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# THE CROSSING NUMBER OF HEXAGONAL GRAPH $H_{3,n}$ IN THE PROJECTIVE PLANE

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# Abstract

Thomassen described all (except finitely many) regular tilings of the torus  $S_1$  and the Klein bottle  $N_2$  into (3,6)-tilings, (4,4)-tilings and (6,3)-tilings. Many researchers made great efforts to investigate the crossing number of the Cartesian product of an *m*-cycle and an *n*-cycle, which is a special kind of (4,4)-tilings, either in the plane or in the projective plane. In this paper we study the crossing number of the hexagonal graph  $H_{3,n}$   $(n \geq 2)$ , which is a special kind of (3,6)-tilings, in the projective plane, and prove that

$$cr_{N_1}(H_{3,n}) = \begin{cases} 0, & n=2, \\ n-1, & n \ge 3. \end{cases}$$

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# 1. INTRODUCTION

The projective plane,  $N_1$ , is a 2-manifold obtained by identifying every point of the 2-sphere with its antipodal point.

Let G be a graph with vertex set V and edge set E. The crossing number of G in a surface  $\Sigma$ , denoted by  $cr_{\Sigma}(G)$ , is the minimum number of pairwise intersections of edges in a drawing of G in the surface  $\Sigma$ . In particular, the crossing number of G in the plane  $S_0$  is denoted by cr(G) for simplicity. It is well known that the crossing number of a graph in the surface  $\Sigma$  is attained only in good drawings of the graph, which are the drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point.

In [16], Thomassen described all (except finitely many) regular tilings of the torus  $S_1$  and the Klein bottle  $N_2$  into hexagons, quadrilaterals and triangles in which the vertices have degree 3, 4 and 6, respectively. To be more specific, let G be a connected d-regular graph ( $d \ge 3$ ) and  $\varphi$  a collection of m-cycles in G. Assume that each edge of G is contained in precisely two cycles in  $\varphi$  and that, for each vertex v in G, the edges incident with v can be labelled  $e_1, e_2, \ldots, e_d$  such that for each  $i = 1, 2, \ldots, d$ , there is a cycle in  $\varphi$  containing  $e_i$  and  $e_{i+1}$  (where  $e_{d+1} = e_1$ ). Then a surface  $\Sigma$  can be obtained by letting the cycles in  $\varphi$  be disjoint convex polygons in the Euclidean plane pasted together by the graph G, and G is said to be a (d, m)-tiling of  $\Sigma$ . Using Euler's formula, Thomassen observed that a regular tiling of the torus or the Klein bottle fit into three categories: (3,6)-tilings, (4,4)-tilings and (6,3)-tilings.

Note that the Cartesian product graph  $C_m \Box C_n$  is a special kind of (4,4)tilings, which can be embedded in orientable surface  $S_k$   $(k \ge 1)$  and non-orientable surface  $N_k$   $(k \ge 2)$ , but cannot be embedded in the plane  $S_0$  or projective plane  $N_1$ . This fact motivates many researchers' intensive interest to determine the exact value of  $cr(C_m \Box C_n)$  or  $cr_{N_1}(C_m \Box C_n)$ . However, computing the crossing number of a given graph is an elusive problem [3], therefore, the results concerning on this topic is quite limited. The crossing number  $cr(C_m \Box C_n)$  has been obtained for all but finitely many n, for each m [1, 2, 4, 9, 11, 13], while  $cr_{N_1}(C_m \Box C_n)$  has been determined only when min  $\{m, n\} = 3$  [14].

Compared with (4,4)-tilings, the crossing number of other regular tilings, either in the plane or in the projective plane, have not been extensively studied in the literature yet. Based on this observation, we began to study this problem.

Very recently, we have determined the crossing number of the hexagonal graph  $H_{3,n}$ , which is a special kind of (3,6)-tilings, in the plane [17]. To further the study, this paper is devoted to investigate the crossing number of  $H_{3,n}$  in the projective plane. The main theorem is the following.

**Theorem 1.** For  $n \ge 2$ , the crossing number of the hexagonal graph  $H_{3,n}$  in the projective plane  $N_1$  is

$$cr_{N_1}(H_{3,n}) = \begin{cases} 0, & n=2, \\ n-1, & n \ge 3. \end{cases}$$

Note that there are only few infinite classes of graphs whose crossing numbers in a surface other than the plane are known exactly, see [5, 6, 7, 8, 12, 14, 15] and the references therein.

This paper is organized as follows. Section 2 is devoted to introduce some basic preliminaries. By investigating the upper and lower bound of  $cr_{N_1}(H_{3,n})$  separately, we give the sketch of the proof of Theorem 1 in Section 3 by assuming the correctness of Lemma 6. We postpone the proof of Lemma 6 on the lower bound of  $cr_{N_1}(H_{3,n})$  to Section 5. In Section 4, we list some lemmas which are used in the proof of Lemma 6.

### 2. Preliminaries

We introduce some basic preliminaries in this section. For  $F \subseteq E(G)$ , we denote by  $G \setminus F$  the graph obtained from G by deleting all edges in F.

Let D be a good drawing of graph G in the surface  $\Sigma$ . We denote the number of crossings in D by  $v_D(G)$ . In a drawing D of graph G, if an edge is not crossed by any other edge, we say that it is *clean* in D, otherwise, we say it is *crossed*. Let A and B be two (not necessary disjoint) subsets of the edge set E(G). The number of crossings involving an edge in A and another edge in B is denoted by  $v_D(A, B)$ . In particular,  $v_D(A, A)$  is denoted by  $v_D(A)$ . By counting the number of crossings in D, we have the following result.

**Lemma 2.** Let A, B, C be mutually disjoint subsets of E(G). Then

$$v_D(A, B \cup C) = v_D(A, B) + v_D(A, C),$$
  
 $v_D(A \cup B) = v_D(A) + v_D(A, B) + v_D(B).$ 

The hexagonal graph  $H_{3,n}$   $(n \geq 2)$  is a special kind of (3,6)-tilings, which can be embedded in the torus such that the number of 6-cycles in the meridional (respectively, longitudinal) direction is 3 (respectively, n). To be exact,  $H_{3,n}$  is the graph with vertex set

$$V(H_{3,n}) = \{a_i, b_i, c_i | i = 1, 2, \dots, 2n\},\$$

and edge set

$$E(H_{3,n}) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1} | i = 1, 2, \dots, 2n\}$$
$$\cup \{a_{2i-1} b_{2i-1}, b_{2i} c_{2i}, a_{2i} c_{2i-1} | i = 1, 2, \dots, n\},\$$

where the indices are expressed modulo 2n, see Figure 1. Note that the graph  $H_{3,n}$  is vertex-transitive.



Figure 1. The graph  $H_{3,n}$ .

Figure 2. The subgraph  $F_i$ .

Let

$$E' = \bigcup_{i=1}^{n} \{a_{2i-1}b_{2i-1}, b_{2i}c_{2i}, a_{2i}c_{2i-1}\}.$$

Observe that, for any edge  $e \in E'$ ,  $H_{3,n} \setminus e$  contains a subdivision of  $H_{3,n-1}$ . For  $1 \leq i \leq n$ , let

$$F_{i} = \{a_{2i-2}a_{2i-1}, a_{2i-1}a_{2i}, b_{2i-2}b_{2i-1}, b_{2i-1}b_{2i}, c_{2i-2}c_{2i-1}, c_{2i-1}c_{2i}, a_{2i-1}b_{2i-1}, b_{2i}c_{2i}, a_{2i}c_{2i-1}\},\$$

where the indices are read modulo 2n, see Figure 2. Then  $F_1, F_2, \ldots, F_n$  is a partition of  $E(H_{3,n})$ , which is to say,

$$E(H_{3,n}) = \bigcup_{i=1}^{n} F_i$$
, and, for  $i \neq j$ ,  $F_i \cap F_j = \emptyset$ .

Let D be a good drawing of  $H_{3,n}$  in the projective plane. We define  $f_D(F_i)$  $(1 \le i \le n)$  to be the function counting the number of crossings related to  $F_i$  in D as follows [8],

(1) 
$$f_D(F_i) = v_D(F_i, F_i) + \frac{1}{2} \sum_{1 \le j \le n, \ j \ne i} v_D(F_i, F_j).$$

By counting the number of crossings in D, we can get the following lemma. Lemma 3.  $v_D(H_{3,n}) = \sum_{i=1}^n f_D(F_i)$ .

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## 3. Sketch of the Proof of Theorem 1

**Lemma 4.** For  $n \ge 2$ ,  $cr_{N_1}(H_{3,n}) \le n-1$ .

**Proof.** Wilson's Lemma [14] states that, for a non-planar graph, its crossing number in the projective plane is strictly less than its crossing number in the plane. Therefore,  $cr_{N_1}(H_{3,n}) \leq cr(H_{3,n}) - 1$ . Combined this fact with the result that  $cr(H_{3,n}) = n$  for  $n \geq 2$  [17], Lemma 4 holds.



 $a_1$   $b_2$   $b_1$   $b_2$   $c_2$   $c_3$   $c_4$   $c_5$ 

Figure 3. A good drawing of  $H_{3,2}$  in the projective plane.

Figure 4. The graph  $H_{3,3} \setminus \{a_2c_1, a_3b_3, b_4b_5, b_6c_6\}$ .

Figure 3 shows that  $H_{3,2}$  is embeddable in the projective plane. That implies the result.

Lemma 5.  $cr_{N_1}(H_{3,2}) = 0.$ 

The following lemma is the key to the proof of Theorem 1.

**Lemma 6.** For  $n \ge 3$ ,  $cr_{N_1}(H_{3,n}) \ge n - 1$ .

We will prove Lemma 6 by induction on n. Thus, the induction basis is needed at first.

# Lemma 7. $cr_{N_1}(H_{3,3}) = 2.$

**Proof.** We only need to prove that  $cr_{N_1}(H_{3,3}) \ge 2$  by Lemma 4. Suppose that there is a good drawing D of  $H_{3,3}$  in the projective plane such that  $v_D(H_{3,3}) \le 1$ . By Figure 4 and Figure 5, we have  $v_D(H_{3,3}) = 1$  since the graph  $H_{3,3} \setminus \{a_2c_1, a_3b_3, b_4b_5, b_6c_6\}$  is a subdivision of  $G_1(10, 15)$ , which is one of the minimal forbidden subgraphs for the projective plane (see Appendix A in [10]).

Thus, we can get a graph which can be embedded in the projective plane from D by removing one of the crossed edges. However, Figure 4 and Figure 5





Figure 5. The graph  $G_1(10, 15)$ .

Figure 6. A possible subdrawing of  $R_i$ .

illustrate that  $H_{3,3} \setminus e$  contains a subdivision of  $G_1(10, 15)$  for any  $e \in E(H_{3,3})$ . A contradiction.

We postpone the proof of Lemma 6 to Section 5. With its proof, we can prove the main result of this paper.

*The proof of Theorem* **1.** Combined with Lemma 4, Lemma 5 and Lemma 6, Theorem 1 follows easily.

### 4. Some Lemmas

Our aim in this section is to provide a definition and some basic lemmas which will be used to prove Lemma 6.

**Definition 8.** An E'-clean drawing of  $H_{3,n}$  is a good drawing of  $H_{3,n}$  in the projective plane such that all of the edges in E' are clean.

We will prove Lemma 6 by induction on n and by contradiction. If there exists a good drawing D of  $H_{3,n}$  in the projective plane such that  $v_D(H_{3,n}) < n - 1$ , then we can obtain that all of the edges in E' are clean in D. Therefore, in this section, we only consider the E'-clean drawing of  $H_{3,n}$ .

For  $1 \leq i \leq n$ , let

$$C_{F_i} = \{a_{2i-1}a_{2i}, a_{2i}c_{2i-1}, c_{2i-1}c_{2i}, c_{2i}b_{2i}, b_{2i}b_{2i-1}, b_{2i-1}a_{2i-1}\},\$$

and

$$R_i = C_{F_i} \cup \{b_{2i-2}b_{2i-1}, b_{2i-2}c_{2i-2}, c_{2i-2}c_{2i-1}\},\$$

where the indices are expressed modulo 2n. Note that  $C_{F_i}$  is the cycle of length 6 in  $F_i$ , and that  $R_i = F_i \cup \{b_{2i-2}c_{2i-2}\} \setminus a_{2i-2}a_{2i-1}$ . The following lemma states that the subdrawing of  $R_i$  may be unique under certain restrictions.

**Lemma 9.** For  $n \ge 3$ , let D be an E'-clean drawing of  $H_{3,n}$  such that, for some  $1 \le i \le n$ ,  $f_D(F_i) < 1$  and  $C_{F_i}$  is contractible. Then the subdrawing of  $R_i$  in D is as shown in Figure 8.

**Proof.** By Equation (1), the edges of  $F_i$  do not have internal crossings in D since  $f_D(F_i) < 1$ . Furthermore, the vertices  $b_{2i-2}$  and  $c_{2i-2}$  lie in the same region since the edge  $b_{2i-2}c_{2i-2} \in E'$  is clean. Hence, there are three possibilities of the subdrawing of  $R_i$  in D, see Figures 6, 7, and 8.





Figure 7. A possible subdrawing of  $R_i$ .

Figure 8. A possible subdrawing of  $R_i$ .

Suppose that  $R_i$  is as drawn in Figure 6. Consider the vertex  $a_{2i-2}$ . This vertex lies either in the region labelled  $f_1$  or in the region  $f_2$ , since the edge  $a_{2i-2}a_{2i-1} \in F_i$  does not have any crossing with  $R_i$ . If  $a_{2i-2}$  lies in  $f_1$ , then each of the edge-disjoint paths  $a_{2i-2}c_{2i-3}\cdots c_1c_{2n}c_{2n-1}\cdots c_{2i}$  and  $a_{2i-2}a_{2i-3}b_{2i-3}b_{2i-4}\cdots b_1b_{2n}b_{2n-1}\cdots b_{2i}$  crosses  $F_i$  at least once, which implies that  $f_D(F_i) \geq 1$  by Equation (1), contradicting that  $f_D(F_i) < 1$ . Finally, if  $a_{2i-2}$  lies in  $f_2$ , then each of the edge-disjoint paths  $a_{2i-2}c_{2i-3}c_{2i-2}$  and  $a_{2i-2}a_{2i-3}b_{2i-2}$  crosses  $F_i$  at least once, which implies that  $f_D(F_i) \geq 1$  by Equation (1), contradicting that  $f_D(F_i) < 1$ . Finally, if  $a_{2i-2}$  lies in  $f_2$ , then each of the edge-disjoint paths  $a_{2i-2}c_{2i-3}c_{2i-2}$  and  $a_{2i-2}a_{2i-3}b_{2i-2}$  crosses  $F_i$  at least once, which implies that  $f_D(F_i) \geq 1$ , a contradiction.

Almost the same argument can be obtained if  $R_i$  is as drawn in Figure 7. Therefore, the subdrawing of  $R_i$  in D is as shown in Figure 8.

In Figure 8, observe that there are two non-contractible cycles  $C(i) \triangleq b_{2i-2}b_{2i-1}b_{2i}c_{2i}c_{2i-1}c_{2i-2}b_{2i-2}$  and  $C(i') \triangleq b_{2i-2}b_{2i-1}a_{2i-1}a_{2i}c_{2i-1}c_{2i-2}b_{2i-2}$  in  $R_i$ , furthermore,  $E(C(i)) \cap E(C(i')) = \{b_{2i-2}b_{2i-1}, c_{2i-2}c_{2i-1}, b_{2i-2}c_{2i-2}\}$ . Thus, the following corollary holds.

**Corollary 10.** For  $n \ge 3$ , let D be an E'-clean drawing of  $H_{3,n}$ . If there exist different integers i and j such that

- (1)  $j \neq i \pm 1 \pmod{n}$ ,
- (2) both  $C_{F_i}$  and  $C_{F_i}$  are contractible,
- (3)  $f_D(F_i) = f_D(F_j) = \frac{1}{2}$ ,
- (4)  $v_D(F_i, F_j) = 1$ ,

then the crossed edge of  $F_i$  (respectively,  $F_j$ ) in D must be either  $b_{2i-2}b_{2i-1}$  or  $c_{2i-2}c_{2i-1}$  (respectively, either  $b_{2j-2}b_{2j-1}$  or  $c_{2j-2}c_{2j-1}$ ).

**Proof.** Firstly, note that  $V(F_i) \cap V(F_j) = \emptyset$  since  $j \neq i \pm 1 \pmod{n}$ . Secondly, by Lemma 9, the subdrawing of  $R_i$  in D is as shown in Figure 8. Similarly, the subdrawing of  $R_j$  is also as shown in Figure 8 by replacing all the indices i by j.

Note that two non-contractible cycles cross each other in the projective plane, therefore, the crossed edge of  $F_i$  (respectively,  $F_j$ ) in D must belong to  $E(C(i)) \cap E(C(i'))$  (respectively,  $E(C(j)) \cap E(C(j'))$ ) since  $f_D(F_i) = f_D(F_j) = \frac{1}{2}$  and  $v_D(F_i, F_j) = 1$ . Hence, the corollary follows because the edge  $b_{2i-2}c_{2i-2} \in E'$  (respectively,  $b_{2j-2}c_{2j-2} \in E'$ ) is clean.

Lemma 11 and Lemma 13 in the following indicate that  $f_D(F_3) \ge 1$  if  $f_D(F_2) = 0$  by considering whether the cycle  $C_{F_2}$  is contractible or not.

**Lemma 11.** For  $n \ge 3$ , let D be an E'-clean drawing of  $H_{3,n}$  such that  $f_D(F_2) = 0$  and the cycle  $C_{F_2}$  is contractible. Then  $f_D(F_3) \ge 1$ .

**Proof.** By the assumption that  $f_D(F_2) = 0$  and by Lemma 9, the subdrawing of  $R_2$  in D is as shown in Figure 8 by replacing all the indices i by 2. Moreover, since  $b_2c_2 \in E'$  is clean, we have the following.

**Claim 12.**  $R_2$  is clean in *D*. In particular, the non-contractible cycle  $C(2) = b_2b_3b_4c_4c_3c_2b_2$  is clean.

We prove the lemma by contradiction. Suppose that  $f_D(F_3) < 1$ . Then the edges of  $F_3$  cannot have internal crossings in D by Equation (1). The cycle  $C_{F_3}$ is contractible, otherwise, two non-contractible cycles  $C_{F_3}$  and C(2) cross each other in the projective plane, contradicting Claim 12. Moreover, all of the vertices of  $C_{F_3}$  lie in the same region labelled  $f_1$  in the subdrawing of  $R_2$  in Figure 8, otherwise, without loss of generality, assume that the vertex  $a_5$  does not lie in  $f_1$ , then the cycle  $C_{F_2}$  will be crossed by the path  $a_5b_5b_6\cdots b_{2n}b_1b_2$   $(n \geq 3)$ , contradicting Claim 12. Therefore, the subdrawing of  $R_2 \cup C_{F_3}$  is as shown either in Figure 9 or in Figure 10.

Case 1. The subdrawing of  $R_2 \cup C_{F_3}$  is as shown in Figure 9. Consider the subdrawing of  $R_2 \cup F_3$ . It is as shown in Figure 11 by Claim 12 and by the assumption that  $f_D(F_3) < 1$ . Note that  $c_2$  and  $c_6$  (respectively,  $b_2$  and  $b_6$ ) do not lie on the boundary of a same region. Thus each of the edge-disjoint



Figure 9. A possible subdrawing of  $R_2 \cup C_{F_3}.$ 





Figure 10. A possible subdrawing of  $R_2 \cup C_{F_3}$ .



Figure 11. The subdrawing of  $R_2 \cup F_3$ . Figure 12. The subdrawing of  $R_2 \cup C_{F_3} \cup C_{F_3}$  $\{a_4a_5, c_4c_5\}.$ 

paths  $c_6 \cdots c_{2n} c_1 c_2$  and  $b_6 \cdots b_{2n} b_1 b_2$  will cross  $F_3$  at least once, which implies that  $f_D(F_3) \ge 1$ , a contradiction.

Case 2. The subdrawing of  $R_2 \cup C_{F_3}$  is as shown in Figure 10. Consider the subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$ . It is as shown in Figure 12 by Claim 12 and by the assumption that  $f_D(F_3) < 1$ . Note that two vertices  $b_4$  and  $b_5$  do not lie on the boundary of a same region. Thus the edge  $b_4b_5 \in F_3$  will cross some edge of  $F_3$ , which implies that  $f_D(F_3) \ge 1$  by Equation (1), a contradiction. 

**Lemma 13.** For  $n \geq 3$ , let D be an E'-clean drawing of  $H_{3,n}$  such that  $f_D(F_2)$ = 0 and the cycle  $C_{F_2}$  is non-contractible. Then  $f_D(F_3) \ge 1$ .

**Proof.** Since  $f_D(F_2) = 0$  and  $b_2c_2 \in E'$  is clean, we conclude the following.

Claim 14. The edges of  $R_2$  are clean in D.

There are two possible subdrawing of  $R_2$  in D since  $C_{F_2}$  is non-contractible, see Figure 13 and Figure 14.



Figure 13. A possible subdrawing of  $R_2$ .

Figure 14. A possible subdrawing of  $R_2$ .

We prove the lemma by contradiction. Suppose that  $f_D(F_3) < 1$ . Then the edges of  $F_3$  do not have internal crossings in D. The cycle  $C_{F_3}$  must be contractible, otherwise, two non-contractible curves  $C_{F_3}$  and  $C_{F_2}$  cross each other in the projective plane, contradicting that  $f_D(F_2) = 0$ . All of the vertices of  $F_3$ lie in the same region in the subdrawing of  $R_2$  by Claim 14, moreover, they locate in the region labelled  $f_1$ , on whose boundary lies all the vertices of  $R_2$ , see Figure 13 and Figure 14.



Figure 15. A possible subdrawing of  $R_2 \cup C_{F_3}$ .

Figure 16. A possible subdrawing of  $R_2 \cup C_{F_3}$ .

Case 1.  $R_2$  is as drawn in Figure 13. As we mentioned above, the subdrawing of  $R_2 \cup C_{F_3}$  is as shown either in Figure 15 or in Figure 16.

Subcase 1.1. The subdrawing of  $R_2 \cup C_{F_3}$  is as shown in Figure 15. Then

the subdrawing of  $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$  is as shown in Figure 17. In this case,  $a_4$  and  $a_5$  do not lie on the boundary of a same region. By Claim 14, there is at least one crossing made by the edge  $a_4a_5 \in F_3$  with some edge of  $F_3$ , which implies that  $f_D(F_3) \geq 1$  by Equation (1), a contradiction.

Subcase 1.2. The subdrawing of  $R_2 \cup C_{F_3}$  is as shown in Figure 16. Then the subdrawing of  $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$  is as shown in Figure 18. Note that the vertices  $b_2$  and  $b_6$  (respectively,  $c_2$  and  $c_6$ ) do not lie on the boundary of a same region, therefore, each of the edge-disjoint paths  $b_6 \cdots b_{2n}b_1b_2$  and  $c_6 \cdots c_{2n}c_1c_2$  crosses  $F_3$  at least once, which enforces that  $f_D(F_3) \geq 1$ , a contradiction.





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Figure 17. The subdrawing of  $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$ .

Figure 18. The subdrawing of  $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$ .

Case 2.  $R_2$  is as drawn in Figure 14. Using the former similar arguments, we assert that the subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5\}$  is as shown either in Figure 19 or in Figure 20.

Subcase 2.1. The subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5\}$  is as shown in Figure 19. Consider the edge  $b_4b_5$ .

Subcase 2.1.1. The edge  $b_4b_5$  is as drawn in Figure 21. By Claim 14, we have  $f_D(F_3) \ge 1$  by the observation that  $c_4$  and  $c_5$  do not lie on the boundary of a same region, a contradiction.

Subcase 2.1.2. The edge  $b_4b_5$  is as drawn in Figure 22. By Claim 14, the edge  $c_4c_5$  cannot cross the edges of  $R_2 \cup F_3$  since  $c_4c_5 \in F_3$ . Then the subdrawig of  $R_2 \cup F_3$  is as drawn in Figure 22.

Now consider the vertex  $a_2$ . It lies either in the region labelled  $f_2$  or in the region  $f_{11}$ , since the edge  $a_2a_3 \in F_2$  is clean, see Figure 22.

Subcase 2.1.2.1. If  $a_2$  lies in  $f_2$ . Notice that the vertices  $a_2$  and  $c_6$  do not lie on the boundary of a same region in the subdrawing of  $R_2$ . Then the path



Figure 19. A possible subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5\}.$ 



Figure 21. The subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, b_4b_5\}.$ 



Figure 20. A possible subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5\}.$ 



Figure 22. The subdrawing of  $R_2 \cup F_3$ .

 $c_6 \cdots c_{2n} c_1 a_2$  will cross  $R_2$  at least once, contradicting Claim 14.

Subcase 2.1.2.2. If  $a_2$  lies in  $f_{11}$ . Notice that the vertices  $a_2$  and  $c_2$  (respectively,  $a_2$  and  $b_2$ ) do not lie on the boundary of a same region. Then each of the edge-disjoint paths  $a_2c_1c_2$  and  $a_2a_1b_1b_2$  will cross  $F_3$  at least once, contradicting that  $f_D(F_3) < 1$ .

Subcase 2.2. The subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5\}$  is as shown in Figure 20. Now consider the edge  $c_4c_5$ . There are two possibilities of the subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$ , see Figure 23 and Figure 24.

Subcase 2.2.1. The subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$  is as shown in Figure 23. By Claim 14, we have  $f_D(F_3) \ge 1$  since the vertices  $b_2$  and  $b_6$  (respectively,  $c_2$  and  $c_6$ ) do not lie on the boundary of a same region, a contradiction.



Figure 23. A possible subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}.$ 



Figure 24. A possible subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}.$ 

Subcase 2.2.2. The subdrawing of  $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$  is as shown in Figure 24. Then there are two possibilities of  $R_2 \cup F_3$ , see Figure 25 and Figure 26.



Figure 25. A possible subdrawing of  $R_2 \cup F_3$ .



Figure 26. A possible subdrawing of  $R_2 \cup F_3$ .

Subcase 2.2.2.1. The subdrawing of  $R_2 \cup F_3$  is as shown in Figure 25. By Claim 14, we have  $f_D(F_3) \ge 1$  since the vertices  $b_2$  and  $b_6$  (respectively,  $c_2$  and  $c_6$ ) do not lie on the boundary of a same region, a contradiction.

Subcase 2.2.2.2. The subdrawing of  $R_2 \cup F_3$  is as shown in Figure 26. Consider the vertex  $a_2$ , contradictions can be made by discussions similar to Subcase 2.1.2. All these contradictions enforce that  $f_D(F_3) \ge 1$ .

Lemma 15 and Lemma 16 in the following imply that  $f_D(F_1) > 0$  if  $f_D(F_2) = 0$ , by considering whether the cycle  $C_{F_2}$  is contractible or not.

**Lemma 15.** For  $n \ge 3$ , let D be an E'-clean drawing of  $H_{3,n}$  such that  $f_D(F_2) = 0$  and the cycle  $C_{F_2}$  is contractible. Then  $f_D(F_1) > 0$ .

**Proof.** By Lemma 9, the subdrawing of  $R_2$  is as drawn in Figure 8 by replacing all the indices i by 2. Moreover, the vertex  $a_2$  lies in the region labelled  $f_1$  in the subdrawing of  $R_2$  in Figure 8, otherwise, the path  $a_2c_1c_2$  will cross  $C_{F_2}$  at least once, contradicting that  $f_D(F_2) = 0$ .

We prove the lemma by contradiction. Suppose that  $f_D(F_1) = 0$ , then all of the vertices of  $F_1$  lie in the same region. There are three possibilities of the subdrawing of  $C_{F_1} \cup F_2$ , see Figures 27, 28 and 29.





Figure 27. A possible subdrawing of  $C_{F_1} \cup F_2$ .

Figure 28. A possible subdrawing of  $C_{F_1} \cup F_2$ .

If  $C_{F_1} \cup F_2$  is as drawn in Figure 27, then either  $C_{F_1}$  or  $F_2$  will be crossed by the path  $a_4a_5 \cdots a_{2n}a_1$  since the vertices  $a_4$  and  $a_1$  do not lie on the boundary of a same region, contradicting that  $f_D(F_1) = f_D(F_2) = 0$ . Similar argument can be made if  $C_{F_1} \cup F_2$  is as drawn in Figure 28. Finally, if  $C_{F_1} \cup F_2$  is as drawn in Figure 29, then either  $C_{F_1}$  or  $F_2$  will be crossed by the path  $a_4a_5 \cdots a_{2n}c_{2n-1}c_{2n}c_1$ (note that  $n \geq 3$  is crucial here for  $c_{2n}$  being not equal to  $c_4$ ), a contradiction.

All these contradictions enforce that  $f_D(F_1) > 0$ .

**Lemma 16.** For  $n \ge 3$ , let D be an E'-clean drawing of  $H_{3,n}$  such that  $f_D(F_2) = 0$  and the cycle  $C_{F_2}$  is non-contractible. Then  $f_D(F_1) > 0$ .

**Proof.** We prove the lemma by contradiction. Suppose that  $f_D(F_1) = 0$ . Then the cycle  $C_{F_1}$  is contractible, otherwise, two non-contractible cycles  $C_{F_2}$  and  $C_{F_1}$  cross each other at least once in the projective plane, contradicting that  $f_D(F_2) = 0$ .

The subdrawings of  $R_2$  is as drawn either in Figure 13 or in Figure 14 by similar arguments in Lemma 13. Moreover, the vertices of  $C_{F_1}$  lie in the region



 $a_{2}$   $a_{1}$   $a_{3}$   $b_{1}$   $b_{3}$   $b_{4}$   $c_{4}$   $c_{3}$ 

Figure 29. A possible subdrawing of  $C_{F_1} \cup F_2$ .

Figure 30. A possible subdrawing of  $C_{F_1} \cup F_2$ .

labelled  $f_1$  if  $R_2$  is as drawn in Figure 13. Therefore, there are four possibilities of the subdrawing of  $C_{F_1} \cup F_2$ , see Figures 30, 31, 32 and 33.



Figure 31. A possible subdrawing of  $C_{F_1} \cup F_2$ .



Figure 32. A possible subdrawing of  $C_{F_1} \cup F_2$ .

Case 1. The subdrawing of  $C_{F_1} \cup F_2$  is as drawn in Figure 30. Since the vertices  $a_4$  and  $a_1$  do not lie on the boundary of a same region, the path  $a_4a_5 \cdots a_{2n}a_1$  will cross either  $C_{F_1}$  or  $F_2$ , contradicting that  $f_D(F_1) = f_D(F_2) = 0$ .

Case 2. The subdrawing of  $C_{F_1} \cup F_2$  is as drawn in Figure 31. Then the path  $c_4c_5 \cdots c_{2n}c_1$  will cross either  $C_{F_1}$  or  $F_2$ , a contradiction.

Case 3. The subdrawing of  $C_{F_1} \cup F_2$  is as drawn in Figure 32. Then the path  $c_1c_{2n}c_{2n-1}a_{2n}a_{2n-1}\cdots a_4$  will cross either  $C_{F_1}$  or  $F_2$  (note that  $n \ge 3$  is crucial here for  $c_{2n}$  being not equal to  $c_4$ ), a contradiction.

Case 4. The subdrawing of  $C_{F_1} \cup F_2$  is as drawn in Figure 33. Then the path  $b_4b_5 \cdots b_{2n}b_1$  will cross either  $C_{F_1}$  or  $F_2$ , a contradiction.

All these contradictions enforce that  $f_D(F_1) > 0$ .







Figure 34. A possible subdrawing of  $R_2 \cup R_j$ .

We end this section with the following lemma which indicates that  $F_i$  do not cross  $F_j$  under certain restrictions.

**Lemma 17.** For  $n \ge 4$ , let D be an E'-clean drawing of  $H_{3,n}$ . If there exist different integers i and j such that

(1)  $j \neq i \pm 1 \pmod{n}$ ,

(2) 
$$f_D(F_i) = f_D(F_j) = \frac{1}{2}$$

(3) at least one of the cycles  $C_{F_i}$  and  $C_{F_j}$  is contractible,

then  $v_D(F_i, F_j) = 0$ .

**Proof.** Suppose to the contrary that  $v_D(F_i, F_j) > 0$ . Without loss of generality, assume that i = 2 and that  $4 \le j \le n$ , moreover, assume that  $C_{F_2}$  is contractible. Then, by Equation (1),  $v_D(F_2, F_j) = 1$  since  $f_D(F_2) = f_D(F_j) = \frac{1}{2}$ .

The edges of  $F_2$  (respectively,  $F_j$ ) do not have internal crossings since  $f_D(F_2) = \frac{1}{2}$  (respectively,  $f_D(F_j) = \frac{1}{2}$ ). By Lemma 9, the subdrawing of  $R_2$  is as drawn in Figure 8 by replacing all the indices *i* by 2. Note that the cycle  $C(2) = b_2 b_3 b_4 c_4 c_3 c_2 b_2$  is non-contractible, furthermore,  $V(F_2) \cap V(F_j) = \emptyset$ .

We consider the following two cases.

Case 1. The cycle  $C_{F_j}$  is contractible. The subdrawing of  $R_j$  is also as shown in Figure 8 by replacing all the indices i by j. Observe that the crossed edge of  $F_2$ (respectively,  $F_j$ ) is either  $c_2c_3$  or  $b_2b_3$  (respectively,  $c_{2j-2}c_{2j-1}$  or  $b_{2j-2}b_{2j-1}$ ) by Corollary 10. We may assume that the crossed edge of  $F_2$  is  $c_2c_3$  (if the crossed edge of  $F_2$  is  $b_2b_3$ , the proof is analogous). Moreover, all of the vertices of  $C_{F_j}$  lie in the region labelled  $f_1$  in the subdrawing of  $R_2$  in Figure 8.

Subcase 1.1. The crossed edge of  $F_j$  is  $c_{2j-2}c_{2j-1}$ . Then the subdrawing of  $R_2 \cup R_j$  is as drawn in Figure 34, either  $F_2$  or  $F_j$  will be crossed by the path  $a_{2j}a_{2j+1}\cdots a_{2n}a_1a_2a_3$ , which enforces that either  $f_D(F_2) \ge 1$  or  $f_D(F_j) \ge 1$ , a contradiction.

Subcase 1.2. The crossed edge of  $F_j$  is  $b_{2j-2}b_{2j-1}$ . Then the subdrawing of  $R_2 \cup R_j$  is as drawn in Figure 35. Observe that the vertices  $b_4$  and  $a_{2j-1}$  do not lie on the boundary of a same region. Thus the path  $b_4b_5a_5a_6\cdots a_{2j-1}$  will have a crossing either with  $F_2$  or  $F_j$ , a contradiction.





Figure 35. A possible subdrawing of  $R_2 \cup R_j$ .

Figure 36. A possible subdrawing of  $R_2 \cup C_{F_i}$ .

Case 2. The cycle  $C_{F_j}$  is non-contractible. Two non-contractible cycles  $C_{F_j}$ and C(2) (respectively,  $C_{F_j}$  and C(2')) should cross each other at least once in the projective plane. Then the crossed edge of  $F_2$  must belong to E(C(2)) $\cap E(C(2'))$  since  $v_D(F_2, F_j) = 1$ . Remember that the edges of E' are clean, therefore, either  $c_2c_3$  or  $b_2b_3$  will have a crossing with an edge e of  $C_{F_j}$ , where  $e \in \{a_{2j-1}a_{2j}, b_{2j-1}b_{2j}, c_{2j-1}c_{2j}\}$ .

We discuss first that  $c_2c_3$  crosses with  $a_{2j-1}a_{2j}$ . For other cases, similar contradictions can be obtained. Note that neither  $F_2$  nor  $F_j$  could be crossed any more.

The subdrawing of  $R_2 \cup C_{F_j}$  is as shown in Figure 36. Consider the vertices  $b_{2j-2}$  and  $c_{2j-2}$ . They must lie in the same region since the edge  $b_{2j-2}c_{2j-2} \in E'$  is clean. There are two possibilities of the subdrawing of  $R_2 \cup R_j$  according to in which region  $b_{2j-2}$  and  $c_{2j-2}$  lie, see Figure 37 and Figure 38.

Subcase 2.1.  $R_2 \cup R_j$  is as drawn in Figure 37. Note that the vertices  $a_4$  and  $b_{2j-2}$  do not lie on the boundary of a same region. Then either  $F_2$  or  $F_j$  will be crossed by the path  $a_4a_5b_5b_6\cdots b_{2j-2}$ , a contradiction.

Subcase 2.2.  $R_2 \cup R_j$  is as drawn in Figure 38, similar argument can be made since either  $F_2$  or  $F_j$  will be crossed by the path  $a_3a_2c_1c_{2n}\cdots c_{2j}$ .

#### 5.The Proof of Lemma 6

Now, we are in a position to prove Lemma 6.

 $^{-2}$ -1  $a_4$ )?,  $b_2$ 



Figure 37. A possible subdrawing of  $R_2 \cup R_j$ .

Figure 38. A possible subdrawing of

The Proof of Lemma 6. We prove the lemma by induction on n. Lemma 7 enforces the inequality holds for n = 3. Suppose that

(2) 
$$cr_{N_1}(H_{3,l}) \ge l-1 \text{ for } l < n,$$

and that there exists a good drawing D of  $H_{3,n}$  satisfying  $v_D(H_{3,n}) < n-1$ . Combined Equation (2) with the fact that  $H_{3,n}$  contains a subdivision of  $H_{3,n-1}$ , we have

(3) 
$$v_D(H_{3,n}) = n - 2.$$

Remember that

 $R_2 \cup R_j$ .

$$E' = \bigcup_{i=1}^{n} \{a_{2i-1}b_{2i-1}, b_{2i}c_{2i}, a_{2i}c_{2i-1}\},\$$

we have the following claim.

**Claim 18.** For any edge  $e \in E'$ , it is clean in D. That means D is an E'-clean drawing of  $H_{3,n}$ .

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**Proof.** Otherwise, a good drawing of  $H_{3,n-1}$  with less than n-2 crossings can be constructed from D by removing the crossed edge e, contradicting Equation (2).  $\Box$ 

Together with Lemma 3, the following two cases are considered.

Case 1. There exists an integer i  $(1 \le i \le n)$  such that  $f_D(F_i) = 0$ . Without loss of generality, let  $f_D(F_2) = 0$ . By Equation (1), we obtain that the edges of  $F_2$  are clean in D.

**Claim 19.** There exists an integer j  $(1 \le j \le n, j \ne 2)$  such that  $f_D(F_j) < 1$ .

**Proof.** Otherwise, by Lemma 3, we have

$$v_D(H_{3,n}) = \sum_{i=1}^n f_D(F_i) \ge n-1,$$

a contradiction to Equation (3).

By Lemma 11 and Lemma 13, it is inferred that  $j \neq 3$ . The following two subcases are studied according to whether j = 1 or not.

Subcase 1.1.  $j \neq 1$ .

Subcase 1.1.1. The cycle  $C_{F_2}$  is contractible. By Lemma 9, the subdrawing of  $R_2$  is as shown in Figure 8 by replacing all the indices *i* by 2. Combined with Claim 18, the non-contractible cycle  $C(2) = b_2 b_3 b_4 c_4 c_3 c_2 b_2$  is clean in D.

Now consider the subgraph  $F_j$ . First of all, the cycle  $C_{F_j}$  is contractible, otherwise, two non-contractible cycles  $C_{F_j}$  and C(2) cross each other in the projective plane, a contradiction. Again, by Lemma 9, the subdrawing of  $R_j$  is as shown in Figure 8 by replacing all the indices *i* by *j*. Thus another non-contractible cycle C(j) occurs, which should cross the non-contractible cycle C(2) in the projective plane, a contradiction.

Subcase 1.1.2. The cycle  $C_{F_2}$  is non-contractible. The cycle  $C_{F_j}$  is contractible, otherwise, two non-contractible curves  $C_{F_j}$  and  $C_{F_2}$  cross each other in the projective plane, contradicting that  $f_D(F_2) = 0$ . According to Lemma 9, the subdrawing of  $R_j$  is as shown in Figure 8 by replacing all the indices *i* by *j*.

Note that a non-contractible cycle C(j) occurs, which should cross  $C_{F_2}$  in the projective plane, contradicting that  $f_D(F_2) = 0$ .

Subcase 1.2. j = 1. By Lemma 15 and Lemma 16, we have  $f_D(F_1) > 0$ . Furthermore, Claim 19 enforces that  $f_D(F_1) = \frac{1}{2}$ .

There exists another integer  $m \ (m \notin \{1,2\})$  such that  $f_D(F_m) < 1$ , otherwise

$$v_D(H_{3,n}) = \sum_{i=1}^n f_D(F_i) \ge \frac{1}{2} + 0 + (n-2) > n-2,$$

contradicting Equation (3). Moreover, we have  $m \neq 3$  by Lemma 11 and Lemma 13. When considering the subgraphs  $F_2$  and  $F_m$ , contradictions can be obtained using the same discussions in Subcase 1.1.

Case 2.  $f_D(F_i) > 0$  for all  $1 \le i \le n$ . By Lemma 3, there exists an integer i  $(1 \le i \le n)$  such that  $f_D(F_i) = \frac{1}{2}$ , otherwise

$$v_D(H_{3,n}) = \sum_{i=1}^n f_D(F_i) \ge n,$$

contradicting Equation (3).

Without loss of generality, let  $f_D(F_2) = \frac{1}{2}$ . We can get that there exists another integer j ( $4 \le j \le n$ ) such that  $f_D(F_j) = \frac{1}{2}$ , otherwise,

$$v_D(H_{3,n}) = \sum_{i=1}^n f_D(F_i) \ge 3 \times \frac{1}{2} + (n-3) > n-2,$$

contradicting Equation (3).

Using Equation (1), the edges of  $F_2$  (respectively,  $F_j$ ) cannot have internal crossings.

Subcase 2.1. The cycle  $C_{F_2}$  is contractible. By Lemma 9, the subdrawing of  $R_2$  is as shown in Figure 8 by replacing all the indices *i* by 2. Note that the cycle  $C(2) = b_2 b_3 b_4 c_4 c_3 c_2 b_2$  is non-contractible.

Lemma 17 enforces that  $v_D(F_2, F_j) = 0$ . Thus, all of the vertices of  $F_j$  lie in the same region in the subdrawing of  $R_2$  in Figure 8, moreover, they lie in the region labelled  $f_1$ , otherwise, each of the edge-disjoint paths  $b_{2j}b_{2j+1}\cdots b_{2n}b_1b_2$ and  $c_{2j}c_{2j+1}\cdots c_{2n}c_1c_2$  crosses  $C_{F_2}$  at least once, which implies that  $f_D(F_2) \ge 1$ by Equation (1), a contradiction. Finally, the cycle  $C_{F_j}$  is contractible, otherwise it will cross the non-contractible cycle C(2) at least once in the projective plane, contradicting that  $v_D(F_2, F_j) = 0$ .

Lemma 9 tells us that the subdrawing of  $R_j$  is as shown in Figure 8 by replacing all the indices *i* by *j*. Thus a non-contractible cycle C(j) occurs, which will cross C(2) at least once in the projective plane. Therefore,  $v_D(F_2, F_j) \ge 1$ by Claim 18, a contradiction to Lemma 17.

Subcase 2.2. The cycle  $C_{F_2}$  is non-contractible. If the cycle  $C_{F_j}$  is contractible, then arguments similar to Subcase 2.1 will lead to a contradiction.

Thus we only need to consider that  $C_{F_j}$  is non-contractible. Since  $f_D(F_2) = f_D(F_j) = \frac{1}{2}$ , we have  $v_D(C_{F_2}, C_{F_j}) = 1$ . Moreover, there must exist another integer  $k \ (k \neq 2, j)$  such that  $f_D(F_k) = \frac{1}{2}$ , otherwise,

$$v_D(H_{3,n}) = \sum_{i=1}^n f_D(F_i) \ge 2 \times \frac{1}{2} + (n-2) > n-2,$$

contradicting Equation (3). Consider the cycle  $C_{F_k}$ , it must be contractible, otherwise, there will be another crossing made by the non-contractible cycles  $C_{F_2}$  and  $C_{F_k}$ , contradicting that  $f_D(F_2) = \frac{1}{2}$ .

Subcase 2.2.1. n = 4. Then j = 4. By Equation (3), we know that  $f_D(F_1) = f_D(F_3) = \frac{1}{2}$ . Furthermore, the cycle  $C_{F_1}$  (respectively,  $C_{F_3}$ ) is contractible in D, otherwise, it will cross the non-contractible cycle  $C_{F_2}$  in the projective plane, contradicting that  $f_D(F_2) = \frac{1}{2}$ . On the other hand, a non-contractible cycle C(1) occurs, which should cross  $C_{F_2}$ , a contradiction.

Subcase 2.2.2.  $n \ge 5$ . Moreover, either k = 3 when j = 4, or k = 1 when j = n. Then by Equation (3), there exists another integer s ( $s \notin \{2, k, j\}$ ) such that  $f_D(F_s) = \frac{1}{2}$ . Therefore, using arguments similar to those in Subcase 2.1, contradictions can be obtained by considering  $F_k$  and  $F_s$ .

Subcase 2.2.3.  $n \ge 5$ . Moreover,  $k \ne 3$  when j = 4. Using arguments similar to those in Subcase 2.1, contradictions can be obtained either by considering  $F_k$  and  $F_j$  when  $k \ne 5$ , or by considering  $F_k$  and  $F_2$  when k = 5.

Subcase 2.2.4.  $n \geq 5$ . Moreover,  $k \neq 1$  when j = n. Analogously, contradictions can be obtained either by considering  $F_k$  and  $F_j$  when  $k \neq n - 1$ , or by considering  $F_k$  and  $F_2$  when k = n - 1.

The proof is complete.

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