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ON HAMILTONIAN CYCLES IN CLAW-FREE CUBIC GRAPHS

Elena Mohr and Dieter Rautenbach

Institute of Optimization and Operations Research
Ulm University, Germany

e-mail: {elena.mohr,dieter.rautenbach}@uni-ulm.de

Abstract

We show that every claw-free cubic graph of order n at least 8 has at most $2^{\lfloor \frac{n}{4} \rfloor}$ Hamiltonian cycles, and we also characterize all extremal graphs.

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1. Introduction

Chia and Thomassen [2] proved that every cubic multigraph of order n at least 4 has at most $2^{n/2}$ Hamiltonian cycles. They asked whether there is a cubic graph of order n at least 6 that has more than $2^{n/3}$ Hamiltonian cycles. As they observed the unique connected cubic graph of order n, where n is a multiple of 6, that arises by adding a perfect matching to the disjoint union of $n/6$ copies of $K_{3,3}-e$ has exactly $2^{n/3}$ Hamiltonian cycles, that is, this bound can be achieved. In the present note, we show that every claw-free cubic graph of order n at least 8 has at most $2^{\lfloor \frac{n}{4} \rfloor}$ Hamiltonian cycles, and we also characterize all extremal graphs, which are structurally similar to the above graphs based on $K_{3,3}-e$. Recall that a graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph.

Figure 1. The diamond D and the graph D'.

Let $\mathcal{G}$ be the set of connected cubic graphs G of some order n such that

- if $n \equiv 0 \pmod 4$, then G arises by adding a matching M to the disjoint union of $n/4$ copies of the diamond D, and
- if $n \equiv 2 \pmod 4$, then G arises by adding a matching M to the disjoint union of one copy of D' and $(n-6)/4$ copies of the diamond D.

Note that, for every even order n at least 4, the set $\mathcal{G}$ contains exactly one graph of order n.

Our result is the following.

Theorem 1. *If G is a claw-free cubic graph of order n at least 8, then G has at most $2^{\lfloor \frac{n}{4} \rfloor}$ Hamiltonian cycles with equality if and only if $G \in \mathcal{G}$.*

We need two preparatory lemmas. For some non-negative integer k, let f_k denote the *k-th Fibonacci number*, that is, $f_0 = 0$, $f_1 = 1$, and $f_k = f_{k-1} + f_{k-2}$ for $k \geq 2$. The following fact is known [1]; in order to keep the paper self-contained, we add a simple proof.

Lemma 2. *For every integer k at least 3, there are $f_{k+1} + f_{k-1}$ sets F of edges of the cycle C_k of order k such that $C_k - F$ has no isolated vertex.*

Proof. Let $C_k = u_1 \cdots u_k u_1$, and let $\mathcal{F}$ be the set of all considered sets F, that is, we need to show that $a_k = f_{k+1} + f_{k-1}$ for $a_k = |\mathcal{F}|$. Let b_k be the number of sets F' of edges of the path $P_{k+1} : v_1 \cdots v_{k+1}$ of order $k+1$ such that none of the vertices $v_1, \ldots, v_k$ is isolated in $P_{k+1} - F'$; note that v_{k+1} is allowed to be isolated in $P_{k+1} - F'$. Clearly, $b_0 = 1$, $b_1 = 1$, and $b_2 = 2$. Furthermore, considering the edge $v_i v_{i+1}$ of P_{k+1} minimizing i that belongs to such a set F', it follows that $b_k = b_{k-2} + b_{k-3} + \cdots + b_0 + 1$, which implies $b_k = b_{k-1} + b_{k-2}$. The initial values and the recursion imply that $b_k = f_{k+1}$. Now, $a_k = 2b_{k-2} + b_{k-1} = 2f_{k-1} + f_k = f_{k+1} + f_{k-1}$, because there are

- b_{k-2} sets in $\mathcal{F}$ that contain $u_1 u_2$ (consider the path $u_2 \cdots u_k$),
- b_{k-2} sets in $\mathcal{F}$ that contain $u_k u_1$ (consider the path $u_1 \cdots u_{k-1}$), and
- b_{k-1} sets in $\mathcal{F}$ that contain neither $u_1 u_2$ nor $u_k u_1$ (consider the path $u_1 \cdots u_k$).

■

Lemma 3. $2^{\lfloor \frac{6\ell}{4} \rfloor} > f_{2\ell+1} + f_{2\ell-1}$ *for every $\ell \in \mathbb{N} \setminus \{1, 3\}$.*

Proof. For small values of ℓ, this is easily verified. For $\ell \geq 5$, this follows easily by induction using

$$2^{\lfloor \frac{6\ell}{4} \rfloor} \geq 2^{\frac{3\ell-1}{2}} > 0.707 \cdot 2.828^{\ell}$$

and

$$f_{2\ell+1} + f_{2\ell-1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2\ell+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2\ell+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{2\ell-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2\ell-1}\right)$$
$$= \left(\frac{1+\sqrt{5}}{2}\right)^{2\ell} + \left(\frac{1-\sqrt{5}}{2}\right)^{2\ell} < 2.619^{\ell} + 0.382. \qquad \blacksquare$$

***Proof of Theorem* 1.** If G lies in $\mathcal{G}$ and has order n at least 8, then every Hamiltonian cycle of G contains each edge of the matching M mentioned in the definition of $\mathcal{G}$. Since the diamond D and the graph D' both contain exactly two paths between their two vertices of degree 2, it follows that G has exactly $2^{\lfloor n/4 \rfloor}$ Hamiltonian cycles. It remains to show that every claw-free cubic graph G of order n at least 8 has at most $2^{\lfloor n/4 \rfloor}$ Hamiltonian cycles with equality only if $G \in \mathcal{G}$. Suppose, for a contradiction, that G is a counterexample of minimum order n. Let $C : u_1u_2 \cdots u_nu_1$ be some Hamiltonian cycle of G, where we identify indices modulo n. The edges in $E(G) \setminus E(C)$ are *chords* of C, and a chord u_iu_j is *short* if u_i and u_j have distance 2 on C.

Suppose, for a contradiction, that u_2u_4 and u_3u_5 are short chords, that is, there are two *crossing* short chords. If u_1 and u_6 are adjacent, then the claw-freeness implies that u_1 and u_6 have a common neighbor, which implies the contradiction that u_7 equals u_n, that is, $n = 7$. Hence, the vertices u_1 and u_6 are not adjacent, and $G' = G - \{u_2, u_3, u_4, u_5\} + u_1u_6$ is a Hamiltonian cubic claw-free graph of order $n' = n - 4$. If $n' = 4$, then G' is K_4, and G arises by adding a matching to the disjoint union of two copies of D, which implies the contradiction $G \in \mathcal{G}$. If $n' = 6$, then the claw-freeness of G implies that G' arises by adding the perfect matching $\{u_1u_6, u_7u_{10}, u_8u_9\}$ to the disjoint union of the two triangles $u_1u_9u_{10}u_1$ and $u_6u_7u_8u_6$. It follows that G arises by adding a matching to the disjoint union of one copy of D' and one copy of D, which implies the contradiction $G \in \mathcal{G}$. Now, if $n' \geq 8$, then the choice of G implies that G' is no counterexample. Note that Hamiltonian cycles of G' that do not contain the edge u_1u_6 do not correspond to Hamiltonian cycles of G, and that Hamiltonian cycles of G' that contain the edge u_1u_6 correspond to two distinct Hamiltonian cycles of G; one using the path $u_1u_2u_3u_4u_5u_6$ and one using the path $u_1u_2u_4u_3u_5u_6$. This implies that G has at most $2 \cdot 2^{\lfloor n'/4 \rfloor} = 2^{\lfloor n/4 \rfloor}$ Hamiltonian cycles with equality only if every Hamiltonian cycle of G' contains the edge u_1u_6 and G' has $2^{\lfloor n'/4 \rfloor}$ Hamiltonian cycles, that is, $G' \in \mathcal{G}$. Since G is claw-free, the edge u_1u_6 is one of the edges of G' that belong to some 2-edge-cut of G', which implies the contradiction $G \in \mathcal{G}$. Altogether, we obtain, that two crossing short chords do not exist.

Since G is claw-free, it follows that, for every two consecutive vertices on C, either one vertex is incident with a short chord and one vertex is incident with a non-short chord, or both vertices are incident with two non-crossing short chords. This implies that n is a multiple of 3, that is, $n = 3k$ for some positive integer k. By symmetry, we may assume that $u_{3i-2}u_{3i}$ is a short chord for every $i \in [k]$, where $[k]$ is the set of positive integers at most k. For every i in $[k]$, every Hamiltonian cycle of G uses one or both of the edges $u_{3i-3}u_{3i-2}$ and $u_{3i}u_{3i+1}$. More precisely, if some Hamiltonian cycle of G uses both these edges, then it contains the subpath $u_{3i-3}u_{3i-2}u_{3i-1}u_{3i}u_{3i+1}$, if it uses $u_{3i-3}u_{3i-2}$ but not $u_{3i}u_{3i+1}$, then it contains the subpath $u_{3i-3}u_{3i-2}u_{3i}u_{3i-1}$, and if it uses $u_{3i}u_{3i+1}$ but not $u_{3i-3}u_{3i-2}$, then it contains the subpath $u_{3i-1}u_{3i-2}u_{3i}u_{3i+1}$. This implies that every Hamiltonian cycle C' of G is uniquely determined by the intersection of $E(C')$ with the set $\{u_{3i}u_{3i+1} : i \in [k]\}$, and that $\{u_{3i}u_{3i+1} : i \in [k]\} \setminus E(C')$ does not contain two of the edges in $\{u_{3i}u_{3i+1} : i \in [k]\}$ that appear consecutively on C. By Lemma 2, it follows that the number of Hamiltonian cycles of G is at most $f_{\frac{n}{3}+1} + f_{\frac{n}{3}-1}$. Since G is cubic, the order n of G is even, which implies that k is even, that is, $k = 2\ell$ and $n = 6\ell$ for some integer $\ell \geq 2$. By Lemma 3, $2^{\lfloor n/4 \rfloor} = 2^{\lfloor 6\ell/4 \rfloor} > f_{2\ell+1} + f_{2\ell-1} = f_{\frac{n}{3}+1} + f_{\frac{n}{3}-1}$ unless $\ell = 3$, that is, $n = 18$. For $n = 18$, there are four different possibilites for the structure of G shown in Figure 2.

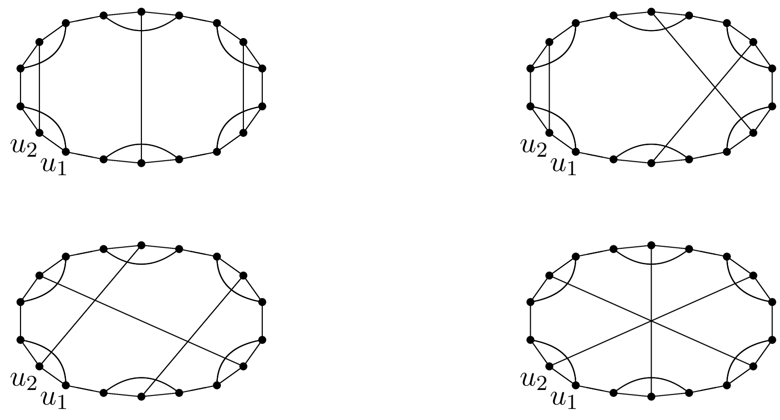

Figure 2. Every Hamiltonian cycle of the top left graph uses the four edges that lie in 2-edge-cuts of this graph, which easily implies that it has exactly four Hamiltonian cycles. Every Hamiltonian cycle of the top right graph uses either both the edges u_8u_{14} and $u_{11}u_{17}$ or none of these two edges, which easily implies that it has exactly four Hamiltonian cycles. The bottom left graph has one Hamiltonian cycle using no non-short chord and two using all three non-short chords, which implies that it has exactly three Hamiltonian cycles. The bottom right graph has one Hamiltonian cycle using no non-short chord, three Hamiltonian cycles using two non-short chords, and two Hamiltonian cycles using all three non-short chord, which implies that it has exactly six Hamiltonian cycles.

Each of these graphs has at most six Hamiltonian cycles. Since this is less than $2^{\lfloor 18/4 \rfloor} = 16$, the proof is complete. ■

References

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