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MINIMUM COVERINGS OF CROWNS WITH CYCLES AND STARS

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Abstract

Let F, G and H be graphs. A (G, H)-*decomposition* of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. For $R \subseteq F$, a (G, H)-*covering* of F with *padding* R is a (G, H)-decomposition of $F + E(R)$. A (G, H)-covering of F with the smallest cardinality is a *minimum* (G, H)-*covering*. This paper gives the solution of finding the minimum (C_k, S_k)-covering of the crown $C_{n,n-1}$.

Keywords: cycle, star, covering, decomposition, crown.

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1. Introduction

Let F, G and H be graphs. A G-*decomposition* of F is a partition of the edge set of F into copies of G. If F has a G-decomposition, we say that F is G-*decomposable*. A (G, H)-*decomposition* of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H. If F has a (G, H)-decomposition, we say that F is (G, H)-*decomposable*.

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A (G, H)-decomposition of F may not exist, a natural question of interest is to see: What is the minimum number of edges needed to be added to the edge set of F so that the resulting graph is (G, H)-decomposable, and what does the collection of added edges look like? For $R \subseteq F$, a (G, H)-*covering* of F with *padding* R is a (G, H)-decomposition of $F + E(R)$. A (G, H)-covering of F with the smallest cardinality is a *minimum* (G, H)-*covering*. Moreover, the cardinality of the minimum (G, H)-covering of F is called the (G, H)-*covering number* of F, denoted by $c(F; G, H)$.

As usual K_n denotes the complete graph with n vertices and $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. A k-*star*, denoted by S_k, is the complete bipartite graph $K_{1,k}$. The vertex of degree k in S_k is the *center* of S_k and any vertex of degree 1 is an *end-vertex* of S_k. Let $\langle y_1, y_2, \ldots, y_k\rangle_x$ denote the k-star with center x and end-vertices $y_1, y_2, \ldots, y_k$. A k-*cycle* (respectively, k-*path*), denoted by C_k (respectively, P_k), is a cycle (respectively, path) with k edges. Let $(v_1, v_2, \ldots, v_k)$ and $v_1v_2\cdots v_k$ denote the k-cycle and $(k-1)$-path through vertices $v_1, \ldots, v_k$ in order, respectively. A *spanning subgraph* H of a graph G is a subgraph of G with $V(H) = V(G)$. A 1-*factor* of G is a spanning subgraph of G with each vertex incident with exactly one edge. For positive integers ℓ and n with $1 \leq \ell \leq n$, the *crown* $C_{n,\ell}$ is a bipartite graph with bipartition (A, B) where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and edge set $\{a_ib_j : i = 0, 1, \ldots, n-1,\ j \equiv i+1, i+2, \ldots, i+\ell \pmod{n}\}$. In the sequel of the paper, (A, B) always means the bipartition of $C_{n,\ell}$ defined here. Note that $C_{n,n-1}$ is the graph obtained from the complete bipartite graph $K_{n,n}$ with a 1-factor removed.

The existence problems for (C_k, S_k)-decomposition of $K_{m,n}$ and $C_{n,n-1}$ have been completely settled by Lee [1] and Lee and Lin [4], respectively. Lee [2] obtained the maximum packing and minimum covering of the balanced complete bipartite multigraph $\lambda K_{n,n}$ with (C_k, S_k). Lee and Chen [3] gave the maximum packing and minimum covering of λK_n with (P_k, S_k). This paper gives the solution of finding the minimum (C_k, S_k)-covering of the crown $C_{n,n-1}$.

2. Preliminaries

Let $G = (V, E)$ be a graph. For sets $A \subseteq V(G)$ and $B \subseteq E(G)$, we use $G[A]$ to denote the subgraph of G induced by A and $G - B$ (respectively, $G + B$) to denote the subgraph obtained from G by deleting (respectively, adding) the edges in B. When $G_1, \ldots, G_t$ are graphs, not necessarily disjoint, we write $G_1 \cup \cdots \cup G_t$ or $\bigcup_{i=1}^t G_i$ for the graph with vertex set $\bigcup_{i=1}^t V(G_i)$ and edge set $\bigcup_{i=1}^t E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^t G_i$ expresses the decomposition of G into $G_1, \ldots, G_t$. For a graph G and a positive integer $\lambda \geq 2$, we use λG to denote the multigraph obtained from G by replacing each edge e by λ edges, each of which has the same ends as e.

The following results are essential to our proof.

Lemma 1 [7]**.** *For integers m and n with $m \geq n \geq 1$, the graph $K_{m,n}$ is S_k-decomposable if and only if $m \geq k$ and*

$$\begin{cases} m \equiv 0 \pmod{k} & if\ n < k, \\ mn \equiv 0 \pmod{k} & if\ n \geq k. \end{cases}$$

Lemma 2 [5]**.** *$\lambda C_{n,\ell}$ is S_k-decomposable if and only if $k \leq \ell$ and $\lambda n\ell \equiv 0 \pmod{k}$.*

Lemma 3 [5]**.** *Let $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$ be the vertex set of the multicrown $\lambda C_{n,\ell}$. Suppose that p and q are positive integers such that $q < p \leq \ell$. If $\lambda nq \equiv 0 \pmod{p}$, then there exists a spanning subgraph G of $\lambda C_{n,\ell}$ such that $\deg_G b_j = \lambda q$ for $0 \leq j \leq n-1$ and G has an S_p-decomposition.*

Lemma 4 [6]**.** *For positive integers k and n, $C_{n,n-1}$ is C_k-decomposable if and only if n is odd, k is even, $4 \leq k \leq 2n$, and $n(n-1) \equiv 0 \pmod{k}$.*

3. Covering Numbers

In this section the covering number of $C_{n,n-1}$ with k-cycles and k-stars is determined.

Lemma 5 [4]**.** *If k is an even integer with $k \geq 4$, then $C_{k+1,k}$ is not (C_k, S_k)-decomposable.*

Lemma 6. *If k is an even integer with $k \geq 4$, then $C_{k+1,k}$ has a (C_k, S_k)-covering with padding S_k.*

Proof. By Lemma 4, we have that $C_{k+1,k}$ is C_k-decomposable. Define a k-star $R = \langle b_1, b_2, \ldots, b_k \rangle_{a_0}$. Clearly, $C_{k+1,k} + E(R)$ is a (C_k, S_k)-covering with padding R. ∎

We obtain the following result by Lemmas 5 and 6.

Corollary 7. *$c(C_{k+1,k}; C_k, S_k) = k + 2$.*

Lemma 8 [4]**.** *If k is an even integer with $k \geq 4$, then $C_{2k,2k-1}$ is (C_k, S_k)-decomposable.*

Lemma 9. *For integers r and k with $r \geq 3$ and $k > r(r+1)$, $C_{k+r+1,k+r}$ can be decomposed into one copy of $r(r+1)$-cycle and $k + 2r + 1$ copies of k-stars.*

Proof. Let $s = r(r+1)/2$. Trivially, $k+r+1 > s$. Let $A_0 = \{a_0, a_1, \ldots, a_{s-1}\}$, $B_0 = \{b_0, b_1, \ldots, b_{s-1}\}$, $H_0 = C_{n,n-1}[A_0 \cup B_0]$, $H_1 = C_{n,n-1}[(A \setminus A_0) \cup B_0]$, and $H_2 = C_{n,n-1}[A \cup (B \setminus B_0)]$. Clearly, $C_{k+r+1,k+r} = H_0 \cup H_1 \cup H_2$. Note that H_0 is isomorphic to $C_{s,s-1}$, H_1 is isomorphic to $K_{k+r+1-s,s}$, and H_2 is isomorphic to $C_{k+r+1-s,k+r-s} \cup K_{s,k+r+1-s}$. Let

$$C = (b_1, a_0, b_2, a_1, b_3, a_2, \ldots, b_{s-1}, a_{s-2}, b_0, a_{s-1})$$

and $H = H_0 - E(C)$. Trivially, C is an $r(r+1)$-cycle in H_0 and $H = C_{s,s-3}$. Note that $r-2 < s-r-1$ for $r \geq 3$ and $s(r-2) = rs - r(r+1) = r(s-r-1)$. By Lemma 3, there exists a spanning subgraph X of H such that $\deg_X b_j = r-2$ for $0 \leq j \leq s-1$ and X has an S_{s-r-1}-decomposition $\mathscr{H}$ with $|\mathscr{H}| = r$. Furthermore, each S_{s-r-1} has its center in A_0 since $\deg_X b_j = r-2 < s-r-1$. Suppose that the centers of the $(s-r-1)$-stars in $\mathscr{H}$ are $a_{i_1}, \ldots, a_{i_r}$. Let $S(u)$ be the $(s-r-1)$-star with center a_{i_u} in $\mathscr{H}$, and let $Y = H - E(X) \cup H_1$. Note that $\deg_Y b_j = (s-3-(r-2)) + (k+r+1-s) = k$ for $0 \leq j \leq s-1$. Hence Y has an S_k-decomposition $\mathscr{H}^{(1)}$ with $\left|\mathscr{H}^{(1)}\right| = s$. For $u \in \{1, \ldots, r\}$, define $S'(u) = H_2[\{a_{i_u}\} \cup (B \setminus B_0)]$ and $Z = H_2 - E(\bigcup_{u=1}^{r} S'(u))$. Clearly, $S'(u)$ is a $(k+r+1-s)$-star with center a_{i_u} in H_2, and $S(u) \cup S'(u)$ is a k-star. There are r copies of such k-stars. Moreover, $\deg_Z b_j = k+r-r = k$ for $s \leq j \leq k+r$, and it follows that Z has an S_k-decomposition $\mathscr{H}^{(2)}$ with $\left|\mathscr{H}^{(2)}\right| = k+r-s+1$. Thus there are $s+r+k+r-s+1 = k+2r+1$ copies of k-stars. This completes the proof. ∎

Lemma 10. *Let k be a positive even integer and let n be a positive integer with $4 \leq k < n-1 < 2k-1$. If $(n-k)(n-k-1) < k$, then $C_{n,n-1}$ has a (C_k, S_k)-covering with padding $P_{k-(n-k)(n-k-1)}$.*

Proof. Let $n-1 = k+r$. From the assumption $k < n-1 < 2k-1$, we have $0 < r < k-1$. The proof is divided into two parts according to the value of r.

Case 1. $r \leq 2$. Let $A_0' = \{a_0, a_1, \ldots, a_{k-1}\}$, $A_1' = \{a_{k+1}, a_{k+2}, \ldots, a_{k+r}\}$, $B_0' = \{b_0, b_1, \ldots, b_{k-1}\}$, $B_1' = \{b_{k+1}, b_{k+2}, \ldots, b_{k+r}\}$. Let $D_0 = C_{n,n-1}[(A_0' \cup \{a_k\}) \cup (B_0' \cup \{b_k\})]$, $D_1 = C_{n,n-1}[A_0' \cup B_1']$, $D_2 = C_{n,n-1}[A_1' \cup B_0']$ and $D_3 = C_{n,n-1}[(A_1' \cup \{a_k\}) \cup (B_1' \cup \{b_k\})]$. Clearly, $C_{n,n-1} = D_0 \cup D_1 \cup D_2 \cup D_3$. Note that D_0 is isomorphic to $C_{k+1,k}$, D_1 is isomorphic to $K_{k,r}$, D_2 is isomorphic to $K_{r,k}$ and D_3 is isomorphic to $C_{r+1,r}$. By Lemma 2, we have that D_0 has a k-star decomposition $\langle b_{j+1}, b_{j+2}, \ldots, b_{j+k}\rangle_{a_j}$ for $0 \leq j \leq k$, where the subscripts of b's are taken modulo $k+1$ in the set of numbers $\{0, 1, \ldots, k\}$. By Lemma 1, we obtain that D_1 and D_2 have k-star decompositions $\langle a_0, a_1, \ldots, a_{k-1}\rangle_{b_j}$ and $\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_i}$ for $k+1 \leq i, j \leq k+r$, respectively.

Subcase 1.1. $r = 1$. Define a $(k-2)$-path R_1 as follows.

$$R_1 = a_{k+1} b_1 a_0 b_2 a_1 b_3 a_2 \cdots a_{\frac{k}{2}-3} b_{\frac{k}{2}-1} a_k,$$

where the subscripts of a's and b's are taken modulo n. Then

$$\begin{aligned}&\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+1}} \cup D_3 \cup R_1\\ &= \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_k} \cup \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+1}} \cup \{a_k b_{k+1}, a_{k+1} b_k\} \cup R_1\\ &= \langle b_0, b_1, \ldots, b_{k-2}, b_{k+1}\rangle_{a_k} \cup \langle b_0, b_1, b_2, \ldots, b_{k-2}, b_k\rangle_{a_{k+1}} \cup a_k b_{k-1} a_{k+1} \cup R_1.\end{aligned}$$

Note that $a_k b_{k-1} a_{k+1} \cup R_1$ is a k-cycle. Hence $C_{k+2,k+1} + E(R_1)$ can be decomposed into $k+3$ copies of k-stars and one copy of k-cycle, that is, $C_{k+2,k+1}$ has a (C_k, S_k)-covering $\mathscr{C}_1$ with $|\mathscr{C}_1| = k+4$ and padding R_1.

Subcase 1.2. $r = 2$. Define a $(k-6)$-path R_2 as follows.

$$R_2 = b_1 a_0 b_2 a_1 \cdots b_{\frac{k}{2}-3} a_{\frac{k}{2}-4} b_{k+1},$$

where the subscripts of a's and b's are taken modulo n. Then

$$\begin{aligned}&\langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+2}} \cup D_3 \cup R_2\\ &= \langle b_0, b_1, \ldots, b_{k-1}\rangle_{a_{k+2}} \cup \{a_k b_{k+1}, a_k b_{k+2}, a_{k+1} b_k, a_{k+1} b_{k+2}, a_{k+2} b_k, a_{k+2} b_{k+1}\} \cup R_2\\ &= \langle b_0, b_2, b_3, \ldots, b_{k-1}, b_{k+1}\rangle_{a_{k+2}} \cup b_{k+1} a_k b_{k+2} a_{k+1} b_k a_{k+2} b_1 \cup R_2.\end{aligned}$$

Note that $b_{k+1} a_k b_{k+2} a_{k+1} b_k a_{k+2} b_1 \cup R_2$ is a k-cycle. Hence $C_{k+3,k+2} + E(R_2)$ can decomposed into $k+5$ copies of k-stars and one copy of k-cycle, that is, $C_{k+3,k+2}$ has a (C_k, S_k)-covering $\mathscr{C}_2$ with $|\mathscr{C}_2| = k+6$ and padding R_2.

Case 2. $r \geq 3$. Let $s = r(r+1)/2$ and H_0, H_1 and H_2 be the graphs defined in the proof of Lemma 9. Define a $(k-2s)$-path R_3 as follows.

$$R_3 = a_{s-1} b_{s+1} a_s b_{s+2} \cdots a_{\frac{k}{2}-2} b_{\frac{k}{2}} a_{k+r},$$

where the subscripts of a's and b's are taken modulo n.

Let S be the k-star with center b_1 and C be the $2s$-cycle mentioned in Lemma 9. Then

$$\begin{aligned}&S \cup C \cup R_3\\ &= (S - a_{k+r} b_1 + a_{s-1} b_1) \cup a_{k+r} b_1 a_0 b_2 a_1 b_3 a_2 \cdots b_{s-1} a_{s-2} b_0 a_{s-1} \cup R_3.\end{aligned}$$

Note that $a_{k+r} b_1 a_0 b_2 a_1 b_3 a_2 \cdots b_{s-1} a_{s-2} b_0 a_{s-1} \cup R_3$ is a k-cycle. Hence $C_{k+r+1,k+r} + E(R_3)$ can be decomposed into $k+2r+1$ copies of k-stars and one copy of k-cycle, that is, $C_{k+r+1,k+r}$ has a (C_k, S_k)-covering $\mathscr{C}_3$ with $|\mathscr{C}_3| = k+2r+2$ and padding R_3. This settles Case 2. ■

Before plunging into the proof of the case of $(n-k)(n-k-1) \geq k$, a result due to Lee and Lin [4] is needed.

Lemma 11 [4]. *If k is an even integer with $k \geq 4$, then there exist $k/2-1$ edge-disjoint k-cycles in $C_{k/2,k/2-1} \cup K_{k/2,k/2}$.*

Lemma 12. *Let k be a positive even integer and let n be a positive integer with $4 \leq k < n-1 < 2k-1$. If $(n-k)(n-k-1) \geq k$, then $C_{n,n-1}$ has a (C_k, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil n(n-1)/k \rceil$.*

Proof. Let $n-1 = k+r$. From the assumption $k < n-1 < 2k-1$, we have $0 < r < k-1$. Since $(n-k)(n-k-1) \geq k$, we assume that $r(r+1) = \alpha k + \beta$, where $\alpha \geq 1$ and $0 \leq \beta \leq k-1$. Let $A_0'' = \left\{a_0, a_1, \ldots, a_{\frac{k}{2}-1}\right\}$, $A_1'' = \left\{a_{\frac{k}{2}}, a_{\frac{k}{2}+1}, \ldots, a_{k-1}\right\}$, $A_2'' = A \setminus (A_0'' \cup A_1'')$, $B_0'' = \{b_0, b_1, \ldots, b_{k-1}\}$, $B_1'' = B \setminus B_0''$. Let $G_i = C_{n,n-1}[A_i'' \cup B_0'']$ for $i \in \{0,1,2\}$ and $G_3 = C_{n,n-1}[A \cup B_1'']$. Clearly, $C_{n,n-1} = G_0 \cup G_1 \cup G_2 \cup G_3$. Note that G_0 and G_1 are isomorphic to $C_{k/2,k/2-1} \cup K_{k/2,k/2}$, G_2 is isomorphic to $K_{r+1,k}$, which is S_k-decomposable by Lemma 1, and G_3 is isomorphic to $K_{k,r+1} \cup C_{r+1,r}$. Let $p_0 = \lceil \alpha/2 \rceil$ and $p_1 = \lfloor \alpha/2 \rfloor$. In the following, we will show that, for each $i \in \{0,1\}$, G_i can be decomposed into p_i copies of C_k and $k/2$ copies of S_{k-2p_i-1}, and G_3 can be decomposed into $k/2$ copies of S_{2p_i+1} and $r+1$ copies of $S_{k'}$, $k' \leq k$, such that the $(k-2p_i-1)$-stars and $(2p_i+1)$-stars have their centers in A_i''.

We first show the required decomposition of G_i for $i \in \{0,1\}$. Since $r < k-1$, we have $r+1 < k$, and in turn $\alpha < r$. Thus, $p_0 = \left\lceil \frac{\alpha}{2} \right\rceil \leq \frac{\alpha+1}{2} \leq \frac{(r-1)+1}{2} \leq \frac{k-2}{2} = \frac{k}{2} - 1$, which implies $p_i \leq k/2-1$ for $i \in \{0,1\}$. This assures us that there exist p_i edge-disjoint k-cycles in G_i by Lemma 11. Suppose that $Q_{i,0}, \ldots, Q_{i,p_i-1}$ are edge-disjoint k-cycles in G_i. Let $F_i = G_i - E\left(\bigcup_{h=0}^{p_i-1} Q_{i,h}\right)$ and $X_{i,j} = F_i\left[\left\{a_{ik/2+j}\right\} \cup B_0''\right]$ where $i \in \{0,1\}$, $j \in \{0, \ldots, k/2-1\}$. Since $\deg_{G_i} a_{ik/2+j} = k-1$ and each $Q_{i,h}$ uses two edges incident with $a_{ik/2+j}$ for each i and j, we have $\deg_{F_i} a_{ik/2+j} = k-2p_i-1$. Hence $X_{i,j}$ is a $(k-2p_i-1)$-star with center $a_{ik/2+j}$.

Next we show the required star decomposition of G_3. For $j \in \{0, \ldots, k/2-1\}$, let

$$X'_{i,j} = \begin{cases} \left\langle b_{k+(2p_0+1)j}, b_{k+(2p_0+1)j+1}, \ldots, b_{k+(2p_0+1)j+2p_0} \right\rangle_{a_j}, & \text{if } i = 0, \\ \left\langle b_{(p_0+3/2)k+(2p_1+1)j}, b_{(p_0+3/2)k+(2p_1+1)j+1}, \right. & \\ \quad \left. \ldots, b_{(p_0+3/2)k+(2p_1+1)j+2p_1} \right\rangle_{a_{k/2+j}}, & \text{if } i = 1, \end{cases}$$

where the subscripts of b's are taken modulo $r+1$ in the set of numbers $\{k, k+1, \ldots, k+r\}$. Since $2p_1+1 \leq 2p_0+1 \leq \alpha+2 \leq r+1$, this assures us that there are enough edges for the construction of $X'_{0,j}$ and $X'_{1,j}$. Note that $X'_{i,j}$ is a $(2p_i+1)$-star and $X_{i,j} \cup X'_{i,j}$ is a k-star for $i \in \{0,1\}$, $j \in \{0, \ldots, k/2-1\}$.

On the other hand, let $k - \beta = \tau(r+1) + \rho$ where $\tau \geq 0$ and $0 \leq \rho \leq r$. We have that

$$
\begin{aligned}
&|E(G_3)| - \left|E\left(\bigcup_{i\in\{0,1\}}\bigcup_{j\in\{0,\ldots,k/2-1\}} X'_{i,j}\right)\right| \\
&= (k+r)(r+1) - (2p_0 + 2p_1 + 2)(k/2) \\
&= (k+r)(r+1) - (\alpha+1)k \\
&= (k+r)(r+1) - r(r+1) - (k-\beta) \\
&= k(r+1) - \tau(r+1) - \rho = (k-\tau)(r+1) - \rho \\
&= (k-\tau-1)\rho + (k-\tau)(r+1-\rho).
\end{aligned}
$$

Hence there exists a decomposition $\mathscr{G}$ of $G_3 - E\left(\bigcup_{i\in\{0,1\}}\bigcup_{j\in\{0,\ldots,k/2-1\}} X'_{i,j}\right)$ into ρ copies of $(k-\tau-1)$-star with center b_w for $w = k, k+1, \ldots, k+\rho-1$ and $r+1-\rho$ copies of $(k-\tau)$-star with center b_w for $w = k+\rho, k+\rho+1, \ldots, k+r$, that is,

$$
Y_w = \begin{cases} S_{k-\tau-1}, & \text{if } w \in \{k, k+1, \ldots, k+\rho-1\}, \\ S_{k-\tau}, & \text{if } w \in \{k+\rho, k+\rho+1, \ldots, k+r\}. \end{cases}
$$

Define a star Y'_w as follows.

$$
Y'_w = \begin{cases} \langle a_{w_1}, a_{w_2}, \ldots, a_{w_\tau}, a_{w_{\tau+1}}\rangle_{b_w}, & \text{if } w \in \{k, k+1, \ldots, k+\rho-1\}, \\ \langle a_{w_1}, a_{w_2}, \ldots, a_{w_\tau}\rangle_{b_w}, & \text{if } w \in \{k+\rho, k+\rho+1, \ldots, k+r\}, \end{cases}
$$

where $b_w a_{w_t} \in E(X'_{i,j})$ for $1 \le t \le \tau+1$. Since $\left|E\left(\bigcup_{i\in\{0,1\}}\bigcup_{j\in\{0,\ldots,k/2-1\}} X'_{i,j}\right)\right| = (\alpha+1)k$, $|B''_1| = r+1$ and $(\tau+1)(r+1) = \tau(r+1)+(r+1) = (k-\beta-\rho)+(r+1) < 2k \le (\alpha+1)k$, it follows that $\tau+1 < (\alpha+1)k/(r+1)$. This assures us that there are enough edges for the construction of Y'_w. Note that $Y_w + E(Y'_w)$ is a k-star. Hence $C_{n,n-1}$ has a (C_k, S_k)-covering $\mathscr{C}_4$ with padding $\bigcup_{w\in\{k,k+1,\ldots,k+r\}} Y'_w$ and $|\mathscr{C}_4| = (k+r+1)+(r+1)+\alpha = k+2r+2+\alpha = \lceil n(n-1)/k\rceil$. This completes the proof. ■

Now, we are ready for the main result of this section.

Theorem 13. *Let k be a positive even integer and let n be a positive integer with $4 \le k \le n-1$. Then*

$$
c(C_{n,n-1}; C_k, S_k) = \begin{cases} \lceil n(n-1)/k\rceil, & \text{if } k < n-1, \\ k+2, & \text{if } k = n-1. \end{cases}
$$

Proof. Since $|E(C_{n,n-1})| = n(n-1)$, we have that $c(C_{n,n-1}; C_k, S_k) \ge \lceil n(n-1)/k\rceil$. Let $n-1 = qk+r$, where q and r are integers with $q \ge 1$, $0 \le r \le k-1$. We consider the following two cases.

Case 1. $q = 1$. For $r = 0$, the result follows from Corollary 7. If $r \ne 0$, by Lemmas 8, 10 and 12, $C_{k+r+1,k+r}$ has a (C_k, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil (k+r+1)(k+r)/k\rceil$.

Case 2. $q \geq 2$. Note that

$$\begin{aligned} C_{n,n-1} &= C_{qk+r+1,qk+r} \\ &= C_{(q-1)k+1,(q-1)k} \cup C_{k+r+1,k+r} \cup K_{(q-1)k,k+r} \cup K_{k+r,(q-1)k}. \end{aligned}$$

Trivially, $|E(C_{(q-1)k+1,(q-1)k})|$, $|E(K_{(q-1)k,k+r})|$ and $|E(K_{k+r,(q-1)k})|$ are multiples of k, by Lemmas 1 and 2, we have that $C_{(q-1)k+1,(q-1)k}$, $K_{(q-1)k,k+r}$ and $K_{k+r,(q-1)k}$ have S_k-decompositions $\mathscr{A}^{(1)}$, $\mathscr{A}^{(2)}$ and $\mathscr{A}^{(3)}$ with $\left|\mathscr{A}^{(1)}\right| = (q-1)((q-1)k+1)$, $\left|\mathscr{A}^{(2)}\right| = \left|\mathscr{A}^{(3)}\right| = (k+r)(q-1)$. For the case of $r=0$, by Lemma 4, $C_{k+1,k}$ has a C_k-decomposition $\mathscr{C}$ with $|\mathscr{C}| = k+1$. Hence $C_{n,n-1}$ is (C_k, S_k)-decomposable, that is, $C_{n,n-1}$ has a (C_k, S_k)-covering $\bigcup_{i=1}^{3} \mathscr{A}^{(i)} \cup \mathscr{C}$ with cardinality $(q-1)((q-1)k+1)+k(q-1)+k(q-1)+k+1 = q(qk+1) = n(n-1)/k$. For the other case of $r \neq 0$, by Lemmas 10 and 12, $C_{k+r+1,k+r}$ has a (C_k, S_k)-covering $\mathscr{C}'$ with $|\mathscr{C}'| = \lceil (k+r+1)(k+r)/k \rceil$. Hence $\bigcup_{i=1}^{3} \mathscr{A}^{(i)} \cup \mathscr{C}'$ is a (C_k, S_k)-covering of $C_{n,n-1}$ with cardinality $(q-1)((q-1)k+1) + (k+r)(q-1) + (k+r)(q-1) + \lceil (k+r+1)(k+r)/k \rceil = \lceil (qk+r+1)(qk+r)/k \rceil = \lceil n(n-1)/k \rceil$. This completes the proof. ■

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