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THE CROSSING NUMBER OF HEXAGONAL GRAPH $H_{3,n}$ IN THE PROJECTIVE PLANE

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Abstract

Thomassen described all (except finitely many) regular tilings of the torus S_1 and the Klein bottle N_2 into (3,6)-tilings, (4,4)-tilings and (6,3)-tilings. Many researchers made great efforts to investigate the crossing number of the Cartesian product of an m-cycle and an n-cycle, which is a special kind of (4,4)-tilings, either in the plane or in the projective plane. In this paper we study the crossing number of the hexagonal graph $H_{3,n}$ ($n \geq 2$), which is a special kind of (3,6)-tilings, in the projective plane, and prove that

$$cr_{N_1}(H_{3,n}) = \begin{cases} 0, & n = 2, \\ n-1, & n \geq 3. \end{cases}$$

Keywords: projective plane, crossing number, hexagonal graph, drawing.
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1. Introduction

The *projective plane*, N_1, is a 2-manifold obtained by identifying every point of the 2-sphere with its antipodal point.

Let G be a graph with vertex set V and edge set E. The *crossing number of G in a surface* Σ, denoted by $cr_\Sigma(G)$, is the minimum number of pairwise intersections of edges in a drawing of G in the surface Σ. In particular, the *crossing number of G in the plane* S_0 is denoted by $cr(G)$ for simplicity. It is well known that the crossing number of a graph in the surface Σ is attained only in *good drawings* of the graph, which are the drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point.

In [16], Thomassen described all (except finitely many) regular tilings of the torus S_1 and the Klein bottle N_2 into hexagons, quadrilaterals and triangles in which the vertices have degree 3, 4 and 6, respectively. To be more specific, let G be a connected d-regular graph ($d \geq 3$) and φ a collection of m-cycles in G. Assume that each edge of G is contained in precisely two cycles in φ and that, for each vertex v in G, the edges incident with v can be labelled $e_1, e_2, \ldots, e_d$ such that for each $i = 1, 2, \ldots, d$, there is a cycle in φ containing e_i and e_{i+1} (where $e_{d+1} = e_1$). Then a surface Σ can be obtained by letting the cycles in φ be disjoint convex polygons in the Euclidean plane pasted together by the graph G, and G is said to be a (d, m)-tiling of Σ. Using Euler's formula, Thomassen observed that a regular tiling of the torus or the Klein bottle fit into three categories: (3,6)-tilings, (4,4)-tilings and (6,3)-tilings.

Note that the Cartesian product graph $C_m \square C_n$ is a special kind of (4,4)-tilings, which can be embedded in orientable surface S_k ($k \geq 1$) and non-orientable surface N_k ($k \geq 2$), but cannot be embedded in the plane S_0 or projective plane N_1. This fact motivates many researchers' intensive interest to determine the exact value of $cr(C_m \square C_n)$ or $cr_{N_1}(C_m \square C_n)$. However, computing the crossing number of a given graph is an elusive problem [3], therefore, the results concerning on this topic is quite limited. The crossing number $cr(C_m \square C_n)$ has been obtained for all but finitely many n, for each m [1, 2, 4, 9, 11, 13], while $cr_{N_1}(C_m \square C_n)$ has been determined only when $\min\{m, n\} = 3$ [14].

Compared with (4,4)-tilings, the crossing number of other regular tilings, either in the plane or in the projective plane, have not been extensively studied in the literature yet. Based on this observation, we began to study this problem.

Very recently, we have determined the crossing number of the hexagonal graph $H_{3,n}$, which is a special kind of (3,6)-tilings, in the plane [17]. To further the study, this paper is devoted to investigate the crossing number of $H_{3,n}$ in the projective plane. The main theorem is the following.

Theorem 1. *For $n \geq 2$, the crossing number of the hexagonal graph $H_{3,n}$ in the projective plane N_1 is*

$$cr_{N_1}(H_{3,n}) = \begin{cases} 0, & n = 2, \\ n-1, & n \geq 3. \end{cases}$$

Note that there are only few infinite classes of graphs whose crossing numbers in a surface other than the plane are known exactly, see [5, 6, 7, 8, 12, 14, 15] and the references therein.

This paper is organized as follows. Section 2 is devoted to introduce some basic preliminaries. By investigating the upper and lower bound of $cr_{N_1}(H_{3,n})$ separately, we give the sketch of the proof of Theorem 1 in Section 3 by assuming the correctness of Lemma 6. We postpone the proof of Lemma 6 on the lower bound of $cr_{N_1}(H_{3,n})$ to Section 5. In Section 4, we list some lemmas which are used in the proof of Lemma 6.

2. PRELIMINARIES

We introduce some basic preliminaries in this section. For $F \subseteq E(G)$, we denote by $G \setminus F$ the graph obtained from G by deleting all edges in F.

Let D be a good drawing of graph G in the surface Σ. We denote the number of crossings in D by $v_D(G)$. In a drawing D of graph G, if an edge is not crossed by any other edge, we say that it is *clean* in D, otherwise, we say it is *crossed.* Let A and B be two (not necessary disjoint) subsets of the edge set $E(G)$. The number of crossings involving an edge in A and another edge in B is denoted by $v_D(A, B)$. In particular, $v_D(A, A)$ is denoted by $v_D(A)$. By counting the number of crossings in D, we have the following result.

Lemma 2. *Let A, B, C be mutually disjoint subsets of $E(G)$. Then*

$$v_D(A, B \cup C) = v_D(A, B) + v_D(A, C),$$
$$v_D(A \cup B) = v_D(A) + v_D(A, B) + v_D(B).$$

The hexagonal graph $H_{3,n}$ ($n \geq 2$) is a special kind of (3,6)-tilings, which can be embedded in the torus such that the number of 6-cycles in the meridional (respectively, longitudinal) direction is 3 (respectively, n). To be exact, $H_{3,n}$ is the graph with vertex set

$$V(H_{3,n}) = \{a_i, b_i, c_i \mid i = 1, 2, \ldots, 2n\},$$

and edge set

$$E(H_{3,n}) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1} |\ i = 1, 2, \ldots, 2n\} \\ \cup\ \{a_{2i-1} b_{2i-1}, b_{2i} c_{2i}, a_{2i} c_{2i-1} |\ i = 1, 2, \ldots, n\},$$

where the indices are expressed modulo $2n$, see Figure 1. Note that the graph $H_{3,n}$ is vertex-transitive.

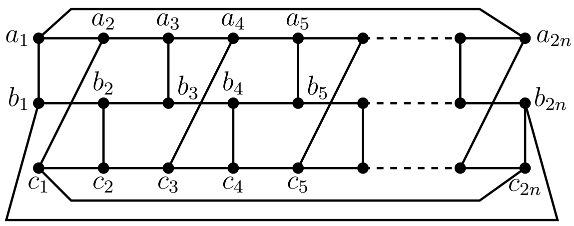

Figure 1. The graph $H_{3,n}$.

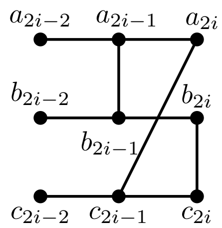

Figure 2. The subgraph F_i.

Let

$$E' = \bigcup_{i=1}^{n} \{a_{2i-1} b_{2i-1}, b_{2i} c_{2i}, a_{2i} c_{2i-1}\}.$$

Observe that, for any edge $e \in E'$, $H_{3,n} \setminus e$ contains a subdivision of $H_{3,n-1}$.

For $1 \leq i \leq n$, let

$$F_i = \{a_{2i-2} a_{2i-1}, a_{2i-1} a_{2i}, b_{2i-2} b_{2i-1}, b_{2i-1} b_{2i}, c_{2i-2} c_{2i-1}, c_{2i-1} c_{2i}, \\ a_{2i-1} b_{2i-1}, b_{2i} c_{2i}, a_{2i} c_{2i-1}\},$$

where the indices are read modulo $2n$, see Figure 2. Then $F_1, F_2, \ldots, F_n$ is a partition of $E(H_{3,n})$, which is to say,

$$E(H_{3,n}) = \bigcup_{i=1}^{n} F_i, \quad \text{and,} \quad \text{for } i \neq j, \quad F_i \cap F_j = \emptyset.$$

Let D be a good drawing of $H_{3,n}$ in the projective plane. We define $f_D(F_i)$ $(1 \leq i \leq n)$ to be the function counting the number of crossings related to F_i in D as follows [8],

$$f_D(F_i) = v_D(F_i, F_i) + \frac{1}{2} \sum_{1 \leq j \leq n,\, j \neq i} v_D(F_i, F_j). \tag{1}$$

By counting the number of crossings in D, we can get the following lemma.

Lemma 3. $v_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i)$.

3. SKETCH OF THE PROOF OF THEOREM 1

Lemma 4. *For $n \geq 2$, $cr_{N_1}(H_{3,n}) \leq n-1$.*

Proof. Wilson's Lemma [14] states that, for a non-planar graph, its crossing number in the projective plane is strictly less than its crossing number in the plane. Therefore, $cr_{N_1}(H_{3,n}) \leq cr(H_{3,n}) - 1$. Combined this fact with the result that $cr(H_{3,n}) = n$ for $n \geq 2$ [17], Lemma 4 holds. ■

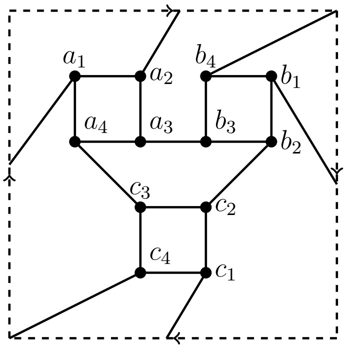

Figure 3. A good drawing of $H_{3,2}$ in the projective plane.

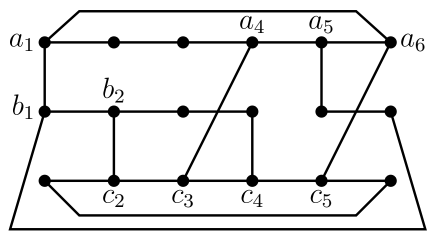

Figure 4. The graph $H_{3,3} \setminus \{a_2c_1, a_3b_3, b_4b_5, b_6c_6\}$.

Figure 3 shows that $H_{3,2}$ is embeddable in the projective plane. That implies the result.

Lemma 5. $cr_{N_1}(H_{3,2}) = 0$.

The following lemma is the key to the proof of Theorem 1.

Lemma 6. *For $n \geq 3$, $cr_{N_1}(H_{3,n}) \geq n-1$.*

We will prove Lemma 6 by induction on n. Thus, the induction basis is needed at first.

Lemma 7. $cr_{N_1}(H_{3,3}) = 2$.

Proof. We only need to prove that $cr_{N_1}(H_{3,3}) \geq 2$ by Lemma 4. Suppose that there is a good drawing D of $H_{3,3}$ in the projective plane such that $v_D(H_{3,3}) \leq 1$. By Figure 4 and Figure 5, we have $v_D(H_{3,3}) = 1$ since the graph $H_{3,3} \setminus \{a_2c_1, a_3b_3, b_4b_5, b_6c_6\}$ is a subdivision of $G_1(10, 15)$, which is one of the minimal forbidden subgraphs for the projective plane (see Appendix A in [10]).

Thus, we can get a graph which can be embedded in the projective plane from D by removing one of the crossed edges. However, Figure 4 and Figure 5

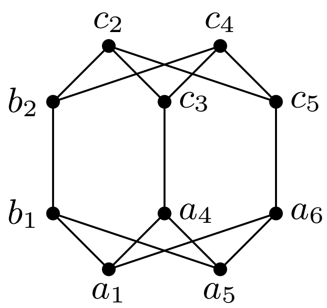

Figure 5. The graph $G_1(10,15)$.

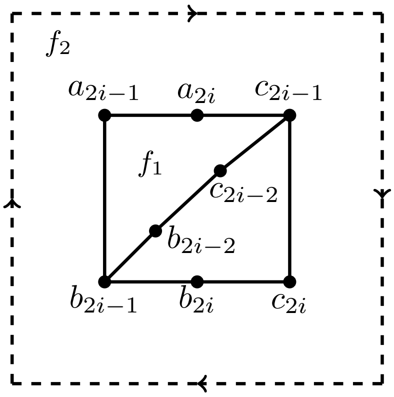

Figure 6. A possible subdrawing of R_i.

illustrate that $H_{3,3} \setminus e$ contains a subdivision of $G_1(10,15)$ for any $e \in E(H_{3,3})$. A contradiction. ■

We postpone the proof of Lemma 6 to Section 5. With its proof, we can prove the main result of this paper.

***The proof of Theorem* 1.** Combined with Lemma 4, Lemma 5 and Lemma 6, Theorem 1 follows easily. ■

4. Some Lemmas

Our aim in this section is to provide a definition and some basic lemmas which will be used to prove Lemma 6.

Definition 8. An *E'-clean drawing* of $H_{3,n}$ is a good drawing of $H_{3,n}$ in the projective plane such that all of the edges in E' are clean.

We will prove Lemma 6 by induction on n and by contradiction. If there exists a good drawing D of $H_{3,n}$ in the projective plane such that $v_D(H_{3,n}) < n-1$, then we can obtain that all of the edges in E' are clean in D. Therefore, in this section, we only consider the E'-clean drawing of $H_{3,n}$.

For $1 \le i \le n$, let

$$C_{F_i} = \{a_{2i-1}a_{2i}, a_{2i}c_{2i-1}, c_{2i-1}c_{2i}, c_{2i}b_{2i}, b_{2i}b_{2i-1}, b_{2i-1}a_{2i-1}\},$$

and

$$R_i = C_{F_i} \cup \{b_{2i-2}b_{2i-1}, b_{2i-2}c_{2i-2}, c_{2i-2}c_{2i-1}\},$$

where the indices are expressed modulo $2n$. Note that C_{F_i} is the cycle of length 6 in F_i, and that $R_i = F_i \cup \{b_{2i-2}c_{2i-2}\} \setminus a_{2i-2}a_{2i-1}$. The following lemma states that the subdrawing of R_i may be unique under certain restrictions.

Lemma 9. *For $n \geq 3$, let D be an E'-clean drawing of $H_{3,n}$ such that, for some $1 \leq i \leq n$, $f_D(F_i) < 1$ and C_{F_i} is contractible. Then the subdrawing of R_i in D is as shown in Figure 8.*

Proof. By Equation (1), the edges of F_i do not have internal crossings in D since $f_D(F_i) < 1$. Furthermore, the vertices b_{2i-2} and c_{2i-2} lie in the same region since the edge $b_{2i-2}c_{2i-2} \in E'$ is clean. Hence, there are three possibilities of the subdrawing of R_i in D, see Figures 6, 7, and 8.

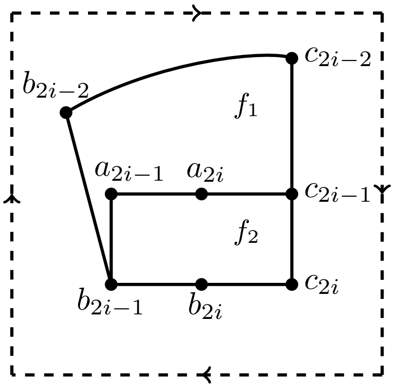

Figure 7. A possible subdrawing of R_i.

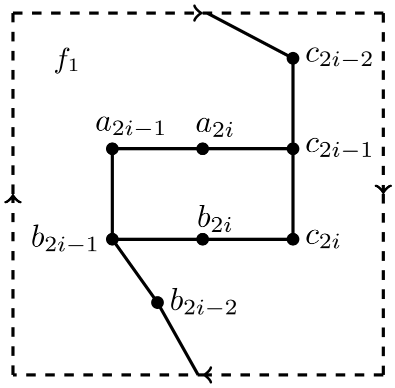

Figure 8. A possible subdrawing of R_i.

Suppose that R_i is as drawn in Figure 6. Consider the vertex a_{2i-2}. This vertex lies either in the region labelled f_1 or in the region f_2, since the edge $a_{2i-2}a_{2i-1} \in F_i$ does not have any crossing with R_i. If a_{2i-2} lies in f_1, then each of the edge-disjoint paths $a_{2i-2}c_{2i-3} \cdots c_1c_{2n}c_{2n-1} \cdots c_{2i}$ and $a_{2i-2}a_{2i-3}b_{2i-3}b_{2i-4} \cdots b_1b_{2n}b_{2n-1} \cdots b_{2i}$ crosses F_i at least once, which implies that $f_D(F_i) \geq 1$ by Equation (1), contradicting that $f_D(F_i) < 1$. Finally, if a_{2i-2} lies in f_2, then each of the edge-disjoint paths $a_{2i-2}c_{2i-3}c_{2i-2}$ and $a_{2i-2}a_{2i-3}b_{2i-3}b_{2i-2}$ crosses F_i at least once, which implies that $f_D(F_i) \geq 1$, a contradiction.

Almost the same argument can be obtained if R_i is as drawn in Figure 7. Therefore, the subdrawing of R_i in D is as shown in Figure 8. ■

In Figure 8, observe that there are two non-contractible cycles $C(i) \triangleq b_{2i-2}b_{2i-1}b_{2i}c_{2i}c_{2i-1}c_{2i-2}b_{2i-2}$ and $C(i') \triangleq b_{2i-2}b_{2i-1}a_{2i-1}a_{2i}c_{2i-1}c_{2i-2}b_{2i-2}$ in R_i, furthermore, $E(C(i)) \cap E(C(i')) = \{b_{2i-2}b_{2i-1}, c_{2i-2}c_{2i-1}, b_{2i-2}c_{2i-2}\}$. Thus, the following corollary holds.

Corollary 10. *For $n \geq 3$, let D be an E'-clean drawing of $H_{3,n}$. If there exist different integers i and j such that*

(1) $j \neq i \pm 1 \pmod{n}$,

(2) *both* C_{F_i} *and* C_{F_j} *are contractible,*

(3) $f_D(F_i) = f_D(F_j) = \frac{1}{2}$,

(4) $v_D(F_i, F_j) = 1$,

then the crossed edge of F_i *(respectively,* F_j*) in* D *must be either* $b_{2i-2}b_{2i-1}$ *or* $c_{2i-2}c_{2i-1}$ *(respectively, either* $b_{2j-2}b_{2j-1}$ *or* $c_{2j-2}c_{2j-1}$*).*

Proof. Firstly, note that $V(F_i) \cap V(F_j) = \emptyset$ since $j \neq i \pm 1 \pmod{n}$. Secondly, by Lemma 9, the subdrawing of R_i in D is as shown in Figure 8. Similarly, the subdrawing of R_j is also as shown in Figure 8 by replacing all the indices i by j.

Note that two non-contractible cycles cross each other in the projective plane, therefore, the crossed edge of F_i (respectively, F_j) in D must belong to $E(C(i)) \cap E(C(i'))$ $\big($respectively, $E(C(j)) \cap E(C(j'))\big)$ since $f_D(F_i) = f_D(F_j) = \frac{1}{2}$ and $v_D(F_i, F_j) = 1$. Hence, the corollary follows because the edge $b_{2i-2}c_{2i-2} \in E'$ (respectively, $b_{2j-2}c_{2j-2} \in E'$) is clean. ■

Lemma 11 and Lemma 13 in the following indicate that $f_D(F_3) \geq 1$ if $f_D(F_2) = 0$ by considering whether the cycle C_{F_2} is contractible or not.

Lemma 11. *For* $n \geq 3$*, let* D *be an* E'*-clean drawing of* $H_{3,n}$ *such that* $f_D(F_2) = 0$ *and the cycle* C_{F_2} *is contractible. Then* $f_D(F_3) \geq 1$.

Proof. By the assumption that $f_D(F_2) = 0$ and by Lemma 9, the subdrawing of R_2 in D is as shown in Figure 8 by replacing all the indices i by 2. Moreover, since $b_2c_2 \in E'$ is clean, we have the following.

Claim 12. R_2 *is clean in* D*. In particular, the non-contractible cycle* $C(2) = b_2b_3b_4c_4c_3c_2b_2$ *is clean.*

We prove the lemma by contradiction. Suppose that $f_D(F_3) < 1$. Then the edges of F_3 cannot have internal crossings in D by Equation (1). The cycle C_{F_3} is contractible, otherwise, two non-contractible cycles C_{F_3} and $C(2)$ cross each other in the projective plane, contradicting Claim 12. Moreover, all of the vertices of C_{F_3} lie in the same region labelled f_1 in the subdrawing of R_2 in Figure 8, otherwise, without loss of generality, assume that the vertex a_5 does not lie in f_1, then the cycle C_{F_2} will be crossed by the path $a_5b_5b_6 \cdots b_{2n}b_1b_2$ ($n \geq 3$), contradicting Claim 12. Therefore, the subdrawing of $R_2 \cup C_{F_3}$ is as shown either in Figure 9 or in Figure 10.

Case 1. The subdrawing of $R_2 \cup C_{F_3}$ is as shown in Figure 9. Consider the subdrawing of $R_2 \cup F_3$. It is as shown in Figure 11 by Claim 12 and by the assumption that $f_D(F_3) < 1$. Note that c_2 and c_6 (respectively, b_2 and b_6) do not lie on the boundary of a same region. Thus each of the edge-disjoint

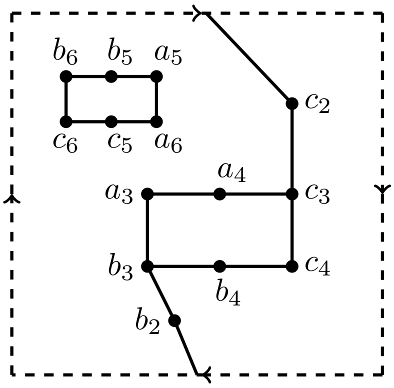

Figure 9. A possible subdrawing of $R_2 \cup C_{F_3}$.

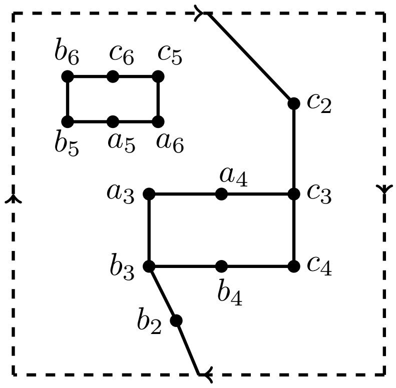

Figure 10. A possible subdrawing of $R_2 \cup C_{F_3}$.

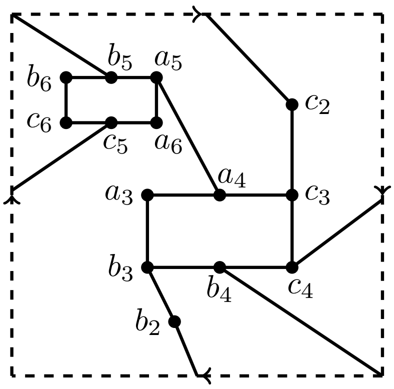

Figure 11. The subdrawing of $R_2 \cup F_3$.

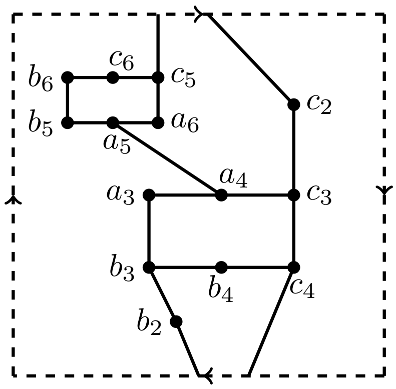

Figure 12. The subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$.

paths $c_6 \cdots c_{2n}c_1c_2$ and $b_6 \cdots b_{2n}b_1b_2$ will cross F_3 at least once, which implies that $f_D(F_3) \geq 1$, a contradiction.

Case 2. The subdrawing of $R_2 \cup C_{F_3}$ is as shown in Figure 10. Consider the subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$. It is as shown in Figure 12 by Claim 12 and by the assumption that $f_D(F_3) < 1$. Note that two vertices b_4 and b_5 do not lie on the boundary of a same region. Thus the edge $b_4b_5 \in F_3$ will cross some edge of F_3, which implies that $f_D(F_3) \geq 1$ by Equation (1), a contradiction. □

Lemma 13. *For $n \geq 3$, let D be an E'-clean drawing of $H_{3,n}$ such that $f_D(F_2) = 0$ and the cycle C_{F_2} is non-contractible. Then $f_D(F_3) \geq 1$.*

Proof. Since $f_D(F_2) = 0$ and $b_2c_2 \in E'$ is clean, we conclude the following.

Claim 14. *The edges of R_2 are clean in D.*

There are two possible subdrawing of R_2 in D since C_{F_2} is non-contractible, see Figure 13 and Figure 14.

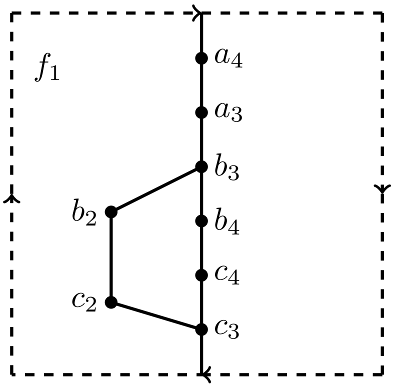

Figure 13. A possible subdrawing of R_2.

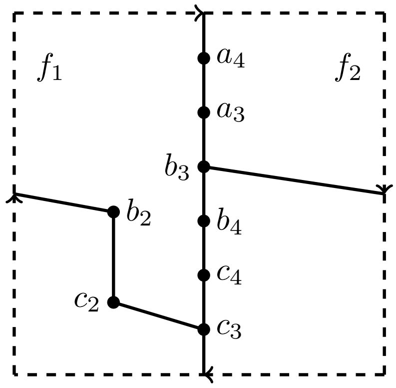

Figure 14. A possible subdrawing of R_2.

We prove the lemma by contradiction. Suppose that $f_D(F_3) < 1$. Then the edges of F_3 do not have internal crossings in D. The cycle C_{F_3} must be contractible, otherwise, two non-contractible curves C_{F_3} and C_{F_2} cross each other in the projective plane, contradicting that $f_D(F_2) = 0$. All of the vertices of F_3 lie in the same region in the subdrawing of R_2 by Claim 14, moreover, they locate in the region labelled f_1, on whose boundary lies all the vertices of R_2, see Figure 13 and Figure 14.

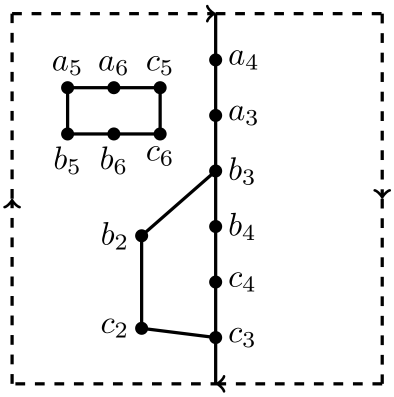

Figure 15. A possible subdrawing of $R_2 \cup C_{F_3}$.

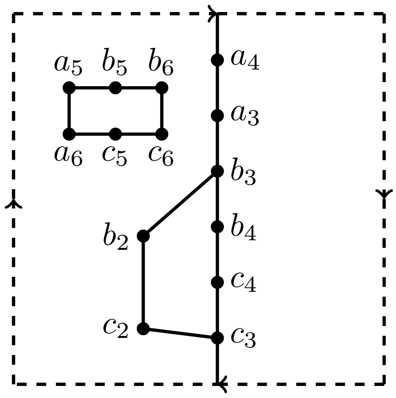

Figure 16. A possible subdrawing of $R_2 \cup C_{F_3}$.

Case 1. R_2 is as drawn in Figure 13. As we mentioned above, the subdrawing of $R_2 \cup C_{F_3}$ is as shown either in Figure 15 or in Figure 16.

Subcase 1.1. The subdrawing of $R_2 \cup C_{F_3}$ is as shown in Figure 15. Then the subdrawing of $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$ is as shown in Figure 17. In this case, a_4 and a_5 do not lie on the boundary of a same region. By Claim 14, there is at least one crossing made by the edge $a_4a_5 \in F_3$ with some edge of F_3, which implies that $f_D(F_3) \geq 1$ by Equation (1), a contradiction.

Subcase 1.2. The subdrawing of $R_2 \cup C_{F_3}$ is as shown in Figure 16. Then the subdrawing of $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$ is as shown in Figure 18. Note that the vertices b_2 and b_6 (respectively, c_2 and c_6) do not lie on the boundary of a same region, therefore, each of the edge-disjoint paths $b_6 \cdots b_{2n}b_1b_2$ and $c_6 \cdots c_{2n}c_1c_2$ crosses F_3 at least once, which enforces that $f_D(F_3) \geq 1$, a contradiction.

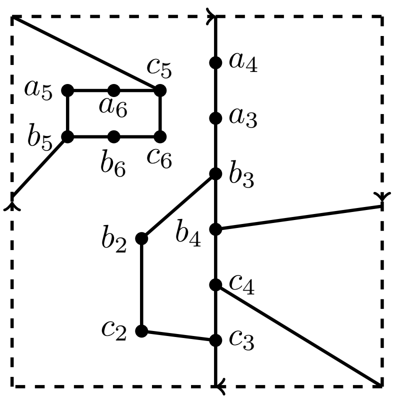

Figure 17. The subdrawing of $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$.

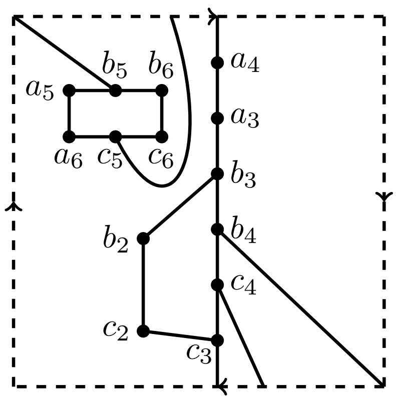

Figure 18. The subdrawing of $R_2 \cup C_{F_3} \cup \{b_4b_5, c_4c_5\}$.

Case 2. R_2 is as drawn in Figure 14. Using the former similar arguments, we assert that the subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5\}$ is as shown either in Figure 19 or in Figure 20.

Subcase 2.1. The subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5\}$ is as shown in Figure 19. Consider the edge b_4b_5.

Subcase 2.1.1. The edge b_4b_5 is as drawn in Figure 21. By Claim 14, we have $f_D(F_3) \geq 1$ by the observation that c_4 and c_5 do not lie on the boundary of a same region, a contradiction.

Subcase 2.1.2. The edge b_4b_5 is as drawn in Figure 22. By Claim 14, the edge c_4c_5 cannot cross the edges of $R_2 \cup F_3$ since $c_4c_5 \in F_3$. Then the subdrawig of $R_2 \cup F_3$ is as drawn in Figure 22.

Now consider the vertex a_2. It lies either in the region labelled f_2 or in the region f_{11}, since the edge $a_2a_3 \in F_2$ is clean, see Figure 22.

Subcase 2.1.2.1. If a_2 lies in f_2. Notice that the vertices a_2 and c_6 do not lie on the boundary of a same region in the subdrawing of R_2. Then the path

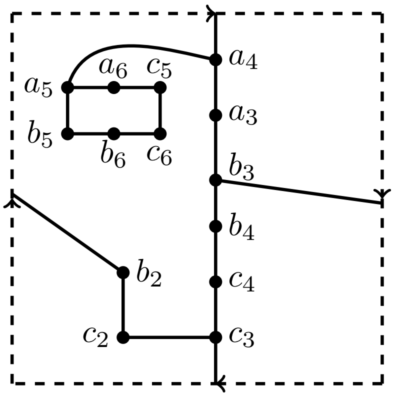

Figure 19. A possible subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5\}$.

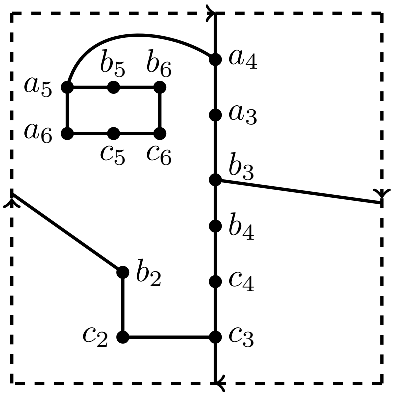

Figure 20. A possible subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5\}$.

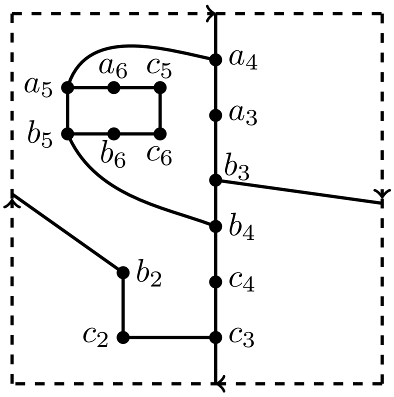

Figure 21. The subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, b_4b_5\}$.

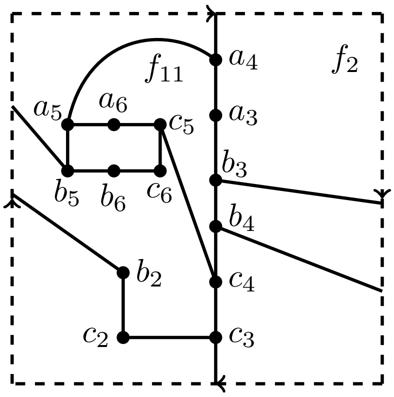

Figure 22. The subdrawing of $R_2 \cup F_3$.

$c_6 \cdots c_{2n}c_1a_2$ will cross R_2 at least once, contradicting Claim 14.

Subcase 2.1.2.2. If a_2 lies in f_{11}. Notice that the vertices a_2 and c_2 (respectively, a_2 and b_2) do not lie on the boundary of a same region. Then each of the edge-disjoint paths $a_2c_1c_2$ and $a_2a_1b_1b_2$ will cross F_3 at least once, contradicting that $f_D(F_3) < 1$.

Subcase 2.2. The subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5\}$ is as shown in Figure 20. Now consider the edge c_4c_5. There are two possibilities of the subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$, see Figure 23 and Figure 24.

Subcase 2.2.1. The subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$ is as shown in Figure 23. By Claim 14, we have $f_D(F_3) \geq 1$ since the vertices b_2 and b_6 (respectively, c_2 and c_6) do not lie on the boundary of a same region, a contradiction.

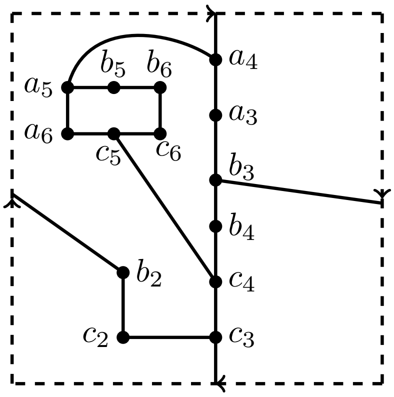

Figure 23. A possible subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$.

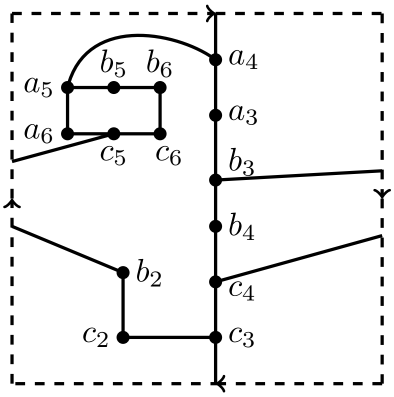

Figure 24. A possible subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$.

Subcase 2.2.2. The subdrawing of $R_2 \cup C_{F_3} \cup \{a_4a_5, c_4c_5\}$ is as shown in Figure 24. Then there are two possibilities of $R_2 \cup F_3$, see Figure 25 and Figure 26.

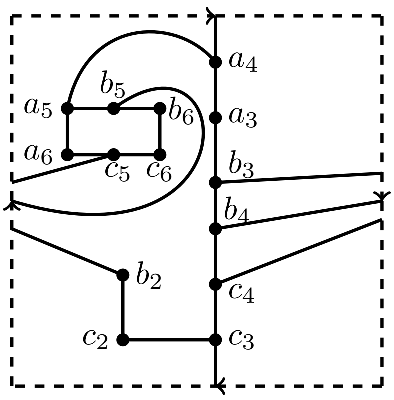

Figure 25. A possible subdrawing of $R_2 \cup F_3$.

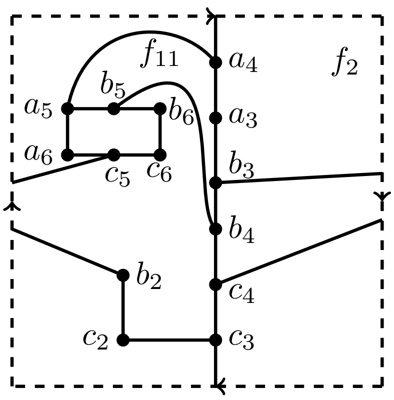

Figure 26. A possible subdrawing of $R_2 \cup F_3$.

Subcase 2.2.2.1. The subdrawing of $R_2 \cup F_3$ is as shown in Figure 25. By Claim 14, we have $f_D(F_3) \geq 1$ since the vertices b_2 and b_6 (respectively, c_2 and c_6) do not lie on the boundary of a same region, a contradiction.

Subcase 2.2.2.2. The subdrawing of $R_2 \cup F_3$ is as shown in Figure 26. Consider the vertex a_2, contradictions can be made by discussions similar to Subcase 2.1.2.

All these contradictions enforce that $f_D(F_3) \geq 1$. □

Lemma 15 and Lemma 16 in the following imply that $f_D(F_1) > 0$ if $f_D(F_2) = 0$, by considering whether the cycle C_{F_2} is contractible or not.

Lemma 15. *For $n \geq 3$, let D be an E'-clean drawing of $H_{3,n}$ such that $f_D(F_2) = 0$ and the cycle C_{F_2} is contractible. Then $f_D(F_1) > 0$.*

Proof. By Lemma 9, the subdrawing of R_2 is as drawn in Figure 8 by replacing all the indices i by 2. Moreover, the vertex a_2 lies in the region labelled f_1 in the subdrawing of R_2 in Figure 8, otherwise, the path $a_2c_1c_2$ will cross C_{F_2} at least once, contradicting that $f_D(F_2) = 0$.

We prove the lemma by contradiction. Suppose that $f_D(F_1) = 0$, then all of the vertices of F_1 lie in the same region. There are three possibilities of the subdrawing of $C_{F_1} \cup F_2$, see Figures 27, 28 and 29.

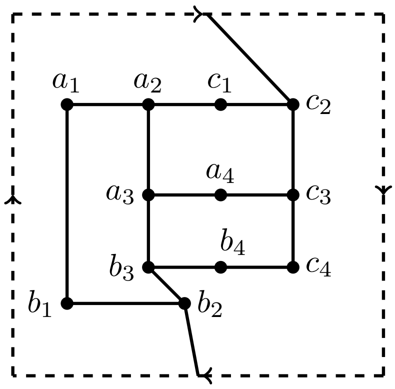

Figure 27. A possible subdrawing of $C_{F_1} \cup F_2$.

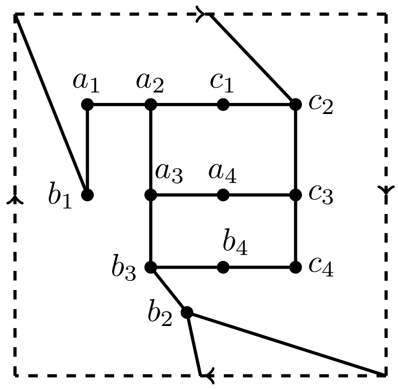

Figure 28. A possible subdrawing of $C_{F_1} \cup F_2$.

If $C_{F_1} \cup F_2$ is as drawn in Figure 27, then either C_{F_1} or F_2 will be crossed by the path $a_4a_5 \cdots a_{2n}a_1$ since the vertices a_4 and a_1 do not lie on the boundary of a same region, contradicting that $f_D(F_1) = f_D(F_2) = 0$. Similar argument can be made if $C_{F_1} \cup F_2$ is as drawn in Figure 28. Finally, if $C_{F_1} \cup F_2$ is as drawn in Figure 29, then either C_{F_1} or F_2 will be crossed by the path $a_4a_5 \cdots a_{2n}c_{2n-1}c_{2n}c_1$ (note that $n \geq 3$ is crucial here for c_{2n} being not equal to c_4), a contradiction.

All these contradictions enforce that $f_D(F_1) > 0$. ■

Lemma 16. *For $n \geq 3$, let D be an E'-clean drawing of $H_{3,n}$ such that $f_D(F_2) = 0$ and the cycle C_{F_2} is non-contractible. Then $f_D(F_1) > 0$.*

Proof. We prove the lemma by contradiction. Suppose that $f_D(F_1) = 0$. Then the cycle C_{F_1} is contractible, otherwise, two non-contractible cycles C_{F_2} and C_{F_1} cross each other at least once in the projective plane, contradicting that $f_D(F_2) = 0$.

The subdrawings of R_2 is as drawn either in Figure 13 or in Figure 14 by similar arguments in Lemma 13. Moreover, the vertices of C_{F_1} lie in the region

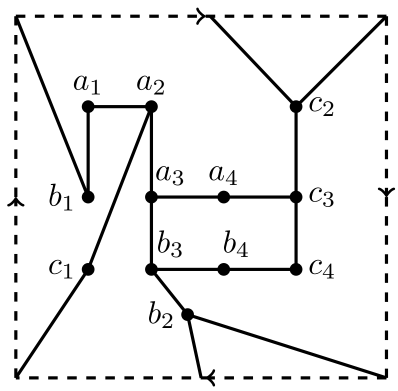

Figure 29. A possible subdrawing of $C_{F_1} \cup F_2$.

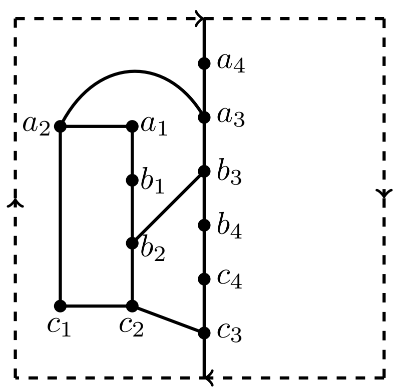

Figure 30. A possible subdrawing of $C_{F_1} \cup F_2$.

labelled f_1 if R_2 is as drawn in Figure 13. Therefore, there are four possibilities of the subdrawing of $C_{F_1} \cup F_2$, see Figures 30, 31, 32 and 33.

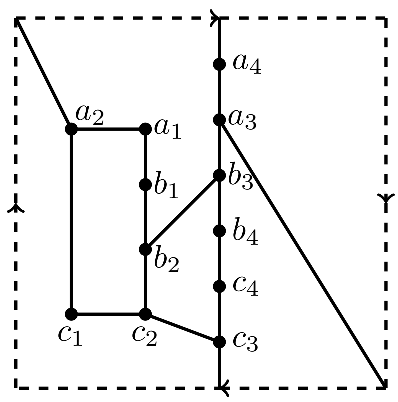

Figure 31. A possible subdrawing of $C_{F_1} \cup F_2$.

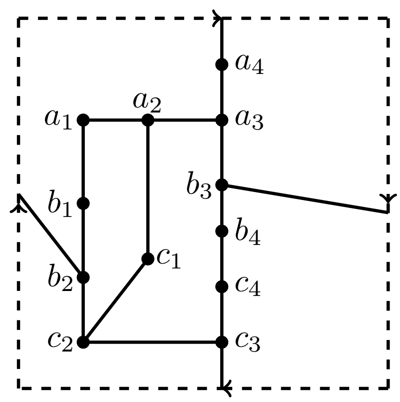

Figure 32. A possible subdrawing of $C_{F_1} \cup F_2$.

Case 1. The subdrawing of $C_{F_1} \cup F_2$ is as drawn in Figure 30. Since the vertices a_4 and a_1 do not lie on the boundary of a same region, the path $a_4a_5 \cdots a_{2n}a_1$ will cross either C_{F_1} or F_2, contradicting that $f_D(F_1) = f_D(F_2) = 0$.

Case 2. The subdrawing of $C_{F_1} \cup F_2$ is as drawn in Figure 31. Then the path $c_4c_5 \cdots c_{2n}c_1$ will cross either C_{F_1} or F_2, a contradiction.

Case 3. The subdrawing of $C_{F_1} \cup F_2$ is as drawn in Figure 32. Then the path $c_1c_{2n}c_{2n-1}a_{2n}a_{2n-1} \cdots a_4$ will cross either C_{F_1} or F_2 (note that $n \geq 3$ is crucial here for c_{2n} being not equal to c_4), a contradiction.

Case 4. The subdrawing of $C_{F_1} \cup F_2$ is as drawn in Figure 33. Then the path $b_4b_5 \cdots b_{2n}b_1$ will cross either C_{F_1} or F_2, a contradiction.

All these contradictions enforce that $f_D(F_1) > 0$. ∎

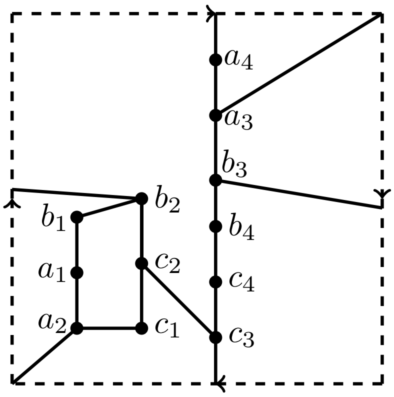

Figure 33. A possible subdrawing of $C_{F_1} \cup F_2$.

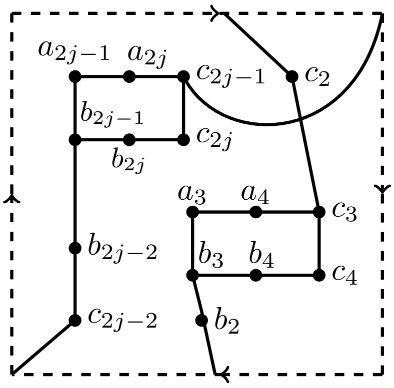

Figure 34. A possible subdrawing of $R_2 \cup R_j$.

We end this section with the following lemma which indicates that F_i do not cross F_j under certain restrictions.

Lemma 17. *For $n \geq 4$, let D be an E'-clean drawing of $H_{3,n}$. If there exist different integers i and j such that*

(1) $j \neq i \pm 1 \pmod{n}$,

(2) $f_D(F_i) = f_D(F_j) = \frac{1}{2}$,

(3) *at least one of the cycles C_{F_i} and C_{F_j} is contractible,*

then $v_D(F_i, F_j) = 0$.

Proof. Suppose to the contrary that $v_D(F_i, F_j) > 0$. Without loss of generality, assume that $i = 2$ and that $4 \leq j \leq n$, moreover, assume that C_{F_2} is contractible. Then, by Equation (1), $v_D(F_2, F_j) = 1$ since $f_D(F_2) = f_D(F_j) = \frac{1}{2}$.

The edges of F_2 (respectively, F_j) do not have internal crossings since $f_D(F_2) = \frac{1}{2}$ (respectively, $f_D(F_j) = \frac{1}{2}$). By Lemma 9, the subdrawing of R_2 is as drawn in Figure 8 by replacing all the indices i by 2. Note that the cycle $C(2) = b_2b_3b_4c_4c_3c_2b_2$ is non-contractible, furthermore, $V(F_2) \cap V(F_j) = \emptyset$.

We consider the following two cases.

Case 1. The cycle C_{F_j} is contractible. The subdrawing of R_j is also as shown in Figure 8 by replacing all the indices i by j. Observe that the crossed edge of F_2 (respectively, F_j) is either c_2c_3 or b_2b_3 (respectively, $c_{2j-2}c_{2j-1}$ or $b_{2j-2}b_{2j-1}$) by Corollary 10. We may assume that the crossed edge of F_2 is c_2c_3 (if the crossed edge of F_2 is b_2b_3, the proof is analogous). Moreover, all of the vertices of C_{F_j} lie in the region labelled f_1 in the subdrawing of R_2 in Figure 8.

Subcase 1.1. The crossed edge of F_j is $c_{2j-2}c_{2j-1}$. Then the subdrawing of $R_2 \cup R_j$ is as drawn in Figure 34, either F_2 or F_j will be crossed by the path $a_{2j}a_{2j+1}\cdots a_{2n}a_1a_2a_3$, which enforces that either $f_D(F_2) \geq 1$ or $f_D(F_j) \geq 1$, a contradiction.

Subcase 1.2. The crossed edge of F_j is $b_{2j-2}b_{2j-1}$. Then the subdrawing of $R_2 \cup R_j$ is as drawn in Figure 35. Observe that the vertices b_4 and a_{2j-1} do not lie on the boundary of a same region. Thus the path $b_4b_5a_5a_6\cdots a_{2j-1}$ will have a crossing either with F_2 or F_j, a contradiction.

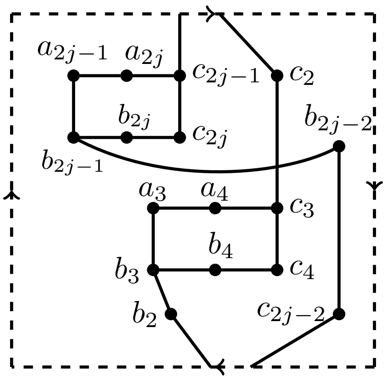

Figure 35. A possible subdrawing of $R_2 \cup R_j$.

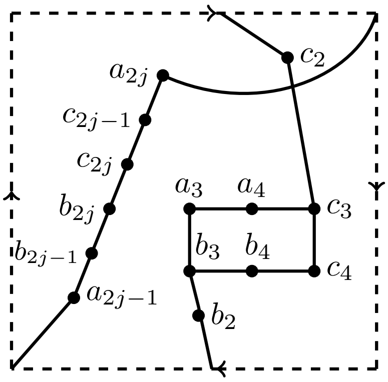

Figure 36. A possible subdrawing of $R_2 \cup C_{F_j}$.

Case 2. The cycle C_{F_j} is non-contractible. Two non-contractible cycles C_{F_j} and $C(2)$ (respectively, C_{F_j} and $C(2'))$ should cross each other at least once in the projective plane. Then the crossed edge of F_2 must belong to $E(C(2)) \cap E(C(2'))$ since $v_D(F_2, F_j) = 1$. Remember that the edges of E' are clean, therefore, either c_2c_3 or b_2b_3 will have a crossing with an edge e of C_{F_j}, where $e \in \{a_{2j-1}a_{2j}, b_{2j-1}b_{2j}, c_{2j-1}c_{2j}\}$.

We discuss first that c_2c_3 crosses with $a_{2j-1}a_{2j}$. For other cases, similar contradictions can be obtained. Note that neither F_2 nor F_j could be crossed any more.

The subdrawing of $R_2 \cup C_{F_j}$ is as shown in Figure 36. Consider the vertices b_{2j-2} and c_{2j-2}. They must lie in the same region since the edge $b_{2j-2}c_{2j-2} \in E'$ is clean. There are two possibilities of the subdrawing of $R_2 \cup R_j$ according to in which region b_{2j-2} and c_{2j-2} lie, see Figure 37 and Figure 38.

Subcase 2.1. $R_2 \cup R_j$ is as drawn in Figure 37. Note that the vertices a_4 and b_{2j-2} do not lie on the boundary of a same region. Then either F_2 or F_j will be crossed by the path $a_4a_5b_5b_6\cdots b_{2j-2}$, a contradiction.

Subcase 2.2. $R_2 \cup R_j$ is as drawn in Figure 38, similar argument can be made since either F_2 or F_j will be crossed by the path $a_3a_2c_1c_{2n}\cdots c_{2j}$. ■

5. The Proof of Lemma 6

Now, we are in a position to prove Lemma 6.

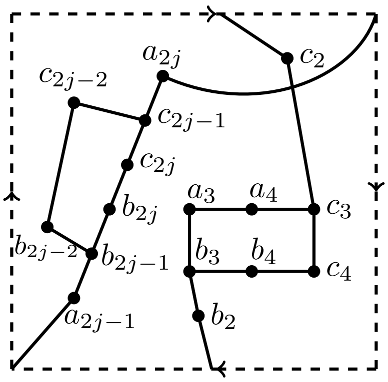

Figure 37. A possible subdrawing of $R_2 \cup R_j$.

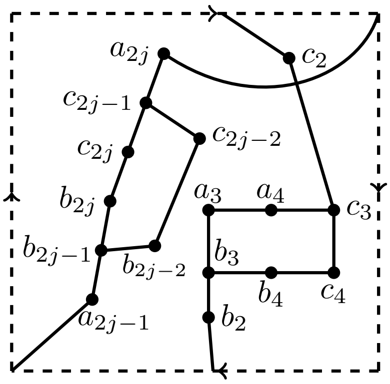

Figure 38. A possible subdrawing of $R_2 \cup R_j$.

The Proof of Lemma 6. We prove the lemma by induction on n. Lemma 7 enforces the inequality holds for $n = 3$. Suppose that

$$cr_{N_1}(H_{3,l}) \geq l - 1 \quad \text{for } l < n, \tag{2}$$

and that there exists a good drawing D of $H_{3,n}$ satisfying $v_D(H_{3,n}) < n - 1$. Combined Equation (2) with the fact that $H_{3,n}$ contains a subdivision of $H_{3,n-1}$, we have

$$v_D(H_{3,n}) = n - 2. \tag{3}$$

Remember that

$$E' = \bigcup_{i=1}^{n} \{a_{2i-1}b_{2i-1}, b_{2i}c_{2i}, a_{2i}c_{2i-1}\},$$

we have the following claim.

Claim 18. *For any edge $e \in E'$, it is clean in D. That means D is an $E'-$clean drawing of $H_{3,n}$.*

Proof. Otherwise, a good drawing of $H_{3,n-1}$ with less than $n-2$ crossings can be constructed from D by removing the crossed edge e, contradicting Equation (2). □

Together with Lemma 3, the following two cases are considered.

Case 1. There exists an integer i $(1 \le i \le n)$ such that $f_D(F_i) = 0$. Without loss of generality, let $f_D(F_2) = 0$. By Equation (1), we obtain that the edges of F_2 are clean in D.

Claim 19. *There exists an integer j $(1 \le j \le n, j \ne 2)$ such that $f_D(F_j) < 1$.*

Proof. Otherwise, by Lemma 3, we have

$$v_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i) \ge n-1,$$

a contradiction to Equation (3). □

By Lemma 11 and Lemma 13, it is inferred that $j \ne 3$. The following two subcases are studied according to whether $j = 1$ or not.

Subcase 1.1. $j \ne 1$.

Subcase 1.1.1. The cycle C_{F_2} is contractible. By Lemma 9, the subdrawing of R_2 is as shown in Figure 8 by replacing all the indices i by 2. Combined with Claim 18, the non-contractible cycle $C(2) = b_2b_3b_4c_4c_3c_2b_2$ is clean in D.

Now consider the subgraph F_j. First of all, the cycle C_{F_j} is contractible, otherwise, two non-contractible cycles C_{F_j} and $C(2)$ cross each other in the projective plane, a contradiction. Again, by Lemma 9, the subdrawing of R_j is as shown in Figure 8 by replacing all the indices i by j. Thus another non-contractible cycle $C(j)$ occurs, which should cross the non-contractible cycle $C(2)$ in the projective plane, a contradiction.

Subcase 1.1.2. The cycle C_{F_2} is non-contractible. The cycle C_{F_j} is contractible, otherwise, two non-contractible curves C_{F_j} and C_{F_2} cross each other in the projective plane, contradicting that $f_D(F_2) = 0$. According to Lemma 9, the subdrawing of R_j is as shown in Figure 8 by replacing all the indices i by j.

Note that a non-contractible cycle $C(j)$ occurs, which should cross C_{F_2} in the projective plane, contradicting that $f_D(F_2) = 0$.

Subcase 1.2. $j = 1$. By Lemma 15 and Lemma 16, we have $f_D(F_1) > 0$. Furthermore, Claim 19 enforces that $f_D(F_1) = \frac{1}{2}$.

There exists another integer m $(m \notin \{1, 2\})$ such that $f_D(F_m) < 1$, otherwise

$$v_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i) \ge \frac{1}{2} + 0 + (n-2) > n-2,$$

contradicting Equation (3). Moreover, we have $m \neq 3$ by Lemma 11 and Lemma 13. When considering the subgraphs F_2 and F_m, contradictions can be obtained using the same discussions in Subcase 1.1.

Case 2. $f_D(F_i) > 0$ for all $1 \leq i \leq n$. By Lemma 3, there exists an integer i ($1 \leq i \leq n$) such that $f_D(F_i) = \frac{1}{2}$, otherwise

$$v_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i) \geq n,$$

contradicting Equation (3).

Without loss of generality, let $f_D(F_2) = \frac{1}{2}$. We can get that there exists another integer j ($4 \leq j \leq n$) such that $f_D(F_j) = \frac{1}{2}$, otherwise,

$$v_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i) \geq 3 \times \frac{1}{2} + (n-3) > n-2,$$

contradicting Equation (3).

Using Equation (1), the edges of F_2 (respectively, F_j) cannot have internal crossings.

Subcase 2.1. The cycle C_{F_2} is contractible. By Lemma 9, the subdrawing of R_2 is as shown in Figure 8 by replacing all the indices i by 2. Note that the cycle $C(2) = b_2b_3b_4c_4c_3c_2b_2$ is non-contractible.

Lemma 17 enforces that $v_D(F_2, F_j) = 0$. Thus, all of the vertices of F_j lie in the same region in the subdrawing of R_2 in Figure 8, moreover, they lie in the region labelled f_1, otherwise, each of the edge-disjoint paths $b_{2j}b_{2j+1}\cdots b_{2n}b_1b_2$ and $c_{2j}c_{2j+1}\cdots c_{2n}c_1c_2$ crosses C_{F_2} at least once, which implies that $f_D(F_2) \geq 1$ by Equation (1), a contradiction. Finally, the cycle C_{F_j} is contractible, otherwise it will cross the non-contractible cycle $C(2)$ at least once in the projective plane, contradicting that $v_D(F_2, F_j) = 0$.

Lemma 9 tells us that the subdrawing of R_j is as shown in Figure 8 by replacing all the indices i by j. Thus a non-contractible cycle $C(j)$ occurs, which will cross $C(2)$ at least once in the projective plane. Therefore, $v_D(F_2, F_j) \geq 1$ by Claim 18, a contradiction to Lemma 17.

Subcase 2.2. The cycle C_{F_2} is non-contractible. If the cycle C_{F_j} is contractible, then arguments similar to Subcase 2.1 will lead to a contradiction.

Thus we only need to consider that C_{F_j} is non-contractible. Since $f_D(F_2) = f_D(F_j) = \frac{1}{2}$, we have $v_D(C_{F_2}, C_{F_j}) = 1$. Moreover, there must exist another integer k ($k \neq 2, j$) such that $f_D(F_k) = \frac{1}{2}$, otherwise,

$$v_D(H_{3,n}) = \sum_{i=1}^{n} f_D(F_i) \geq 2 \times \frac{1}{2} + (n-2) > n-2,$$

contradicting Equation (3). Consider the cycle C_{F_k}, it must be contractible, otherwise, there will be another crossing made by the non-contractible cycles C_{F_2} and C_{F_k}, contradicting that $f_D(F_2) = \frac{1}{2}$.

Subcase 2.2.1. $n = 4$. Then $j = 4$. By Equation (3), we know that $f_D(F_1) = f_D(F_3) = \frac{1}{2}$. Furthermore, the cycle C_{F_1} (respectively, C_{F_3}) is contractible in D, otherwise, it will cross the non-contractible cycle C_{F_2} in the projective plane, contradicting that $f_D(F_2) = \frac{1}{2}$. On the other hand, a non-contractible cycle $C(1)$ occurs, which should cross C_{F_2}, a contradiction.

Subcase 2.2.2. $n \geq 5$. Moreover, either $k = 3$ when $j = 4$, or $k = 1$ when $j = n$. Then by Equation (3), there exists another integer s ($s \notin \{2, k, j\}$) such that $f_D(F_s) = \frac{1}{2}$. Therefore, using arguments similar to those in Subcase 2.1, contradictions can be obtained by considering F_k and F_s.

Subcase 2.2.3. $n \geq 5$. Moreover, $k \neq 3$ when $j = 4$. Using arguments similar to those in Subcase 2.1, contradictions can be obtained either by considering F_k and F_j when $k \neq 5$, or by considering F_k and F_2 when $k = 5$.

Subcase 2.2.4. $n \geq 5$. Moreover, $k \neq 1$ when $j = n$. Analogously, contradictions can be obtained either by considering F_k and F_j when $k \neq n - 1$, or by considering F_k and F_2 when $k = n - 1$.

The proof is complete. ■

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