

## NON-1-PLANARITY OF LEXICOGRAPHIC PRODUCTS OF GRAPHS

NAOKI MATSUMOTO

*Department of Computer and Information Science*  
*Faculty of Science and Technology*  
*Seikei University, Tokyo, Japan*

**e-mail:** naoki.matsumo10@gmail.com

AND

YUSUKE SUZUKI

*Department of Mathematics*  
*Niigata University*  
*8050 Ikarashi 2-no-cho, Nishi-ku, Niigata, 950-2181, Japan*

**e-mail:** y-suzuki@math.sc.niigata-u.ac.jp

### Abstract

In this paper, we show the non-1-planarity of the lexicographic product of a theta graph and  $K_2$ . This result completes the proof of the conjecture that a graph  $G \circ K_2$  is 1-planar if and only if  $G$  has no edge belonging to two cycles.

**Keywords:** 1-planar graph, lexicographic product.

**2010 Mathematics Subject Classification:** 05C10.

### 1. INTRODUCTION

In this paper, we only consider simple and connected graphs, which have neither loops nor multiple edges. We denote the vertex set and the edge set of a graph  $G$  by  $V(G)$  and  $E(G)$ , respectively. For a graph  $H$ , an  $H$ -subgraph of  $G$  is a subgraph of  $G$  isomorphic to  $H$ .

A *drawing* of a graph  $G$  on the Euclidean plane or the sphere  $S^2$  is a representation of  $G$  on  $S^2$  where vertices are distinct points of  $S^2$ , and edges are curves on the sphere joining the points corresponding to their end vertices; we

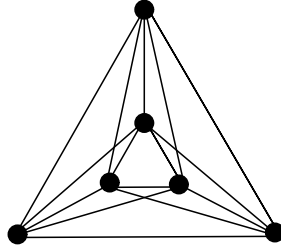
are preferential to “sphere” in the paper since we do not distinguish between the outer region and the inner regions of the plane. A drawing is *proper* if edges are simple curves without vertices of the graph in their interiors. A *crossing point* is a transversal intersection of two curves on the sphere. In this paper, we consider only proper drawings such that no two adjacent edges cross, no two edges touch each other tangentially and no more than two edges cross at the same point.

A graph  $G$  is *1-planar* if it can be properly drawn on the sphere  $S^2$  so that each of its edges crosses at most one other edge. By the above definition, notice that every planar graph is 1-planar. We can also regard the drawing as a continuous map  $f : G \rightarrow S^2$  which may not be injective where  $G$  is regarded as a 1-dimensional topological space. In this paper, we call the above map  $f$  a *1-embedding* of  $G$  into the sphere. In this case, we say that the image  $f(G)$  is a *1-plane graph*; similarly to the difference between “planar graph” and “plane graph”. Sometimes, for simplicity, we denote a given 1-plane graph by  $G$ , instead of  $f(G)$ . Similarly, for simplicity, we use  $v$  (respectively,  $e$ ) instead of  $f(v)$  (respectively,  $f(e)$ ) for  $v \in V(G)$  (respectively,  $e \in E(G)$ ) in a 1-plane graph  $f(G)$  (or simply  $G$ ).

An edge of a 1-plane graph  $G$  is *crossing* if it crosses another edge, and is *non-crossing* otherwise. If an edge  $v_0v_2$  of a 1-plane graph  $G$  crosses an edge  $v_1v_3$  at a crossing point  $z$ , then we say that the arc  $v_i z$  is a *half-edge* of  $G$  for each  $i \in \{0, 1, 2, 3\}$ . The crossing point of two edges  $ab$  and  $cd$  can be denoted by  $z_{\{ab, cd\}}$  with the subscript to clarify such a pair of crossing edges. A connected component  $D$  of  $S^2 - G$  whose boundary contains no crossing point is a *face* of the 1-plane graph  $G$ ; the boundary of the face  $D$  is a closed walk consisting of non-crossing edges only. A connected component  $D$  of  $S^2 - G$  whose boundary contains a crossing point is a *fake face*. For the number of edges of a 1-planar graph  $G$ , the following tight upper bound is well-known (e.g., see [1]):  $|E(G)| \leq 4|V(G)| - 8$ . A 1-planar graph  $G$  is *optimal* if it satisfies the equality in the above inequality.

Recently, 1-planar graphs have been widely researched in the literature (e.g., see the survey paper [5]). Especially, in contrast with general planar embeddings (without crossings) of graphs, it was shown in [7] that testing 1-planarity of a given graph is an NP-complete problem. As a relevant fact, we can easily check that an edge contraction may not preserve the 1-planarity. Hence, the 1-planarity of a graph cannot be characterized in terms of forbidden minors. Furthermore, it is known [6, 7] that there are infinitely many non-1-planar graphs  $G$  with minimum degree at least 3 such that  $G - e$  is 1-planar for any edge  $e$  of  $G$ . This implies that we cannot establish a Kuratowski-like theorem for 1-planar graphs.

As stated above, it is not easy to test the 1-planarity of a given graph. However, there are results for some special classes of graphs. It is known that the complete graph  $K_n$  is 1-planar if and only if  $n \leq 6$ . Figure 1 shows the unique (unlabeled) 1-embedding of  $K_6$  on the sphere; the “uniqueness” is discussed in [8].

Figure 1. Unique 1-embedding of  $K_6$  on the sphere.

Furthermore in [3], the characterization of complete multi-partite 1-planar graphs was given.

Recently, the 1-planarity of “joins” and “products” of graphs have been discussed in [2, 4]. The *lexicographic product*  $G \circ H$  of two graphs  $G$  and  $H$  is a graph such that the vertex set of  $G \circ H$  is the Cartesian product  $V(G) \times V(H)$ , and two vertices  $(u, v)$  and  $(x, y)$  are adjacent if and only if either  $u$  is adjacent to  $x$  in  $G$  or  $u = x$  and  $v$  is adjacent to  $y$  in  $H$ . For example, the complete graph with 6 vertices shown in Figure 1 is the lexicographic product of  $C_3$  and  $K_2$ ; where  $C_k$  denotes a cycle with  $k$  edges. By definition,  $G \circ H \neq H \circ G$  in general. It was shown in [2] that  $K_2 \circ H$ , where  $|V(H)| \leq 4$ , is 1-planar if and only if  $H$  is a subgraph of either  $C_3$  or  $C_4$ , and the following conjecture was proposed in the same paper. A graph  $G$  is a *cactus* if  $G$  is connected and if every edge of  $G$  belongs to at most one cycle.

**Conjecture 1** [2]. *The lexicographic product of a graph  $G$  and  $K_2$  is 1-planar if and only if  $G$  is a cactus.*

The “if”-part of the conjecture was proved in [2]. In the present paper, we prove the “only if”-part of the conjecture by proving the following theorem.

**Theorem 2.** *If a lexicographic product of a graph  $G$  and  $K_2$  is 1-planar, then  $G$  is a cactus.*

In the next section, we introduce the notion of a “barrier loop”, which plays an important role to discuss 1-embeddability of graphs in our argument. To prove Theorem 2, we first consider in Section 3 the ways to 1-embed the lexicographic product of  $C_n$  and  $K_2$  on the sphere. Then we prove that the lexicographic product of a theta graph and  $K_2$  is not 1-planar. Our main result is proved in the end of Section 4.

## 2. BARRIER LOOP

Let  $G$  be a 1-planar graph and let  $f$  be a 1-embedding of  $G$ . Suppose that the 1-plane graph  $f(G)$  has a simple closed curve  $L = v_0 z_0 v_1 z_1 v_2 z_2 \cdots v_{k-1} z_{k-1} v_0$  on

$S^2$  where each  $v_i$  is a vertex of  $G$  and each  $z_j$  is a crossing point of  $f(G)$  such that  $v_i z_i$  and  $z_i v_{i+1}$  are half-edges of  $f(G)$  for each  $i \in \{0, \dots, k-1\}$  with indices taken modulo  $k$ . We call the above simple closed curve on  $S^2$  a *barrier loop* of  $f(G)$ . In the above  $L$ , if  $v_i z_i v_{i+1}$  in the sequence corresponds to an edge  $v_i v_{i+1}$  of  $G$ , then we sometimes omit the crossing point  $z_i$  and write  $L = \dots v_i v_{i+1} \dots$ . The following proposition plays an important role when discussing re-1-embeddings of 1-planar graphs. For two graphs  $G$  and  $H$ , we define  $G \cap H = (V(G) \cap V(H)) \cup (E(G) \cap E(H))$ .

**Proposition 3.** *Let  $G$  be a 1-planar graph and  $f : G \rightarrow S^2$  be a 1-embedding of  $G$  such that  $f(H)$  has a barrier loop  $L$  for a subgraph  $H$  of  $G$ . If  $u$  and  $v$  locate in distinct regions separated by  $L$  for  $u, v \in V(G)$ , then there exists no path  $P$  joining  $u$  and  $v$  such that  $P \cap H \subseteq \{u, v\}$ .*

**Proof.** Suppose, for a contradiction, that there exists a path  $P$  joining  $u$  and  $v$  such that  $P \cap H \subseteq \{u, v\}$ . Then,  $P$  crosses  $L$  by the above assumption. Since  $u, v \notin V(L)$ , we have  $V(P) \cap V(L) = \emptyset$ , hence  $P$  crosses  $L$  transversally at a crossing point  $z$  in  $f$  where  $xz$  and  $yz$  are half-edges contained in  $L$  for  $x, y \in V(G)$ . In this case,  $xy$  should be an edge of  $G$  and  $xy$  crosses another edge  $e \in E(P)$ . Since  $xy$  is a crossing edge in  $f(H)$ ,  $e$  is contained in  $E(H)$ ; otherwise,  $L$  would not be a barrier loop in  $f(H)$ . This implies that  $P \cap H$  contains the edge  $e$ , a contradiction. ■

It easily follows from the above proposition that there exists no edge  $uv \in E(G) \setminus E(H)$  such that  $u$  and  $v$  are in the different regions separated by the barrier loop  $L$  in  $f(H)$ .

### 3. POSSIBLE RE-1-EMBEDDINGS OF $C_n \circ K_2$

It was proved in [8] that there exist exactly two ways to 1-embed  $K_4$  into the sphere as shown in Figure 2. We say that the left-hand side 1-embedding of  $K_4$  in the figure is *tetrahedral* and the right-hand side one is *pyramidal*.



Figure 2. Two 1-embeddings of  $K_4$ .

**Lemma 4.** *Let  $G$  be a disjoint union of  $H_1$  and  $H_2$  where  $H_i \cong K_4$  for each  $i \in \{1, 2\}$  and let  $f$  be a 1-embedding of  $G$  into the sphere. If there is a crossing point created by  $e_1 \in E(H_1)$  and  $e_2 \in E(H_2)$ , then each of  $f(H_1)$  and  $f(H_2)$  is tetrahedral and they are dual to each other.*

**Proof.** We assume that  $H_1$  (respectively,  $H_2$ ) has vertices  $a_0, a_1, a_2$  and  $a_3$  (respectively,  $b_0, b_1, b_2$  and  $b_3$ ). First, suppose that  $f(H_1)$  is pyramidal. In  $f(G)$ , we may assume that  $a_0a_1a_2a_3$  is the unique cycle consisting of non-crossing edges without loss of generality. Note that  $\{a_0a_2, a_1a_3\}$  forms a pair of crossing edges. Furthermore, we may assume that  $a_0a_1$  crosses  $b_0b_1$  by the assumption in the statement. In this case,  $b_2$  cannot be adjacent to  $b_0$  or  $b_1$  by Proposition 3, since  $a_0a_1 \cup \{a_0a_2, a_1a_3\}$  forms a barrier loop. Thus, we have got a contradiction.

Secondly, suppose that both  $f(H_1)$  and  $f(H_2)$  are tetrahedral. We denote the region corresponding to the triangular face bounded by  $a_i a_{i+1} a_{i+2}$  in  $f(H_1)$  by  $R_i$  where the indices are taken modulo 4. In this case, we may assume that  $a_0a_1$  crosses  $b_0b_3$  and that  $b_0$  lies in  $R_0$ . Observe that  $b_3$  is in the interior of  $R_3$ . Now, if  $b_1$  is also in  $R_0$  (respectively,  $R_3$ ), then  $b_1$  cannot be adjacent to  $b_3$  (respectively,  $b_0$ ) by the 1-planarity of  $G$ . Therefore,  $b_1$  lies in  $R_1$  or  $R_2$ , say  $R_1$ . Similarly, the remaining vertex  $b_2$  must be in  $R_2$ , and we get our desired conclusion by the 1-planarity of  $G$ . ■

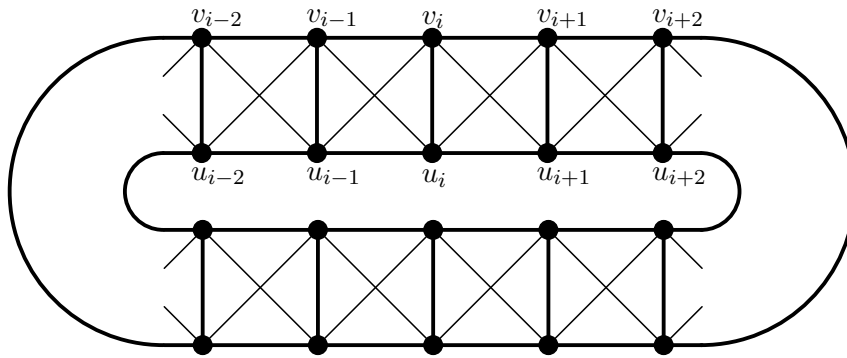


Figure 3. 1-embedding of  $X_n$ .

Consider a 1-embedding of  $X_n = C_n \circ K_2$  for  $n \geq 3$ , shown in Figure 3. In the figure, there are two cycles  $C = u_0 u_1 \dots u_{n-1}$  and  $C' = v_0 v_1 \dots v_{n-1}$  of length  $n$  which consist of non-crossing edges only. Then non-crossing edges  $u_i v_i$  called *rungs* are placed between  $C$  and  $C'$  in parallel for  $0 \leq i \leq n-1$ . In the 1-embedding,  $u_i v_{i+1}$  and  $u_{i+1} v_i$  forms a pair of crossing edges for  $0 \leq i \leq n-1$  with indices taken modulo  $n$ . Denote by  $H_i$  a subgraph of  $X_n$  isomorphic to  $K_4$  such that  $V(H_i) = \{u_i, v_i, u_{i+1}, v_{i+1}\}$  and  $E(H_i) = \{u_i v_i, v_i v_{i+1}, v_{i+1} u_{i+1},$

$u_{i+1}u_i, u_iv_{i+1}, u_{i+1}v_i\}$ , where indices are taken modulo  $n$ . In the following arguments,  $X_n$  is supposed to have the above vertices and edges, and indices are taken modulo  $n$  unless otherwise stated. First, we prove the following propositions giving basic properties of  $X_n$ .

**Proposition 5.**  $X_n$  ( $n \geq 3$ ) is 4-connected. Furthermore, if  $n \geq 4$ , then the connectivity of  $X_n$  is exactly 4.

**Proof.** Since  $X_3$  is isomorphic to  $K_6$  and the connectivity of  $G \circ H$  for two connected graphs  $G$  (which is not a complete graph) and  $H$  is exactly the connectivity of  $G$  times the order of  $H$  [9], the proposition follows. ■

**Proposition 6.** In  $X_n$  ( $n \geq 4$ ), there is no subgraph isomorphic to  $K_4$  other than  $H_i$ .

**Proof.** In  $X_n$ ,  $u_i$  has degree 5, and we have to choose three vertices adjacent to  $u_i$  from  $\{u_{i-1}, v_{i-1}, v_i, u_{i+1}, v_{i+1}\}$ . However, there is no edge between  $\{u_{i-1}, v_{i-1}\}$  and  $\{u_{i+1}, v_{i+1}\}$  if  $n \geq 4$ . Therefore, the proposition follows. ■

**Proposition 7.** In  $X_n$  ( $n \geq 4$ ), each rung  $u_iv_i$  is included in exactly two  $K_4$ -subgraphs  $H_{i-1}$  and  $H_i$  while each of the other edges is included in only one  $K_4$ -subgraph.

**Proof.** It easily follows from Proposition 6. ■

Now we consider local structures of 1-embedded  $X_n$  on the sphere with  $n \geq 5$ .

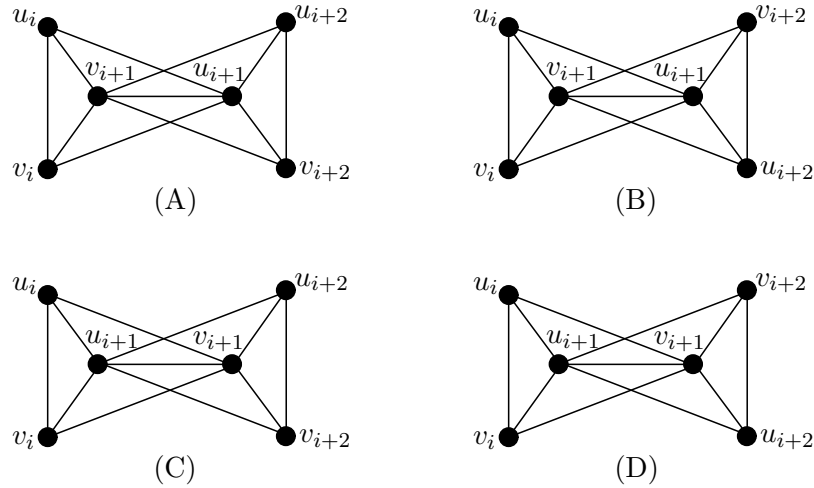
**Lemma 8.** In any 1-embedding  $f : X_n \rightarrow S^2$  with  $n \geq 5$ , edges  $e \in E(H_i)$  and  $e' \in E(H_j)$  cannot cross if  $j$  equals neither  $i - 1$  nor  $i + 1$ .

**Proof.** Suppose that  $i < j$  and  $j$  equals neither  $i - 1$  nor  $i + 1$ . Suppose, for a contradiction, that  $H_i$  and  $H_j$  have a pair of crossing edges  $e \in E(H_i)$  and  $e' \in E(H_j)$ . By Lemma 4,  $f(H_i)$  and  $f(H_j)$  are dual to each other; note that each of them is tetrahedral. In this case, we can easily find a barrier loop corresponding to a cycle in  $f(H_j)$  which separates any two vertices of  $f(H_i)$ . If  $j \neq i + 2$ , the path  $u_{i+1}u_{i+2}v_{i+1}$  cannot exist by Proposition 3. Also in the case when  $j = i + 2$ , the path  $u_iu_{i-1}v_i$  cannot exist by the same reason; note that  $i - 1 \neq j + 1$  since  $n \geq 5$ . ■

**Lemma 9.** In any 1-embedding  $f : X_n \rightarrow S^2$  with  $n \geq 5$ ,  $\{u_iv_i, u_{i+1}v_{i+1}\}$  is not a pair of crossing edges.

**Proof.** This is just a corollary of Lemma 8. ■

**Lemma 10.** In any 1-embedding  $f : X_n \rightarrow S^2$  with  $n \geq 5$ , if edges  $e \in E(H_i)$  and  $e' \in E(H_{i+1})$  cross, then each of  $f(H_i)$  and  $f(H_{i+1})$  is tetrahedral and  $f(H_i \cup H_{i+1})$  is one of the 1-plane graphs (A), (B), (C) and (D) shown in Figure 4.

Figure 4.  $H_i$  and  $H_{i+1}$  having crossing edges.

**Proof.** First, suppose, for a contradiction, that  $f(H_i)$  is pyramidal. By Lemma 9, one of  $u_i v_i v_{i+1} u_{i+1}$  and  $u_i v_i u_{i+1} v_{i+1}$ , say  $u_i v_i v_{i+1} u_{i+1}$  bounds the unique quadrangular face of  $f(H_i)$ . If an edge between  $\{u_{i+1}, v_{i+1}\}$  and  $\{u_{i+2}, v_{i+2}\}$  crosses an edge of  $H_i$ , then exactly one of  $u_{i+2}$  and  $v_{i+2}$ , say  $u_{i+2}$ , lies in the fake face  $u_i v_i z_{\{u_i v_{i+1}, v_i u_{i+1}\}}$  of  $f(H_i)$ . In this case,  $u_i v_i z_{\{u_i v_{i+1}, v_i u_{i+1}\}}$  forms a barrier loop and  $u_{i+2}$  cannot be adjacent to  $v_{i+1}$  by Proposition 3, a contradiction. On the other hand, if  $u_{i+2} v_{i+2}$  cross an edge of  $H_i$ , then  $f(H_i \cup H_{i+2})$  has a pair of crossing edges  $e \in E(H_i)$  and  $e' \in E(H_{i+2})$ , contrary to Lemma 8.

Hence  $f(H_i)$  is tetrahedral. By Lemma 8,  $u_{i+2}$  and  $v_{i+2}$  lie in the same face of  $f(H_i)$ . If both  $u_{i+2}$  and  $v_{i+2}$  lie in the face  $u_i u_{i+1} v_{i+1}$  or  $v_i u_{i+1} v_{i+1}$  of  $f(H_i)$ , then  $f(H_i \cup H_{i+1})$  does not have a pair of crossing edges  $e \in E(H_i)$  and  $e' \in E(H_{i+1})$ , a contradiction.

Hence both  $u_{i+2}$  and  $v_{i+2}$  lie in the same face  $u_i v_i u_{i+1}$  or  $u_i v_i v_{i+1}$  of  $f(H_i)$ . Suppose that the edge  $u_{i+2} v_{i+2}$  lies inside the face  $u_i v_i u_{i+1}$  of  $f(H_i)$ . If  $u_i v_i$  crosses an edge of  $H_{i+1}$ , then the edge is  $v_{i+1} u_{i+2}$  or  $v_{i+1} v_{i+2}$ , say  $v_{i+1} u_{i+2}$  (see the left-hand side of Figure 5). Then since  $u_{i-1}$  is adjacent to both  $u_i$  and  $v_i$ , the vertex  $u_{i-1}$  lies inside the face  $u_{i+1} u_{i+2} v_{i+2}$  of  $f(H_i \cup H_{i+1})$  (see the right-hand side of Figure 5) and now a barrier loop  $v_{i+1} u_{i+2} v_{i+2}$  separating  $u_i$  and  $v_i$  prevents the path  $u_i v_{i-1} v_i$  to be placed on the sphere. Therefore,  $u_i v_i$  is not crossing in  $f(H_i \cup H_{i+1})$  and hence  $v_{i+1} u_{i+2}$  crosses  $u_i u_{i+1}$  or  $v_i u_{i+1}$ , and consequently,  $f(H_i \cup H_{i+1})$  is the 1-plane graph (A) or (B), respectively, shown in Figure 4. Similarly, if we suppose that  $u_i v_i v_{i+1}$  contains the edge  $u_{i+2} v_{i+2}$ , then we obtain (C) and (D) in Figure 4. Thus, the lemma follows. ■

In a 1-embedding  $f : X_n \rightarrow S^2$ , the 1-plane graph  $f(H_i \cup H_{i+1})$  is called a *bow* if it is one of the 1-plane graphs (A), (B), (C) and (D) shown in Figure 4.

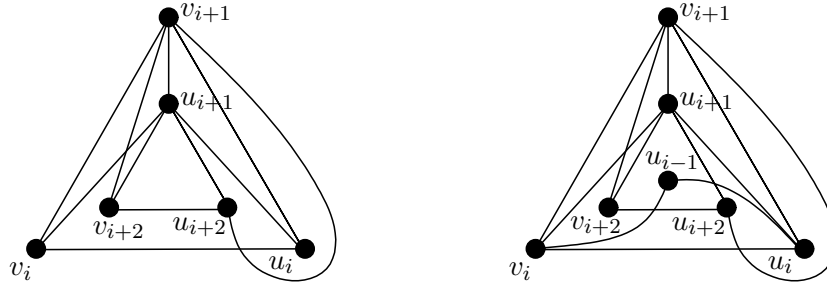


Figure 5. Configurations in the proof of Lemma 10.

**Lemma 11.** *In any 1-embedding  $f : X_n \rightarrow S^2$  with  $n \geq 5$ , if  $f(H_i)$  is tetrahedral, then either  $f(H_{i-1})$  or  $f(H_{i+1})$ , say  $f(H_{i+1})$ , is also tetrahedral and  $f(H_i \cup H_{i+1})$  is a bow.*

**Proof.** Suppose, for a contradiction, that no edge of  $H_{i-1}$  and  $H_{i+1}$  crosses an edge of  $H_i$ . This implies that  $\{u_{i-1}, v_{i-1}\}$  and  $\{u_{i+2}, v_{i+2}\}$  lie in different faces of  $f(H_i)$ . Then the path  $P = u_{i+2}u_{i+3} \cdots u_{i-1}$  of  $X_n$  has an edge  $u_s u_{s+1}$  for some  $s \in \{i+2, \dots, i-2\}$  that crosses an edge of  $H_i$ , contrary to Lemma 8. Hence one of  $f(H_{i-1})$  and  $f(H_{i+1})$  has a crossing edge that crosses an edge of  $f(H_i)$  and, by Lemma 10, the lemma follows. ■

**Lemma 12.** *In any 1-embedding  $f : X_n \rightarrow S^2$  with  $n \geq 5$ , if  $f(H_i \cup H_{i+1})$  is a bow, then each of two triangular faces and six triangular fake faces of  $f(H_i \cup H_{i+1})$  contains no vertex of  $V(X_n) \setminus \{u_i, v_i, u_{i+1}, v_{i+1}, u_{i+2}, v_{i+2}\}$ .*

**Proof.** It suffices to prove the lemma when  $f(H_i \cup H_{i+1})$  is of the form (A) in Figure 4. Suppose that the triangular face  $u_{i+1}u_{i+2}v_{i+2}$  of  $f(H_i \cup H_{i+1})$  contains a vertex  $u_s$  for some  $s \notin \{i, i+1, i+2\}$ . Similar to the proof of the above lemma, we consider a path  $u_s u_{s+1} \cdots u_i$  and obtain a contradiction. The same argument works for the other triangular face and six triangular fake faces. ■

**Lemma 13.** *In any 1-embedding  $f : X_n \rightarrow S^2$  with  $n \geq 5$ , if  $f(H_i)$  is pyramidal, then each of the four triangular fake faces of  $f(H_i)$  contains no vertex of  $V(X_n) \setminus \{u_i, v_i, u_{i+1}, v_{i+1}\}$ .*

**Proof.** The proof is analogous to the proof of Lemma 12. ■

By Lemmas 8, 9, 10, 11, 12 and 13, every 1-embedding of  $X_n$  is like a chain consisting of pyramidal  $K_4$ 's which contain no vertices in their four triangular fake faces and pairs of tetrahedral  $K_4$ 's each of which contains no vertex in its two triangular faces and six triangular fake faces (see Figure 6 which represents a 1-embedding of  $X_{14}$  with three bows). If  $H_i$  is pyramidal for any  $i \in \{0, \dots, n-1\}$ ,

then we call  $f(X_n)$  *canonical*. Any  $f(X_n)$  with  $n \geq 5$  has exactly two *big fake faces*, or two *big faces* if  $f(X_n)$  is canonical, each of whose boundaries contains exactly  $n$  vertices and crossing points in total. Observe that each of the other faces and fake faces is triangular in the 1-embedding.

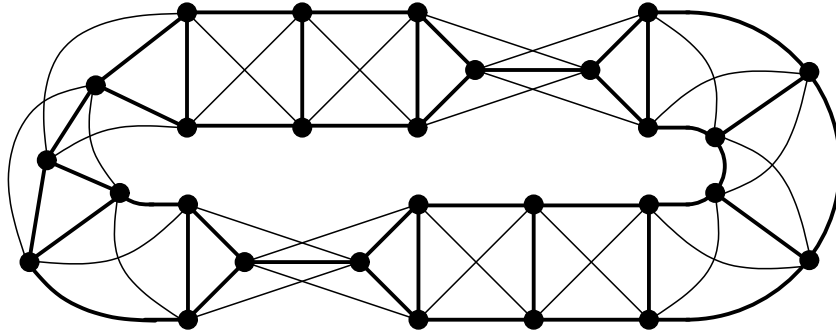


Figure 6. 1-embedding of  $X_{14}$  with three bows.

#### 4. PROOF OF THE MAIN RESULT

Let  $\theta_{i,j,k}$  denote the *theta graph* consisting of three inner disjoint paths joining two vertices having length  $i, j$  and  $k$ , respectively. Such a theta graph  $\theta_{i,j,k}$  is expected to be simple, hence if  $i \leq j \leq k$ , then  $j \geq 2$ . For our purpose, we first prove the following theorem.

**Theorem 14.** *A lexicographic product of a theta graph  $\theta_{i,j,k}$  and  $K_2$  is not 1-planar.*

**Proof.** We consider a theta graph  $\theta_{i,j,k}$  with  $i \leq j \leq k$ . First we show that the graphs  $\theta_{1,2,2} \circ K_2$  and  $\theta_{2,2,2} \circ K_2$  are not 1-planar. Since  $\theta_{1,2,2} \circ K_2$  has 8 vertices and 24 edges, if it has a 1-embedding on the sphere, then the 1-embedding is optimal. However, this graph has a vertex having odd degree, a contradiction, since every vertex of every optimal 1-planar graph has even degree (see [8]). Also it is easy to see that  $\theta_{2,2,2} \circ K_2$  contains a non-1-planar graph  $K_{4,6}$  (see [3]) as a subgraph.

Since  $j \geq 2$ , we may assume that  $j + k \geq 5$  in what follows. The theta graph  $\theta_{i,j,k}$  is a cycle  $C = x_0 x_1 \cdots x_{n-1}$  ( $n = j + k$ ) with a path of length  $i$  linking two vertices  $x_0$  and  $x_j$ . Then the graph  $G = \theta_{i,j,k} \circ K_2$  has a subgraph  $H$  isomorphic to  $X_n$  with  $n \geq 5$ . Suppose, for a contradiction, that  $G$  has a 1-embedding  $f$ . All possible  $f(H)$  are described in Section 3. In  $f(H)$  there are either two big fake faces or two big faces whose boundaries are denoted by  $L$  and  $L'$ , respectively, such that each of  $L$  and  $L'$  has exactly  $n$  points each of which is

either a vertex of  $H$  or a crossing point of  $f(H)$ . Relabel vertices of  $L$  and  $L'$  such that the vertices of  $H$  lying on  $L$  (respectively,  $L'$ ) belong to  $\{u_0, u_1, \dots, u_{n-1}\}$  (respectively,  $\{v_0, v_1, \dots, v_{n-1}\}$ ). If  $f(H_s \cup H_{s+1})$  is a bow, then the two vertices  $u_{s+1}$  and  $v_{s+1}$  are placed so that  $u_s v_s u_{s+1}$  is a triangular face of  $H$ . In the above labeling of vertices of  $H$ ,  $\{u_s, v_s\}$  corresponds to  $x_s \in V(C)$ .

In  $f(H)$  with vertices labeled as above,  $f(H_{n-1} \cup H_0)$  is not a bow. For otherwise, either  $u_1 u_0 v_1 z_{\{u_1 v_2, v_1 u_2\}}$  (when  $f(H_1)$  is pyramidal) or  $u_1 u_0 v_1 v_2$  (when  $f(H_1)$  is tetrahedral) is a barrier loop in  $f(H)$  that separates  $v_0$  and  $v_j$ . This contradicts (by Proposition 3) the fact that  $G$  has a path  $P$  joining  $v_0$  and  $v_j$  such that  $P \cap H = \{v_0, v_j\}$ .

Now we choose half edges from  $f(H)$  to form a barrier loop denoted by  $L_0$ . First, we take a part of such a barrier loop around  $v_0$  so as not to pass through  $v_0$ .

- (i) If each of  $f(H_{n-1})$  and  $f(H_0)$  is pyramidal, then  $v_{n-1} u_0 v_1$  are taken (from  $f(H_{n-1} \cup H_0)$ ).
- (ii) If one of  $f(H_{n-1})$  and  $f(H_0)$ , say  $f(H_{n-1})$ , is pyramidal and  $f(H_0 \cup H_1)$  is a bow, then  $v_{n-1} u_0 z_{\{u_0 v_1, u_1 u_2\}} u_1 v_2$  are taken (from  $f(H_{n-1} \cup H_0 \cup H_1)$ ).
- (iii) If each of  $f(H_{n-2} \cup H_{n-1})$  and  $f(H_0 \cup H_1)$  is a bow, then  $v_{n-2} v_{n-1} z_{\{u_{n-2} v_{n-1}, u_{n-1} u_0\}} u_0 z_{\{u_1 u_2, u_0 v_1\}} u_1 v_2$  are taken (from  $f(H_{n-2} \cup H_{n-1} \cup H_0 \cup H_1)$ ).

For each  $s \in \{0, \dots, n-1\}$  such that no half edge is chosen from  $f(H_s)$  so far, we do as follows.

- (iv) If  $f(H_s)$  is pyramidal, then  $v_s z_{\{v_s u_{s+1}, u_s v_{s+1}\}} v_{s+1}$  are taken.
- (v) If  $f(H_s)$  is tetrahedral and either  $f(H_{s-1} \cup H_s)$  or  $f(H_s \cup H_{s+1})$ , now say  $f(H_s \cup H_{s+1})$ , is a bow, then  $v_s z_{\{v_s v_{s+1}, u_{s+1} v_{s+2}\}} v_{s+2}$  are taken.

The above obtained  $L_0$  is clearly a barrier loop separating  $v_0$  and  $u_j$ . This contradicts (by Proposition 3) the fact that  $G$  has a path  $P$  joining  $v_0$  and  $u_j$  such that  $P \cap H = \{v_0, u_j\}$ . ■

Finally, we prove Theorem 2 to solve Conjecture 1.

**Proof of Theorem 2.** Suppose that  $G$  is not a cactus. Then, it is easy to see that  $G$  contains a simple theta graph as its subgraph. By Theorem 14, the lexicographic product of a graph  $G$  and  $K_2$  is not 1-planar and hence the theorem follows. ■

## 5. REMARKS

In the paper, we mainly discussed the way to 1-embed  $X_n = C_n \circ K_2$  for our purpose. The obtained result can be useful when considering re-1-embedding

of other 1-plane graphs since we can find canonical  $f(X_n)$ 's in some classes of 1-plane graphs; e.g., a class of optimal 1-plane graphs. However, the condition  $n \geq 5$  in the argument is necessary. As we can see, the 1-embedding of  $X_4$  in Figure 7 does not satisfy Lemma 9. As we stated before,  $X_3$  is isomorphic to  $K_6$  and hence any four vertices of  $X_3$  can induce  $K_4$ , which is  $H_i$  in our argument.

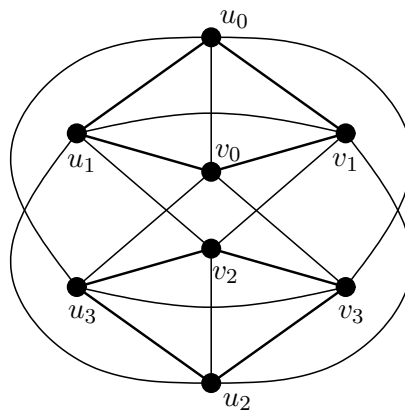


Figure 7. 1-embedding of  $X_4$  not satisfying Lemma 9.

### Acknowledgments

This work was supported by JSPS KAKENHI Grant Number 16K05250.

### REFERENCES

- [1] M.O. Albertson and B. Mohar, *Coloring vertices and faces of locally planar graphs*, Graphs Combin. **22** (2006) 289–295.  
<https://doi.org/10.1007/s00373-006-0653-4>
- [2] J. Bucko and J. Czap, *1-planar lexicographic products of graphs*, Appl. Math. Sci. **9** (2015) 5441–5449.  
<https://doi.org/10.12988/ams.2015.56439>
- [3] J. Czap and D. Hudák, *1-planarity of complete multipartite graphs*, Discrete Appl. Math. **160** (2012) 505–512.  
<https://doi.org/10.1016/j.dam.2011.11.014>
- [4] J. Czap, D. Hudák and T. Madaras, *Joins of 1-planar graphs*, Acta Math. Sin. (Engl. Ser.) **30** (2014) 1867–1876.  
<https://doi.org/10.1007/s10114-014-4017-3>
- [5] S.G. Kobourov, G. Liotta and F. Montecchiani, *An annotated bibliography on 1-planarity*, Comput. Sci. Rev. **25** (2017) 49–67.  
<https://doi.org/10.1016/j.cosrev.2017.06.002>

- [6] V.P. Korzhik, *Minimal non-1-planar graphs*, Discrete Math. **308** (2008) 1319–1327.  
<https://doi.org/10.1016/j.disc.2007.04.009>
- [7] V.P. Korzhik and B. Mohar, *Minimal obstructions for 1-immersions and hardness of 1-planarity testing*, J. Graph Theory **72** (2013) 30–71.  
<https://doi.org/10.1002/jgt.21630>
- [8] Y. Suzuki, *Re-embeddings of maximum 1-planar graphs*, SIAM J. Discrete Math. **24** (2010) 1527–1540.  
<https://doi.org/10.1137/090746835>
- [9] C. Yang and J. Xu, *Connectivity of lexicographic product and direct product of graphs*, Ars Combin. **111** (2013) 3–12.

Received 5 February 2019

Revised 21 May 2019

Accepted 21 May 2019