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TOUGHNESS, FORBIDDEN SUBGRAPHS, AND HAMILTON-CONNECTED GRAPHS[1]

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Abstract

A graph G is called Hamilton-connected if for every pair of distinct vertices $\{u, v\}$ of G there exists a Hamilton path in G that connects u and v. A graph G is said to be t-tough if $t \cdot \omega(G - X) \leq |X|$ for all $X \subseteq V(G)$ with $\omega(G - X) > 1$. The toughness of G, denoted $\tau(G)$, is the maximum value of t such that G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). It is known that a Hamilton-connected graph G has toughness $\tau(G) > 1$, but that the reverse statement does not hold in general. In this paper, we investigate all possible forbidden subgraphs H such that every H-free graph G with $\tau(G) > 1$ is Hamilton-connected. We find that the results are completely analogous to the Hamiltonian case: every graph H such that any 1-tough H-free graph is Hamiltonian also ensures that every H-free graph with toughness larger than one is Hamilton-connected. And similarly, there is no other forbidden subgraph having this property, except possibly for the graph $K_1 \cup P_4$ itself. We leave this as an open case.

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1. Introduction

We use standard graph terminology and notation adopted from the textbook [4], and consider simple graphs only. Let G be a graph with vertex set $V(G)$, and let H be a subgraph of G. For a vertex $u \in V(G)$, the *neighborhood* of u in H is denoted by $N_H(u) = \{v \in V(H) \mid uv \in E(G)\}$ and the *degree* of u in H is denoted by $d_H(u) = |N_H(u)|$. When it is understood from the context, we use $N(u)$ and $d(u)$ instead of $N_G(u)$ and $d_G(u)$, respectively. For two distinct vertices $u, v \in V(G)$, a (u, v)-path is a path with end vertices u and v. We use K_n and P_n to denote the complete graph and the path with n vertices, respectively. For a nonempty subset S of $V(G)$, we use $\langle S \rangle$ to denote the subgraph of G induced by S, and for a proper subset S of $V(G)$, we use $G - S$ to denote the subgraph induced by $V(G) \setminus S$. For a given graph H, we say G is *H-free* if G does not contain an induced copy of H. Let $\omega(G)$ denote the number of components of the graph G. As introduced in [7], a connected graph G is said to be *t-tough* if $t \cdot \omega(G - X) \leq |X|$ for all $X \subseteq V(G)$ with $\omega(G - X) > 1$. The *toughness* of G, denoted by $\tau(G)$, is the maximum value of t such that G is t-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$).

A cycle in a graph G is called a *Hamilton cycle* if it contains all vertices of G, and a graph is said to be *Hamiltonian* if it contains a Hamilton cycle. A *Hamilton path* in a graph G is a path that contains all vertices of G, and a graph G is *Hamilton-connected* if every pair of vertices of G occurs as the two end vertices of a Hamilton path of G. It is easy to verify and a well-known fact that a Hamiltonian graph is 1-tough, and that a Hamilton-connected graph has toughness strictly larger than one. It is also known that the reverse statements do not hold, i.e., there exist infinitely many non-Hamiltonian 1-tough graphs, and there exist infinitely many graphs with toughness strictly larger than one that are not Hamilton-connected. More specifically, to answer Chvátal's Conjecture [7] which states that there exists a constant t_0 such that every t_0-tough graph on $n \geq 3$ vertices is Hamiltonian, the authors in [2] proved that $t_0 \geq 9/4$ by constructing an infinite family of non-Hamiltonian graphs with toughness arbitrarily close to $9/4$ from below. It is natural and interesting to investigate under which additional conditions the reverse statements do hold. In other words, under which additional conditions are the properties of being 1-tough and being Hamiltonian equivalent, and similarly for the stronger properties of having toughness strictly larger than one and being Hamilton-connected. The type of additional conditions we focus on here are forbidden subgraph conditions. For Hamiltonicity this type of problem was addressed by the authors of [10]. More relations between different Hamiltonian properties and toughness conditions have been studied in [1], leading to several equivalent conjectures, some seemingly stronger and some seemingly weaker than Chvátal's Conjecture. The survey paper [3] deals with a large number of results that have been established until more than ten years ago. A more recent survey of results and open problems appeared a few years ago [5].

We recall two results of [10] that motivated the research of this paper. Here $G_1 \cup G_2$ denotes the disjoint union of two vertex-disjoint graphs G_1 and G_2, and kG denotes the disjoint union of k copies of the graph G.

Theorem 1 (Li *et al.* [10])**.** *Let R be an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free 1-tough graph on at least three vertices is hamiltonian.*

Note that every induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$ is also an induced subgraph of $K_1 \cup P_4$, and that $K_1 \cup P_4$ is the only induced subgraph of $K_1 \cup P_4$ that is not an induced subgraph of P_4, $K_1 \cup P_3$ or $2K_1 \cup K_2$. The following complementary result in [10] shows that there is no graph H other than the induced subgraphs of $K_1 \cup P_4$ that can ensure every 1-tough H-free graph is Hamiltonian.

Theorem 2 (Li *et al.* [10])**.** *Let R be a graph on at least three vertices. If every R-free 1-tough graph on at least three vertices is Hamiltonian, then R is an induced subgraph of $K_1 \cup P_4$.*

The two theorems together clearly leave $K_1 \cup P_4$ as the only open case in characterizing all the graphs H such that every H-free 1-tough graph is Hamiltonian, and it seems to be a very hard case. In fact, this was the conjecture of Nikoghosyan in [12] that motivated the work in [10].

To date it is even unknown whether there exists some constant t such that every t-tough $K_1 \cup P_4$-free graph is Hamiltonian.

A Hamiltonian graph is 1-tough, and hence 2-connected, so a Hamilton-connected graph G on at least three vertices is also 2-connected. It is even clearly 3-connected: if there exists a cut set $\{u, v\}$ in G, then u and v cannot be connected by a Hamilton path in G, because only the vertices of one component of $G - \{u, v\}$ can be picked up between u and v. It is almost equally easy to show that a Hamilton-connected graph has toughness strictly larger than one. This can be seen by considering an arbitrary cut set S in a Hamilton-connected graph G, and a Hamilton path P between two distinct vertices u and v of S (noting that $|S| \geq 3$ since G is 3-connected). Now, obviously $\omega(G - S) \leq \omega(P - S) \leq |S| - 1$, hence $\tau(G) > 1$.

In 1978, Jung [8] obtained the following result, in which he showed that for P_4-free graphs, the necessary condition $\tau(G) > 1$ is also a sufficient condition for Hamilton-connectivity.

Theorem 3 (Jung [8]). *Let G be a P_4-free graph. Then G is Hamilton-connected if and only if $\tau(G) > 1$.*

In a paper of 2000 [6], Chen and Gould concluded that if $\{S, T\}$ is a pair of graphs such that every 2-connected $\{S, T\}$-free graph is Hamiltonian, then every 3-connected $\{S, T\}$-free graph is Hamilton-connected. Following up on this idea, we considered the following question. Suppose R is a graph such that every 1-tough R-free graph is Hamiltonian. Is then every R-free graph G with $\tau(G) > 1$ Hamilton-connected? For the purpose of answering this question, we tried to prove each of the forbidden subgraph cases analogous to the statement in Theorem 1. Of course Theorem 3 has already given us a partial positive answer. And indeed, we get a positive answer for each of these cases, as indicated in the following result.

Theorem 4. *Let R be an induced subgraph of $K_1 \cup P_3$ or $2K_1 \cup K_2$. Then every R-free graph G with $\tau(G) > 1$ on at least three vertices is Hamilton-connected.*

We note here that from the proof of this result, it can be observed that the toughness condition $\tau(G) > 1$ in the above result cannot be weakened to the condition that the graph is 3-connected. We also proved the following analogue of Theorem 2, showing that except for the proper induced subgraphs of $K_1 \cup P_4$, there are no other forbidden induced subgraphs that can ensure every graph with toughness larger than one is Hamilton-connected.

Theorem 5. *Let R be a graph on at least three vertices. If every R-free graph G with $\tau(G) > 1$ on at least three vertices is Hamilton-connected, then R is an induced subgraph of $K_1 \cup P_4$.*

We conclude this section with the left unknown case as an open problem.

Problem 1. Is every $K_1 \cup P_4$-free graph G with $\tau(G) > 1$ on at least three vertices Hamilton-connected?

As remarked earlier, we do not even know whether such graphs are Hamiltonian, even if the condition on the toughness is replaced by $\tau(G) > t$ for any constant $t \geq 1$.

The next two sections are devoted to the proofs of Theorem 4 and Theorem 5, respectively.

2. Proof of Theorem 4

For a path P in G with a given orientation and a vertex x on P, x^+ and x^- denote the immediate successor and the immediate predecessor of x on P (if they exist), respectively. For any subset $I \subseteq V(P)$, let $I^- = \{x^- \mid x \in I\}$ and

$I^+ = \{x^+ \mid x \in I\}$. For two vertices $x, y \in V(P)$, xPy denotes the subpath of P from x to y, and $y\overline{P}x$ denotes the path from y to x in the opposite direction. For a subgraph H disjoint from P in G, when $N(x) \cap V(H) \neq \emptyset$ and $N(y) \cap V(H) \neq \emptyset$, we use xHy to denote a path in G from x to y with all internal vertices in H.

Now, let P be a longest (u, v)-path in a graph G, and let H be a component of $G - V(P)$. Furthermore, let $I = N_P(H) = \{x_1, x_2, \ldots, x_s\}$, and let w be a vertex of H. Then we start with the following lemma.

Lemma 1. *Both* $\{w\} \cup I^+$ *and* $\{w\} \cup I^-$ *are independent sets.*

Proof. Suppose, to the contrary, that there is an edge in $\{w\} \cup I^+$. If the edge appears between w and a vertex of I^+, say $wx_i^+ \in E(G)$ with $x_i^+ \in I^+$, then $uPx_iHx_i^+Pv$ is a (u, v)-path longer than P, contradicting the choice of P. If the edge appears between two vertices of the set I^+, say $x_i^+x_j^+ \in E(G)$ with $x_i^+, x_j^+ \in I^+$, then $uPx_iHx_j\overline{P}x_i^+x_j^+Pv$ is a (u, v)-path longer than P, a contradiction. Hence $\{w\} \cup I^+$ is an independent set. Similarly, by symmetry $\{w\} \cup I^-$ is also an independent set. ■

Next we complete the proof for the two choices of R in Theorem 4, respectively. Note that we do not have to consider proper induced subgraphs of R, since a graph is R-free if it is S-free for an induced subgraph S of R.

The case $R = K_1 \cup P_3$.
Assume that G is a $K_1 \cup P_3$-free graph with $\tau(G) > 1$. Suppose to the contrary that G is not Hamilton-connected, and that u, v is a pair of distinct vertices of G that is not connected by a Hamilton path in G. Let P be a longest (u, v)-path in G. Since P is not a Hamilton path, $V(G) \setminus V(P) \neq \emptyset$. Assume that H is a component of $G - V(P)$. Then $|N_P(H)| \geq 3$ since $\tau(G) > 1$. Assume that $N_P(H) = \{v_1, v_2, \ldots, v_s\}$ with $s \geq 3$, in this order according to the fixed chosen orientation of P. We denote the segment of P from v_i^+ to v_{i+1}^- by Q_i for all i with $1 \leq i \leq s-1$. If $v_1 \neq u$, then let $Q_0 = uPv_1^-$. If $v_s \neq v$, then let $Q_s = v_s^+Pv$.

Before completing the proof for this case, we first prove the following two claims.

Claim 1. *At least two of the segments of* P *are connected by a path* (*possibly an edge*) *that is internally-disjoint with* P.

Proof. By Lemma 1, the neighbors of H on P are not consecutive vertices on P. If none of the segments of P is connected to another segment of P by a path (or edge) internally-disjoint with P, then every segment is in a separate component after removal of the vertices of $N_P(H)$. Then there will be at least s components after deleting the s vertices of $N_P(H)$, contradicting the fact that $\tau(G) > 1$. □

Using Claim 1, we assume that Q_i and Q_j $(0 \leq i < j \leq s)$ are connected by a path (edge) that is internally-disjoint with P. In fact, the next claim shows that we may assume that this path is actually an edge.

Claim 2. *Q_i and Q_j are connected by an edge.*

Proof. Supposing the statement is false, we consider a shortest path that connects Q_i and Q_j and is internally-disjoint with P, and denote it as $Q = q_1q_2 \cdots q_r$ (with $q_1 \in V(Q_i)$ and $q_r \in V(Q_j)$). Obviously, Q is an induced path, and $N(Q) \cap V(H) = \emptyset$. If $r \geq 3$, then $\{w, q_1, q_2, q_3\}$ induces a copy of $K_1 \cup P_3$, where w is a vertex of $V(H)$, a contradiction. Hence, the shortest path connecting Q_i and Q_j is an edge, and the claim holds. □

We use Claim 2 and distinguish two cases, depending on the value of the indices i and j, as follows.

Case A. Q_i and Q_j are connected by an edge, for some i and j with $1 \leq i < j \leq s-1$. Suppose that xy is an edge with $x \in V(Q_i)$ and $y \in V(Q_j)$, and chosen such that $|v_i^+ P xy \overline{P} v_j^+|$ is as small as possible. Using Lemma 1, we know that either $x \neq v_i^+$ or $y \neq v_j^+$. Without loss of generality, say $x \neq v_i^+$. By the choice of xy, we have that $x^-y \notin E(G)$. Then an arbitrary vertex w of $V(H)$ together with the three vertices of $\{x^-, x, y\}$ induces a copy of $K_1 \cup P_3$, a contradiction.

Case B. All edges connecting two different segments of P have at least one end vertex in Q_0 or Q_s. By Claim 2, the assumption of this case implies that any two of the $s-1$ segments Q_i ($i \in \{1, 2, \ldots, s-1\}$) of P are not connected by a path internally-disjoint with P. Then there must be a segment Q_i ($i \in \{1, 2, \ldots, s-1\}$) that has a neighbor in Q_0 or Q_s. Otherwise, P has $s+1$ segments and only Q_0 and Q_s among all these segments are connected by such a path. Then, by deleting the s neighbors of H on P, we obtain $s+1$ components, contradicting the fact that $\tau(G) > 1$. Without loss of generality, we assume that Q_i ($i \in \{1, 2, \ldots, s-1\}$) is connected to Q_0 by an edge. We use xy to denote an edge between $V(Q_0)$ and $V(Q_i)$, chosen in such a way that $|v_1^- \overline{P} xy P v_{i+1}^-|$ is as small as possible. Using Lemma 1, we know that either $x \neq v_1^-$ or $y \neq v_{i+1}^-$. Without loss of generality, say $x \neq v_1^-$. By the choice of xy, we have that $x^+y \notin E(G)$. Then an arbitrary vertex w of $V(H)$ together with the three vertices of $\{x^+, x, y\}$ induces a copy of $K_1 \cup P_3$, a contradiction.

This completes the proof for the case $R = K_1 \cup P_3$. We now turn to the remaining case that $R = 2K_1 \cup K_2$.

The case $R = 2K_1 \cup K_2$.

Suppose that G is a $2K_1 \cup K_2$-free graph with $\tau(G) > 1$, and assume that G is not Hamilton-connected. Let u, v be a pair of distinct vertices of G that is not connected by a Hamilton path in G, and let P be a longest (u, v)-path in G. Then $V(G) \setminus V(P) \neq \emptyset$. Assume that H is a component of $G - V(P)$. Since $\tau(G) > 1$, we have $|N_P(H)| \geq 3$. Similarly as in the case $R = K_1 \cup P_3$, we use $N_P(H) = \{v_1, v_2, \ldots, v_s\}$ to denote all neighbors of H on P, so with $s \geq 3$ and in this order according to the chosen orientation of P.

We continue with first proving three useful claims.

Claim 3. *H is trivial, i.e., $|V(H)| = 1$.*

Proof. Suppose H contains an edge w_1w_2. Using Lemma 1, we get that $\{w_1, v_1^+, v_2^+\}$ and $\{w_2, v_1^+, v_2^+\}$ are independent sets. Then $\{v_1^+, v_2^+, w_1, w_2\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. □

Let $H = \{w\}$. Then $N_P(w) = \{v_1, v_2, \ldots, v_s\}$, with $s \geq 3$. Let Q_i be the segment of P from v_i^+ to v_{i+1}^- for $1 \leq i \leq s-1$, denoted as $Q_i = x_{i_1}x_{i_2}\cdots x_{i_{r_i}}$, with $x_{i_1} = v_i^+$ and $x_{i_{r_i}} = v_{i+1}^-$. If $v_1 \neq u$, then let $Q_0 = ux_{0_1}x_{0_2}\cdots x_{0_{r_0}}$, with $x_{0_{r_0}} = v_1^-$. If $v_s \neq v$, then let $Q_s = x_{s_1}x_{s_2}\cdots x_{s_{r_s}}v$, with $x_{s_1} = v_s^+$. Now we prove the following useful facts.

Claim 4. *For all $i \in \{1, 2, \ldots, s-1\}$, we have $v_1^+x_{i_j} \notin E(G)$ for every odd j, $v_1^+x_{i_j} \in E(G)$ for every even j, and r_i is odd. In addition, if Q_0 and Q_s exist, then v_1^+ is alternately adjacent and nonadjacent to the vertices of the segments Q_0 and Q_s with $v_1^+x_{0_{r_0}} \notin E(G)$ and $v_1^+x_{s_1} \notin E(G)$.*

Proof. We divide the proof into two cases according to the length of the segment Q_1.

Case A. $|Q_1| = 1$, i.e., $v_2 = v_1^{++}$. In this case, the claim holds for the segment Q_1 itself. For the segments Q_i ($i = 0, 2, 3, \ldots, s$), we first prove that if $v_1^+x_{i_j} \in E(G)$, then $v_1^+x_{i_{j+1}} \notin E(G)$, and if $v_1^+x_{i_j} \notin E(G)$, then $v_1^+x_{i_{j+1}} \in E(G)$ for all $j \in \{1, 2, \ldots, i_{r_i} - 1\}$. Suppose that there is a segment Q_i with $i = 0, 2, 3, \ldots, s$ such that $v_1^+x_{i_j} \in E(G)$ and $v_1^+x_{i_{j+1}} \in E(G)$ for some $j \in \{1, 2, \ldots, i_{r_i} - 1\}$. Then there exists a longer (u, v)-path $P' = uPv_1wv_2Px_{i_j}v_1^+x_{i_{j+1}}Pv$ (if $i \neq 0$), or $P' = uPx_{i_j}v_1^+x_{i_{j+1}}Pv_1wv_2Pv$ (if $i = 0$), a contradiction. Suppose that Q_i ($i = 0, 2, 3, \ldots, s$) is a segment with $v_1^+x_{i_j} \notin E(G)$ and $v_1^+x_{i_{j+1}} \notin E(G)$ for some $j \in \{1, 2, \ldots, i_{r_i} - 1\}$. Then $\{w, v_1^+, x_{i_j}, x_{i_{j+1}}\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus the neighbors of v_1^+ occur on every segment Q_i alternately along the path. By Lemma 1 we have $v_1^+x_{i_1} \notin E(G)$ for $i = 2, 3, \ldots, s$ and $v_1^+x_{i_{r_i}} \notin E(G)$ for $i = 0, 2, 3, \ldots, s-1$. Hence r_i is odd for $i \in \{1, 2, \ldots, s-1\}$, and the claim holds.

Case B. $|Q_1| \geq 2$, i.e., $v_1^{++} \notin N_P(w)$. Firstly, we consider the case that $i \in \{0, 2, 3, \ldots, s\}$. By Lemma 1, $v_1^+x_{i_1} \notin E(G)$ for $i \in \{2, 3, \ldots, s\}$. To avoid that $\{w, x_{i_1}, v_1^+, v_1^{++}\}$ induces a copy of $2K_1 \cup K_2$, we have $v_1^{++}x_{i_1} \in E(G)$. If there exists an index $j \in \{1, 2, \ldots, i_{r_i} - 1\}$ such that $v_1^+x_{i_j} \in E(G)$ and $v_1^+x_{i_{j+1}} \in E(G)$, then we have a longer (u, v)-path $P' = uPv_1wv_i\overline{P}v_1^{++}x_{i_1}Px_{i_j}v_1^+x_{i_{j+1}}Pv$ (if $i \neq 0$), or a longer (u, v)-path $P' = uPx_{i_j}v_1^+x_{i_{j+1}}Pv_1wv_2\overline{P}v_1^{++}x_{2_1}Pv$ (if $i = 0$), a contradiction. If $v_1^+x_{i_j} \notin E(G)$ and $v_1^+x_{i_{j+1}} \notin E(G)$ for some $j \in \{1, 2, \ldots, i_{r_i} - 1\}$, then $\{w, v_1^+, x_{i_j}, x_{i_{j+1}}\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus the neighbors of v_1^+ occur on every segment Q_i ($i = 0, 2, 3, \ldots, s$) alternately along the path. We know $v_1^+ x_{i_1} \notin E(G)$ for $i = 2, 3, \ldots, s$, by Lemma 1. Now $v_1^+ x_{i_{r_i}} \notin E(G)$ for $i = 0, 2, 3, \ldots, s-1$; otherwise $P' = uPv_1^+ x_{i_{r_i}} \overline{P} x_{i_1} v_1^{++} P v_i w v_{i+1} P v$ is a longer (u,v)-path (if $i \neq 0$), or $P' = uPx_{i_{r_i}} v_1^+ v_1 w v_2 \overline{P} v_1^{++} x_{2_1} P v$ is a longer (u,v)-path (if $i = 0$). Hence r_i is odd for $i \in \{2, \ldots, s-1\}$.

Secondly, we consider the remaining case that $i = 1$. If $v_1^+ x_{1_j} \in E(G)$ and $v_1^+ x_{1_{j+1}} \in E(G)$ for some $j \in \{2, 3, \ldots, i_{r_i} - 1\}$, then we obtain a contradiction by the longer (u,v)-path $P' = uPv_1 w v_2 \overline{P} x_{1_{j+1}} v_1^+ x_{1_j} \overline{P} v_1^{++} x_{2_1} P v$ (if $x_{1_j} \neq v_1^{++}$), or $P' = uPv_1 w v_2 \overline{P} x_{1_{j+1}} v_1^+ x_{1_j} x_{2_1} P v$ (if $x_{1_j} = v_1^{++}$). If $v_1^+ x_{1_j} \notin E(G)$ and $v_1^+ x_{1_{j+1}} \notin E(G)$ for some $j \in \{2, 3, \ldots, i_{r_i} - 1\}$, then $\{w, v_1^+, x_{1_j}, x_{1_{j+1}}\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus the neighbors of v_1^+ occur on the segment Q_1 alternately along the path. Suppose that $v_1^+ x_{1_{r_1}} \in E(G)$. If $w x_{2_2} \in E(G)$, i.e., $x_{2_2} = v_3$, then $uPv_1^+ x_{1_{r_1}} \overline{P} v_1^{++} x_{2_1} v_2 w v_3 P v$ is a longer (u,v)-path. If $w x_{2_2} \notin E(G)$, then $v_1^+ x_{2_2} \in E(G)$ and $uPv_1 w v_2 x_{2_1} v_1^{++} P x_{1_{r_1}} v_1^+ x_{2_2} P v$ is a longer (u,v)-path. Hence, $v_1^+ x_{1_{r_1}} \notin E(G)$ and r_1 is odd. Therefore the claim holds for all cases. □

We need one more claim which is easy to prove.

Claim 5. $N(v_1^+) \subseteq V(P)$.

Proof. If there is a vertex $z \in V(G) \setminus V(P)$ such that $v_1^+ z \in E(G)$, then the vertex set $\{w, x_{2_1}, v_1^+, z\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Therefore, $N(v_1^+) \subseteq V(P)$. □

Let $S = N(v_1^+) \cup N_P(H)$ and $|S| = s'$. By Claim 4, the vertices of S occur on the path P alternately. If $|V(P)|$ is odd, then $s' = \left\lceil \frac{|V(P)|}{2} \right\rceil$; if $|V(P)|$ is even, then $s' = \frac{|V(P)|}{2}$. Moreover, S is a cut set whose deletion yields at least three components, including the two trivial ones with vertices w and v_1^+. If one of the other components contains an edge $z_1 z_2$, then $\{w, v_1^+, z_1, z_2\}$ induces a copy of $2K_1 \cup K_2$, a contradiction. Thus all components of $G - S$ are trivial, meaning that every vertex of $V(P) \setminus S$ is a component. Hence, $\omega(G - S) \geq s' - 1 + 1 = |S|$, contradicting the fact that $\tau(G) > 1$.

This completes the proof of Theorem 4. ■

3. Proof of Theorem 5

For our proof that there is no graph H, apart from the induced subgraphs of $K_1 \cup P_4$, that can ensure every H-free graph with toughness larger than one is Hamilton-connected, we make use of the following lemma.

Lemma 2 (Li *et al.* [10]). *Let R be a graph on at least three vertices. If R is not an induced subgraph of $K_1 \cup P_4$, then R contains one of the graphs in $\mathcal{H} = \{C_3, C_4, C_5, K_{1,3}, 2K_2, 4K_1\}$ as an induced subgraph.*

Using Lemma 2, we can complete our proof of Theorem 5 by showing that not every R-free graph with toughness larger than one is Hamilton-connected, for each of the graphs $R \in \mathcal{H}$. To show this, we continue by giving suitable counterexamples; some of these graphs are even not Hamiltonian. The only class for which we cannot refer to known results, is the class of $4K_1$-free graphs. It is not difficult to check that the graphs sketched in Figure 1 are examples of $4K_1$-free graphs that are not Hamilton-connected but have toughness larger than one. In this sketch, the middle three vertices in the figure are supposed to be joined to all the vertices of the complete graph on the left, and u and v are also joined to all vertices of the complete graphs on the right; the other middle vertex is only joined to the two indicated vertices on the right; these indicated vertices are not joined to u or v.

Note that between u and v there is no Hamilton path (even if $m = s = t = 1$), since the deletion of $\{u, v\}$ leaves a graph with a cut vertex z (the other vertex in the middle), and one cannot pick up all the vertices in both components that result from deleting z.

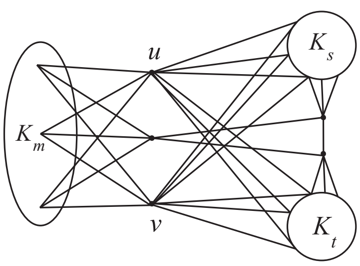

Figure 1. $4K_1$-free non-Hamilton-connected graphs.

For $R = C_3$, the well-known non- Hamiltonian Petersen graph is a suitable counterexample, since it is C_3-free and has toughness 4/3.

For $R \in \{C_4, C_5, 2K_2\}$, we can find suitable split graphs as counterexamples. Split graphs consist of a clique C and an independent set I with some (or possibly all or none) of the edges joining a vertex of C and a vertex of I (but no edges joining pairs of vertices of I). Split graphs are known to be $\{C_4, C_5, 2K_2\}$-free. It was proved in [9] that every $\frac{3}{2}$-tough split graph is Hamiltonian, and that there is a sequence $\{G_n\}_{n=1}^{\infty}$ of split graphs with no 2-factor (a 2-regular spanning subgraph, not necessarily connected) and $\tau(G_n) \to 3/2$. The latter graphs clearly serve as suitable examples for our purposes.

For $R = K_{1,3}$, we use the known fact that for a claw-free noncomplete graph G, $2\tau(G) = \kappa(G)$, where $\kappa(G)$ denotes the (vertex) connectivity of G. In [11], the authors conjectured that every 4-connected claw-free graph is Hamiltonian, and they showed examples of 3-connected claw-free graphs that are not Hamiltonian. These examples have toughness 3/2 and clearly serve our purposes.

This completes our proof of Theorem 5. ■

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