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EQUITABLE TOTAL COLORING OF CORONA OF CUBIC GRAPHS

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Abstract

The minimum number of total independent partition sets of $V \cup E$ of a graph $G = (V, E)$ is called the *total chromatic number* of G, denoted by $\chi''(G)$. If the difference between cardinalities of any two total independent sets is at most one, then the minimum number of total independent partition sets of $V \cup E$ is called the *equitable total chromatic number*, and is denoted by $\chi''_{=}(G)$.

In this paper we consider equitable total coloring of coronas of cubic graphs, $G \circ H$. It turns out that independently on the values of equitable total chromatic number of factors G and H, equitable total chromatic number of corona $G \circ H$ is equal to $\Delta(G \circ H) + 1$. Thereby, we confirm Total Coloring Conjecture (TCC), posed by Behzad in 1964, and Equitable Total Coloring Conjecture (ETCC), posed by Wang in 2002, for coronas of cubic graphs. As a direct consequence we get that all coronas of cubic graphs are of Type 1.

Keywords: equitable coloring, total coloring, equitable total coloring, cubic graphs.

2010 Mathematics Subject Classification: 05C15, 05C76.

1. Introduction

Graph coloring is one of the most important problems in graph theory. As an extension of proper vertex and edge coloring, the concept of total coloring is developed. In the paper, we consider one of non-classical models of total coloring, namely equitable total coloring.

A *k-total-coloring* of G is an assignment of k colors to the edges and vertices of G, so that the adjacent or incident elements obtain different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total-coloring. Clearly, $\chi''(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. The well known Total Coloring Conjecture [1, 16] states that the total chromatic number of any graph G is at most $\Delta(G) + 2$.

Conjecture 1 (TCC, [1, 16]). *For any graph G the following inequalities hold*

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2.$$

Although the hypothesis has been known since 1964, it has been proven only for some specific classes of graphs, in particular for cubic graphs [15]. Graphs with $\chi''(G) = \Delta(G)+1$ are said to be *Type* 1, and graphs with $\chi''(G) = \Delta(G)+2$ are said to be *Type* 2. The problem of deciding whether a graph is Type 1 has been shown to be NP-complete even for cubic bipartite graphs [13].

In this paper one of non-classical models of total coloring is considered. A k-total-coloring is *equitable* if the cardinalities of any two color classes differ by at most one (ref. Figure 1). The smallest k for which G has an equitable k-total-coloring is the *equitable total chromatic number* of G, and it is denoted by $\chi''_=(G)$. The concept of equitable total coloring was first presented in [6]. This model of graph coloring has many practical applications. Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring. In particular, one motivation for equitable coloring suggested by Meyer [12] concerns scheduling problems. Furmańczyk [8] mentions a specific application of this type of scheduling problem, namely, assigning university courses to time slots in a way that avoids scheduling incompatible courses at the same time and spreads the courses evenly among the available time slots. The topic of equitable coloring, also its total version, was widely discussed in literature. Similarly to the situation with proper total coloring, it was conjectured that the equitable total chromatic number of any graph is at most $\Delta(G) + 2$.

Conjecture 2 (ETCC, [14]). *For any graph G the following inequalities hold*

$$\Delta(G) + 1 \leq \chi''_=(G) \leq \Delta(G) + 2.$$

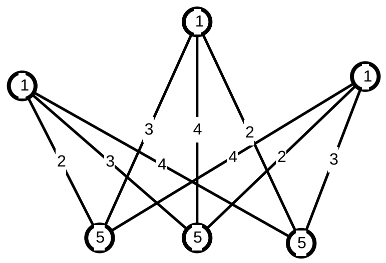

Figure 1. An exemplary equitable total 5-coloring of $K_{3,3}$.

This conjecture was proven among others for cubic graphs in [14]. Wang [14] proved that every cubic graph has an equitable total coloring with 5 colors. Recently, it has been shown that the problem of deciding whether the equitable total chromatic number of a bipartite cubic graph is 4 is NP-complete [4].

One can ask whether there exist graphs with equitable total chromatic number greater than total chromatic number. It turns out the answer to this question is positive. There are known examples of cubic graphs such that their total chromatic number is strictly less than their equitable total chromatic number [4, 6].

In this paper we ask about the value of the equitable total coloring number of graph products. The problem was considered for some Cartesian products of graphs [3]. Moreover, graph products are interesting and useful in many situations. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors. We continue the research on graph products, but this time as a factor we take cubic graphs and we consider corona product of graphs.

Given two simple graphs G and H, the *corona product of G and H* is the graph $G \circ H$ obtained by taking one copy of G, $|V(G)|$ copies of H, and making the i-th vertex of G adjacent to every vertex of the i-th copy of H, H_i (ref. Figure 2). This graph product was introduced by Frucht and Harary in 1970 [5].

In this paper we focus on coronas of two arbitrary cubic graphs. This kind of graph product seems to be interesting because corona graphs lie often close to the boundary between easy and hard problems [9]. Here, we ask whether the fact of being the cubic graph of Type 1 or 2 has the influence on the value of the equitable total chromatic number of the corona of such factors. It turns out that the answer is negative. Let G and H be two cubic graphs with $|V(G)| = n_G$ and $|V(H)| = n_H$ vertices, respectively. It is easy to see that the maximum degree of the corona graph $G \circ H$ is $\Delta(G \circ H) = n_H + 3$. We prove that the equitable total chromatic number of the corona graph $G \circ H$ is equal to $\Delta(G \circ H) + 1 = n_H + 4$, independently of the type of factors G and H. As a consequence, we get that the total chromatic number of $G \circ H$ is equal to $\Delta(G \circ H) + 1 = n_H + 4$, i.e., they are all of Type 1.

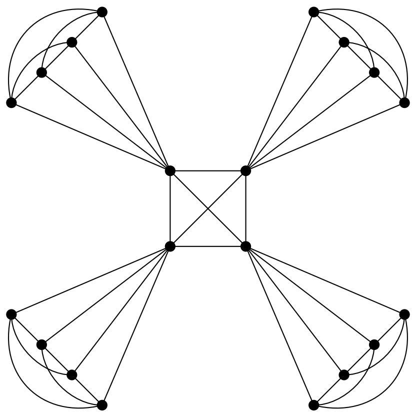

Figure 2. Corona $K_4 \circ K_4$.

2. Notation and Definitions

Definition 1. A *semi-graph* is a triple $G = (V, E, S)$, where $V(G)$ is a set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a multiset of *semi-edges* having one endpoint in $V(G)$.

Note that if $S(G) = \emptyset$, then a semi-graph G is a simple graph. All definitions given below for semi-graphs, that do not require the existence of semi-edges, are also valid for graphs. When it could be confusing we explicitly write graph or semi-graph. We write edges having endpoints v and w shortly as vw and semi-edges having endpoint v as $v\cdot$. When vertex v is an endpoint of $e \in E \cup S$ we say that v and e are *incident*. Two elements of $E \cup S$ incident with the same vertex, respectively two vertices incident with the same edge, are called *adjacent*. $N(v)$ denotes the *open neighborhood* of a vertex $v \in V(G)$, i.e., the set of adjacent vertices for v. $N[v] = N(v) \cup \{v\}$ is the *close neighborhood* of v. The *degree* $\deg(v)$ of a vertex v of G is the number of elements of $E \cup S$ that are incident with v. We say that G is *r-regular* if the degree of each vertex is equal to r. An exemplary semi-graph G is given in Figure 3, where $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$, and $S(G) = \{v_1\cdot, v_1\cdot, v_1\cdot, v_2\cdot, v_2\cdot, v_2\cdot, v_3\cdot, v_3\cdot, v_3\cdot, v_4\cdot, v_4\cdot, v_4\cdot\}$.

For a given graph $G = (V(G), E(G))$ with $V(G) = \{v_1, \ldots, v_{n_G}\}$, and graph $H = (V(H), E(H))$ with $V(H) = \{u_1, \ldots, u_{n_H}\}$, $|V(H)| = n_H$, for any $i \in \{1, \ldots, n_G\}$ we define the *open fan* of $v_i \in V(G)$ as a set of n_H semi-edges with common endvertex v_i and we denote it by $F_H(v_i)$. The *close fan* $F_H[v_i]$ is a set $F_H(v_i) \cup \{v_i\}$. For any $j \in \{1, \ldots, n_H\}$, we define the *open claw* of $u_j \in V(H)$

as a set of edges in H incident with u_j, and we denote it by $I_H(u_j)$. We have $I_H(u_j) \subset E(H)$. The *close claw* of u_j, $I_H[u_j]$, is a set $I_H(u_j) \cup \{u_j\}$.

Definition 2. A *semi-corona* $G \circ_s H$ of a graph $G = (V(G), E(G))$ and a graph $H = (V(H), E(H))$ is the semi-graph $G' = (V(G'), E(G'), S(G'))$, where

$$V(G') = V(G),$$
$$E(G') = E(G),$$
$$S(G') = \bigcup_{v \in V(G)} F_H(v).$$

A semi-corona $G \circ_s H$ may be also defined as the semi-graph obtained from graph G by adding n_H semi-edges to each vertex of G. It is easy to see that semi-corona $G \circ_s H$ of a cubic graph G and n_H-vertex cubic graph H is (n_H+3)-regular semi-graph. An example of semi-corona is given in Figure 3.

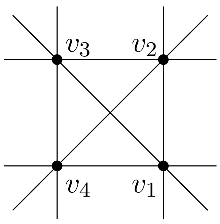

Figure 3. Semi-corona $K_4 \circ_s H$, where H is a 3-vertex graph.

Now, we will define the operation $+_{v_i}$. For a given semi-graph $G \circ_s H = G_0 = (V(G_0), E(G_0), S(G_0))$ with $V(G) = \{v_1, \ldots, v_{n_G}\}$ and graph $H = (V(H), E(H))$, we define $G_i = (V(G_i), E(G_i), S(G_i))$ as a semi-graph $G_{i-1} +_{v_i} H$, where

$$V(G_i) = V(G_{i-1}) \cup V(H_i),$$
$$E(G_i) = E(G_{i-1}) \cup E(H_i) \cup \{v_i w : w \in V(H_i)\},$$
$$S(G_i) = S(G_{i-1}) \backslash F_{H_i}(v_i).$$

It is easy to see that $G_{n_G} = G \circ H$. Of course, H_i denotes the i-th copy of H. We will name graphs $G_1, \ldots, G_{n_G}$ as *extended semi-coronas* (ref. Figure 4).

For $k \in \mathbb{N}^+$ and given semi-graph $G = (V, E, S)$, a *proper vertex k-coloring* of G is a map $c_V : V \to \{1, \ldots, k\}$ such that $c_V(x) \neq c_V(y)$ for any two adjacent vertices x and y. The smallest number of colors admitting such a coloring is named as the *chromatic number* and it is denoted by $\chi(G)$.

Similarly, a *proper edge k-coloring* of G is a map $c_{E \cup S} : E \cup S \to \{1, \ldots, k\}$ such that $c_{E \cup S}(e_1) \neq c_{E \cup S}(e_2)$ for any two adjacent elements e_1, e_2 of $E \cup S$. If

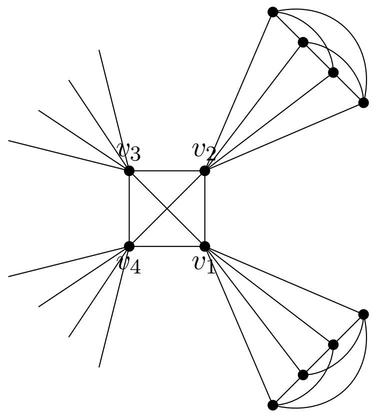

Figure 4. Extended semi-corona $G_2 = ((K_4 \circ_s K_4) +_{v_1} K_4) +_{v_2} K_4$.

$S = \emptyset$, then we will write $c_E : E \to \{1, \ldots, k\}$. The smallest number of colors admitting such a coloring is named as the *chromatic index* and it is denoted by $\chi'(G)$.

A *total k-coloring* of G is a map $c_T : V \cup E \cup S \to \{1, \ldots, k\}$ such that

- $c_T|_V$ is a proper vertex coloring,
- $c_T|_{E \cup S}$ is a proper edge coloring,
- $c_T(e) \neq c_T(v)$ whenever $e \in E \cup S$, $v \in V$ and e is incident with v.

A vertex (edge (total)) k-coloring is *equitable* if the cardinalities of any two color classes differ by at most one.

For a given vertex (edge (total)) k-coloring of a graph G, this means for a partition of the appropriate set into k independent color classes $\{P_1, P_2, \ldots, P_k\}$, the *vertex* (*edge* (*total*)) *coloring sequence* $C_V(G)$ $(C_E(G)$ $(C_T(G)))$ is a sequence of their cardinalities, i.e., $(|P_1|, |P_2|, \ldots, |P_k|)$. For the total 5-coloring of $K_{3,3}$ given in Figure 1, $C_T(K_{3,3}) = (3, 3, 3, 3, 3)$. For the vertex coloring being restriction of the total coloring to V the sequence $C_V(K_{3,3}) = (3, 0, 0, 0, 3)$. Similarly, we get $C_E(K_{3,3}) = (0, 3, 3, 3, 0)$.

3. Equitable Coloring of Cubic Graphs

Let us remind some known results concerning coloring of cubic graphs that will be useful in the further part of this work. First of all, let us notice that, when we consider only vertex coloring, the chromatic number is equal to the equitable chromatic number for all connected cubic graphs [2]. This means that every proper vertex coloring of connected cubic graph G with $\chi(G)$ colors can be made equitable without adding new colors.

Theorem 3 [2]**.** *If G is a connected cubic graph, then*

$$\chi(G) = \chi_=(G).$$

Corollary 4. *If G is a connected cubic graph, then*

$$2 \leq \chi_=(G) \leq 4.$$

In the case of equitable edge coloring, it is known, as an easy application of Kempe chains, that the equitable chromatic index for any graph is equal to its chromatic index

$$\chi'_=(G) = \chi'(G).$$

Theorem 5 [17]**.** *Every graph G has an equitable edge k-coloring for each $k \geq \chi'_=(G)$.*

Let us recall also Vizing theorem.

Theorem 6 [16]**.** *Let G be a graph. Then*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

For more information about equitable vertex and edge colorings we refer to [7].

As we have already mentioned, cubic graphs are one of graph classes that the Equitable Total Coloring Conjecture holds for. We have the following result.

Theorem 7 [10, 14]**.** *Every cubic graph G can be equitably total colored with k colors for every $k \geq 5$.*

In the further part of our paper we will use an equitable total $(n+4)$-coloring of an n-vertex cubic graphs G. Such a coloring exists due to Theorem 7. Now, we give some properties of such a coloring.

Proposition 8. *In any equitable total $(n+4)$-coloring of n-vertex cubic graph G the cardinalities of color classes are between 1 and 3.*

Proof. By the contrary, let us assume that there is at least one color class of the cardinality at least 4. Since the coloring is equitable, the remaining color classes are of cardinality at least 3. This means that the number of elements (vertices and edges) in G is not less than $4 + 3(n+3) = 3n + 13$, while we know that this number is equal to $\frac{5}{2}n$, a contradiction.

On the other hand, since $\frac{5}{2}n > n + 4$, the cardinalities of color classes are obviously greater or equal to 1. ■

Let $C_T(G)$ be the total coloring sequence of an equitable total $(n+4)$-coloring of n-vertex cubic graph G. Let $\#_j(C_T(G))$ denote the number of terms (color classes) in $C_T(G)$ of cardinality j, $j = 1, 2, 3$.

Proposition 9. *Let G be an n-vertex cubic graph colored in an equitable total way with $n + 4$ colors.*

(i) *If $4 \le n \le 14$, then*

$$\#_1(C_T(G)) = 8 - \frac{n}{2},$$
$$\#_2(C_T(G)) = \frac{3}{2}n - 4,$$
$$\#_3(C_T(G)) = 0.$$

(ii) *If $n \ge 16$, then*

$$\#_1(C_T(G)) = 0,$$
$$\#_2(C_T(G)) = \frac{n}{2} + 12,$$
$$\#_3(C_T(G)) = \frac{n}{2} - 8.$$

Proof. It is easy to observe that the only value of n when all terms in $C_T(G)$ are equal to 2 is $n = 16$. It is enough to solve equation $2(n + 4) = \frac{5}{2}n$. For smaller number of vertices, terms in a sequence $C_T(G)$ are equal to 1 and 2, for bigger ones — to 2 and 3. Now, the values of $\#_j(C_T(G))$, $j = 1, 2, 3$, are the solutions of system of equations for $j = 1$ in Case (i) and for $j = 2$ in Case (ii),

$$\begin{cases} j \cdot \#_j(C_T(G)) + (j+1) \cdot \#_{j+1}(C_T(G)) & = \frac{5}{2}n, \\ \#_j(C_T(G)) + \#_{j+1}(C_T(G)) & = n + 4. \end{cases}$$

■

4. Equitable Total Coloring of Semi-Coronas

Lemma 10. *Let $G \circ_s H$ be a semi-corona of cubic graphs: n_G-vertex graph G and n_H-vertex graph H. Then*

$$\chi''_{=}(G \circ_s H) = \Delta(G \circ_s H) + 1 = n_H + 4.$$

Proof. Since $\chi''_{=}(T) \ge \Delta(T) + 1$ for any graph T, all we need is to construct an equitable total $(n_H + 4)$-coloring of $G \circ_s H$. Do as follows.

1. Color equitably edges of cubic graph G with $n_H + 4$ colors. The corresponding edge color sequence $C_E(G)$ is equal to a non-increasing sequence of integers

$(l_e(1), l_e(2), \ldots, l_e(n_H+4))$ such that their sum is the number of edges of G and the difference of any two entries is at most 1. Of course, $l_e(i)$ denotes the number of edges in G colored with i. In cases when n_H is extremely larger than n_G, some colors are unused.

Since $3 \leq \chi'_{=}(G) \leq 4$ for every cubic graph G and we color edges of G with at least 8 colors, this step is possible due to Theorem 5.

2. Extend this coloring into any proper total (n_H+4)-coloring of G. Let us assume that all edges and some vertices of G have been already colored. Notice that for every uncolored vertex $v \in V(G)$ at most six colors are forbidden — the colors assigned to three incident edges and at most three adjacent vertices, if they have already been colored. Since we have $n_H + 4 \geq 8$ colors, there are at least two allowed colors for every vertex v. We can choose one of them. Let $C_V(G) = (l_v(1), l_v(2), \ldots, l_v(n_H+4))$ be the corresponding vertex coloring sequence. Of course, $C_E(G) + C_V(G)$ is the total coloring sequence of G.

3. Extend the total coloring of G into an equitable total (n_H+4)-coloring of semi-corona $G \circ_s H$ by coloring properly semi-edges of $G \circ_s H$, i.e., elements of an open fan $F(v)$ for every $v \in V(G)$.

Note that exactly 4 colors are not allowed to color semi-edges from $F(v)$. Let $c(F(v))$ denote the set of all allowed colors for semi-edges from $F(v)$. Since $|c(F(v))| = n_H$, the coloring of $F(v)$ is determined, with an accuracy to any permutation of $c(F(v))$.

We claim that the total coloring of $G \circ_s H$ obtained in the way described above is equitable. Indeed, let us notice that a color i used to color the vertex $v \in V(G)$ implies $i \notin c(F(v))$ while a color i used to color the edge $e = uv \in E(G)$ implies $i \notin c(F(u))$ and $i \notin c(F(v))$. Thus, the fact that a color i is used to color $l_v(i)$ vertices and $l_e(i)$ edges in G means that the color i will appear in $n_G - l_v(i) - 2l_e(i)$ sets of available colors $c(F(v))$ and this means that it can be used to color $n_G - l_v(i) - 2l_e(i)$ semi-edges. Thus, the color i is used $l_v(i) + l_e(i) + n_G - l_v(i) - 2l_e(i) = n_G - l_e(i)$ times. Since the sequence $(l_e(1), l_e(2), \ldots, l_e(n_H+4))$ from the first step was equitable, then the sequence $(n_G - l_e(1), \ldots, n_G - l_e(n_H+4))$ is also equitable. Thus the extended total coloring of $G \circ_s H$ is equitable. ■

5. MAIN RESULT

Before we prove the main theorem of this paper, let us recall the Hall's theorem.

Theorem 11 [11]. *Let $\mathcal{S}$ be a family of finite subsets of S, $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ and $S_i \subset S$. The family $\mathcal{S}$ has a transversal if and only if for each subset I of $\{1, 2, \ldots, k\}$ we have $|\bigcup_{i \in I} S_i| \geq |I|$.*

Lemma 12. *Let H be a cubic graph on $n_H \geq 4$ vertices, and let c be an equitable total (n_H+4)-coloring of H with a color sequence $C_T(H)$. For every color $x \in \{1, 2, \ldots, n_H+4\}$ there is an equitable total coloring (n_H+4)-coloring c' of H with the same color sequence $C_T(H)$ such that $c'(u) \neq x$ for every $u \in V(H)$.*

Proof. Let x be any color from $\{1, 2, \ldots, n_H+4\}$. If $c(u) \neq x$ for every $u \in V(H)$, then c' is equal to c. So, we may assume that there is a vertex u colored with x in the coloring c, i.e., $c(u) = x$. Let u_1, u_2 and u_3 be vertices adjacent to u in H, i.e., $N(u) = \{u_1, u_2, u_3\}$. We have $c(u_i) \neq x$ and $c(uu_i) \neq x$, $i = 1, 2, 3$.

We denote $c(s) \Leftrightarrow c(r)$ if we exchange the color of an element s into the color of r and vice versa, $s, r \in V(H) \cup E(H)$. After this exchange, we have a new coloring c' with $c'(s) = c(r)$ and $c'(r) = c(s)$.

If $4 \leq n_H \leq 8$, we have eight possible cubic graphs and it can be easily verified that it is always possible to recolor graph H to achieve the desirable property. For the reader convenience, we have put the Appendix with all cubic graphs on $n_H \leq 8$ vertices and their exemplary equitable total (n_H+4)-coloring to enable the reader easily verification.

If $n_H \geq 10$, we have $|E(H)| \geq 15$ and we consider three cases, dependly on the cardinality of color class P_x.

Case 1. If $|P_x| = 1$, we have the following two subcases.

Subcase 1.1. $\{c(u_1), c(u_2), c(u_3)\} \neq \{c(uu_1), c(uu_2), c(uu_3)\}$. Then there exists an edge among uu_1, uu_2, uu_3, say e, such that $c(e) \notin \{c(u_1), c(u_2), c(u_3)\}$ and we may do the exchange $c(u) \Leftrightarrow c(e)$.

Subcase 1.2. $\{c(u_1), c(u_2), c(u_3)\} = \{c(uu_1), c(uu_2), c(uu_3)\}$. Since the coloring c is equitable and $|P_x| = 1$, each color is used at most twice. This imply that colors $c(uu_1), c(uu_2)$, and $c(uu_3)$ are not used more to color other edges. This means that we can do an exchange $c(u) \Leftrightarrow c(e)$ for any edge $e \in E(H) \backslash \{uu_1, uu_2, uu_3\}$.

Case 2. If $|P_x| = 2$, suppose there exists an edge $e \notin \{uu_1, uu_2, uu_3\}$ such that $c(e) = x$ and e is incident to four edges e_1, e_2, e_3, e_4. Consider the set of edges $A = E(H) \backslash \{uu_1, uu_2, uu_3, e, e_1, e_2, e_3, e_4\}$. Since $n_H \geq 10$, we have $|A| \geq 7$.

- If $10 \leq n_H \leq 16$, then $|P_i| \leq 2$, for all colors i. So there exists an edge e' in A, colored with $y \notin \{c(u_1), c(u_2), c(u_3), c(uu_1), c(uu_2), c(uu_3)\}$, we do the exchange $c(u) \Leftrightarrow c(e')$.

- If $n_H \geq 18$, then $2 \leq |P_i| \leq 3$, for all colors i, and $|E(H)| \geq 27$, and hence $|A| \geq 19$. There are at most 12 edges in A colored with color in $\{c(u_1), c(u_2), c(u_3), c(uu_1), c(uu_2), c(uu_3)\}$, so there are at least 7 edges in A whose colors can be exchanged with the color of u.

If the second element colored with x is a vertex, we have to exclude, in addition, only three edges incident to them. Then $|A| \geq 9$ and we may easily choose an edge whose color will be exchanged with the color of u. We repeat the process to exclude all vertices colored with x.

Case 3. If $|P_x| = 3$, then $n_H \geq 18$ and $2 \leq |P_i| \leq 3$ for all colors i, $|E(H)| \geq 27$. Suppose, in the worst case, there are two edges e_1, e_2 colored with x adjacent to the edges $\{e_{11}, e_{12}, e_{13}, e_{14}, e_{21}, e_{22}, e_{23}, e_{24}\}$. Consider $A = E(H) \backslash \{uu_1, uu_2, uu_3, e_1, e_2, e_{11}, e_{12}, e_{13}, e_{14}, e_{21}, e_{22}, e_{23}, e_{24}\}$, $|A| \geq 14$. There are at most 12 edges in A colored with colors from $\{c(u_1), c(u_2), c(u_3), c(uu_1), c(uu_2), c(uu_3)\}$. So, there are at least 2 edges in A whose colors can be exchanged with the color of u.

Thus, in all the cases we have obtained a new coloring c' of H with the same cardinalities of color classes as it was in the coloring c and none of vertices in H is colored with x. ■

Theorem 13. *Let G and H be cubic graphs on n_G and n_H vertices, respectively. Then*

$$\chi''_{=}(G \circ H) = \Delta(G \circ H) + 1 = n_H + 4.$$

Proof. Let $n_H \geq 6$. For such cases the main idea of the proof is the following.

1. To color semi-corona $G \circ_s H$ in an equitable total way with $n_H + 4$ colors; we get the total equitable coloring sequence $C_T(G \circ_s H)$.
2. To color graph H_1 in an equitable total way with $n_H + 4$ colors such that
   - (a) none of colors assigned to vertices in H_1 is equal to $c(v_1)$,
   - (b) the coloring of H_1 together with the coloring of $G \circ_s H$ form an equitable total (n_H+4)-coloring of $H_1 \cup (G \circ_s H)$, i.e., $C_T(G \circ_s H) + C_T(H_1)$ will be an equitable total coloring sequence $C_T(G_1)$ of the extended semi-corona G_1.
3. To show that the coloring of H_1 may be "joined" with the coloring of semi-corona to obtain a proper total coloring of G_1.

Given the equitable total coloring of the extended semi-corona G_i, $i = 1, \ldots, n_G - 1$ with the corresponding total coloring sequence $C_T(G_i)$.

4. Color graph H_{i+1} in an equitable total way with $n_H + 4$ colors such that
   - (a) none of colors assigned to vertices in H_{i+1} is equal to $c(v_{i+1})$,
   - (b) the coloring of H_{i+1} together with the coloring of G_i form an equitable total $(n_H + 4)$-coloring of $H_{i+1} \cup G_i$, i.e., $C_T(G_i) + C_T(H_{i+1})$ will be an equitable total coloring sequence $C_T(G_{i+1})$ of the extended semi-corona G_{i+1}.
5. "Join" the coloring of H_{i+1} with the coloring of G_i to obtain a proper total coloring of G_{i+1}.

Since $G_{n_G} = G \circ H$, finally we get an equitable total coloring of the whole corona $G \circ H$. Now, all we need is to clarify the above steps and to show that they are possible to do.

Ad Step 1. We color semi-corona $G \circ_s H$ with $n_H + 4$ colors due to the way given in the proof of Lemma 10.

Ad Steps 2 and 4. An equitable total $(n_H + 4)$-coloring of H fulfilling the assumptions is possible to achieve due to Lemma 12.

Ad Steps 3 and 5. Let $V(G) = \{v_1, \dots, v_{n_G}\}$ and $V(H) = \{u_1, \dots, u_{n_H}\}$. To show that the equitable total coloring of a copy of H may be 'joined" with the coloring of the appropriate semi-graph we will prove that for every close claw $I_H[u_j]$, $1 \leq j \leq n_H$ there exists a semi-edge $v_i\cdot$ in $F_H(v_i)$ colored with $c(v_i\cdot)$ such that if we assign the same color $c(v_i\cdot)$ to the edge $v_i u_j$, then we get proper (partial) total coloring of $G \circ H$, $1 \leq i \leq n_G$.

The procedure described below should be repeated for all copies of H. For our convenience, let us consider one copy of H whose vertices $\{u_1, \dots, u_{n_H}\}$ should be joined with a vertex $v \in V(G)$.

Let $c(F_H(v)) = (x_1, \dots, x_{n_H})$ with $|x_j| \geq |x_{j+1}|$, $1 \leq j \leq n_H - 1$. We define $w_1 \in V(H)$ such that $c(I_H[w_1]) \cap \{x_1\} = \emptyset$, and for $2 \leq j \leq n_H - 6$, define $w_j \in V(H) \backslash \{w_1, w_2, \dots, w_{j-1}\}$ such that $c(I_H[w_j]) \cap \{x_j\} = \emptyset$. That is possible because in the set $V(H) \backslash \{w_1, w_2, \dots, w_{j-1}\}$ there are at most 6 vertices u such that $c(I_H[u]) \cap \{x_j\} \neq \emptyset$. Finally, we change the one semi-edge $v\cdot$ with $x_j = c(v\cdot)$ into an edge vw_j with $c(vw_j) = x_j$ in $E(G \circ H)$, for $1 \leq j \leq n_H - 6$.

Therefore, we have six "unjoined" vertices, let us denote them as $w_{n_H - 5}, \dots, w_{n_H}$ in H and six unassigned semi-edges $e_1, \dots, e_6$ in the appropriate open fan $F_H(v)$, such that $\mathcal{C} = \{c(e_1), \dots, c(e_6)\} = \{x_{n_H-5}, x_{n_H-4}, \dots, x_{n_H}\}$, and the cardinalty of any element in $\mathcal{C}$ is at most two. In addition, if $I_H[u_j] = \{u_j, e_{j1}, e_{j2}, e_{j3}\}$, then $c(I_H[u_j]) = \{c(u_j), c(e_{j1}), c(e_{j2}), c(e_{j3})\}$. Let $Y_\alpha = \mathcal{C} \cap c(I_H[u_\alpha])$ and $X_\alpha = \mathcal{C} \setminus Y_\alpha$, for $n_H - 5 \leq \alpha \leq n_H$. Observe some facts.

Fact (i): $|X_\alpha| \geq 2$, for each α.

Fact (ii): Since each color from $\mathcal{C}$ is used at most twice in H, each such a color is included in at least two sets X_α's.

Claim 14. *For each set of indices $K \subseteq \{n_H - 5, \dots, n_H\}$ it holds*

$$\left| \bigcup_{\alpha \in K} X_\alpha \right| \geq |K|.$$

Proof. • If $|K| \in \{1, 2\}$, then $\left|\bigcup_{\alpha\in K} X_\alpha\right| \geq 2 \geq |K|$ by Fact (i).

• If $|K| \in \{5, 6\}$, then $\left|\bigcup_{\alpha\in K} X_\alpha\right| = 6 \geq |K|$ by Fact (ii).

• If $|K| \in \{3, 4\}$ and $\left|\bigcup_{\alpha\in K} X_\alpha\right| = 2$, then $|X_\alpha| = 2$, for each $\alpha \in K$ and $X_\alpha = X_\beta$. Moreover, $c(I_H[u_\alpha]) = c(I_H[u_\beta]) = \mathcal{C} \setminus X_\alpha$ $\alpha, \beta \in K$. Since $\left|\bigcup_{\alpha\in K} I_H[u_\alpha]\right| \geq 9$, we have that at least one color in $\mathcal{C} \setminus X_\alpha$ has been used more than twice in H, a contradiction. Therefore, $\left|\bigcup_{\alpha\in K} X_\alpha\right| \geq 3$ for $|K| \in \{3, 4\}$. So, the only case that remains to be considered is $|K| = 4$ with $\left|\bigcup_{\alpha\in K} X_\alpha\right| = 3$.

• If $|K| = 4$ and $\left|\bigcup_{\alpha\in K} X_\alpha\right| = 3$, then $|c(I_H[u_\alpha]) \cap \mathcal{C}| = |Y_\alpha| \geq 3$ for every $\alpha \in K$ and we have three colors from $\mathcal{C}$ that are not in $\bigcup_{\alpha\in K} X_\alpha$. These three colors must be used at every close claw $I_H[u_\alpha]$, $\alpha \in K$. Thus, the number of elements of $\bigcup_{\alpha\in K} I_H[u_\alpha]$ that must be colored with colors from $\bigcup_{\alpha\in K} Y_\alpha$ is at least seven (for example seven edges from Figure 5). Thus, at least one color must be used at least three times, a contradiction.

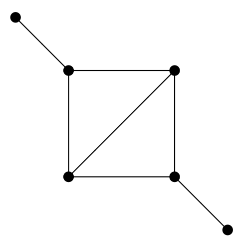

Figure 5. An example of subgraph of cubic graph H. □

Claim 14 means that we can apply Theorem 11 and get a transversal for $\mathcal{X}$. Thus we are able to select one representative for each set in $\mathcal{X}$ in such a way that no two sets from $\mathcal{X}$ get the same representative. Therefore, the equitable total coloring of a copy of H might be 'joined" with the coloring of the appropriate semi-graph, if only $n_H \geq 6$.

Now, we need to prove only the correctness of the theorem for $n_H = 4$. The algorithm of the equitable total coloring of $G \circ H$ where $n_H = 4$ is as follows.

1. Color semi-corona $G \circ_s K_4$ in an equitable total way with 8 colors due to the algorithm given in the proof of Lemma 10; we get the total equitable coloring sequence $C_T(G \circ_s K_4)$.

2. Determine an equitable total coloring sequence of a length 8 for H_1: $C_T(H_1)$, such that $C_T(G \circ_s K_4) + C_T(H_1)$ results in an equitable total coloring sequence of the extended semi-corona G_1.

3. Transform colored $G \circ_s K_4$ into the partially colored extended semi-corona G_1 in such a way that colors of vertices and edges of G are not changed, while the colors of $F_H(v_1)$ are assigned arbitrarily to the edges joining H_1 with G in G_1.

4. Color H_1 due to $C_T(H_1)$. Since only two its terms (colors) are of value 2, we color $H_1 = K_4$ in such a way that color of cardinality 2 are assigned to one vertex and one edge in H_1. The rest of colors are used only once in the coloring of H_1. Certainly, we may get proper coloring of H_1. Since we color H_1 to the guidelines of point 1, we end up with an equitable total coloring of G_1.

We generalize Steps 2–4 for next copies of K_4 and execute them until we get an equitable total 8-coloring of the whole corona $G \circ K_4$. ■

6. Final Remarks

Since in the proof of Lemma 10 we did not use the fact that G is cubic, we may generalize the lemma to the following one.

Corollary 15. *Let G be an r-regular graph and let H be a cubic graph on n_H vertices where $n_H \geq r - 3$. Then*

$$\chi''_{=}(G \circ_s H) = \Delta(G \circ_s H) + 1 = n_H + 4.$$

Finally we get the result.

Corollary 16. *Let G be an r-regular graph and let H be a cubic graph on n_H vertices where $n_H \geq r - 3$. Then*

$$\chi''_{=}(G \circ H) = \Delta(G \circ H) + 1 = n_H + 4.$$

Many interesting questions remain still open, for instance the equitable total colorability of coronas of r-regular graphs with $r > 3$. We hope that our paper will be a source of inspiration to answer this question.

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7. Appendix

In this section we show an example of equitable total $(n_H + 4)-$coloring for all cubic graphs with $4 \leq n_H \leq 8$ vertices.

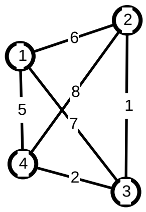

Figure 6. An exemplary equitable total 8-coloring of K_4.

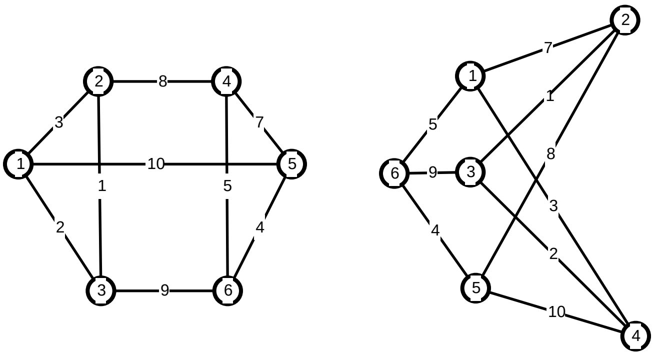

Figure 7. An exemplary equitable total 10-colorings of all 6-vertex cubic graphs.

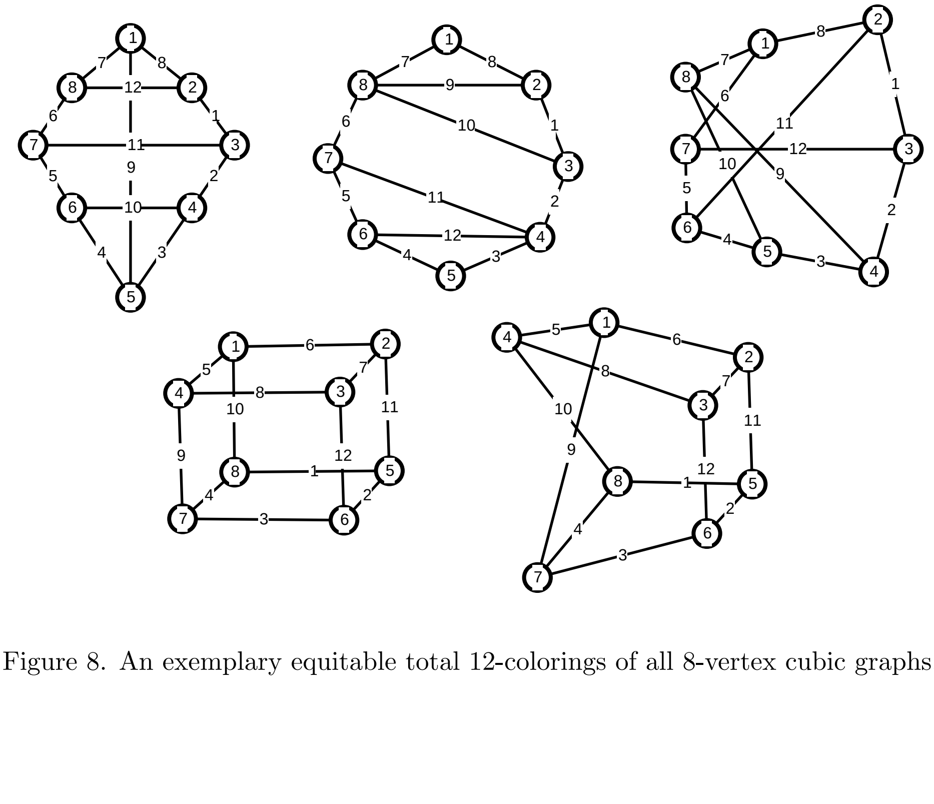

Figure 8. An exemplary equitable total 12-colorings of all 8-vertex cubic graphs.

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