Discussiones Mathematicae
Graph Theory 42 (2022) 63–79
https://doi.org/10.7151/dmgt.2239

BOUNDS ON WATCHING AND WATCHING GRAPH PRODUCTS

Danny Dyer[1]

St. John's Campus
Memorial University of Newfoundland
e-mail: dyer@mun.ca

and

Jared Howell

Grenfell Campus
Memorial University of Newfoundland
e-mail: jahowell@grenfell.mun.ca

Abstract

A watchman's walk for a graph G is a minimum-length closed dominating walk, and the length of such a walk is denoted (G). We introduce several lower bounds for such walks, and apply them to determine the length of watchman's walks in several grids.

Keywords: watchman's walk, domination, graph products.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

The watchman's walk problem was introduced in [11], and concerned finding a walk through a museum such that a guard, along this walk, would either visit, or be able to peer into, every room; would begin and end at the same point, so that the walk could be easily repeated; and finally, of all such walks, be the shortest possible. In graph theoretic terms, this is to find a *minimum length closed dominating walk* (MCDW) of a graph G. We denote the length of such a walk as $w(G)$, and call it the *watchman number* of G.

[1]Supported by NSERC.

Obviously, this parameter is heavily related to the domination number of a graph, as the vertices of the walk make up a dominating set. However, these parameters may be quite different. Since we are interested in the length of a walk, it is possible that we will have to visit "useless" vertices (not required in any dominating set) many times, which can significantly add to $w(G)$. Also, it is possible to construct graphs for which no MCDW contains any minimum dominating set. As noted in [12], one such family of graphs is $C_n \square K_2$, for $n \geq 8$. Thus, we get a sense that while related, these parameters are fundamentally quite different.

Likewise, there are similarities to two domination variants: connected domination and dominating cycles. Certainly, the vertices of every MCDW form a connected dominating set, though again, due to the need to revisit vertices, the difference between these two parameters can be arbitrarily large. (Even in a path, the length of an MCDW is essentially double that of a minimum connected dominating set.) And, while a minimum length dominating cycle is similar to an MCDW, it may be shorter (as an MCDW may take a "short cut" back through previously visited vertices), and an MCDW will exist in every graph, while a dominating cycle may not exist even in non-trees.

Previous work with the idea of the watchman's walk has touched on a variety of graph families. The parameter $w(G)$ has been studied for trees and cycles [11]; some circulant graphs [3]; planar and outerplanar graphs [7, 13]; and the block intersection graphs of Steiner Triple systems [8]. (Allowing multiple watchmen and minimizing the time a vertex is unguarded has also been considered [1, 5, 6].)

In many cases, the intuitively "correct" watchman's walk is not difficult to find. However, it is difficult to prove such a closed dominating walk is minimum. In this paper, we will concentrate on finding lower bounds for the watchman number. We will particularly consider our results on the Cartesian products of graphs, and prove a number of results on these families. Such families, particularly grids ($P_n \square P_m$), have long been a part of the domination literature. Recently, Chang's conjecture [2] was verified.

Theorem 1 [10]**.** *If* $16 \leq n \leq m$, *then*

$$\gamma(P_n \square P_m) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$

A similar result for independent domination followed [4]. The examination of the eternal domination number on grids is on-going, with results for $P_k \square P_n$ grids when $k \leq 5$; see, for example, [9].

There has also been previous work on the watchman's walk of the Cartesian product of graphs [12]. In that work, bounds are given for $w(T \square K_n)$ which depend highly on the structure of the tree T. In a tree T, we call every vertex adjacent to a leaf a stem. Let T_0 be the subtree of T obtained by deleting all the leaves. An external stem will be a stem that is also a leaf in T_0; otherwise, the stem is internal. Further, given a set Y of stems, let $L(Y)$ be the set of leaves adjacent to any vertex in Y.

Theorem 2 [12]**.** *Let T be a tree, S be the set of stems adjacent to at least $\frac{n}{2}$ leaves, X be the set of external stems adjacent to at least $\frac{n}{2}$ leaves, and Y be the remaining external stems of T. Let $T^\star$ be the subtree of T induced by $V(T)\setminus L(S)$. Then*

$$2|E(T_0)| + (n-1)|X| + 2|L(Y)| \le w(T\Box K_n) \le 2|E(T^\star)| + (n-1)|S| + 1.$$

Further work is done on the particular case when $n = 2$. The following bounds are proved, and shown to be sharp.

Theorem 3 [12]**.** *Let T be a tree with $\gamma(T) \ge 2$. Then*

$$2|E(T_0)| + 2\left\lceil \frac{|L(T_0)|}{2} \right\rceil \le w(T\Box K_2) \le 2|E(T_0)| + 2|L(T_0)| - 1.$$

In this paper we improve on the work of Hartnell and Whitehead on the watchman's walks of the Cartesian products of graphs. To that end, in Section 2 we introduce a number of lower bounds on the watchman's walk based on classical graph parameters such as diameter and maximum degree. In Section 3, we consider the special case of graph products, and produce a number of bounds on these products in the general setting. In Section 4, we consider the special case of grids ($P_n\Box P_m$) and toroidal grids ($C_n\Box C_m$), and conclude with some remarks on strong grids in Section 5.

2. Bounds

Recall that the diameter of a graph G, $\mathrm{diam}(G)$, is the maximum distance between any two vertices of G, and let u and v be a pair of vertices that are this maximum distance apart. While neither u nor v must be on a watchman's walk, some neighbour u' of u must be visited. Then, any watchman's walk that begins at u' must continue to, at least, a neighbour of v, and then return to u'. That is, $w(G) \ge 2(\mathrm{diam}(G) - 2)$. We can generalise this result. Let $d(u, v)$ be the distance between a pair of vertices in a graph G. Then for any set $K \subseteq V(G)$, define $\rho(K) = \min_{u,v\in K, u\ne v}\{d(u, v)\}$.

Theorem 4. *If G is a connected graph and $K \subseteq V(G)$, then*

$$w(G) \ge \max_{K\subseteq V(G)} \{|K|(\rho(K) - 2)\}.$$

Proof. Let $K \subseteq V(G)$ for a connected graph G with $d(u, v) \geq 3$ for all distinct $u, v \in K$. We will prove that $w(G) \geq |K|(\rho(K)-2)$. (Those cases where $\rho(K) \leq 2$ will only be the maximum for graphs with a universal vertex, in which case $w(G) = 0$, as required.) Let W be an MCDW of G. Certainly, W dominates each vertex of K. If $K = \{v_1, v_2, \ldots, v_k\}$, then let the first vertex of W that dominates v_i be w_i. Then W can be considered a walk $w_1, \sigma_1, w_2, \sigma_2, w_3, \ldots, w_k, \sigma_k$, where σ_i is a walk from w_i to w_{i+1}, and σ_k is a walk from w_k to w_1. If $\ell(\sigma_i)$ is the length of such a walk, then we know that $\ell(\sigma_i) \geq d(w_i, w_{i+1}) \geq d(v_i, v_{i+1}) - 2$. Then $w(G) = \sum_{i=1}^{k} \ell(\sigma_i) \geq \sum_{i=1}^{k} (d(v_i, v_{i+1}) - 2) \geq |K|(\rho(K) - 2)$. ■

In the case where $|K| = 2$, we see that Theorem 4 reduces to the diameter bound previously discussed. This bound is tight for caterpillars. Alternatively, consider the k-star, $K_{1,k}$, with each of the edges subdivided m times. Considering the set K given by the leaves of this subdivided graph, we again see the bound is tight. (We know that $w(T)$ of a tree T is twice the number of edges of its leaf deleted subtree [11].)

Connected graphs with MCDWs of length 0 and 2 are easy to characterize, as they are those graphs with a dominating vertex and a dominating edge (without a dominating vertex), respectively. Considering graphs without a dominating edge, we obtain the following lower bound. Recall that the maximum degree of a graph is denoted $\Delta(G)$.

Theorem 5. *If G is a connected graph with $n \geq 3$ vertices and no dominating edge, then $w(G) \geq \frac{n}{\Delta(G)-1}$.*

Proof. Let $\Delta = \Delta(G)$. Consider traversing an MCDW of G, keeping tally of which vertices are dominated as each vertex of the walk is reached. Since G has no dominating edge, the walk is of length at least 3, and at some point, passes through three distinct vertices in sequence. Consider the last of these vertices as the first of the walk.

The first vertex will dominate at most $\Delta + 1$ vertices. The next $w(G) - 3$ vertices in the watchman's walk will subsequently each dominate at most $\Delta - 1$ new vertices, as each vertex (and the vertex previous in the MCDW) would have been dominated by the previous vertex. This leaves the last two vertices of the walk. The penultimate vertex dominates at most $\Delta - 2$ vertices, as it and its predecessor has been dominated by the predecessor, and the final vertex of the walk had previously been dominated by the first vertex. Similarly for the final vertex of the walk. Since every vertex of G must be dominated, $n \leq (\Delta + 1) + (\Delta - 1)(w(G) - 3) + 2(\Delta - 2) = (\Delta - 1)w(G)$, or $w(G) \geq \frac{n}{\Delta-1}$, as required. ■

If we know more about the structure of the graph, we can do a little bit better. Given a graph G, define $N(G) = \min_{uv \in E(G)}\{|N[u] \cap N[v]|\}$, that is, the smallest intersection of the closed neighbourhoods of adjacent vertices.

Theorem 6. *If G is a connected graph with $n \geq 3$ vertices, no dominating edge, and maximum degree $\Delta(G) \geq N(G)$, then $w(G) \geq \frac{n}{\Delta(G)-N(G)+1}$.*

Proof. We follow the proof of Theorem 5. Again, consider traversing an MCDW of G, keeping tally of which vertices are dominated as each vertex in the walk is reached. Since G has no dominating edge, the walk is of length at least 3, and at some point, passes through three distinct vertices in sequence. Consider the last of these vertices as the first of the walk.

The first vertex will dominate at most $\Delta + 1$ vertices. The next $w(G) - 3$ vertices in the watchman's walk will subsequently each dominate at most $\Delta + 1 - N(G)$ new vertices, as the vertices that were in the closed neighbourhood of the previous vertex were counted at the previous step. The penultimate vertex dominates at most at most $\Delta - N(G)$ as the final vertex in the walk was dominated by the first vertex. The final vertex itself dominates at most $\Delta + 1 - N(G) - (N(G) - 1) = \Delta - 2N(G) + 2$ as its neighbourhood intersects that of the first vertex and the penultimate vertex, and both of their shared neighbourhoods have at least one vertex in common. Since every vertex is dominated in an MCDW, $n \leq (\Delta + 1) + (\Delta - N(G) + 1)(w(G) - 3) + \Delta - N(G) + \Delta - 2N(G) + 2 = (\Delta - N(G) + 1)w(G)$, or $w(G) \geq \frac{n}{\Delta - N(G)+1}$, as required. ■

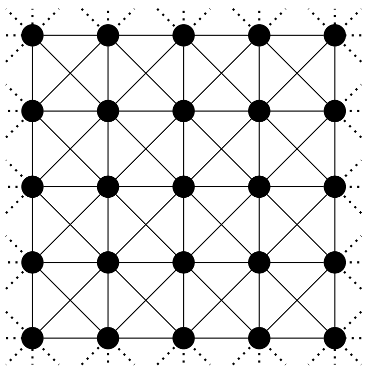

Figure 1. $C_5 \boxtimes C_5$.

To illustrate the improvement of this bound, consider $C_5 \boxtimes C_5$ as illustrated in Figure 1. The bound from Theorem 5 results in $w(C_5 \boxtimes C_5) \geq 4 > 25/7$ and from Theorem 6 results in $w(C_5 \boxtimes C_5) \geq 5 = 25/5$. In fact, $w(C_5 \boxtimes C_5) = 5$ and can be realised by any diagonal cycle.

There are also some simple bounds that relate directly to the number of edges or vertices in a graph. To prove the following bounds, we first prove a structural lemma.

Lemma 7. *If W is a watchman's walk in a connected graph G, then* (i) *no edge of W is traversed more than twice; and* (ii) *no edge of a cycle contained in W is ever traversed more than once.*

Proof. Let W be a watchman's walk in G, and assume that some edge of W, uv, is traversed at least three times. (If more than three, we need consider only the first three traversals.) Either the traversals are all in the same direction, or two are in one direction, and one in the other.

In the first case, we may assume that the structure of the walk is as follows: beginning at u, then v, then a subwalk W_1 from v back to u; then uv again, and a subwalk W_2 from v to u; then finally uv, followed by a walk W_3 which returns to u. We dominate the same vertices with the following walk: begin at u, along uv, then follow W_1 to u; reverse the walk W_2 from u to v; then take the walk W_3 back to u. This is a closed walk which visits the same vertices as W, hence is dominating, but is shorter, a contradiction.

On the other hand, consider if edge uv is traversed twice, and once as vu. Then the walk W, beginning at uv, must be of the form uv; then a walk, W_1 from v to u; then uv; then a walk W_2 from v back to v; then vu, followed by a walk W_3 from u to u. (Both W_2 and W_3 might contain no edges.) Instead, consider the walk beginning at u that begins uv, then is followed by W_2; then by W_1; then by W_3. This walk is again dominating, and using fewer edges than W, a contradiction.

If W is a watchman's walk in G whose edges induce a cycle, let uv be an edge that is traversed twice in that cycle. If uv is traversed twice in the same direction, then we may consider the walk as beginning with u; then uv; then following W_1 until u is next reached; then uv again; then following W_2 until u is reached the final time. Instead, consider the walk starting at u; then following W_2 in reverse, to v; then following W_1 to u. Again, this walk uses the same vertices, so a shorter dominating walk than W.

If uv is traversed in both directions, we can consider the walk W as follows: first uv, then a walk W_1 which returns to v; then vu, followed by a walk W_2 that returns to u. Since W contains a cycle including uv, W_1 and W_2 must meet at some vertex w. Then we may consider W_1 the union of two walks: W_1' from v to w, then W_1'' from w to v. Similarly, we may define W_2' and W_2''. Then the walk, beginning at w, given by W_1'', then W_1', then W_2'', then W_2' is a shorter dominating walk than W. ■

Lemma 8. *If G is a connected graph on n vertices and T is a spanning tree for G with the maximum possible number of leaves, ℓ, then $w(G) \leq 2(n-1-\ell)$.*

Proof. A closed dominating walk of G can be formed by walking the edges of the leaf deleted subtree of T, with each edge being walked twice. This consists of the desired number of edges. ■

Knowing nothing else about a graph other than the number of edges, we first consider what bounds may be placed on the length of an MCDW as a result. With those bounds in mind, we then consider what ratios in that range are actually achievable.

Theorem 9. *If G is a connected graph on m edges, then $0 \leq w(G) \leq 2m-4$.*

Proof. Certainly, by Lemma 7, no edge is used 3 times. Since G is connected, it has at most $m+1$ vertices. If T is a spanning tree of G, then it will have at least two leaves. By Lemma 8, we have $w(G) \leq 2((m+1)-1-2) = 2m-4$. ■

Corollary 10. *If $\frac{a}{b} \in [0,2)$ is a fraction in lowest terms, then there exists a graph G with m edges such that $\frac{w(G)}{m} = \frac{a}{b}$.*

Proof. We know that $w(K_n) = 0$, and $w(C_7) = 7$, so the ratios 0 and 1 are achievable. For $\frac{a}{b} \in (0,1)$ in lowest terms, form G by taking a $7a$-cycle and append $7(b-a)$ pendent vertices to the cycle. This gives $w(G) = 7a$ and $|E(G)| = 7a + 7(b-a) = 7b$, hence $\frac{w(G)}{m} = \frac{a}{b}$, as desired.

For $\frac{a}{b} \in (1,2)$ in lowest terms, pick c such that $0 < c < b$ and $a + c = 2b$. Then $\frac{a}{b} = \frac{2b-c}{b} = \frac{8b-4c}{4b}$. Form G by taking a path of length $4b-2c+2$ and from one of the non-leaf vertices, append $2c-2$ pendent vertices. The watchman's walk is of length $2(4b-2c+2-2) = 8b-4c$ and there are $4b-2c+2+2c-2 = 4b$ edges. Thus, $\frac{w(G)}{m} = \frac{8b-4c}{4b} = \frac{2b-c}{b} = \frac{a}{b}$, as desired. ■

Having consider the number of edges m as a bound for the watchman's walk, it is natural to next consider the number of vertices n.

Theorem 11. *If G is a connected graph on $n \geq 3$ vertices, then $0 \leq w(G) \leq 2n-6$, with equality in the upper bound only if $G = P_n$ for $n \geq 7$.*

Proof. The inequality follows from Lemma 8, as any spanning tree has at least two leaves. Let T be a spanning tree of G with the maximum number of leaves. Certainly, if $G = T = P_n$, then the bound is achieved.

Consider if $T \neq P_n$. T will have at least 3 leaves and by Lemma 8, $w(G) \leq 2n-8$.

If $T = P_n$, then since T is the spanning tree with the most leaves, every vertex of G is of degree at most 2. Thus, G is either a cycle, and hence $w(G) = n < 2n-6$ for $n \geq 7$, or G is a path, as required. ■

Corollary 12. *If G is a graph on $n \geq 6$ vertices and $w(G)$ is odd, then $3 \leq w(G) \leq 2n-9$. Moreover, if $w(G) = 2n-9$, then $\Delta(G) \leq 3$.*

Proof. Certainly, there are no watchman's walks of length 1, so $w(G) \geq 3$. Consider a watchman's walk. Since $w(G)$ is odd, it must contain an odd cycle of length at least 3 as a subgraph. By Lemma 7, no edge in such a cycle is traversed twice. We know that the upper bound in Theorem 11 is $2n-6$. Assume to the contrary that there is a graph G on $n \geq 6$ vertices with a watchman's walk W of length $2n-7$. If there are 3 or more vertices in G not visited by W, then $w(G) \leq 2((n-3)-1) = 2n-8$; a contradiction. If there are exactly 2 vertices in G not visited by W, then exactly 3 edges are traversed once. These must form a cycle. Since we have to visit each of the vertices of this cycle, they must have private neighbours. Since all other edges in W are traversed twice, W must be a 3-cycle with trees appended to each vertex. This is a contradiction, as there then must be at least 3 vertices not visited by W. Finally, if there is exactly 1 vertex in G not visited by W, then exactly 5 edges are traversed once. Using a similar argument to the above, we contradict the number of vertices not visited by W. Thus $w(G) \leq 2n-9$.

If $w(G) = 2n-9$, by Lemma 8, a spanning tree of G with the maximum number of leaves has at most 3 leaves. It follows that $\Delta(G) \leq 3$. ■

Note that for any path P, $w(P)$ is even, and the only cycle that meets the upper bound is C_9. Thus all other graphs that meet the bound have maximum degree 3. A collection of these is illustrated in Figure 2, where the dashed lines represent paths.

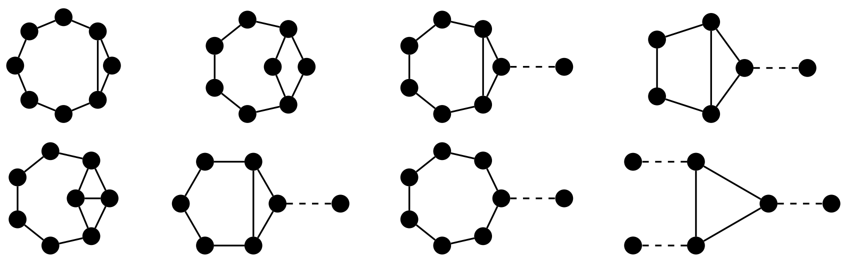

Figure 2. A collection of graphs meeting the upper bound in Corollary 12.

3. Bounding Graph Products

We begin this section by giving a very general upper bound on the watchman's walk of the Cartesian product of two graphs, then consider some cases in which the bound can be simplified or minimized. Recall that, for a graph G, $\gamma(G)$ is the order of a minimum dominating set, and we call such a set a γ-set of G. Let D_G be the set of all γ-sets of G.

We will be interested in walks, open and closed, through the vertices of a graph. If a walk W is the shortest walk in G that visits all the vertices in

$S \subseteq V(G)$, we call W a *minimum covering walk of S in G*, and denote its length by $T_G(S)$. We similarly define a *minimum closed covering walk of S in G*, and denote its length by $\overline{T}_G(S)$. When $S = V(G)$, we write $T(G) = T_G(V(G))$.

Finally, let $\text{par}(k)$ be defined as being 1 if k is an odd integer, and 0 if k is an even integer.

Theorem 13. *If G and H are connected graphs, then*

$$w(G\square H) \leq \gamma(G) \cdot T(H) + \min_{S\in D_G} \{\overline{T}_G(S)\} + \text{par}(\gamma(G)) \cdot \text{diam}(H).$$

Proof. Let W' be a minimum covering walk of H and W'' be a minimum closed covering walk of a γ-set of G that is of length $\min_{S\in D_G}\{\overline{T}_G(S)\}$. (Certainly, such a walk also dominates G.) Using these walks as a base, we will form a walk W in $G\square H$ that is closed and dominating.

Let G_1 and G_2 be the subgraphs of $G\square H$ isomorphic to G that correspond to the endpoints of the walk W'. Consider the walks W'' in each copy of G_i. These walks cover a γ-set of G, by construction. Pick a vertex in the γ-set of G_1, and call it $v_{1,1}$, and the analogous vertex in G_2, $v_{2,1}$. Label all of the vertices in the γ-sets of G_1 and G_2 as $v_{i,j}$, in the order they occur in the walk W''.

Starting at $v_{1,1}$, walk through the copy of H containing it along the walk W' to $v_{2,1}$. Then, in G_2, begin following the walk W'' until $v_{2,2}$ is reached. Then, walk along W' in the copy of H containing $v_{2,2}$ until $v_{1,2}$ is reached. Then, in G_1, continue the walk W'' until $v_{1,3}$ is reached, at which point use W' to return to G_2, and so on. This walk will continue to use copies of W' at each vertex of the γ-set, and the walks between vertices of the γ-set will be derived from W'' — in the end, using an edge that corresponds to each edge in W''.

At this point, the walk will terminate at either $v_{1,1}$, if $\gamma(G)$ is even, or $v_{2,1}$, if $\gamma(G)$ is odd. In the former case, the walk is closed. In the latter, we traverse the copy of H that contains both $v_{1,1}$ and $v_{2,1}$, and this distance is at most $\text{diam}(H)$.

Finally, we consider if the closed walk we have constructed is dominating. Let $(u, v) \in V(G\square H)$. By construction, vertex u is dominated by the γ-set covered by W'. Since every vertex in this γ-set is visited in every copy of G, the vertex (u, v) is also dominated. ∎

We can replace some of the terms of Theorem 13 with more commonly used expressions.

Corollary 14. *If G and H are connected graphs, then*

$$w(G\square H) \leq \gamma(G) \cdot (2|V(H)| + 2) + |V(H)| - 7.$$

Proof. Let $n = |V(H)|$. For any connected H, every vertex of H could be visited by forming a spanning tree of H, then walking each edge of it twice. However, there would be no need to close this walk — that is, the farthest vertices in the spanning tree need only be visited once each. Thus, $T(H) \leq 2(n-1)-2$, in the case that the spanning tree has diameter 2.

A similar scheme may be employed to estimate the length of a closed walk through a γ-set, S, of G. Consider the subgraph of G formed by S and the shortest paths between the vertices of the γ-set. Again, forming a spanning tree in this subgraph, we see that we can form a closed walk that contains S by doubling all the edges of the spanning tree. As vertices in S may each be distance 3, this means that we may traverse $(\gamma(G)-1)\cdot 2\cdot 3$ edges, and hence that $\overline{T}_G(S) \leq 6\gamma(G)-6$.

Finally, it is possible that if the parity of $\gamma(G)$ is odd, following the method of Theorem 13 will land us in a different copy of G, then the one we started in. Thus, a traversal in H could be between two vertices with distance at most $n-1$.

Then $w(G\Box H) \leq \gamma(G)(2n-4)+6\gamma(G)-6+n-1 = \gamma(G)(2n+2)+n-7$, as required. ■

In the case that $\gamma(G)$ is odd, we may improve the result slightly. (If the graph H is Hamiltonian, we may even replace the values $T(H)$ and $\overline{T}(H)$ by $|V(H)|-1$ and $|V(H)|$, respectively.)

Corollary 15. *If G and H are connected graphs and $\gamma(G)$ is odd, then*

$$w(G\Box H) \leq (\gamma(G)-1)\cdot T(H) + \min_{S\in D_G}\{\overline{T}(S)\} + \overline{T}(H).$$

Proof. We follow the construction of W as in Theorem 13. However, when the final vertex of the γ-set is reached in G_1, rather than doing a walk through a copy of H to G_2, instead do a closed walk through H, ending back in G_1. Then follow the remaining portion of W' through G_1 to complete the walk. ■

In the special case when $w(G)=2$, we know that $\gamma(G)=2$, and hence the final term in Theorem 13 will vanish. In these cases, we may consider for what graphs this bound is tight; that is, when is this the "right" way to watch these graphs?

Theorem 16. *If G is a multipartite graph, each part having at least $|V(H)|$ vertices, where $w(G)=2$ and H is connected, then $w(G\Box H) \geq 2|V(H)|$.*

Proof. Let G be as described. Consider $G\Box H$, and an MCDW, W, through its vertices. If two parts of every copy of G is visited, then we are considering a walk with at least $2|V(H)|$ vertices, as required.

If, in some copy of G only a single part is visited, we may consider how the vertices of this part of this copy of G are dominated. Namely, some version of each these vertices must be visited in an adjacent copy of G. Since each part is of order $|V(H)|$ and to move between any two vertices that dominate vertices of the unvisited part takes at least two steps, this gives the desired result. ■

Corollary 17. *If G is a multipartite graph, each part having at least $|V(H)|$ vertices, where $w(G) = 2$ and H has a Hamiltonian path, then $w(G \square H) = 2|V(H)|$.*

Proof. By Theorem 16, we need only exhibit an MCDW of length $2|V(H)|$. Let $v_{a,b}$ be the vertices of $G \square H$, where $a \in V(G)$ and $b \in V(H)$. Since $w(G) = 2$, there exists an edge $xy \in E(G)$, where $\{x, y\}$ dominates G. Let $|V(H)| = m$ and $P = z_1, z_2, \ldots, z_m$ be a Hamiltonian path in H. The walk $(v_{x,z_1}, v_{x,z_2}, \ldots, v_{x,z_m}, v_{y,z_m}, v_{y,z_{m-1}}, \ldots, v_{y,z_1})$ is a watchman's walk of the desired length as it is clearly closed and $\{v_{x,z_i}, v_{y,z_i}\}$ dominates the i^{th} copy of G, for each $1 \leq i \leq m$. ■

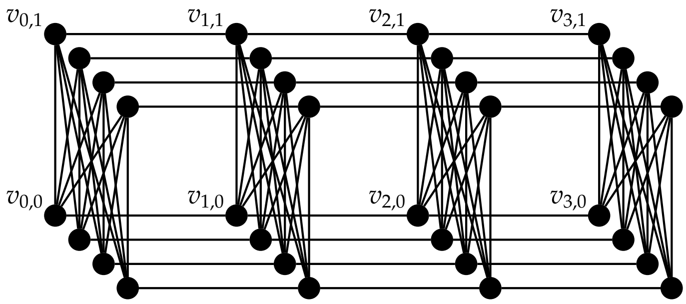

Figure 3. $K_{4,4} \square P_4$.

To illustrate this equality, consider $K_{4,4} \square P_4$, as given in Figure 3. A watchman's walk of length 8 is given by $(v_{0,0}, v_{1,0}, v_{2,0}, v_{3,0}, v_{3,1}, v_{2,1}, v_{1,1}, v_{0,1})$.

Define $\mathcal{I}$ to be the set of graphs created by taking any two graphs G and H, adding a dominating vertex u to G and v to H, then joining u and v by a path of length $p \in \{0, 1, 2, 3\}$. For example, the graph in Figure 4 is in $\mathcal{I}$.

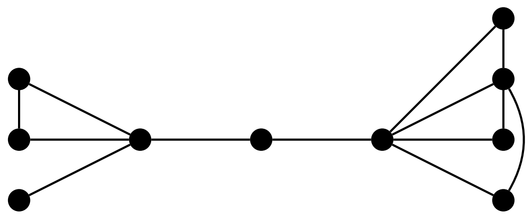

Figure 4. A graph in $\mathcal{I}$.

It is interesting to note that for any $G = H \square P_n$, where $H \in \mathcal{I}$, the bound from Theorem 4, with $|K| = 2$ and $\rho(K) = \text{diam}(G)$, equals the bound from Theorem 13. Thus, in this case we can find $w(G)$. If H is the graph in Figure 4, then $w(G) = 2n + 2$. We obtain the following theorem for grids in the flavour of much of the early work on domination numbers.

Theorem 18. *If $1 \leq k \leq 6$, $k \leq n$ and $3 \leq n$, then $w(P_k \square P_n) = 2n + 2k - 8$.*

Proof. Following the previous argument, the lower bound in Theorem 4 is equal to the upper bound in Theorem 13 for $k \geq 3$; the cases when $k = 1$ and $k = 2$ are straightforward. ∎

4. Watching Grids

First let us consider how Theorem 13 can be improved when considering the product of cycles. In the nicest case, one of these cycles will have length that is a multiple of six.

Lemma 19. *Let $m \geq n \geq 6$ and $n \in \mathbb{Z}$;*

$$w(C_m \square C_n) \leq \frac{n(m+1)}{3}.$$

Proof. The proof consists of exhibiting a closed dominating walk of the given length. Label the grid with vertices from $\mathbb{Z}_m \times \mathbb{Z}_n$ in the typical way, with $(0,0)$ in the bottom left corner. Consider the subgraph consisting of the vertices induced by the vertices labelled $\mathbb{Z}_m \times \{r, r+1, r+2, r+3, r+4, r+5\}$. Let F_r denote the walk of length $2m+1$ as follows

$(0,r), (0,r+1), (1,r+1), (2,r+1),$
$\ldots, (m-2,r+1), (m-2,r+2), (m-2,r+3), (m-2,r+4), (m-3,r+4),$
$\ldots, (0,r+4), (0,r+5).$

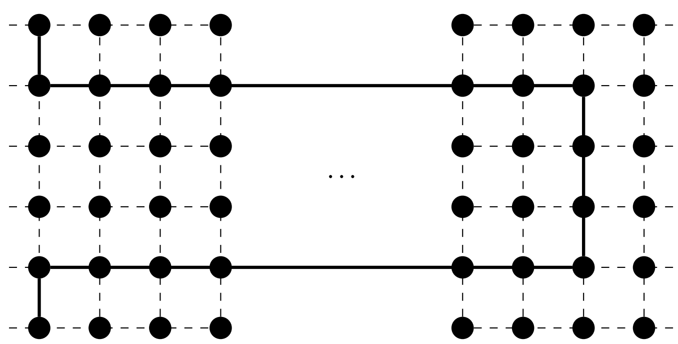

Figure 5. F-walk.

Let e_x denote the edge from $(0, x)$ to $(0, x+1)$ with the operation in $\mathbb{Z}_n$. The following gives a closed dominating walk: $F_0, e_5, F_6, e_{11}, F_{12}, e_{17}, \ldots, F_{n-6}, e_{n-1}$. Each F_r has $2m+1$ edges; there are $n/6$ of these. As well, there are $n/6$ edges e_x. Thus the total length of the walk is $n(m+1)/3$, as required. ■

Applying Theorem 5 to this case gives $w(C_m \square C_n) \geq mn/3$. Thus the upper and lower bounds differ by only $n/3$. We conjecture that the upper bound is the correct value. Considering vertices that are dominated more than once by non-sequential vertices, we see that our construction has exactly $n/3$ such vertices, which occur at the "corners" of our F-walks. We suspect that these are required, but have been unable to prove their necessity.

If n is not a multiple of six, we can still find a closed dominating walk that we believe to be minimum. Define

$$f(m, 6q+r) = 2q(m+1) + \begin{cases} 0 & \text{if } r = 0, \\ m+1 & \text{if } r = 1, \\ m+2 & \text{if } r = 2, \\ m+3 & \text{if } r = 3, \\ 2m & \text{if } r = 4, \\ 2m+1 & \text{if } r = 5. \end{cases}$$

Theorem 20. *If $m \geq n \geq 6$, then $w(C_m \square C_n) \leq f(m, n)$.*

Proof. The proof follows in the same way as that of Lemma 19. If n is divisible by 6, it is exactly the same. Consider the case when $n = 6q + 5$. We follow the same collection of F-walks q times, joined by an edge sequentially. For the remaining 5 rows, we instead follow a modified F-walk (again joined to the first and last F-walk), as depicted in Figure 1. Thus, we have constructed a closed dominating walk using q walks of length $2(m+1)$ (including the joining edges), and one walk of length $2m+1$ (also including the joining edge), as required. A similar argument holds for $n = 6q + 4$.

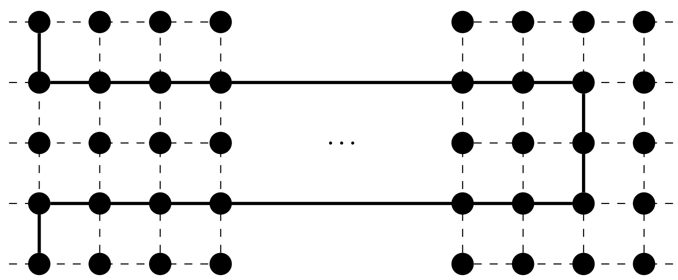

Figure 6. Modified F-walk.

When $n = 6q + 1$, we follow the same collection of q F-walks, joined by an edge sequentially. This leaves one row, which we will dominate by walking all the vertices in that row, returning to the same vertex (i.e., an m-cycle), followed by an edge to join to the first F-walk. This is a closed dominating walk using q walks of length $2(m + 1)$ (including the joining edges), and one walk of length $m + 1$ (including the joining edge), as required. Noting that the m-cycle also dominates the rows above and below it, a similar construction will work for $n = 6q + 2$ and $n = 6q + 3$. ■

Next let us consider the Cartesian product of paths. Define

$$g(m, 6q + r) = 2q(m + 2) + \begin{cases} 2 & \text{if } r = 0, \\ 2m - 4 & \text{if } r = 1, \\ 2m - 2 & \text{if } r = 2, \\ 2m & \text{if } r = 3, \\ 2m + 2 & \text{if } r = 4, \\ 2m + 4 & \text{if } r = 5. \end{cases}$$

Theorem 21. *If $m \geq n \geq 6$, then $w(P_m \Box P_n) \leq g(m, n)$.*

Proof. Consider first the case when $n = 6q$. We exhibit a closed dominating walk of length $2qm + 4q + 2$. Label the grid with vertices from $\mathbb{Z}_m \times \mathbb{Z}_n$ in the typical way with $(0, 0)$ in the bottom left corner. Consider the subgraph consisting of the vertices induced by the vertices labelled $\mathbb{Z}_m \times \{r, r + 1, r + 2, r + 3, r + 4, r + 5\}$. Let T_r denote the edges of a walk as follows

$$\begin{aligned}&(3, r), (3, r + 1), (4, r + 1), (5, r + 1),\\ &\ldots, (m - 1, r + 1), (m - 1, r + 2), (m - 1, r + 3), (m - 1, r + 4), (m - 2, r + 4),\\ &\ldots, (3, r + 4), (3, r + 5).\end{aligned}$$

Let T'_r denote the edges of a walk as follows

$$(1, r + 5), (1, r + 4), \ldots, (1, r).$$

Let $e_{i,y}$ denote the edge from (i, y) to $(i, y + 1)$, with the operation in $\mathbb{Z}_n$, and $e'_{x,j}$ denote the edge from (x, j) to $(x + 1, j)$. The following gives a closed dominating walk

$$T_0, e_{3,5}, T_6, e_{3,11}, T_{12}, e_{3,17}, \ldots, T_{n-6}, e'_{2,n-1}, e'_{1,n-1}, T'_{n-6}, e_{1,n-7}, \ldots, T'_0, e'_{1,0}, e'_{2,0}.$$

Each T_r has $2m - 3$ edges and each T'_r has 5 edges; there are q of each of these. There are $2(q - 1)$ e-edges and 4 e'-edges. Thus the total length of the walk is $2qm + 4q + 2$, as required.

Note that in each subsequent case, we need only add $2r$ vertical edges and $2(m - 4)$ horizontal edges to the walk to remain closed and dominating. The result follows. ■

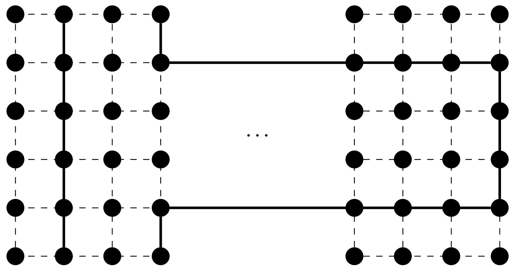

Figure 7. T-walks.

Again, using the result of Theorem 5, we see that $\frac{mn}{3} \leq w(P_m \Box P_n) \leq \frac{n(m+2)}{3} + 2$ when n is divisible by 6. We conjecture that the upper bound is the correct one.

5. Conclusions and Future Directions

Obviously, while the bounds produced for the grid and the toroidal grid are close, work remains to be done to close the gaps between them. As the authors believe they have found the "right" way to walk the grids, this means that new theoretical lower bounds are needed to tackle the problem of the watchman's walk.

While products are a natural extension to the current literature on the watchman's walk, early discussions of the problem were considered by the authors in relation to the popular app game *Pokémon Go*. Pokémon Go is a game in which players attempt to catch fantastic creatures called Pokémon. Unlike previous versions of the Pokémon franchise, this augmented reality version is location-based — to catch certain Pokémon or interact with other aspects of the game, players must move in the real world to different locations. If a player was playing in a large, grid-based city, such as Manhattan, to guarantee capture of all the Pokémon in an area, the player would follow a watchman's walk on a grid such as those described in this paper.

Playing in the flat country-side, however, with many fields, would present a different option. Rather than a grid of the form $P_n \Box P_m$, players would be able to conceive of their environment as $P_n \boxtimes P_m$, where $\boxtimes$ denotes the strong product of two graphs.

Theorem 22. *If G and H are connected graphs on n and m vertices, respectively, with $\overline{T}(G) \leq \overline{T}(H)$ and $\overline{T}(G)$ divisible by 5, then*

$$w(G \boxtimes H) \leq \frac{\overline{T}(G)}{5} \left(\overline{T}(H) + \operatorname{diam}(G)\right) + w(G).$$

Proof. Consider minimum closed covering walks W_G and W_H in G and H, respectively. In $G \boxtimes H$, there are m copies of G and n copies of H. We will construct a walk in $G \boxtimes H$ based on the idea of walking along W_G and W_H simultaneously.

Let the vertices of $W_G = u_0, u_1, u_2, \ldots, u_k (= u_0)$ and $W_H = v_0, v_1, v_2, \ldots, v_\ell$. Certainly $k \geq n$ and $\ell \geq n$, as these walks may revisit vertices. Let W_j be the walk $(u_j, v_0), (u_{j+1}, v_1), (u_{j+2}, v_2), \ldots, (u_{j+\ell}, v_\ell) = (u_{j+\ell}, v_0)$, where the indices of the u_i are considered modulo k. Thus, the first and last vertices of this walk are in the same copy of G; call this copy G_0.

Let $W' = \bigcup_{j=0}^{\frac{k}{5}} V(W_{5j})$. Consider a vertex $(u_a, v_b) \in W'$, with $a \neq 0$. Then the vertex (u_{a+5}, v_b) is in W', by construction. These vertices dominate (u_{a+1}, v_b) and (u_{a+4}, v_b), respectively. Finally, if $(u_a, v_b) \in W'$, then $(u_{a+1}, v_{b+1}), (u_{a+4}, v_{b-1}) \in W'$. These dominate vertices (u_{a+2}, v_b) and (u_{a+3}, v_b), respectively. Thus, every vertex in $G \square H$ is dominated except possibly those of the form (u_0, v_b).

Form a walk W as follows. First walk W_0. This will end at a vertex in G_0. Walk through G_0 to (u_5, v_0); this walk will have length at most diam(G). Then walk W_5, and repeat this process, walking to (u_{10}, v_0), then walking W_{10}, and so on. Repeat until returning to (u_0, v_0). As it is possible that some vertex in G_0 has not been dominated, walk a watchman's walk in G_0. This walk will be closed and dominating, and be of length $\frac{k}{5} \cdot \ell + \frac{k}{5}\mathrm{diam}(G) + w(G)$, as required. ■

Finally, we look at the strong product of cycles. (This could be considered as playing Pokémon on the inside of a toroidal space station.) The following result is an improvement upon Theorem 22 in this case.

Theorem 23. *If m and n are relatively prime, then*

$$w(C_{5m} \boxtimes C_{5n}) = 5mn.$$

Proof. First note that $w(C_{5m} \boxtimes C_{5n}) \geq 5mn$ by Theorem 6. Label the grid with vertices from $\mathbb{Z}_{5m} \times \mathbb{Z}_{5n}$ in the typical way. Consider the closed diagonal walk $((0,0), (1,1), (2,2), \ldots)$. Since m and n are relatively prime, this walk contains the vertex (x, y) if and only if $x - y \equiv 0 \pmod 5$. This means the walk is of length $5mn$. Now consider any vertex (x, y) not on the walk. If $x - y \equiv 1 \pmod 5$, then $(x-1, y)$ dominates (x, y) and is on the walk. If $x - y \equiv 2 \pmod 5$, then $(x-1, y+1)$ dominates (x, y) and is on the walk. If $x - y \equiv 3 \pmod 5$, then $(x+1, y-1)$ dominates (x, y) and is on the walk. If $x - y \equiv 4 \pmod 5$, then $(x+1, y)$ dominates (x, y) and is on the walk. Since this closed walk is dominating, it is a watchman's walk and hence the result holds. ■

References

[1] I. Beaton, R. Begin, S. Finbow and C.M. van Bommel,, *Even time constraints on the watchman's walk*, Australas. J. Combin. **56** (2013) 113–121.

[2] T.Y. Chang, Domination Numbers of Grid Graphs (PhD. Thesis, ProQuest LLC, Ann Arbor, MI, University of South Florida, 1992).

[3] K. Clarke, The Watchman's Walk on Cartesian Products and Circulants, Thesis (Honours)—Memorial University of Newfoundland.

[4] S. Crevals and P.R.J. Östergård, *Independent domination of grids*, Discrete Math. **338** (2015) 1379–1384.
https://doi.org/10.1016/j.disc.2015.02.015

[5] C. Davies, S. Finbow, B.L. Hartnell, Q. Li and K. Schmeisser, *The watchman problem on trees*, Bull. Inst. Combin. Appl. **37** (2003) 35–43.

[6] D. Dyer and R. Milley, *A time-constrained variation of the watchman's walk problem*, Australas. J. Combin. **53** (2012) 97–108.

[7] D. Dyer and J. Howell, *The multiplicity of watchman's walks*, Congr. Numer. **226** (2016) 301–317.

[8] D. Dyer and J. Howell, *Watchman's walks of Steiner triple system block intersection graphs*, Australas. J. Combin. **68** (2017) 23–34.

[9] S. Finbow, M.E. Messinger and M.F. van Bommel, *Eternal domination on $3 \times n$ grid graphs*, Australas. J. Combin. **61** (2015) 156–174.

[10] D. Gonçalves, A. Pinlou, M. Rao and S. Thomassé, *The domination number of grids*, SIAM J. Discrete Math. **25** (2011) 1443–1453.
https://doi.org/10.1137/11082574

[11] B.L. Hartnell, D.F. Rall and C.A. Whitehead, *The watchman's walk problem: An introduction*, Congr. Numer. **130** (1998) 149–155.

[12] B.L. Hartnell and C.A. Whitehead, *Minimum dominating walks in Cartesian product graphs*, Util. Math. **65** (2004) 73–82.

[13] B.L. Hartnell and C.A. Whitehead, *Minimum dominating walks on graphs with large circumference*, Util. Math. **78** (2009) 33–40.

Received 17 September 2018
Revised 7 May 2019
Accepted 3 July 2019