)$ time. If $|A(v)|=1$, then $L_{2}(F[i])$ is made 2 and $L_{2}(x)$ is made 2 for every $x \in N_{G}\left(c_{t}\right)$ such that $L_{2}(x)=1$, where $F[i]=c_{t}$. This can be done in $O\left(d_{G}(v)+d_{G}\left(F_{i}\right)\right)$ time.

For $i \in[r-1]$, at the $i$-th iteration, the algorithm takes

$$
O\left(\sum_{v \in V\left(B_{i}\right) \backslash\left\{F_{i}\right\}} d_{G}(v)+d_{G}\left(F_{i}\right)\right)=O\left(\sum_{v \in V\left(B_{i}\right)} d_{G}(v)\right)
$$

time. Now consider the $r$-th iteration of the algorithm. If $L_{1}(v)=0=L_{2}(v)$ for some $v \in V\left(B_{r}\right)$, then $L_{2}\left(c_{j}\right)$ is made 2 and the $L$-label of the neighbors of $c_{j}$ is updated. This takes $O\left(d_{G}\left(c_{j}\right)\right)$ time. If $L_{1}(u)=1=L_{2}(u)$ for some $u \in N_{G}(v)$, then a vertex $w$ of $V\left(B_{r}\right)$ is chosen and the $L$-labels of the neighbors of $w$ and $v$ are updated. This takes $O\left(d_{G}(v)+d_{G}(w)\right)$ time. So in total at the $r$-th iteration, the algorithm takes

$$
O\left(\sum_{v \in V\left(B_{r}\right)} d_{G}(v)\right)
$$

time. From the above discussion, we conclude that the algorithm takes at most $O(|V(G)|+|E(G)|)$ time. Therefore, we have the following theorem.

Theorem 11. A minimum semitotal dominating set of a given block graph can be computed in linear time.

## 4. NP-Completeness

In this section, we show that the semitotal domination problem is NP-complete for undirected path graphs, a subclass of chordal graphs. The semitotal domination problem is shown to be NP-complete for chordal graphs [14]. Let $\mathcal{F}$ be a finite family of nonempty sets. A graph $G=(V, E)$ is called an intersection graph for $\mathcal{F}$ if there exists a one-to-one correspondence between $\mathcal{F}=\left\{A_{1}, A_{2}, \ldots, A_{n}\right\}$ and $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ such that $v_{i} v_{j} \in E$ if and only if $A_{i} \cap A_{j} \neq \emptyset$. A graph $G$ is called an undirected path graph if $G$ is an intersection graph of a family of undirected paths of a tree.

Given a graph $G$ and a positive integer $k$, the domination problem is to decide whether $G$ has a dominating set of cardinality at most $k$. We describe next a polynomial time reduction from the domination problem to the semitotal domination problem. Given a graph $G=(V, E)$, we construct another graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, where $V^{\prime}=V \cup\left\{x_{i}, y_{i}, z_{i}, p_{i}, q_{i} \mid i \in[n]\right\}$ and $E^{\prime}=E \cup\left\{v_{i} x_{i}, x_{i} y_{i}, y_{i} z_{i}, y_{i} p_{i}, p_{i} q_{i} \mid\right.$ $i \in[n]\}$. The construction of the graph $G^{\prime}$ from the graph $G$ is illustrated in Figure 2.


Figure 2. The constructed graph $G^{\prime}$ from the graph $G$.
Lemma 12. The graph $G$ has a dominating set of cardinality at most $k$ if and only if the graph $G^{\prime}$ has a semi-TD-set of cardinality at most $k+2 n$.

Proof. Let $D$ be a dominating set of cardinality at most $k$. Consider the set $D^{\prime}=D \cup\left\{y_{i}, p_{i} \mid i \in[n]\right\}$. We note that $D^{\prime}$ is a dominating set of $G^{\prime}$ with cardinality at most $k+2 n$. Since $d_{G}\left(y_{i}, p_{i}\right)=1$ and $d_{G}\left(y_{i}, v_{i}\right)=2$ for all $i \in[n]$, the set $D^{\prime}$ is a semi-TD-set of $G^{\prime}$.

To prove the converse, we first show that there is a semi-TD-set $D^{\prime}$ of $G$ of cardinality at most $k+2 n$ such that $y_{i}, p_{i} \in D^{\prime}$ and $x_{i}, z_{i}, q_{i} \notin D^{\prime}$ for all $i \in[n]$. Assume that $D^{\prime}$ is a minimum semi-TD-set of $G^{\prime}$ with cardinality at most $k+2 n$. Since $D^{\prime}$ is a semi-TD-set, $q_{i}$ or $p_{i} \in D^{\prime}$ in order to dominate $q_{i}$ and also $z_{i}$ or $y_{i} \in D^{\prime}$ in order to dominate $z_{i}$. Without loss of generality, we may assume that $y_{i}, p_{i} \in D^{\prime}$ for each $i \in[n]$. Also we may assume that $q_{i}, z_{i} \notin D^{\prime}$, for otherwise we can obtain another smaller semi-TD-set of $G^{\prime}$ of cardinality at most $k+2 n$
by removing $q_{i}$ and $z_{i}$. Now suppose that $x_{i} \in D^{\prime}$. We may assume that $v_{i} \notin D^{\prime}$, for otherwise we get another semi-TD-set of $G^{\prime}$ of cardinality at most $k+2 n$ by removing $x_{i}$ from $D^{\prime}$ as desired. With this assumption, the set $\left(D^{\prime} \backslash\left\{x_{i}\right\}\right) \cup\left\{v_{i}\right\}$ is also a semi-TD-set of $G$ with cardinality at most $k+2 n$. Hence without loss of generality, we assume that $x_{i}, z_{i}, q_{i} \notin D^{\prime}$ for all $i \in[n]$. Consider the set $D^{\prime \prime}=D^{\prime} \backslash\left\{y_{i}, p_{i} \mid i \in[n]\right\}$. The resulting set $D^{\prime \prime}$ is a dominating set of $G$ such that $\left|D^{\prime \prime}\right| \leq k$. This completes the proof of the lemma.

We now prove that the constructed graph $G^{\prime}$ is an undirected path graph. Suppose that $G$ is an undirected path graph having $n$ vertices. So by definition of undirected path graphs, there exists a tree $T$ and a family $\mathcal{P}$ of paths of $T$ such that $G$ is the intersection graph of the family of paths $\mathcal{P}$ of $T$. Let $T$ be a tree and $\mathcal{P}=\left\{P_{v_{i}} \mid i \in[n]\right\}$ be the family of distinct paths of $T$ such that $G$ is the intersection graph of the family of paths $\mathcal{P}$ of $T$. For each path $P_{v_{i}}$ of $T$, let $v_{i}^{*}$ be an end vertex of the path $P_{v_{i}}$. We construct two sets of paths by extending each $P_{v_{i}}$ at $v_{i}^{*}$. We extend $P_{v_{i}}$ at $v_{i}^{*}$ to $q_{i}$ and $z_{i}$ by attaching paths $v_{i}^{*} u_{i} x_{i} y_{i} a_{i} p_{i} q_{i}$ and $v_{i}^{*} u_{i} x_{i} y_{i} z_{i}$, respectively. Let $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ be the sets of paths obtained from each $P_{v_{i}}$ where $i \in[n]$ by extending $P_{v_{i}}$ at $v_{i}^{*}$ to $q_{i}$ and $z_{i}$, respectively. Suppose $T^{\prime}$ is the tree obtained from $T$ by introducing the sets of paths $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$. Let $P_{v_{i}}^{*}=P_{v_{i}} \cup\left\{v_{i}^{*} u_{i}\right\}$ for every $i \in[n]$ and let $\mathcal{P}^{*}=\left\{P_{v_{i}}^{*} \mid i \in[n]\right\}$. The graph $G^{\prime}$ is now the intersection graph of the family of paths $\mathcal{P}^{*} \cup\left\{x_{i} y_{i} a_{i} \mid i \in[n]\right\} \cup\left\{u_{i} x_{i}, a_{i} p_{i}, p_{i} q_{i}, y_{i} z_{i} \mid i \in[n]\right\}$ of $T^{\prime}$. Therefore, $G^{\prime}$ is an undirected path graph. We note that the path $P_{v_{i}}^{*}$ in $T^{\prime}$ corresponds to the vertex $v_{i}$, the path $x_{i} y_{i} a_{i}$ in $T^{\prime}$ corresponds to the vertex $y_{i}$, and the paths $u_{i} x_{i}, a_{i} p_{i}, p_{i} q_{i}, y_{i} z_{i}$ in $T^{\prime}$ correspond to the vertices $x_{i}, p_{i}, q_{i}, z_{i}$, respectively.

The domination problem is shown to be NP-complete for undirected path graphs [2]. Therefore as an immediate consequence of Lemma 12, we have the following theorem.

Theorem 13. The semitotal domination problem is NP-complete for undirected path graphs.

## 5. Conclusion

In this paper, we considered the complexity of finding a minimum semi-TD-set in block graphs and present a linear time algorithm for this problem. On the other hand, we proved that the decision version of finding a minimum semi-TD-set is NP-complete in undirected path graphs, which is a superclass of block graphs. We note that strongly chordal graphs form a superclass of the block graphs. It would therefore be interesting to raise the problem to study the complexity of finding a minimum semitotal dominating set in strongly chordal graphs.

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