# DISTRIBUTION OF CONTRACTIBLE EDGES AND THE STRUCTURE OF NONCONTRACTIBLE EDGES HAVING ENDVERTICES WITH LARGE DEGREE IN A 4-CONNECTED GRAPH 

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#### Abstract

Let $G$ be a 4-connected graph $G$, and let $E_{c}(G)$ denote the set of 4contractible edges of $G$. We prove results concerning the distribution of edges in $E_{c}(G)$. Roughly speaking, we show that there exists a set $\mathcal{K}_{0}$ and a mapping $\varphi: \mathcal{K}_{0} \rightarrow E_{c}(G)$ such that $\left|\varphi^{-1}(e)\right| \leq 4$ for each $e \in E_{c}(G)$.


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## 1. Introduction

In this paper, we consider only finite undirected simple graphs with no loops and no multiple edges.

Let $G=(V(G), E(G))$ be a graph. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of $e$. For $x \in V(G), N_{G}(x)$ denotes the neighborhood of $x$ and $\operatorname{deg}_{G}(x)$ denotes the degree of $x$; thus $\operatorname{deg}_{G}(x)=\left|N_{G}(x)\right|$. For $X \subseteq V(G)$, we let $N_{G}(X)=\bigcup_{x \in X} N_{G}(x)$, and the subgraph induced by $X$ in $G$ is denoted by $G[X]$. For an integer $i \geq 0$, we let $V_{i}(G)$ denote the set of vertices $x$ of $G$ with $\operatorname{deg}_{G}(x)=i$ and we let $V_{\geq i}(G)=\bigcup_{j \geq i} V_{j}(G)$. A subset $S$ of $V(G)$ is called a cutset if $G-S$ is disconnected. A cutset with cardinality $i$ is simply referred to as an $i$-cutset. For an integer $k \geq 1$, we say that $G$ is $k$-connected if $|V(G)| \geq k+1$ and $G$ has no $(k-1)$-cutset.

Let $G$ be a 4-connected graph. For two distinct 4-cutsets $S, T$, we say that $S$ crosses $T$ if $S$ intersects with every component of $G-T$. It is easy to see that $S$ crosses $T$ if and only if $T$ crosses $S$, which is in turn equivalent to saying that $S$ intersects at least two components of $G-T$. Furthermore, we call a family of 4 -cutsets $\mathcal{S}$ cross free if no two members of $\mathcal{S}$ cross. A 4 -cutset $S$ of $G$ is said to be trivial if there exists a vertex $z$ of degree 4 such that $N_{G}(z)=S$; otherwise it is said to be nontrivial. For $e \in E(G)$, we let $G / e$ denote the graph obtained from $G$ by contracting $e$ into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that $e$ is 4-contractible or 4-noncontractible according as $G / e$ is 4-connected or not. A 4-noncontractible edge $e=a b$ is said to be trivially 4-noncontractible if there exists a vertex $z$ of degree 4 such that $z a, z b \in E(G)$. We let $E_{c}(G), E_{n}(G)$ and $E_{t n}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively. Note that if $|V(G)| \geq 6$, then $e \in E_{n}(G)$ if and only if there exists a 4 -cutset $S$ such that $V(e) \subseteq S$, and $e \in E_{t n}(G)$ if and only if there exists a trivial 4-cutset $S$ such that $V(e) \subseteq S$.

The following theorem concerning the number of 4 -contractible edges in a 4 -connected graph was proved in [2].

Theorem A. If $G$ is a 4-connected graph, then $\left|E_{c}(G)\right| \geq(1 / 68) \sum_{u \in V(G)}$ $\left(\operatorname{deg}_{G}(u)-4\right)$.

The coefficient $1 / 68$ in Theorem A seems far from best possible. The purpose of this paper is to prove two results which will be useful in refining Theorem A. Our results can be also seen as a "large-degree version" of the two structure theorems proved in [1] concerning edges not contained in triangles (see Theorems C and D below).

Throughout the rest of this paper, we let $G$ be a 4 -connected graph. Set

$$
\begin{aligned}
& \mathcal{L}=\{(S, A) \mid S \text { is a } 4 \text {-cutset, } A \text { is the union of the vertex set of } \\
&\text { some components of } G-S, \emptyset \neq A \neq V(G)-S\} \\
& \mathcal{L}_{0}=\{(S, A) \in \mathcal{L} \mid S \text { is a nontrivial 4-cutset }\}
\end{aligned}
$$

For $(S, A) \in \mathcal{L}$, we let $\bar{A}=V(G)-S-A$. Thus if $(S, A) \in \mathcal{L}$, then $(S, \bar{A}) \in \mathcal{L}$ and $N_{G}(A)-A=N_{G}(\bar{A})-\bar{A}=S$.

Let $F$ be a subset of $E_{n}(G)-E_{t n}(G)$. Let $\tilde{V}(G)$ denote the set of those vertices of $G$ which are incident with an edge in $F$, and let $\tilde{G}$ denote the spanning subgraph of $G$ with edge set $F$; that is to say, $\tilde{V}(G)=\bigcup_{e \in F} V(e)$ and $\tilde{G}=$ $(V(G), F)$. Now take $\left(S_{1}, A_{1}\right), \ldots,\left(S_{k}, A_{k}\right) \in \mathcal{L}$ so that for each $e \in F$, there exists $S_{i}$ such that $V(e) \subseteq S_{i}$. We choose $\left(S_{1}, A_{1}\right), \ldots,\left(S_{k}, A_{k}\right)$ so that $k$ is minimum and so that $\left(\left|A_{1}\right|, \ldots,\left|A_{k}\right|\right)$ is lexicographically minimum, subject to the condition that $k$ is minimum (thus if $F=\emptyset$, then $k=0$ ). Note that the
minimality of $k$ implies that for each $1 \leq i \leq k$, we have $E\left(G\left[S_{i}\right]\right) \cap F \neq \emptyset$ and hence $\left(S_{i}, A_{i}\right) \in \mathcal{L}_{0}$. Set $\mathcal{S}=\left\{S_{1}, \ldots, S_{k}\right\}$. Further set

$$
\begin{aligned}
& \mathcal{K}=\left\{(u, S, A) \mid u \in \tilde{V}(G), S \in \mathcal{S},(S, A) \in \mathcal{L}_{0},\right. \text { there exists } \\
&e \in F \text { such that } u \in V(e) \subseteq S\}, \\
& \mathcal{K}^{*}=\{(u, S, A) \in \mathcal{K} \mid \text { there is no }(v, T, B) \in \mathcal{K} \text { with } \\
&v=u \text { and }(T, B) \neq(S, A) \text { such that } B \subseteq A\} .
\end{aligned}
$$

Moreover let $\mathcal{K}_{0}$ be the set of those members $(u, S, A) \in \mathcal{K}^{*}$ which satisfy one of the following two conditions:
(1) $\operatorname{deg}_{G}(u) \geq 5$; or
(2) $\operatorname{deg}_{G}(u)=4,\left|N_{G}(u) \cap A\right|=1$ and, if we write $N_{G}(u) \cap A=\{a\}$, then $u a \in E_{c}(G)$.

We say that $F$ is admissible if the following statement is true (note that this definition implies that if $F=\emptyset$, then $F$ is admissible).

Statement B. Let $u v \in F$, and let $S$ be a 4-cutset with $u, v \in S$, and let $A$ be the vertex set of a component of $G-S$. Then there exists $e \in E_{c}(G)$ such that either $e$ is incident with $u$ or there exists $a \in N_{G}(u) \cap(S \cup A) \cap V_{4}(G)$ such that $e$ is incident with $a$.

Now we let $\tilde{E}(G)$ denote the set of those edges of a 4-connected graph $G$ which are not contained in a triangle. The following result appears as Theorem 1 in [1].

Theorem C. The set $\tilde{E}(G) \cap E_{n}(G)$ is admissible.
Let $L$ be the set of edges $e$ such that both endvertices of $e$ have degree 4. In this paper, we prove the following theorem.

Theorem 1. Let $F=E_{n}(G)-E_{t n}(G)-L$. Let $\mathcal{S}$ be as above, and suppose that $\mathcal{S}$ is cross free. Then $F$ is admissible.

Note that in the case where $F=\tilde{E}(G) \cap E_{n}(G)$, we can show that $\mathcal{S}$ is cross free (see Claim 4.1 in [1]), and this is why we do not need the assumption that $\mathcal{S}$ is cross free in Theorem C.

The following theorem appears as Theorem 2 in [1].
Theorem D. Let $\mathcal{K}_{0}$ be as above with $F=\tilde{E}(G) \cap E_{n}(G)$. Then we can assign to each $(u, S, A) \in \mathcal{K}_{0}$ a 4-contractible edge $\varphi(u, S, A)$ having the property stated in Statement B, so that for each $e \in E_{c}(G)$ there are at most two members $(u, S, A)$ of $\mathcal{K}_{0}$ such that $\varphi(u, S, A)=e$.

The following theorem is our main result.

Theorem 2. Let $\mathcal{S}$ and $\mathcal{K}_{0}$ be as above with $F=E_{n}(G)-E_{t n}(G)-L$, and suppose that $\mathcal{S}$ is cross free. Then we can assign to each $(u, S, A) \in \mathcal{K}_{0} a 4$ contractible edge $\varphi(u, S, A)$ having the property stated in Statement B, so that for each $e \in E_{c}(G)$ there are at most four members $(u, S, A)$ of $\mathcal{K}_{0}$ such that $\varphi(u, S, A)=e$.

We remark that in Theorem 2, situations in which there are three or four members $(u, S, A)$ of $\mathcal{K}_{0}$ such that $\varphi(u, S, A)=e$ are rather limited (see Claim 4.17).

Recall that Theorems 1 and 2 will be useful in refining Theorem A. The reasons are as follows. Let $k$ be a maximum value with $\left|E_{c}(G)\right| \geq k \sum_{u \in V(G)}\left(\operatorname{deg}_{G}(u)\right.$ -4 ) for a 4 -connected graph $G$. Note that we know that $1 / 68 \leq k \leq 1 / 13$, and hence assume $1 / 68 \leq k \leq 1 / 13$ throughout the rest of this argument. If $\left|V_{\geq 5}(G)\right|=0$, then the above inequality holds immediately. Thus we now assume that $\left|V_{\geq 5}(G)\right| \geq 1$. Let $\mathcal{S}$ be as above with $F=E_{n}(G)-E_{t n}(G)-L$. If $\left|E_{c}(G)\right|<\bar{k} \sum_{u \in V(G)}\left(\operatorname{deg}_{G}(u)-4\right)$, then we can show that $\mathcal{S}$ is cross free by Theorem 1 in [4]. Suppose that $\left|E_{c}(G)\right|<(1 / 28) \sum_{u \in V(G)}\left(\operatorname{deg}_{G}(u)-4\right)$. Then $\mathcal{S}$ is cross free by the above argument. Hence we can use Theorem 2, and we can show that $\left|E_{c}(G)\right| \geq(1 / 28) \sum_{u \in V(G)}\left(\operatorname{deg}_{G}(u)-4\right)$ by Theorem 2 (the verification of this statement involves lengthy calculations), which is a contradiction. Thus we have $\left|E_{c}(G)\right| \geq(1 / 28) \sum_{u \in V(G)}\left(\operatorname{deg}_{G}(u)-4\right)$. However, it is likely that the coefficient $1 / 28$ can further be improved in view of the fact that situations in which there are three or four members $(u, S, A)$ of $\mathcal{K}_{0}$ such that $\varphi(u, S, A)=e$ are limited. Thus matters concerning refinements of Theorem A will be discussed in a separate paper.

Our notation is standard, and is mostly taken from Diestel [3]. The organization of this paper is as follows. In Section 2, we introduce known results proved in [1], and prove some preliminary results. We prove Theorem 1 in Section 3, and Theorem 2 in Section 4.

## 2. Preliminaries

Throughout the rest of this paper, we let $G$ denote a 4-connected graph with $F=E_{n}(G)-E_{t n}(G)-L \neq \emptyset$ (note that in proving Theorems 1 and 2, we may clearly assume $F \neq \emptyset$ ). Thus $|V(G)| \geq 6$. Also let $\mathcal{L}, \mathcal{L}_{0}$ be as in the second paragraph following the statement of Theorem A.

In this section, we state several results which we use in the proof of Theorems 1 and 2.

### 2.1. Known results

In this subsection, we state results about the distribution of 4-contractible edges. The following lemmas follow from Lemmas 2.2 through 2.13, respectively, in [1].

Lemma 2.1. Let $(S, A),(T, B) \in \mathcal{L}_{0}$, and suppose that $S \cap T \neq \emptyset$. Then either $A \cap B \neq \emptyset$ and $\bar{A} \cap \bar{B} \neq \emptyset$, or $A \cap \bar{B} \neq \emptyset$ and $\bar{A} \cap B \neq \emptyset$.

Lemma 2.2. Let $(S, A),(T, B) \in \mathcal{L}$, and suppose that $A \cap B \neq \emptyset$ and $\bar{A} \cap \bar{B} \neq \emptyset$. Then $((S \cap T) \cup(S \cap B) \cup(A \cap T), A \cap B) \in \mathcal{L}$ and $((S \cap T) \cup(S \cap \bar{B}) \cup(\bar{A} \cap T)$, $\bar{A} \cap \bar{B}) \in \mathcal{L}$.

Lemma 2.3. Let $(S, A) \in \mathcal{L}$.
(i) If $W \subseteq S$ and $|W| \leq|A|$, then $\left|N_{G}(W) \cap A\right| \geq|W|$. Further if $|W|<|A|$ and $\left|N_{G}(W) \cap A\right|=|W|$, then $\left((S-W) \cup\left(N_{G}(W) \cap A\right), A-\left(N_{G}(W) \cap A\right)\right) \in \mathcal{L}$.
(ii) If $x \in S$, then $N_{G}(x) \cap A \neq \emptyset$. Further if $(S, A) \in \mathcal{L}_{0}$ and $\left|N_{G}(x) \cap A\right|=1$, then $\left((S-\{x\}) \cup\left(N_{G}(x) \cap A\right), A-\left(N_{G}(x) \cap A\right)\right) \in \mathcal{L}$.

Lemma 2.4. Let $a b \in E(G)$ with $\operatorname{deg}_{G}(a)=\operatorname{deg}_{G}(b)=4$. Then $N_{G}(a)-\{b\} \neq$ $N_{G}(b)-\{a\}$.

Lemma 2.5. Let $u, a, b, w$ be four distinct vertices with $u a, u b, a b, a w, b w \in E(G)$ and $\operatorname{deg}_{G}(a)=\operatorname{deg}_{G}(b)=4$, and write $N_{G}(a)=\{u, b, w, x\}$ and $N_{G}(b)=\{u, a$, $w, y\}$. Then $x \neq y$, and we have $a x, b y \in E_{c}(G) \cup E_{t n}(G)$.

Lemma 2.6. Under the notation of Lemma 2.5, suppose that $\operatorname{deg}_{G}(u), \operatorname{deg}_{G}(w)$ $\geq 5$. Then $a x, b y \in E_{c}(G)$.

Lemma 2.7. Under the notation of Lemma 2.5, suppose that $\operatorname{deg}_{G}(u) \geq 5$ and $\operatorname{deg}_{G}(w)=4$. Then one of the following holds:
(i) $x w \notin E(G)$ and $a x \in E_{c}(G)$; or
(ii) $y w \notin E(G)$ and $b y \in E_{c}(G)$.

Lemma 2.8. Let $(P, X) \in \mathcal{L}_{0}$ and $u \in P$. Suppose that $X$ is minimal, subject to the condition that $u \in P$ (i.e., there is no $(R, Z) \in \mathcal{L}_{0}$ with $(P, X) \neq(R, Z)$ such that $u \in R$ and $Z \subseteq X)$. Then $u a \in E_{c}(G) \cup E_{t n}(G)$ for each $a \in N_{G}(u) \cap X$.

Lemma 2.9. Let $(R, Z) \in \mathcal{L}_{0}$ and $a \in R$. Suppose that $\left|N_{G}(a) \cap Z\right|=1$, and write $N_{G}(a) \cap Z=\{x\}$. Then ax $\in E_{c}(G) \cup E_{t n}(G)$.

Lemma 2.10. Let $u, a, b$ be three distinct vertices with $u a, u b, a b \in E(G)$ and $\operatorname{deg}_{G}(a)=4$, and write $N_{G}(a)=\{u, b, x, y\}$. Suppose that there exists $(R, Z) \in$ $\mathcal{L}_{0}$ such that $u, a \in R, b, y \in Z$ and $x \in \bar{Z}$. Suppose further that $Z$ is minimal, subject to the condition that $u, a \in R$ and $b \in Z$. Then the following hold.
(i) $x y \notin E(G)$.
(ii) $a x \in E_{c}(G) \cup E_{t n}(G)$.
(iii) $a y \in E_{c}(G) \cup E_{t n}(G)$.

Lemma 2.11. Under the notation of Lemma 2.10, suppose that $\operatorname{deg}_{G}(b) \geq 5$. Then $a x \in E_{c}(G)$ or $a y \in E_{c}(G)$.

Lemma 2.12. Under the notation of Lemma 2.10, suppose that $\operatorname{deg}_{G}(b), \operatorname{deg}_{G}(u)$ $\geq 5$. Then ax, ay $\in E_{c}(G)$.

### 2.2. Vertices not contained in $\tilde{V}(G)$

Recall that $F=E_{n}(G)-E_{t n}(G)-L$ and $\tilde{V}(G)=\bigcup_{e \in F} V(e)$. In this subsection, we prove results concerning conditions for a vertex not to belong to $\tilde{V}(G)$.
Lemma 2.13. Under the notation of Lemma 2.5, $a, b \notin \tilde{V}(G)$.
Proof. In view of the symmetry of the roles of $a$ and $b$, it suffices to prove $a \notin \tilde{V}(G)$. Suppose that $a \in \tilde{V}(G)$. Then there exists $e \in F$ such that $e$ is incident with $a$. Since $a u, a w \in E_{t n}(G)$ and $a b \in L, e \neq a u, a b, a w$. Hence $e=a x$. By Lemma 2.5, we get $e \in E_{c}(G) \cup E_{t n}(G)$, a contradiction.

Lemma 2.14. Under the notation of Lemma 2.10, suppose that $\operatorname{deg}_{G}(u)=4$ or $\operatorname{deg}_{G}(b)=4$. Then $a \notin \tilde{V}(G)$.

Proof. Suppose that $a \in \tilde{V}(G)$. Then there exists $e \in F$ such that $e$ is incident with $a$. Since $a u, a b \in E_{t n}(G) \cup L, e \neq a u, a b$. Consequently $e=a x$ or $a y$, which contradicts Lemma 2.10(ii) or (iii).

## 3. Proof of Theorem 1

In the rest of this paper, we establish Theorems 1 and 2 by proving several claims. The proofs of most of the claims in this paper are quite similar to the proofs of the claims in [1] having virtually the same statements. However, considering that we are dealing with $E_{n}(G)-E_{t n}(G)-L$ instead of $\tilde{E}(G) \cap E_{n}(G)$, we have decided to include the details of the proofs in this paper. In this section, we prove Theorem 1.

### 3.1. Neighborhood of a vertex of degree 5

In this subsection, we prove that Statement B is true if $\operatorname{deg}_{G}(u) \geq 5$. Specifically, we prove the following proposition in a series of claims.

Proposition 3.1. Let $(P, X) \in \mathcal{L}_{0}$ and $u \in P$, and suppose that $\operatorname{deg}_{G}(u) \geq 5$. Then one of the following holds:
(1) there exists $a \in N_{G}(u) \cap X$ such that $u a \in E_{c}(G)$; or
(2) there exists $a \in N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$ for which there exists $e \in E_{c}(G)$ such that $e$ is incident with $a$.

Note that Proposition 3.1 implies that in Theorem 1, the assumption that $\mathcal{S}$ is cross free is not necessary for vertices $u$ with $\operatorname{deg}_{G}(u) \geq 5$. Throughout this subsection, let $(P, X), u$ be as in Proposition 3.1. We may assume that $X$ is minimal, subject to the condition that $u \in P$ (i.e., there is no $(R, Z) \in \mathcal{L}_{0}$ with $(R, Z) \neq(P, X)$ such that $u \in R$ and $Z \subseteq X)$.

The following four claims are virtually the same as Claims 3.2 through 3.5 in [1].

Claim 3.2. Suppose that there exists an edge e joining a vertex in $N_{G}(u) \cap X \cap$ $V_{4}(G)$ and a vertex $N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$. Suppose that $e \in E_{n}(G)$, and write $e=a b$. Then $a$ or $b$, say $a$, satisfies the following conditions.
(i) If we write $N_{G}(a)=\{u, b, x, y\}$, then $x y \notin E(G)$.
(ii) $a \notin \tilde{V}(G)$.
(iii) There exists $e^{\prime} \in E_{c}(G)$ such that $e^{\prime}$ is incident with a.

Proof. If $a b \in E_{t n}(G)$, then there exists $w \in V_{4}(G)$ such that $w a, w b \in E(G)$, and hence the desired conclusions follows from Lemmas 2.7 and 2.13. Thus we may assume that $a b \in E_{n}(G)-E_{t n}(G)$. Then there exists $(R, Z) \in \mathcal{L}_{0}$ with $a, b \in R$. We first show that $u \notin R$. Suppose that $u \in R$. Then by Lemma 2.1, we may assume $X \cap Z \neq \emptyset$ and $\bar{X} \cap \bar{Z} \neq \emptyset$. Since $a, b \in(P \cup X) \cap R$, it follows from Lemma 2.2 that $((P \cap R) \cup(P \cap Z) \cup(X \cap R), X \cap Z) \in \mathcal{L}_{0}$, which contradicts the minimality of $X$. Thus $u \notin R$. We may assume $u \in Z$. We may also assume that we have chosen $(R, Z)$ so that $Z$ is minimal, subject to the condition that $a, b \in R$ and $u \in Z$. By Lemma 2.3(i), we have $N_{G}(a) \cap Z \neq\{u\}$ or $N_{G}(b) \cap Z \neq\{u\}$. We may assume $N_{G}(a) \cap Z \neq\{u\}$. Since $N_{G}(a) \cap \bar{Z} \neq \emptyset$ by Lemma 2.3(ii), we have $\left|N_{G}(a) \cap Z\right|=2$ and $\left|N_{G}(a) \cap \bar{Z}\right|=1$. Write $N_{G}(a) \cap Z=\{u, y\}$ and $N_{G}(a) \cap \bar{Z}=\{x\}$. Then $b, a, u, x, y$ satisfy the assumptions of Lemmas 2.10, 2.11 and 2.14 with the roles of $b$ and $u$ replaced by each other. Consequently the desired conclusions follow from (i) of Lemma 2.10 and Lemmas 2.11 and 2.14.

Claim 3.3. Let $a \in X$, and suppose that $u a \in E_{n}(G)$. Then $u a \in E_{t n}(G)$.
Proof. This follows from Lemma 2.8.
Claim 3.4. Suppose that each edge joining $u$ and a vertex in $X$ is 4-noncontractible, and that there is no edge which joins a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$. Then $N_{G}(u) \cap X \cap V_{4}(G)=\emptyset$.

Proof. Suppose that $N_{G}(u) \cap X \cap V_{4}(G) \neq \emptyset$, and take $a \in N_{G}(u) \cap X \cap V_{4}(G)$. We have $u a \in E_{t n}(G)$ by Claim 3.3. Hence there exists $b \in V_{4}(G)$ such that $u b, a b \in$ $E(G)$. From $a \in X$ and $a b \in E(G)$, it follows that $b \in P \cup X$. Thus $a b$ is an edge joining a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$, a contradiction.

Claim 3.5. Suppose that each edge joining $u$ and a vertex in $X$ is 4-noncontractible, and that there is no edge which joins a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$. Then there exists $a \in N_{G}(u) \cap P \cap V_{4}(G)$ and $b \in N_{G}(u) \cap X$ such that $a b \in E(G),\left|N_{G}(a) \cap X\right|=2$ and $\left|N_{G}(a) \cap \bar{X}\right|=1$.
Proof. Take $z \in N_{G}(u) \cap X$. Then $u z \in E_{t n}(G)$ by Claim 3.3, and hence there exists $a_{z} \in V_{4}(G)$ such that $a_{z} u, a_{z} z \in E(G)$. Since $N_{G}(u) \cap X \cap V_{4}(G)=\emptyset$ by Claim 3.4, $a_{z} \in P$. Since $\operatorname{deg}_{G}\left(a_{z}\right)=4$ and $u \in N_{G}\left(a_{z}\right) \cap P,\left|N_{G}\left(a_{z}\right) \cap X\right|+$ $\left|N_{G}\left(a_{z}\right) \cap \bar{X}\right| \leq 3$, and hence it follows from Lemma 2.3(ii) that $1 \leq \mid N_{G}\left(a_{z}\right) \cap$ $X \mid \leq 2$. Now by way of contradiction, suppose that the claim is false. Then $\left|N_{G}\left(a_{z}\right) \cap X\right|=1$, i.e., $N_{G}\left(a_{z}\right) \cap X=\{z\}$. Since $z \in N_{G}(u) \cap X$ is arbitrary, this means that $a_{y} \neq a_{z}$ for any $y, z \in N_{G}(u) \cap X$ with $y \neq z$ and if we set $W=\left\{a_{z} \mid z \in N_{G}(u) \cap X\right\}$, then we have $|W|=\left|N_{G}(u) \cap X\right|$ and $N_{G}(\{u\} \cup W) \cap$ $X=N_{G}(u) \cap X$, and hence $|N(\{u\} \cup W) \cap X|=|W|=|\{u\} \cup W|-1$. In view of Lemma 2.3(i), this implies $|\{u\} \cup W| \geq|X|+1$, i.e., $|W| \geq|X|$. Again fix $z \in N_{G}(u) \cap X$. Since $N_{G}\left(a_{y}\right) \cap X=\{y\}$ for each $y \in\left(N_{G}(u) \cap X\right)-\{z\}, N_{G}(z) \subseteq$ $\left(P-\left(W-\left\{a_{z}\right\}\right)\right) \cup(X-\{z\})$. Consequently $\operatorname{deg}_{G}(z) \leq|P|-|W|+|X| \leq|P|=4$, which implies $z \in N_{G}(u) \cap X \cap V_{4}(G)$. But this contradicts Claim 3.4, completing the proof.

The following claim corresponds to Claim 3.6 in [1].
Claim 3.6. Suppose that each edge joining $u$ and a vertex in $X$ is 4-noncontractible, and that there is no edge which joins a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$. Further let $a, b$ be as in Claim 3.5, and write $N_{G}(a) \cap X=\{b, y\}$ and $N_{G}(a) \cap \bar{X}=\{x\}$. Then $x y \notin E(G)$, and ax, ay $\in E_{c}(G)$.
Proof. Note that $\operatorname{deg}_{G}(b) \geq 5$ by Claim 3.4, and $\operatorname{deg}_{G}(u) \geq 5$ by the assumption of Proposition 3.1. Thus the desired conclusions follows from (i) of Lemma 2.10 and Lemma 2.12.

Proposition 3.1 now follows from Claims 3.2 and 3.6.

### 3.2. Non-crossing 4-cutsets

In this subsection, we complete the proof of Theorem 1. Throughout the rest of this paper, we let $\mathcal{S}, \mathcal{K}, \mathcal{K}^{*}$ and $\mathcal{K}_{0}$ be as in the paragraph preceding Statement B with $F=E_{n}(G)-E_{t n}(G)-L$, and suppose that $\mathcal{S}$ is cross free.

The following claim immediately follows from the definition of $\mathcal{K}^{*}$.

Claim 3.7. Let $u \in \tilde{V}(G)$. Then for each $(u, S, A) \in \mathcal{K}$, there exists a member $(v, T, B)$ of $\mathcal{K}^{*}$ with $v=u$ and $B \subseteq A$. In particular, there exist at least two members $(v, T, B)$ of $\mathcal{K}^{*}$ with $v=u$.

The following claim is virtually the same as Claim 4.3 in [1].
Claim 3.8. Let $(u, S, A),(v, T, B) \in \mathcal{K}^{*}$ with $u=v$ and $(S, A) \neq(T, B)$. Then $(S \cup A) \cap B=A \cap(T \cup B)=\emptyset$.

Proof. If $S=T$, the desired conclusion clearly holds. Thus we may assume that $S \neq T$. Since $\mathcal{S}$ is cross free, we have that $S \cap \bar{B}=T \cap \bar{A}=\emptyset, S \cap B=T \cap \bar{A}=\emptyset$, $S \cap \bar{B}=T \cap A=\emptyset$, or $S \cap B=T \cap A=\emptyset$. Suppose that $S \cap \bar{B}=T \cap \bar{A}=\emptyset$. Then since $S \neq T$, we have $A \cap T \neq \emptyset$ and $|(S \cap T) \cup(\bar{A} \cap T) \cup(S \cap \bar{B})|=|T|-|A \cap T|<4$, and hence $\bar{A} \cap \bar{B}=\emptyset$. Since $S \cap \bar{B}=\emptyset$ and $A \cap T \neq \emptyset$, this implies $\bar{B}$ is a proper subset of $A$. But since $(u, T, \bar{B}) \in \mathcal{K}$ and $(u, S, A) \in \mathcal{K}^{*}$, this contradicts the definition of $\mathcal{K}^{*}$. If $S \cap B=T \cap \bar{A}=\emptyset$ or $S \cap \bar{B}=T \cap A=\emptyset$, then we obtain $B \subseteq A$ or $A \subseteq B$, respectively, and hence we similarly get a contradiction. Thus $S \cap B=T \cap A=\emptyset$. Since $S \neq T$, this also implies $A \cap B=\emptyset$, as desired.

Recall that $\tilde{G}=(V(G), F)$. The following claim corresponds to Claim 4.4 in [1].

Claim 3.9. Let $u \in \tilde{V}(G)$. Then the following hold.
(i) There exists a member $(v, T, B)$ of $\mathcal{K}_{0}$ with $v=u$.
(ii) Suppose that $\operatorname{deg}_{G}(u) \geq 5$, or $\operatorname{deg}_{\tilde{G}}(u) \geq 2$, or there exist three members $(v, T, B)$ of $\mathcal{K}^{*}$ with $v=u$. Then for each $(u, S, A) \in \mathcal{K}^{*}$, we have $(u, S, A) \in$ $\mathcal{K}_{0}$. In particular, if $\operatorname{deg}_{G}(u)=4$ and $\operatorname{deg}_{\tilde{G}}(u) \geq 2$, then $\operatorname{deg}_{\tilde{G}}(u)=2$ and there exist precisely two members $(v, T, B)$ of $\mathcal{K}_{0}$ with $v=u$.

Proof. If $\operatorname{deg}_{G}(u) \geq 5$, the desired conclusion immediately follows from Claim 3.7 and the definition of $\mathcal{K}_{0}$. Thus we may assume that $\operatorname{deg}_{G}(u)=4$. We first prove (ii). Thus let $u$ be as in (ii) with $\operatorname{deg}_{G}(u)=4$. Then by Lemma 2.3(ii) and Claim 3.8, it follows that $\left|N_{G}(u) \cap A\right|=1$ for each $(u, S, A) \in \mathcal{K}^{*}$, and that for each $a \in N_{G}(u)-N_{\tilde{G}}(u)$, there exists $(u, S, A) \in \mathcal{K}^{*}$ such that $a \in A$. Again by Claim 3.8, this implies that for each $(u, S, A) \in \mathcal{K}^{*}, N_{G}(u) \cap S=N_{\tilde{G}}(u) \cap S$. Note that this also implies that if $\operatorname{deg}_{\tilde{G}}(u) \geq 2$, then we have $\operatorname{deg}_{\tilde{G}}(u)=2$ and there exist precisely two members $(v, T, B)$ of $\mathcal{K}^{*}$ with $v=u$. Now let $(u, S, A) \in \mathcal{K}^{*}$, and write $N_{G}(u) \cap A=\{a\}$. To complete the proof of (ii), it suffices to show that $(u, S, A) \in \mathcal{K}_{0}$. Suppose that $(u, S, A) \notin \mathcal{K}_{0}$. Then $u a \in E_{n}(G)$, and hence $u a \in E_{t n}(G)$ by Lemma 2.9, which implies that there exists $c \in V_{4}(G)$ such that $c u, c a \in E(G)$. Since $N_{G}(u) \cap A=\{a\}$, this forces $c \in S$. But since $u c \in L$, $c \notin N_{\tilde{G}}(u)$, which contradicts the earlier assertion that $N_{G}(u) \cap S=N_{\tilde{G}}(u) \cap S$. Thus (ii) is proved.

We now prove (i). We may assume that there exists $(u, S, A) \in \mathcal{K}^{*}$ such that $(u, S, A) \notin \mathcal{K}_{0}$. Then arguing as above, we see that $\left|N_{G}(u) \cap(S \cup A)\right| \geq 3$ (note that if $\left|N_{G}(u) \cap A\right| \geq 2$, we clearly have $\left.\left|N_{G}(u) \cap(S \cup A)\right| \geq 3\right)$. Take $(u, T, B) \in \mathcal{K}^{*}$ with $B \subseteq \bar{A}$. Then $\left|N_{G}(u) \cap B\right|=1$. Write $N_{G}(u) \cap B=\{b\}$. Suppose that $(u, T, B) \notin \mathcal{K}_{0}$. Then there exists $c^{\prime} \in V_{4}(G)$ such that $c^{\prime} u, c^{\prime} b \in E(G)$. This in turn implies $\left|N_{G}(u) \cap A\right|=1$. Write $N_{G}(u) \cap A=\{a\}$. Then there exists $c \in V_{4}(G)$ such that $c u, c a \in E(G)$. Since $\operatorname{deg}_{G}(u)=4, \operatorname{deg}_{\tilde{G}}(u) \geq 1$ and $a b \notin E(G)$, this forces $c=c^{\prime}$. But then applying Lemma 2.13 with $a$ and $b$ replaced by $u$ and $c$, we obtain $u \notin \tilde{V}(G)$, which contradicts the assumption that $u \in \tilde{V}(G)$. Thus (i) is also proved.

We are now in a position to complete the proof of Theorem 1.
Let $u, S, A$ be as in Statement B. Then $(S, A) \in \mathcal{L}_{0}$. Hence if $\operatorname{deg}_{G}(u) \geq 5$, then the desired conclusion follows from Proposition 3.1. Thus we may assume $\operatorname{deg}_{G}(u)=4$. But then from Claim 3.9(i) and the definition of $\mathcal{K}_{0}$, we see that there exists $e \in E_{c}(G)$ such that $e$ is incident with $u$. Consequently $F=E_{n}(G)-$ $E_{t n}(G)-L$ is admissible, as desired.

## 4. Proof of Theorem 2

In this section, we prove Theorem 2. We continue with the notation of Subsection 3.2. In particular, we suppose that $\mathcal{S}$ is cross free, which is the assumption of Theorem 2.

### 4.1. Definition of $\lambda(u, S, A), \alpha(u, S, A)$ and $\varphi(u, S, A)$

In this subsection, to each $(u, S, A) \in \mathcal{K}_{0}$, we assign an edge $\lambda(u, S, A)$, and an endvertex $\alpha(u, S, A)$ of $\lambda(u, S, A)$, and a 4-contractible edge $\varphi(u, S, A)$ incident with $\alpha(u, S, A)$. The following claim corresponds to Claim 5.1 in [1].
Claim 4.1. Let $(u, S, A) \in \mathcal{K}_{0}$, and set $W=\left\{z \in S-\{u\}-N_{\tilde{G}}(u)| | N_{G}(z) \cap A \mid\right.$ $=1\}$. Then $\left((S-W) \cup\left(N_{G}(W) \cap A\right), A-\left(N_{G}(W) \cap A\right)\right) \in \mathcal{L}_{0}$.
Proof. By the definition of $\mathcal{K}$, there exists $e \in F$ such that $u \in V(e) \subseteq S$. Hence $W \subseteq S-V(e)$, which implies $|W| \leq 2$. On the other hand, since $(S, A) \in \mathcal{L}_{0}$, $|A| \geq 2$. Thus $|W| \leq|A|$. Suppose that $|W|=|A|$. Then $|W|=|A|=2$. By Lemma 2.3(i), $N_{G}(\{x, z\}) \cap A=A$ for each $x \in V(e)$ and $z \in W$. Since we also have $N_{G}(W) \cap A=A$ by Lemma 2.3(i) and since $\left|N_{G}(z) \cap A\right|=1$ for each $z \in W$, this means that $N_{G}(x) \cap A=A$ for each $x \in V(e)$. Consequently $\operatorname{deg}_{G}(a)=4$ and $V(e) \subseteq N_{G}(a)$ for each $a \in A$, which implies $e \in E_{t n}(G)$, a contradiction. Thus $|W|<|A|$. Therefore it follows from Lemma 2.3(i) that $\left((S-W) \cup\left(N_{G}(W) \cap A\right), A-\left(N_{G}(W) \cap A\right)\right) \in \mathcal{L}$, which implies the desired conclusion because $V(e) \subseteq S-W$.

Now let $(u, S, A) \in \mathcal{K}_{0}$, and let $W$ be as in Claim 4.1. We let $\left(P_{u, S, A}, X_{u, S, A}\right)$ be a member of $\mathcal{L}_{0}$ with $u \in P_{u, S, A}$ and $X_{u, S, A} \subseteq A-\left(N_{G}(W) \cap A\right)$ such that $X_{u, S, A}$ is minimal, i.e., there is no $(R, Z) \in \mathcal{L}_{0}$ with $(R, Z) \neq\left(P_{u, S, A}, X_{u, S, A}\right)$ such that $u \in R$ and $Z \subseteq X_{u, S, A}$. We remark that we do not require that there should exist an edge $e \in E_{n}(G)$ with $u \in V(e) \subseteq P_{u, S, A}$. The following claim immediately follows from the definition of ( $P_{u, S, A}, X_{u, S, A}$ ).

Claim 4.2. Let $(u, S, A) \in \mathcal{K}_{0}$. Let $z \in S-\{u\}-N_{\tilde{G}}(u)$ and suppose that $\left|N_{G}(z) \cap A\right|=1$. Then $z \notin P_{u, S, A}$.

Let again $(u, S, A) \in \mathcal{K}_{0}$, and let $(P, X)=\left(P_{u, S, A}, X_{u, S, A}\right)$ be as above. We define the type of $(u, S, A)$ as follows: $(u, S, A)$ is of type 1 if there exists a 4 contractible edge joining $u$ and a vertex in $X ;(u, S, A)$ is of type 2 if it is not of type 1 and there exists a 4-contractible edge joining a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G) ;(u, S, A)$ is of type 3 if it is not of type 1 or 2 but there exists an edge joining a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G) ;(u, S, A)$ is of type 4 if it is not of type $i$ for any $i=1,2,3$. We let $\mathcal{K}_{i}$ denote the set of those members of $\mathcal{K}_{0}$ which are the type $i$ ( $i=1,2,3,4$ ). The following claim, which will be used implicitly throughout the rest of this paper, is virtually the same as Claim 5.3 in [1].

Claim 4.3. Let $(u, S, A) \in \mathcal{K}_{0}-\mathcal{K}_{1}$. Then $\operatorname{deg}_{G}(u) \geq 5$.
Proof. Suppose that $\operatorname{deg}_{G}(u)=4$. Then by the definition of $\mathcal{K}_{0},\left|N_{G}(u) \cap A\right|=1$ and, if we write $N_{G}(u) \cap A=\{a\}$, then $u a \in E_{c}(G)$. By Lemma 2.3(ii), $a \in X$. Consequently $(u, S, A) \in \mathcal{K}_{1}$ by definition, which contradicts the assumption that $(u, S, A) \in \mathcal{K}_{0}-\mathcal{K}_{1}$.

We first define $\lambda(u, S, A)$. If $(u, S, A) \in \mathcal{K}_{1}$, let $\lambda(u, S, A)$ be a 4 -contractible edge joining $u$ and a vertex in $X$; if $(u, S, A) \in \mathcal{K}_{2}$, let $\lambda(u, S, A)$ be a 4 contractible edge joining a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap$ $(P \cup X) \cap V_{4}(G)$; if $(u, S, A) \in \mathcal{K}_{3}$, let $\lambda(u, S, A)$ be an edge joining a vertex in $N_{G}(u) \cap X \cap V_{4}(G)$ and a vertex in $N_{G}(u) \cap(P \cup X) \cap V_{4}(G)$; if $(u, S, A) \in \mathcal{K}_{4}$, let $\lambda(u, S, A)=a b$ where $a, b$ are as in Claim 3.5. The following claim follows from the definition of $\lambda(u, S, A)$.

Claim 4.4. Let $2 \leq i, j \leq 4$ with $i \neq j$, and let $\left(u_{1}, S_{1}, A_{1}\right) \in \mathcal{K}_{i}$ and $\left(u_{2}, S_{2}, A_{2}\right)$ $\in \mathcal{K}_{j}$. Then $\lambda\left(u_{1}, S_{1}, A_{1}\right) \neq \lambda\left(u_{2}, S_{2}, A_{2}\right)$.

The following claims are virtually the same as Claims 5.5 and 5.6 , respectively, in [1].

Claim 4.5. Let $\left(u_{1}, S_{1}, A_{1}\right),\left(u_{2}, S_{2}, A_{2}\right) \in \mathcal{K}_{0}$ with $u_{1}=u_{2}$ and $\left(S_{1}, A_{1}\right) \neq$ $\left(S_{2}, A_{2}\right)$. Then $\lambda\left(u_{1}, S_{1}, A_{1}\right) \neq \lambda\left(u_{2}, S_{2}, A_{2}\right)$.

Proof. By Claim 3.8, $A_{1} \cap A_{2}=\emptyset$. Hence $X_{u_{1}, S_{1}, A_{1}} \cap X_{u_{2}, S_{2}, A_{2}} \subseteq A_{1} \cap A_{2}=\emptyset$. Since at least one of the endvertices of $\lambda\left(u_{j}, S_{j}, A_{j}\right)$ is in $X_{u_{j}, S_{j}, A_{j}}$, this implies $\lambda\left(u_{1}, S_{1}, A_{1}\right) \neq \lambda\left(u_{2}, S_{2}, A_{2}\right)$.

Claim 4.6. Let e be an edge joining two vertices of degree 4. Then there exist at most two members $(u, S, A)$ of $\mathcal{K}_{2} \cup \mathcal{K}_{3}$ for which $\lambda(u, S, A)=e$.
Proof. Suppose that there exist three members $\left(u_{j}, S_{j}, A_{j}\right)(1 \leq j \leq 3)$ of $\mathcal{K}_{2} \cup \mathcal{K}_{3}$ such that $\lambda\left(u_{j}, S_{j}, A_{j}\right)=e$. By Claim 4.5, the $u_{j}$ are all distinct. But this contradicts Lemma 2.4.

We prove two more claims concerning properties of $\lambda(u, S, A)$. The following claim corresponds to Claim 6.1 in [1].

Claim 4.7. Let $(u, S, A),(v, T, B) \in \mathcal{K}_{0}-\mathcal{K}_{1}$ with $u=v$ and $(S, A) \neq(T, B)$. Then $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_{4}(G)=\emptyset$.

Proof. Suppose that $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_{4}(G) \neq \emptyset$, and let $a \in$ $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_{4}(G)$, and let $(P, X)=\left(P_{u, S, A}, X_{u, S, A}\right)$. Then $a \in P \cup X \subseteq S \cup A$. Similarly $a \in T \cup B$. Hence $a \in(S \cup A) \cap(T \cap B) \subseteq S \cap T$ by Claim 3.8. Since $\operatorname{deg}_{G}(a)=4$ and $u \in N_{G}(a) \cap S \cap T,\left|N_{G}(a) \cap(A \cup B)\right| \leq 3$. Since $A \cap B=\emptyset$ by Claim 3.8, this together with Lemma 2.3(ii) implies that we have $\left|N_{G}(a) \cap A\right|=1$ or $\left|N_{G}(a) \cap B\right|=1$. We may assume $\left|N_{G}(a) \cap A\right|=1$. If $(u, S, A) \in \mathcal{K}_{4}$, then by the definition of $\lambda(u, S, A), a$ coincides with the vertex $a$ in Claim 3.5, and hence $\left|N_{G}(a) \cap A\right| \geq\left|N_{G}(a) \cap X\right|=2$ by Claim 3.5, a contradiction. Thus $(u, S, A) \in \mathcal{K}_{2} \cup \mathcal{K}_{3}$. Consequently $u a \in E_{\text {tn }}(G)$ by the definition of types 2 and 3, and hence $a \notin N_{\tilde{G}}(u)$. By Claim 4.2, this implies $a \notin P$, which contradicts the fact that $a \in(P \cup X) \cap S \subseteq P$.

The following claim is virtually the same as Claim 6.2 in [1].
Claim 4.8. Let $(u, S, A),(v, T, B) \in \mathcal{K}_{4}$ with $(u, S, A) \neq(v, T, B)$. Then $\lambda(u, S$, $A) \neq \lambda(v, T, B)$.
Proof. Suppose that $\lambda(u, S, A)=\lambda(v, T, B)$. Let $(P, X)=\left(P_{u, S, A}, X_{u, S, A}\right)$, and let $a, b, x, y$ be as in Claims 3.5 and 3.6. Then $\lambda(u, S, A)=\lambda(v, T, B)=a b$, and hence $v \in N_{G}(a) \cap N_{G}(b)$. In particular, $v \in N_{G}(a)-\{b\}=\{u, x, y\}$. Since we get $x b \notin E(G)$ from $x \in \bar{X}$ and $b \in X, v \neq x$. We also have $v \neq u$ by Claim 4.5. Thus $v=y$, and hence $y, a \in P_{v, T, B}$. Consequently $y a \in E_{n}(G)$, which contradicts Claim 3.6.

We now define $\alpha(u, S, A)$. If $(u, S, A) \in \mathcal{K}_{1}$, let $\alpha(u, S, A)=u$. Now assume $(u, S, A) \in \mathcal{K}_{2}$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$. If $\lambda(u, S, A)$ has an endvertex in $P$ and there is no $(w, R, Z) \in \mathcal{K}_{2}$ with $(w, R, Z) \neq$ $(u, S, A)$ such that $\lambda(w, R, Z)=\lambda(u, S, A)$, then we let $\alpha(u, S, A)$ be the endvertex
of $\lambda(u, S, A)$ in $X$. Next assume $(u, S, A) \in \mathcal{K}_{3}$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$ which satisfies (ii) and (iii) of Claim 3.2. If there is no $(w, R, Z) \in \mathcal{K}_{3}$ with $(w, R, Z) \neq(u, S, A)$ such that $\lambda(w, R, Z)=\lambda(u, S, A)$, then we choose $\alpha(u, S, A)$ so that it also satisfies (i) of Claim 3.2. Finally, if $(u, S, A) \in \mathcal{K}_{4}$, let $\alpha(u, S, A)=a$, where $a$ is as in Claim 3.5. Note that if $\left(u_{1}, S_{1}, A_{1}\right),\left(u_{2}, S_{2}, A_{2}\right) \in \mathcal{K}_{3}$ with $\left(u_{1}, S_{1}, A_{1}\right) \neq\left(u_{2}, S_{2}, A_{2}\right)$ and $\lambda\left(u_{1}, S_{1}, A_{1}\right)=$ $\lambda\left(u_{2}, S_{2}, A_{2}\right)$, then $u_{1} \neq u_{2}$ by Claim 4.5, and hence it follows from Lemmas 2.6 and 2.13 that both endvertices of $\lambda\left(u_{1}, S_{1}, A_{1}\right)$ satisfy (ii) and (iii) of Claim 3.2. Thus in view of Claim 4.6, we can define $\alpha(u, S, A)$ so that the following claim holds.

Claim 4.9. Let $\left(u_{1}, S_{1}, A_{1}\right),\left(u_{2}, S_{2}, A_{2}\right) \in \mathcal{K}_{2} \cup \mathcal{K}_{3}$ with $\left(u_{1}, S_{1}, A_{1}\right) \neq\left(u_{2}, S_{2}, A_{2}\right)$ and $\lambda\left(u_{1}, S_{1}, A_{1}\right)=\lambda\left(u_{2}, S_{2}, A_{2}\right)$. Then $\alpha\left(u_{1}, S_{1}, A_{1}\right) \neq \alpha\left(u_{2}, S_{2}, A_{2}\right)$.

Finally we define $\varphi(u, S, A)$. If $(u, S, A) \in \mathcal{K}_{1} \cup \mathcal{K}_{2}$, simply let $\varphi(u, S, A)=$ $\lambda(u, S, A)$; if $(u, S, A) \in \mathcal{K}_{3}$, let $\varphi(u, S, A)$ be a 4 -contractible edge incident with $\alpha(u, S, A)$, whose existence is guaranteed by Claim 3.2 (iii) or Lemma 2.6 (it is possible that the other endvertex of $\varphi(u, S, A)$ lies in $\bar{X})$; if $(u, S, A) \in \mathcal{K}_{4}$, let $\varphi(u, S, A)=a x$, where $a, x$ are as in Claim 3.6.

### 4.2. Properties of $\varphi(u, S, A)$

In this subsection, we complete the proof of Theorem 2 by showing that for any pair $(e, a)$ of a 4-contractible edge $e$ and an endvertex $a$ of $e$, there are at most two members $(u, S, A)$ of $\mathcal{K}_{0}$ for which $(\varphi(u, S, A), \alpha(u, S, A))=(e, a)$. The first two claims immediately follow from Claims 4.5 and 4.9 , respectively.

Claim 4.10. Let $(u, S, A),(v, T, B) \in \mathcal{K}_{1}$ with $(u, S, A) \neq(v, T, B)$. Then $(\varphi(u$, $S, A), \alpha(u, S, A)) \neq(\varphi(v, T, B), \alpha(v, T, B))$.

Claim 4.11. Let $(u, S, A),(v, T, B) \in \mathcal{K}_{2}$ with $(u, S, A) \neq(v, T, B)$. Then $(\varphi(u$, $S, A), \alpha(u, S, A)) \neq(\varphi(v, T, B), \alpha(v, T, B))$.

The following claims are virtually the same as Claims 7.3 and 7.4, respectively, in [1].

Claim 4.12. Let $(u, S, A) \in \mathcal{K}_{2}$ and $(v, T, B) \in \mathcal{K}_{1}$, and suppose that $\varphi(u, S, A)=$ $\varphi(v, T, B)$. Then $v \in P_{u, S, A}$, and there is no $(w, R, Z) \in \mathcal{K}_{2}$ with $(w, R, Z) \neq$ $(u, S, A)$ such that $\varphi(w, R, Z)=\varphi(u, S, A)$.

Proof. Write $\varphi(u, S, A)=\varphi(v, T, B)=v b$. Also let $v z$ be an edge in $F$ such that $v, z \in T$. Let $(P, X)=\left(P_{u, S, A}, X_{u, S, A}\right)$. Suppose that $v \in X$. Then since $v z \in E(G)$, we have $z \in P \cup X$, and hence $z \in(P \cup X) \cap T$. Since $\operatorname{deg}_{G}(v)=4$, it follows from the definition of $\mathcal{K}_{0}$ that $N_{G}(v) \cap B=\{b\}$. Since $u \in N_{G}(v) \cap N_{G}(b)$, this implies $u \in T$, and hence $u \in P \cap T$. Thus by Lemmas 2.1 and 2.2, there
exists a 4-cutset $U$ with $U \supseteq(P \cup X) \cap T$ such that $G-U$ has a component $H$ with $V(H) \subseteq X-(X \cap T) \subseteq X-\{v\}$. But then since $v \in X \cap T \subseteq U$, $z \in(P \cup X) \cap T \subseteq U$ and $v z \in F \subseteq E_{n}(G)-E_{t n}(G), U$ is a nontrivial 4cutset, which contradicts the minimality of $X$ because $u \in P \cap T \subseteq U$ (see the remark made in the paragraph preceding Claim 4.2). Thus $v \in P$. Now suppose that there exists $(w, R, Z) \in \mathcal{K}_{2}$ with $(w, R, Z) \neq(u, S, A)$ such that $\varphi(w, R, Z)=\varphi(u, S, A)$. Then $w \neq u$ by Claim 4.5. Hence applying Lemma 2.13 with $a=v$, we see that $v \notin \tilde{V}(G)$. But this contradicts the assumption that $(v, T, B) \in \mathcal{K}_{1}$. Thus there no such $(w, R, Z)$.

Claim 4.13. Let $(u, S, A) \in \mathcal{K}_{2}$ and $(v, T, B) \in \mathcal{K}_{1}$. Then $(\varphi(u, S, A), \alpha(u, S, A))$ $\neq(\varphi(v, T, B), \alpha(v, T, B))$.

Proof. We may assume $\varphi(u, S, A)=\varphi(v, T, B)$. Write $\varphi(u, S, A)=v b$. We have $\alpha(v, T, B)=v$ by definition. On the other hand, in view of Claim 4.12, $\alpha(u, S, A)=b$ by the choice of $\alpha(u, S, A)$ described in Subsection 4.1. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$.

The following claim corresponds to Claim 7.5 in [1].
Claim 4.14. Let $(u, S, A) \in \mathcal{K}_{3}$ and $(v, T, B) \in \mathcal{K}_{1}$. Then $\alpha(u, S, A) \neq \alpha(v, T, B)$.
Proof. By Lemma 2.13 and Claim 3.2, $\alpha(u, S, A) \notin \tilde{V}(G)$. On the other hand, $\alpha(v, T, B)=v \in \tilde{V}(G)$. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$.

The following claims are virtually the same as Claims 7.6 and 7.7 , respectively, in [1].

Claim 4.15. Let $(u, S, A) \in \mathcal{K}_{3} \cup \mathcal{K}_{4}$ and $(v, T, B) \in \mathcal{K}_{2}$. Then $\varphi(u, S, A) \neq$ $\varphi(v, T, B)$.

Proof. Suppose that $\varphi(u, S, A)=\varphi(v, T, B)$. Write $\lambda(u, S, A)=a b$ with $\alpha(u$, $S, A)=a$. Then $\operatorname{deg}_{G}(a)=4$. Also write $\varphi(v, T, B)=a x$. Then $v \in N_{G}(a) \cap$ $N_{G}(x)$. First assume that there exists $(w, R, Z) \in \mathcal{K}_{3}$ with $(w, R, Z) \neq(u, S, A)$ such that $\lambda(w, R, Z)=\lambda(u, S, A)$. Then $\operatorname{deg}_{G}(b)=4$. By Claim 4.5, $w \neq u$. Thus $N_{G}(a)=\{u, b, w, x\}$. Since $\operatorname{deg}_{G}(v) \geq 5$ and $\operatorname{deg}_{G}(b)=4, v \neq b$. Since $v \in N_{G}(a) \cap N_{G}(x) \subseteq N_{G}(a)-\{x\}$, this implies $v=u$ or $w$. On the other hand, $\operatorname{deg}_{G}(a)=4$ and $a$ is a common endvertex of $\varphi(v, T, B)$ and $\lambda(u, S, A)=\lambda(w, R, Z)$. Since $\varphi(v, T, B)=\lambda(v, T, B)$, this contradicts Claim 4.7. Next assume that there is no such $(w, R, Z)$. Write $N_{G}(a)=\{u, b, x, y\}$. Suppose that $(u, S, A) \in \mathcal{K}_{3}$. Then $x y \notin E(G)$ by the choice of $\alpha(u, S, A)$, which implies $v \neq y$. Also we have $\operatorname{deg}_{G}(b)=4$ by the definition of $\lambda(u, S, A)$, which implies $v \neq b$. Consequently, $v=u$, which contradicts Claim 4.7. Suppose that $(u, S, A) \in \mathcal{K}_{4}$. By Claim 3.6, $x y \notin E(G)$, which implies $v \neq y$. Again by

Claim 3.6, $x b \notin E(G)$, and hence $v \neq b$. Thus $v=u$, which again contradicts Claim 4.7.

Claim 4.16. Let $(u, S, A),(v, T, B) \in \mathcal{K}_{3}$ with $(u, S, A) \neq(v, T, B)$. Then $(\varphi(u$, $S, A), \alpha(u, S, A)) \neq(\varphi(v, T, B), \alpha(v, T, B))$.
Proof. Suppose that $(\varphi(u, S, A), \alpha(u, S, A))=(\varphi(v, T, B), \alpha(v, T, B))$. Write $\lambda(u, S, A)=a b, \varphi(u, S, A)=\varphi(v, T, B)=a x$, and $N_{G}(a)=\{u, b, x, y\}$. Then $\alpha(u, S, A)=\alpha(v, T, B)=a$, and $v \in N_{G}(a)-\{x\}$. Since $\operatorname{deg}_{G}(a)=4$ and $a$ is a common endvertex of $\lambda(u, S, A)$ and $\lambda(v, T, B), v \neq u$ by Claim 4.7. Since $\operatorname{deg}_{G}(b)=4, v \neq b$. Thus $v=y$, and hence $\lambda(v, T, B)=a u$ or $a b$. On the other hand, since $\operatorname{deg}_{G}(u) \geq 5, \lambda(v, T, B) \neq a u$. Consequently $\lambda(v, T, B)=a b$, which contradicts Claim 4.9.

The following claim shows that in most cases, we have $(\varphi(u, S, A), \alpha(u, S, A))$ $\neq(\varphi(v, T, B), \alpha(v, T, B))$ for $(u, S, A),(v, T, B) \in \mathcal{K}_{0}$ with $(u, S, A) \neq(v, T, B)$.

Claim 4.17. The following hold.
(i) Let $(u, S, A),(v, T, B) \in \mathcal{K}_{0}-\mathcal{K}_{4}$ with $(u, S, A) \neq(v, T, B)$. Then $(\varphi(u, S, A)$, $\alpha(u, S, A)) \neq(\varphi(v, T, B), \alpha(v, T, B))$.
(ii) Let $(u, S, A) \in \mathcal{K}_{4},(v, T, B) \in \mathcal{K}_{0}-\mathcal{K}_{1}$ with $(u, S, A) \neq(v, T, B)$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq(\varphi(v, T, B), \alpha(v, T, B))$.

Proof. Statement (i) follows from Claims 4.10, 4.11 and 4.13 through 4.16. Thus we prove (ii). By Claim 4.15, we may assume that $(v, T, B) \in \mathcal{K}_{3} \cup \mathcal{K}_{4}$. Suppose that $(\varphi(u, S, A), \alpha(u, S, A))=(\varphi(v, T, B), \alpha(v, T, B))$. Let $(P, X)=$ $\left(P_{u, S, A}, X_{u, S, A}\right)$ and let $a, b, x, y$ be as in Claims 3.5 and 3.6. Also let $(Q, Y)=$ $\left(P_{v, T, B}, X_{v, T, B}\right)$. Note that $N_{G}(a)=\{u, b, x, y\}$ and $v \in N_{G}(a)-\{x\}$. If $v=y$, then $a, y \in Q$, and hence $a y \in E_{n}(G)$, which contradicts Claim 3.6. Thus $v \neq y$. We also have $v \neq u$ by Claim 4.7. Consequently $v=b$, which implies $\lambda(v, T, B)=$ $a u$ or ay. Suppose that $(v, T, B) \in \mathcal{K}_{3}$. Then since $V(\lambda(v, T, B)) \subseteq V_{4}(G)$, $\lambda(v, T, B)=a y$. But then $a y \in E_{n}(G)$ by the definition of $\mathcal{K}_{3}$, which contradicts Claim 3.6. Thus we have $(v, T, B) \in \mathcal{K}_{4}$. Applying Claim 3.6 to $(Q, Y)$, we now obtain $b, a \in Q, x \in \bar{Y}$ and $y, u \in Y$. In particular, $x u \notin E(G)$. Set $U=(P \cap Q) \cup(P \cap Y) \cup(X \cap Q)$. Since $y \in X \cap Y$ and $x \in \bar{X} \cap \bar{Y}$, it follows from Lemma 2.2 that $(U, X \cap Y) \in \mathcal{L}$. Since $u \in P \cap Y \subseteq U$, it follows from the minimality of $X$ that $(U, X \cap Y) \notin \mathcal{L}_{0}$, i.e., $U$ is a trivial 4-cutset. Hence there exists $c \in V_{4}(G)$ such that $N_{G}(c)=U$. Since $a, b, u \in U, c \in N_{G}(a)-\{b, u\}=\{x, y\}$. On the other hand, since $x u \notin E(G), c \neq x$. Consequently $c=y$, which implies $y \in N_{G}(u) \cap X \cap V_{4}(G)$. But since $(u, S, A) \in \mathcal{K}_{4}$, this contradicts Claim 3.4.

The following claim, together with Claim 4.17, shows that for each $e \in E_{c}(G)$ and for each endvertex $a$ of $e$, there are at most two members $(u, S, A)$ of $\mathcal{K}_{0}$ such that $(\varphi(u, S, A), \alpha(u, S, A))=(e, a)$.

Claim 4.18. Let $(u, S, A) \in \mathcal{K}_{4},(v, T, B) \in \mathcal{K}_{1}$ with $(\varphi(u, S, A), \alpha(u, S, A))=$ $(\varphi(v, T, B), \alpha(v, T, B))$. Then $(\varphi(w, R, Z), \alpha(w, R, Z)) \neq(\varphi(u, S, A), \alpha(u, S, A))$ for $(w, R, Z) \in \mathcal{K}_{0}-\{(u, S, A),(v, T, B)\}$.

Proof. Suppose that there exists $(w, R, Z) \in \mathcal{K}_{0}-\{(u, S, A),(v, T, B)\}$ such that

$$
(\varphi(w, R, Z), \alpha(w, R, Z))=(\varphi(u, S, A), \alpha(u, S, A))
$$

By Claim 4.17(ii), we have $(w, R, Z) \in \mathcal{K}_{1}-\{(v, T, B)\}$. On the other hand, since

$$
(\varphi(v, T, B), \alpha(v, T, B))=(\varphi(u, S, A), \alpha(u, S, A))=(\varphi(w, R, Z), \alpha(w, R, Z))
$$

it follows from Claim 4.17 (i) that $(w, R, Z) \in \mathcal{K}_{4}-\{(u, S, A)\}$, which is a contradiction.

In view of the remark made before the statement of Claim 4.18, it follows from Claims 4.17 and 4.18 that for each $e \in E_{c}(G)$, there are at most four members $(u, S, A)$ of $\mathcal{K}_{0}$ such that $\varphi(u, S, A)=e$. This completes the proof of Theorem 2.

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