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DEGREE SUM CONDITION FOR THE EXISTENCE OF SPANNING k-TREES IN STAR-FREE GRAPHS

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Abstract

For an integer $k \geq 2$, a *k-tree* T is defined as a tree with maximum degree at most k. If a k-tree T spans a graph G, then T is called a *spanning k-tree* of G. Since a spanning 2-tree is a Hamiltonian path, a spanning k-tree is an extended concept of a Hamiltonian path. The first result, implying the existence of k-trees in star-free graphs, was by Caro, Krasikov, and Roditty in 1985, and independently, Jackson and Wormald in 1990, who proved that for any integer k with $k \geq 3$, every connected $K_{1,k}$-free graph contains a spanning k-tree. In this paper, we focus on a sharp condition that guarantees the existence of a spanning k-tree in $K_{1,k+1}$-free graphs. In particular, we show that every connected $K_{1,k+1}$-free graph G has a spanning k-tree if the degree sum of any $3k-3$ independent vertices in G is at least $|G|-2$, where $|G|$ is the order of G.

Keywords: spanning tree, k-tree, star-free, degree sum condition.

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1. Introduction and Main Result

In this paper, we consider only finite and simple graphs. Let G be a graph. We denote the order of G by $|G|$. For a vertex $x \in V(G)$, we denote the degree of x in G by $\deg_G(x)$ and the set of vertices adjacent to x in G by $N_G(x)$. The independence number of a graph G is denoted by $\alpha(G)$. For an integer $k \geq 2$ and a graph G, we define

$$\sigma_k(G) = \min_{S \subseteq V(G)} \left\{ \sum_{x \in S} \deg_G(x) \;\middle|\; S \text{ is an independent set of } k \text{ vertices} \right\}$$

if $\alpha(G) \geq k$, and define $\sigma_k(G) = \infty$ if $\alpha(G) < k$.

Let $K_{1,m}$ denote the star with m leaves. For a graph G and a given graph H, G is called *H-free* if G contains no induced subgraph isomorphic to H.

The existence of a Hamiltonian path in a given graph has been much studied. In particular, if a graph satisfies any of a number of density conditions, a Hamiltonian path is guaranteed to exist. The following result is one of the best known among these density conditions.

Theorem 1 (Ore [5]). *Let G be a graph. If $\sigma_2(G) \geq |G| - 1$, then G has a Hamiltonian path.*

This theorem has led to many new results and conjectures concerning paths and cycles in graphs. One direction is motivated by the fact that a Hamiltonian path is a spanning tree with small maximum degree. So it is natural to ask how

Theorem 1 might be generalized to guarantee the existence of a spanning tree with maximum degree at most $k \geq 3$.

For an integer $k \geq 2$, a *k-tree* T is defined as a tree of maximum degree at most k. If a k-tree T spans a graph G, then T is called a *spanning k-tree* of G. Note that a spanning 2-tree is a Hamiltonian path.

For general graphs, Win gave a degree sum condition for the existence of a spanning k-tree as a generalization of Theorem 1 as follows.

Theorem 2 (Win [6]). *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_k(G) \geq |G| - 1$, then G has a spanning k-tree.*

By restricting graphs to be star-free, Caro, Krasikov, and Roditty in 1985 and independently, Jackson and Wormald in 1990, obtained the following result, which guarantees the existence of a spanning k-tree.

Theorem 3 (Caro, Krasikov, and Roditty [2], Jackson and Wormald [3]). *For an integer $k \geq 3$, every connected $K_{1,k}$-free graph contains a spanning k-tree.*

Theorem 3 is best possible in the sense that there exist infinitely many connected $K_{1,k+1}$-free graphs which have no spanning k-tree. Thus some additional conditions are needed for connected $K_{1,k+1}$-free graphs to have a spanning k-tree. In fact, for the case when $k = 2$, Liu and Tian in 1986 and independently, Broersma in 1988, obtained the following result.

Theorem 4 (Liu, Tian, and Wu [4], Broersma [1]). *Let G be a connected $K_{1,3}$-free graph. If*

$$\sigma_3(G) \geq |G| - 2,$$

then G has a Hamiltonian path.

The purpose of this paper is to give a degree sum condition for connected $K_{1,k+1}$-free graphs to have a spanning k-tree. Our main result is the following.

Theorem 5. *Let k be an integer with $k \geq 2$. If a connected $K_{1,k+1}$-free graph G satisfies*

$$\sigma_{3k-3}(G) \geq |G| - 2,$$

then G has a spanning k-tree.

Theorem 5 gives a generalization of Theorem 4. By Theorem 5, we also obtain an upper bound on the independence number $\alpha(G)$ for $K_{1,k+1}$-free graphs to have a spanning k-tree.

Corollary 6. *Let k be an integer with $k \geq 2$. If a connected $K_{1,k+1}$-free graph G satisfies*

$$\alpha(G) \leq 3k - 4,$$

then G has a spanning k-tree.

The degree sum condition of Theorem 5 is sharp as shown in Section 2 and the example also shows the sharpness of the independence number in Corollary 6.

2. Sharpness of Main Theorem

Let K_n denote the complete graph of order n. For two graphs G and H, let $G \cup H$ be the union of G and H.

We show that the lower bounds of $\sigma_{3k-3}(G)$ in Theorem 5 and the independence number in Corollary 6 are best possible. In fact, we give the following example.

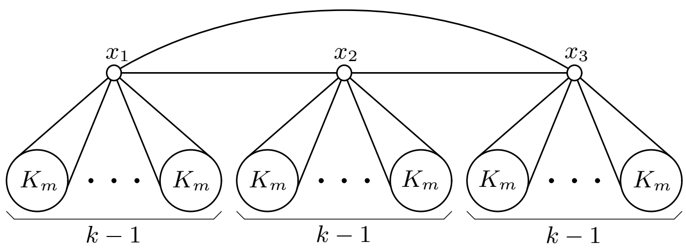

Figure 1. An infinite family of connected $K_{1,k+1}$-free graphs G having no spanning k-tree and satisfying $\sigma_{3k-3}(G) = |G| - 3$.

Let $k \geq 2$ and $m \geq 1$ be integers. Let T be a triangle with $V(T) = \{x_1, x_2, x_3\}$. For each $i = 1, 2, 3$, define a graph H_i as $k-1$ disjoint copies of K_m. The graph G is obtained by joining x_i and all the vertices in $V(H_i)$ for each $i = 1, 2, 3$. Then G has no induced subgraph isomorphic to $K_{1,k+1}$ and $|G| = 3m(k-1)+3$. Since $\alpha(H_1 \cup H_2 \cup H_3) = 3k-3$, we can choose $3k-3$ independent vertices one by one from each complete graph K_m. Then $\sigma_{3k-3}(G) = 3m(k-1) = |G| - 3$. For any spanning tree T of G, one of three vertices x_1, x_2 and x_3 must have degree more than k in T. Hence G has no spanning k-tree, and thus the lower bounds of $\sigma_{3k-3}(G)$ in Theorem 5 and the independence number in Corollary 6 are sharp.

Note that the graphs in Figure 1 show that $K_{1,k}$-freeness in Theorem 3 cannot be replaced by $K_{1,k+1}$-freeness.

3. Proof of Theorem 5

Let T be a tree and let v be a vertex of T. The *outdirected tree* with respect to (T, v) is the directed tree obtained from T in which all the edges are directed away from v. The *out-neighborhood* $N^+_{T,v}(x)$ of a vertex x of T is the set of vertices adjacent from x in the outdirected tree with respect to (T, v).

Proof of Theorem 5

Let k be an integer with $k \geq 2$, and let G be a connected $K_{1,k+1}$-free graph satisfying $\sigma_{3k-3}(G) \geq |G| - 2$. The case $k = 2$ follows from Theorem 4. Thus we consider the case when $k \geq 3$. Let T be a maximal k-tree of G. Suppose that T is not a spanning tree of G. Then G has a vertex u_0 not contained in T which is adjacent to a vertex v in $V(T)$.

Claim 7. $\deg_T(v) = k$.

Proof. Suppose that $\deg_T(v) \neq k$. Since T is a k-tree, $\deg_T(v) < k$. Then $T + vu_0$ is a k-tree of order $|V(T)| + 1$. This contradicts the maximality of T. Hence $\deg_T(v) = k$. ■

Let $S_1, S_2, \ldots, S_k$ denote the components of $T - v$. For each $1 \leq i \leq k$, let s_i be the vertex of S_i which is adjacent to v in T. Note that $\deg_{S_i}(s_i) \leq k - 1$ for each i.

Claim 8. *For each $1 \leq i \leq k$, u_0 is nonadjacent to s_i in G.*

Proof. Suppose that $u_0 s_i \in E(G)$ for some $1 \leq i \leq k$. Then $T + vu_0 + u_0 s_i - vs_i$ is a k-tree of order $|V(T)| + 1$, which contradicts the maximality of T. ■

Since v is a common neighbor of $u_0, s_1, s_2, \ldots, s_k$ in G, by the $K_{1,k+1}$-freeness of G and Claim 8, s_i and s_j are adjacent in G for some $1 \leq i < j \leq k$. Without loss of generality, we may assume that $s_{k-1}s_k \in E(G) \setminus E(T)$.

Claim 9. $\deg_T(s_{k-1}) = \deg_T(s_k) = k$.

Proof. By symmetry, it suffices to show that $\deg_T(s_k) = k$. If $\deg_T(s_k) \neq k$, then $\deg_T(s_k) < k$ because T is a k-tree, and hence $T + s_{k-1}s_k + u_0 v - vs_k$ is a k-tree of order $|V(T)| + 1$. This contradicts the maximality of T. ■

As seen in Figure 2, we redefine $T_i = S_i$ and $t_i = s_i$ for each $1 \leq i \leq k-2$ and let $T_{k-1}, \ldots, T_{2k-3}$ and $T_{2k-2}, \ldots, T_{3k-4}$ be the components of $S_{k-1} - s_{k-1}$ and $S_k - s_k$, respectively. Let $t_{k-1}, \ldots, t_{2k-3}$ (respectively, $t_{2k-2}, \ldots, t_{3k-4}$) denote the vertices of $T_{k-1}, \ldots, T_{2k-3}$ (respectively, $T_{2k-2}, \ldots, T_{3k-4}$) which are adjacent to s_{k-1} (respectively, s_k) in T. Since $T_1, T_2, \ldots, T_{3k-4}$ are vertex-disjoint k-trees, we can choose a leaf $u_i \in V(T_i)$ of T for each $1 \leq i \leq 3k - 4$. By the maximality of T and $\deg_T(u_i) = 1$, $N_G(u_i) \subseteq V(T)$ for each $1 \leq i \leq 3k - 4$.

Claim 10. *The set $\{u_0, u_1, \ldots, u_{3k-4}\}$ is an independent set of G.*

Proof. For $1 \leq i \leq 3k-4$, since $N_G(u_i) \subseteq V(T)$, we have $u_0 u_i \notin E(G)$. Suppose that $u_i u_j \in E(G)$ for some $1 \leq i < j \leq 3k - 4$. Consider the following tree T_A,

$$T_A = \begin{cases} T + u_i u_j + u_0 v - vt_i & \text{if } 1 \leq i \leq k-2, \\ T + u_i u_j + u_0 v + s_{k-1}s_k - vs_k - s_{k-1}t_i & \text{if } k-1 \leq i \leq 2k-3, \\ T + u_i u_j + u_0 v + s_{k-1}s_k - vs_{k-1} - s_k t_i & \text{if } 2k-2 \leq i \leq 3k-4. \end{cases}$$

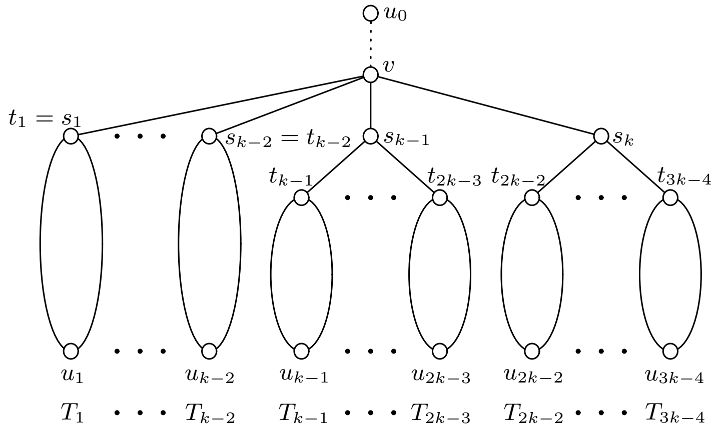

Figure 2. A maximal k-tree T.

Then T_A is a k-tree of order $|V(T)| + 1$, which contradicts the maximality of T. Hence the claim holds. ■

For each $1 \le i \le 3k - 4$, define

$$W_i = \left(\bigcup_{0 \le j \le 3k-4, j \ne i} N_G(u_j) \right) \cap V(T_i).$$

Claim 11. *For each $1 \le i \le 3k - 4$, $t_i \notin W_i$.*

Proof. If $t_i \in W_i$ for some $1 \le i \le 3k - 4$, then t_i is adjacent to a leaf u_j of T_j with $j \ne i$ or to the vertex u_0. Consider the following tree T_B,

$$T_B = \begin{cases} T + t_i u_j + u_0 v - v t_i & \text{if } 1 \le i \le k-2, \\ T + t_i u_j + s_{k-1} s_k + u_0 v - s_{k-1} t_i - v s_k & \text{if } k-1 \le i \le 2k-3, \\ T + t_i u_j + s_{k-1} s_k + u_0 v - s_k t_i - v s_{k-1} & \text{if } 2k-2 \le i \le 3k-4. \end{cases}$$

Then T_B is a k-tree of order $|V(T)| + 1$, which contradicts the maximality of T. Consequently, $t_i \notin W_i$ for each $1 \le i \le 3k - 4$. ■

Claim 12. *For each $1 \le i \le 3k - 4$, any vertex $w \in W_i$ satisfies the following three statements:*

(i) $\deg_T(w) = k$;

(ii) *no vertex u_j with $1 \le j \le 3k - 4$ is adjacent to any vertex of $N^+_{T_i, u_i}(w)$ in G; and*

(iii) $|(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}| \le k - 1$.

Proof. (i) Suppose that $\deg_T(w) \neq k$ for some $w \in W_i$ with $1 \leq i \leq 3k-4$. Since T is a k-tree, $\deg_T(w) < k$. By the definition of W_i, w is adjacent to a vertex u_j with $j \neq i$ in G (possibly, $j = 0$). Consider the following tree T_C,

$$T_C = \begin{cases} T + u_j w + u_0 v - v t_i & \text{if } 1 \leq i \leq k-2, \\ T + u_j w + s_{k-1}s_k + u_0 v - s_{k-1}t_i - v s_k & \text{if } k-1 \leq i \leq 2k-3, \\ T + u_j w + s_{k-1}s_k + u_0 v - s_k t_i - v s_{k-1} & \text{if } 2k-2 \leq i \leq 3k-4. \end{cases}$$

Then T_C is a k-tree of order $|V(T)|+1$, which contradicts the maximality of T. Hence $\deg_T(w) = k$ as desired.

(ii) Suppose that for some $1 \leq j \leq 3k-4$, u_j is adjacent to a vertex $w^+ \in N^+_{T_i,u_i}(w)$ in G. By the definition of W_i, w is adjacent to a leaf u_ℓ with $\ell \neq i$ or to the vertex u_0. Note that $w \neq t_i$ by Claim 11. Consider the following k-tree T_D,

$$T_D = \begin{cases} T + u_\ell w + u_j w^+ + u_0 v - v t_i - w w^+ & \text{if } 1 \leq i \leq k-2, \\ T + u_\ell w + u_j w^+ + s_{k-1}s_k + u_0 v - v s_k - s_{k-1}t_i - w w^+ & \text{if } k-1 \leq i \leq 2k-3, \\ T + u_\ell w + u_j w^+ + s_{k-1}s_k + u_0 v - v s_{k-1} - s_k t_i - w w^+ & \text{if } 2k-2 \leq i \leq 3k-4. \end{cases}$$

Then T_D is a k-tree of order $|V(T)|+1$. This contradicts the maximality of T.

(iii) To the contrary, assume that $|(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}| \geq k$. Since $\deg_T(w) = k \geq 3$ by Claim 12(i), a vertex $w_1 \in N^+_{T_i,u_i}(w)$ exists. Note that both w_1 is different from any u_j with $j \neq i$ because $w_1 \in V(T_i)$ and $(\{u_0, u_1, u_2, \ldots, u_{3k-4}\} \setminus \{u_i\}) \cap V(T_i) = \emptyset$. Then w_1 and k vertices in $(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}$ are all neighbors of w in G. Moreover, Claims 10 and 12(ii) assart that w_1 and k vertices in $(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}$ are independent in G. This contradicts the assumption that G is $K_{1,k+1}$-free. Hence $|(N_G(w) \cap \{u_0, u_1, u_2, \ldots, u_{3k-4}\}) \setminus \{u_i\}| \leq k-1$. ■

Claim 13. *We have* $|N_G(s_i) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \leq k-1$ *for each* $i = k-1$ *and* $i = k$.

Proof. We first prove that

$$N_G(s_k) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_{2k-2}, \ldots, u_{3k-4}\}.$$

By Claim 8, $s_k u_0 \notin E(G)$. If $s_k u_i \in E(G)$ for some $1 \leq i \leq 2k-3$, then $T + s_k u_i + u_0 v - v s_k$ is a k-tree of order $|V(T)|+1$. This contradicts the maximality of T. Hence $N_G(s_k) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_{2k-2}, \ldots, u_{3k-4}\}$ as desired. This implies that $|N_G(s_k) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \leq k-1$. By symmetry, applying the preceding argument, we obtain the claim for the case when $i = k-1$. ■

Claim 14. $|N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \leq k-1$.

Proof. We show that $N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_0, \ldots, u_{k-2}\}$. Suppose that $vu_i \in E(G)$ for some $k-1 \le i \le 3k-4$. Then $T + u_iv + s_{k-1}s_k + u_0v - s_{k-1}v - s_kv$ is a k-tree of order $|V(T)|+1$. This contradicts the maximality of T. Hence $N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\} \subseteq \{u_0, \ldots, u_{k-2}\}$. Thus $|N_G(v) \cap \{u_0, u_1, \ldots, u_{3k-4}\}| \le k-1$. ■

By Claim 12(i), $|N^+_{T_i,u_i}(w)| = k-1$ for any $w \in W_i$ with $1 \le i \le 3k-4$. It follows from Claim 12(ii) that

$$\begin{aligned} |N_G(u_i) \cap V(T_i)| &\le |V(T_i)| - (k-1)|W_i| - |\{u_i\}| \\ &= |V(T_i)| - (k-1)|W_i| - 1. \end{aligned} \tag{1}$$

For each $0 \le j \le 3k-4$ with $j \ne i$, Claim 12(iii) asserts that

$$\sum_{0 \le j \le 3k-4, j \ne i} |N_G(u_j) \cap V(T_i)| \le (k-1)|W_i|. \tag{2}$$

By (1) and (2), we obtain

$$\sum_{0 \le j \le 3k-4} |N_G(u_j) \cap V(T_i)| \le |V(T_i)| - 1.$$

Hence we obtain

$$\begin{aligned} \sum_{1 \le i \le 3k-4} \sum_{0 \le j \le 3k-4} |N_G(u_j) \cap V(T_i)| &\le \sum_{1 \le i \le 3k-4} (|V(T_i)| - 1) \\ &\le |T| - |\{s_{k-1}, s_k, v\}| - (3k-4) \\ &= |T| - 3k + 1. \end{aligned} \tag{3}$$

By (3), Claims (13) and (14),

$$\begin{aligned} \sum_{0 \le i \le 3k-4} \deg_G(u_i) &\le |T| - 3k + 1 + (k-1)|\{s_{k-1}, s_k, v\}| + |N_{G-V(T)}(u_0)| \\ &\le |T| - 2 + |G| - |T| - |\{u_0\}| = |G| - 3. \end{aligned}$$

This contradicts the degree sum condition of Theorem 5 and hence the proof of Theorem 5 is completed. ■

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