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AN ANALOGUE OF DP-COLORING FOR VARIABLE DEGENERACY AND ITS APPLICATIONS

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Abstract

A graph G is list vertex k-arborable if for every k-assignment L, one can choose $f(v) \in L(v)$ for each vertex v so that vertices with the same color induce a forest. In [6], Borodin and Ivanova proved that every planar graph without 4-cycles adjacent to 3-cycles is list vertex 2-arborable. In fact, they proved a more general result in terms of variable degeneracy. Inspired by these results and DP-coloring which is a generalization of list coloring and has become a widely studied topic, we introduce a generalization on variable degeneracy including list vertex arboricity. We use this notion to extend a general result by Borodin and Ivanova. Not only this theorem implies results about planar graphs without 4-cycles adjacent to 3-cycle by Borodin and Ivanova, it also implies other results including a result by Kim and Yu [S.-J. Kim and X. Yu, *Planar graphs without* 4*-cycles adjacent to triangles are DP-*4*-colorable*, Graphs Combin. 35 (2019) 707–718] that every planar graph without 4-cycles adjacent to 3-cycles is DP-4-colorable.

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1. Introduction

Every graph in this paper is finite, simple, and undirected. We let $V(G)$ denote the vertex set and $E(G)$ denote edge set of a graph G. For $U \subseteq V(G)$, we let $G[U]$ denote the subgraph of G induced by U. For $X, Y \subseteq V(G)$ where X and Y are disjoint, we let $E_G(X, Y)$ be the set of all edges in G with one endpoint in X and the other in Y.

The *vertex-arboricity* $va(G)$ of a graph G is the minimum number of subsets in which $V(G)$ can be partitioned so that each subset induces a forest. This concept was introduced by Chartrand, Kronk, and Wall [9] as *point-arboricity*. They also proved that $va(G) \leq 3$ for every planar graph G. Later, Chartrand and Kronk [10] proved that this bound is sharp by providing an example of a planar graph G with $va(G) = 3$. It was shown that determining the vertex-arboricity of a graph is NP-hard by Garey and Johnson [14] and determining whether $va(G) \leq 2$ is NP-complete for maximal planar graphs G by Hakimi and Schmeichel [15]. Some results on this topic are as follows.

Raspaud and Wang [20] showed that $va(G) \leq \lceil \frac{k+1}{2} \rceil$ for every k-degenerate graph G. It was proved that every planar graph G has $va(G) \leq 2$ when G is without k-cycles for $k \in \{3, 4, 5, 6\}$ (Raspaud and Wang [20]), without 7-cycles (Huang, Shiu, and Wang [16]), without intersecting 3-cycles (Chen, Raspaud, and Wang [11]), without chordal 6-cycles (Huang and Wang [17]), or without intersecting 5-cycles (Cai, Wu, and Sun [8]).

The concept of list coloring was independently introduced by Vizing [22] and by Erdős, Rubin, and Taylor [13]. A *k-assignment* L of a graph G assigns a list $L(v)$ (a set of colors) with $|L(v)| = k$ to each vertex v of G. A graph G is *L-colorable* if there is a proper coloring c where $c(v) \in L(v)$. If G is L-colorable for each k-assignment L, then we say G is *k-choosable*. The *list chromatic number* of G, denoted by $\chi_l(G)$, is the minimum number k such that G is k-choosable.

Borodin, Kostochka, and Toft [7] introduced list vertex arboricity which is a list version of vertex arboricity. We say that G has an *L-forested-coloring* f for a set $L = \{L(v) | v \in V(G)\}$ if one can choose $f(v) \in L(v)$ for each vertex v so that the subgraph induced by vertices with the same color is a forest. We say that G is *list vertex k-arborable* if G has an L-forested-coloring for each k-assignment L. The *list vertex arboricity* $a_l(G)$ is defined to be the minimum k such that G is list vertex k-arborable. Obviously, $a_l(G) \geq va(G)$ for every graph G.

It was proved that every planar graph G is list vertex 2-arborable when G is without k-cycles for $k \in \{3, 4, 5, 6\}$ (Xue and Wu [25]), with no 3-cycles at distance less than 2 (Borodin and Ivanova [4]), or without 4-cycles adjacent to 3-cycles (Borodin and Ivanova [6]).

Dvořák and Postle [12] introduced a generalization of list coloring in which they called a *correspondence coloring*. Following Bernshteyn, Kostochka, and Pron [2], we call it a *DP-coloring*.

Definition. Let L be an assignment of a graph G. We call H an *L-cover* of G if it satisfies all the followings conditions.

(i) The vertex set of H is $\bigcup_{u \in V(G)}(\{u\} \times L(u)) = \{(u, c) \,|\, u \in V(G), c \in L(u)\}$;

(ii) $H[\{u\} \times L(u)]$ is a complete graph for each $u \in V(G)$;

(iii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (maybe empty);

(iv) If $uv \notin E(G)$, then no edges of H connect$\{u\} \times L(u)$ and $\{v\} \times L(v)$.

Definition. An (H, L)-coloring of G is an independent set in an L-cover H of G with size $|V(G)|$. We say that a graph is *DP-k-colorable* if G has an (H, L)-coloring for each k-assignment L and each L-cover H of G. The *DP-chromatic number* of G, denoted by $\chi_{DP}(G)$, is the minimum number k such that G is DP-k-colorable.

If we define edges on H to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$, then G has an (H, L)-coloring if and only if G is L-colorable. Thus DP-coloring is a generalization of list coloring and $\chi_{DP}(G) \geq \chi_l(G)$.

Dvořák and Postle [12] observed that $\chi_{DP}(G) \leq 5$ for every planar graph G. This extends a seminal result by Thomassen [21] on list colorings. Voigt [23] gave an example of a planar graph which is not 4-choosable (thus not DP-4-colorable). Kim and Ozeki [18] showed that planar graphs without k-cycles are DP-4-colorable for each $k \in \{3, 4, 5, 6\}$. Kim and Yu [19] extended the result on 3- and 4-cycles by showing that planar graphs without 3-cycles adjacent to 4-cycles are DP-4-colorable.

Inspired by DP-coloring and list-forested-coloring, we define a generalization of list-forested-coloring as follows.

Definition. Let H be a an L-cover of a graph G with a list assignment L. A *representative set* S of G is a set of vertices in H such that

(1) $|S| = |V(G)|$ and

(2) $u \neq v$ for any two different members (u, c) and (v, c') in S.

A *representative graph* G_S is defined to be the graph obtained from G and a representative set S such that vertices u and v are adjacent in G_S if and only if (u, i) and (v, j) are in S and both are adjacent in H.

A *DP-forested-coloring* of (G, H) is a representative set S such that the representative graph G_S is a forest. We say that a graph is *DP-vertex-k-arborable* if G has a DP-forested-coloring of (G, H) for each k-assignment L and each L-cover H of G.

If we define edges on H to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$, then G has a DP-forested-coloring for G and H if and only if G has an L-forested-coloring. Note that G has an (H, L)-coloring if and only if G has a representative set S such that G_S has no edges.

In [6], Borodin and Ivanova proved that every planar graph without 4-cycles adjacent to 3-cycle is list vertex 2-arborable. In fact, they proved a more general result which we explain later. Inspired by these results, we prove that every planar graph without 4-cycles adjacent to 3-cycles is DP-vertex-2-arborable. We also prove a theorem that extends a general result by Borodin and Ivanova. Among many consequences, this theorem implies a result by Kim and Yu [19] that every planar graph without 4-cycles adjacent to 3-cycle is DP-4-colorable.

We note that results in [6] are proved by means of a partition of the vertex set into desired sets. But representative sets and representative graphs cannot be considered as partitions. Thus we need different techniques to prove our results.

2. Main Results

Some definitions are required to understand the main results and the proofs. Let $\delta(G)$ for a graph G denote the minimum degree of G. A graph G is *strictly k-degenerate* for a positive integer k if every subgraph G' has a vertex v with $d_G(v) < k$. Thus a strictly 1-degenerate graph is an edgeless graph and a strictly 2-degenerate graph is a forest. Note that vertices in a strictly k-degenerate graph can be removed in an order so that each vertex at the time of removal is adjacent to less than k remaining vertices. Now, let f be a function from $V(G)$ to the set of positive integers. A graph G is *strictly f-degenerate* if every subgraph G' has a vertex v with $d_G(v) < f(v)$.

Now, let $f_i, i \in \{1, \ldots, s\}$, be a function from $V(G)$ to the set of nonnegative integers. An *$(f_1, \ldots, f_s)$-partition* of a graph G is a partition of $V(G)$ into $V_1, \ldots, V_s$ such that an induced subgraph $G[V_i]$ is strictly f_i-degenerate for each $i \in \{1, \ldots, s\}$. A *$(k_1, \ldots, k_s)$-partition* where k_i is a constant for each $i \in \{1, \ldots, s\}$ is an $(f_1, \ldots, f_s)$-partition such that $f_i(v) = k_i$ for each vertex v. We say that G is *$(f_1, \ldots, f_s)$-partitionable* if G has an $(f_1, \ldots, f_s)$-partition. Let c be a function from $V(G)$ to the set of positive integers. Define f_c from $f_i, i \in \{1, \ldots, s\}$, and c by $f_c(v) = f_{c(v)}(v)$. Define G_c to be a graph obtained from G and c such that $V(G_c) = V(G)$ while vertices u and v are adjacent in G_c if and only if u and v are adjacent in G and $c(u) = c(v)$. Thus a graph G is $(f_1, \ldots, f_s)$-partitionable if and only if there is a function c such that G_c is strictly f_c-degenerate. By Four Color Theorem [1], every planar graph is $(1, 1, 1, 1)$-partitionable. Chartrand and Kronk [10] constructed planar graphs which are not $(2, 2)$-partitionable. Even stronger, Wegner [24] showed that there exists a planar graph which is not $(2, 1, 1)$-partitionable. Thus it is of interest to find sufficient conditions for planar graphs to be $(1, 1, 1, 1)$-, $(2, 1, 1)$-, or $(2, 2)$-partitionable.

Borodin, Kostochka, and Toft [7] observed that the notion of $(f_1, \ldots, f_s)$-partition can be applied to problems in list coloring and list vertex arboricity. Since v cannot be strictly 0-degenerate, the condition that $f_i(v) = 0$ is equivalent to v cannot be colored by i. In other words, i is not in the list of v. Thus the case of $f_i \in \{0, 1\}$ corresponds to list coloring, and the one of $f_i \in \{0, 2\}$ corresponds to L-forested-coloring. Voigt [23] showed that there exists a planar graph that is not 4-choosable. Naturally, it is also interesting to find sufficient conditions for planar graphs to be 4-choosable or list vertex 2-arborable. Borodin and Ivanova [6] obtained a general result which implies planar graphs without 4-cycles adjacent to 3-cycles are 4-choosable and list vertex 2-arborable.

Theorem 1 (Theorem 6 in [6]). *Every planar graph without 4-cycles adjacent to 3-cycles is $(f_1, \ldots, f_s)$-partitionable if $s \geq 2$, $f_1(v) + \cdots + f_s(v) \geq 4$ for each vertex v, and $f_i(v) \in \{0, 1, 2\}$ for each v and i.*

We extend the concept of DP-coloring to $(f_1, \ldots, f_s)$-partition as follows. Let H be an L-cover of G with the list $\{1, \ldots, s\}$ for every vertex and R be a representative set. Define $f_R(v)$ to equal $f_i(v)$ where $(v, i) \in R$. We say that a graph G is *DP-$(f_1, \ldots, f_s)$-colorable* if we can find a representative set R for every L-cover H of G such that G_R is strictly f_R-degenerate. Such R is called a *DP-$(f_1, \ldots, f_s)$-coloring*. If we define edges on H to match exactly the same colors for each $uv \in E(G)$, then a $(f_1, \ldots, f_s)$-partition exists if and only if a DP-$(f_1, \ldots, f_s)$-coloring exists. Thus $(f_1, \ldots, f_s)$-partition is a special case of DP-$(f_1, \ldots, f_s)$-coloring.

To prove our results, we use two following lemmas.

Lemma 2 (Theorem 2 in [3]). *Every planar graph G without two adjacent 3-cycles has $\delta(G) \leq 4$.*

Lemma 3 (Theorem 2 in [5]). *If a planar graph G without 4-cycles adjacent to 3-cycles has $\delta(G) = 4$, then G contains a configuration, say F, which is a 6-cycle $x_1 \cdots x_6$ with a chord x_1x_5 such that $d(x_i) = 4$ for each $i \in \{1, \ldots, 6\}$.*

Using these two lemmas, we obtain the following corollary.

Corollary 4. *If a planar graph G without 4-cycles adjacent to 3-cycles has $\delta(G) \geq 4$, then G contains a configuration F as in Lemma 3.*

Proof. Since G does not contain 4-cycles adjacent to 3-cycles, we have that G does not contain two adjacent 3-cycles. By Lemma 2, $\delta(G) \leq 4$. Combining with $\delta(G) \geq 4$, we have $\delta(G) = 4$. The proof is complete by Lemma 3. ■

Note that a DP-$(2, 2)$-coloring is equivalent to a DP-forested-coloring.

Theorem 5. *Every planar graph without 4-cycles adjacent to 3-cycles is DP-vertex-2-arborable.*

Proof. Suppose that G with an L-cover H is a minimal counterexample. First, we show that $\delta(G) \geq 4$. Suppose to the contrary that G contains a vertex v

with degree at most 3. By minimality, $G - v$ has a DP-$(2,2)$-coloring R_v. Since v has degree at most 3, there is (v,i) in H with at most one neighbor in R'. Adding (v,i) to R_v completes a DP-$(2,2)$-coloring of G, a contradiction. Thus $\delta(G) \geq 4$. From Corollary 4, we have a configuration F. Since G does not contain 4-cycles adjacent to 3-cycles, we obtain that F is an induced subgraph of G. By minimality, there is a DP-$(2,2)$-coloring R' on $G - \{x_1, \ldots, x_6\}$. It remains to show that we can extend a DP-$(2,2)$-coloring to G.

For each $x_k \in V(F)$ and $i \in \{1,2\}$, we put $f_i^*(x_k)$ equal to 2 minus the number of $(v,j) \in R'$ such that (v,j) and (x_k,i) are adjacent in H.

If F has a DP-(f_1^*, f_2^*)-coloring R^*, then one can obtain a desired DP-$(2,2)$-coloring on G which can be seen from the removal such that we remove vertices in $\{x_1, \ldots, x_6\}$ (in an order according to R^*), and then we remove the vertices in $G - \{x_1, \ldots, x_6\}$ (in an order according to R').

Observe that each of x_1 and x_5 has at most one neighbor outside F and x_j has at most two neighbors outside F for $j \in \{2,3,4,6\}$. From $(f_1(x_j), f_2(x_j)) = (2,2)$ for each j and the definition of $f_i^*(x_j)$, we have $\{f_1^*(x_1), f_2^*(x_1)\} = \{1,2\} = \{f_1^*(x_5), f_2^*(x_5)\}$. Also, we have $f_1^*(x_j) + f_2^*(x_j) \geq 2$ for $j \in \{2,3,4,6\}$.

We will consider only the case that $f_1^*(x_j) + f_2^*(x_j) = 2$ for $j \in \{2,3,4,6\}$ by the following reason. For each set of f_i^*, we can find a set of f_i' with $f_i'(v) \leq f_i^*(v)$ for each vertex v and each $i \in \{1, \ldots, s\}$ such that $f_1'(x_j) + f_2'(x_j) = 2$ for $j \in \{2,3,4,6\}$. If we have a partition of $V(G)$ into $V_1, \ldots, V_s$ such that an induced subgraph $G[V_i]$ is strictly f_i'-degenerate, then this partition is also f_i^*-degenerate. It follows that G is $(f_1', \ldots, f_s')$-partitionable implies G is $(f_1^*, \ldots, f_s^*)$-partitionable. Thus the case that satisfies the equality implies the remaining case of f^*.

Case 1. $f_i^*(x_k) \geq 1$ for each $i \in \{1,2\}$ and $k \in \{1, \ldots, 6\}$. From above, we have $(f_1^*(x_1), f_2^*(x_1)) = (1,2)$ or $(2,1)$ and $(f_1^*(x_i), f_2^*(x_i)) = (1,1)$ for each $i \in \{2,3,4,6\}$. By symmetry, we assume $(f_1^*(x_5), f_2^*(x_5)) = (1,2)$. Since the names of colors can be interchanged, we assume further that (x_k, i) and (x_{k+1}, i) are adjacent in H^* for each $k \in \{1, \ldots, 4\}$ and $i \in \{1,2\}$. However, the matchings from $\{(x_1,1),(x_1,2)\}$ to $\{(x_5,1),(x_5,2)\}$ and to $\{(x_6,1),(x_6,2)\}$ are arbitrary. Thus there are four non-isomorphic structures of H^*. To illustrate desired colorings for all four structures, we use Figure 1 to demonstrate the representation on a vertex x_k. The single cycle means $(x_k,1)$ and the double cycle means $(x_k,2)$. The shade at $(x_k,1)$ indicates that we choose $(x_k,1)$ to be in a coloring R^*. Figures 2–5 show all four structures of H^* with desired colorings.

Case 2. There exists k such that $f_i^*(x_k) = 0$ but $f_j^*(x_{k+1}) \geq 1$ where (x_k, i) and (x_{k+1}, j) are adjacent. Note that all subscripts in this case are taken modulo 6. We will apply a greedy coloring (in which we described later) to $x_{k+1}, x_{k+2}, \ldots, x_6, x_1, x_2, \ldots, x_k$, respectively. If we choose (x_p, i) to be in R^*

in the process of a coloring, we update $f_1^*(x_q)$ and $f_2^*(x_q)$ of an uncolored vertex x_q by $f_j^*(x_q) = \max\{0, f_j^*(x_q) - 1\}$ if (x_p, i) and (x_q, j) are adjacent in H^*.

First, we choose (x_{k+1}, j) to be in R^*. By the condition of the case, $(f_1^*(x_k), f_2^*(x_k))$ remains the same after an update. Next apply greedy coloring to $x_{k+2}, \ldots, x_6, x_1, x_2, \ldots, x_{k-1}$ by choosing (x_m, i) such that $f_i^*(x_m) > 0$ to be in R^*. Since $f_1^*(x_j)+f_2^*(x_j) \geq d_F(x_j)$, one can see that a greedy coloring can be attained. Now at x_k, we have that $(f_1^*(x_k), f_2^*(x_k)) \neq (0,0)$ by the choice of (x_{k+1}, j) in the beginning. Thus we can choose $(x_k, 1)$ or $(x_k, 2)$ to be in R^* to complete the coloring.

Now it remains to show that every (f_1^*, f_2^*) of F in the beginning is similar to one in Case 1 or Case 2. From the observation before Case 1 that $\{f_1^*(x_1), f_2^*(x_1)\} = \{f_1^*(x_5), f_2^*(x_5)\} = \{1, 2\}$ and $f_1^*(x_j) + f_2^*(x_j) = 2$ for $j \in \{2, 3, 4, 6\}$. Suppose (f_1^*, f_2^*) is not as in Case 2. Considering $(f_1^*(x_1), f_2^*(x_1))$, we have $f_1^*(x_6) = f_2^*(x_6) = 1$. Similarly, considering $(f_1^*(x_5), f_2^*(x_5))$, we have $f_1^*(x_4) = f_2^*(x_4) = 1$. Recursively, we obtain that $f_1^*(x_i) = f_2^*(x_i) = 1$ for $i = 3$ and $i = 2$, respectively. Thus we have the situation as in Case 1. ■

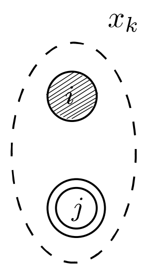

Figure 1. $(x_k, 1)$ with ${f_1}^*(x_k) = i, (x_k, 2)$ with ${f_2}^*(x_k) = j$ and we choose $(x_k, 1)$ in a coloring.

Now we are ready to prove a general result.

Theorem 6. *Every planar graph without* 4*-cycles adjacent to* 3*-cycles is DP-*$(f_1, \ldots, f_s)$*-colorable if* $s \geq 2$, $f_1(v) + \cdots + f_s(v) \geq 4$ *for each vertex* v, *and* $f_i(v) \in \{0, 1, 2\}$ *for each* v *and* i.

Proof. Suppose that G with an L-cover H is a minimal counterexample. First, we show that $\delta(G) \geq 4$. Suppose to the contrary that G contains a vertex v with degree at most 3. By minimality, $G - v$ has a DP-$(f_1, \ldots, f_s)$-coloring R_v. Since v has degree at most 3, there is (v, i) in H with less than $f_i(v)$ neighbors in R'. Adding (v, i) to R_v completes a DP-$(2, 2)$-coloring of G, a contradiction. Thus $\delta(G) \geq 4$. By Corollary 4, we have a configuration F. By minimality, there is a DP-$(f_1, \ldots, f_s)$-coloring R' on $G - \{x_1, \ldots, x_6\}$.

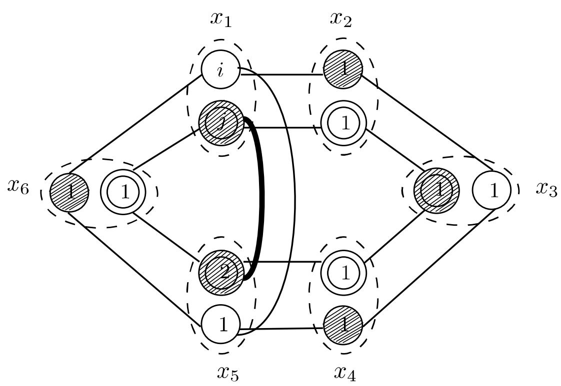

Figure 2. A desired coloring of F with respect to this structure.

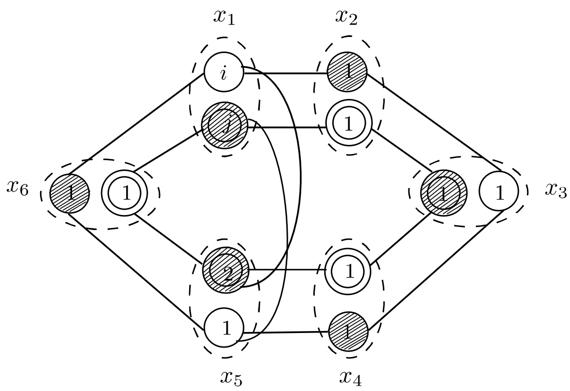

Figure 3. A desired coloring of F with respect to this structure.

For each $x_k \in V(F)$ and $k \in \{1, \ldots, s\}$, we put $f_i^*(x_k)$ equal to $f_i(x_k)$ minus the number of $(v, j) \in R'$ such that (v, j) and (x, i) are adjacent in H.

Similarly to the proof of Theorem 5, if we have a DP-$(f_1^*, \ldots, f_s^*)$-coloring of F, then one can obtain a desired DP-$(f_1, \ldots, f_s)$-coloring on G.

Note that vertices x_i may have different sizes of their list of colors. To make all x_ks have comparable $(f_1^*(x_k), \ldots, f_s^*(x_k))$, we fill out illegal color i for x_k by using $f_i^*(x_k) = 0$. Observe that each of x_1 and x_5 has at most one neighbor outside F and x_j has at most two neighbors outside F for $j \in \{2, 3, 4, 6\}$. Since $f_1(x_i) + \cdots + f_s(x_i) \geq 4$, we have $f_1^*(x_i) + \cdots + f_s^*(x_i) \geq 3$ for $i \in \{1, 5\}$ and $f_1^*(x_i) + \cdots + f_s^*(x_i) \geq 2$ for $i \in \{2, 3, 4, 6\}$. We will consider an inequality as an equality by the reason similar to one in the proof of Theorem 5. Combining with the fact that $f_i(v) \in \{0, 1, 2\}$ for each i and each vertex v, we obtain that $(f_1^*(x_k), \ldots, f_s^*(x_k))$ has two or three positive coordinates when $k \in \{1, 5\}$ and

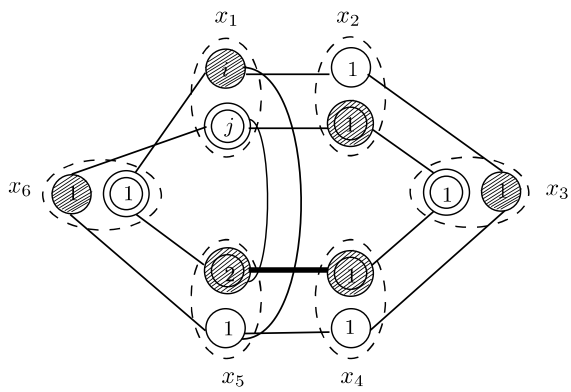

Figure 4. A desired coloring of F with respect to this structure.

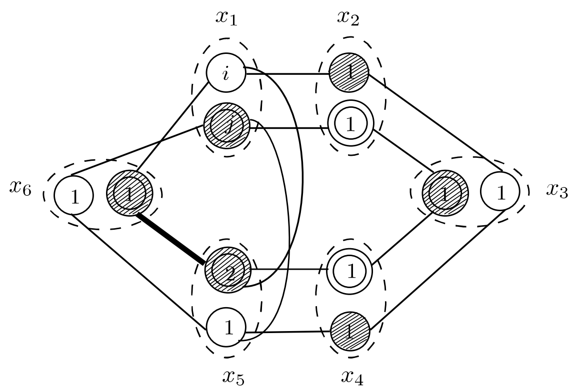

Figure 5. A desired coloring of F with respect to this structure.

$(f_1^*(x_k), \ldots, f_s^*(x_k))$ has one or two positive coordinates when $k \in \{2, 3, 4, 6\}$. If $(f_1^*(x_k), \ldots, f_s^*(x_k))$ and $(f_1^*(x_{k+1}), \ldots, f_s^*(x_{k+1}))$ have different numbers of positive coordinates, then we can complete the coloring by a method similar to Case 2 in the proof of Theorem 5.

Thus we assume that each $(f_1^*(x_k), \ldots, f_s^*(x_k))$ has exactly two positive coordinates. Since color i in which $f_i^*(x_k) = 0$ can be discarded from consideration, we arrive that each $(f_1^*(x_k), \ldots, f_s^*(x_k))$ can be reduced to $(f_{i_1}^*(x_k), f_{i_2}^*(x_k))$. Thus the proof can be completed by a method similar to Case 1 in the proof of Theorem 5. ∎

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