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DISTRIBUTION OF CONTRACTIBLE EDGES AND THE STRUCTURE OF NONCONTRACTIBLE EDGES HAVING ENDVERTICES WITH LARGE DEGREE IN A 4-CONNECTED GRAPH

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Abstract

Let G be a 4-connected graph G, and let $E_c(G)$ denote the set of 4-contractible edges of G. We prove results concerning the distribution of edges in $E_c(G)$. Roughly speaking, we show that there exists a set $\mathcal{K}_0$ and a mapping $\varphi : \mathcal{K}_0 \to E_c(G)$ such that $|\varphi^{-1}(e)| \leq 4$ for each $e \in E_c(G)$.

Keywords: 4-connected graph, contractible edge, cross free.

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1. Introduction

In this paper, we consider only finite undirected simple graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of e. For $x \in V(G)$, $N_G(x)$ denotes the neighborhood of x and $\deg_G(x)$ denotes the degree of x; thus $\deg_G(x) = |N_G(x)|$. For $X \subseteq V(G)$, we let $N_G(X) = \bigcup_{x\in X} N_G(x)$, and the subgraph induced by X in G is denoted by $G[X]$. For an integer $i \geq 0$, we let $V_i(G)$ denote the set of vertices x of G with $\deg_G(x) = i$ and we let $V_{\geq i}(G) = \bigcup_{j\geq i} V_j(G)$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected. A cutset with cardinality i is simply referred to as an *i-cutset*. For an integer $k \geq 1$, we say that G is *k-connected* if $|V(G)| \geq k+1$ and G has no $(k-1)$-cutset.

Let G be a 4-connected graph. For two distinct 4-cutsets S, T, we say that S *crosses* T if S intersects with every component of $G-T$. It is easy to see that S crosses T if and only if T crosses S, which is in turn equivalent to saying that S intersects at least two components of $G-T$. Furthermore, we call a family of 4-cutsets $\mathcal{S}$ *cross free* if no two members of $\mathcal{S}$ cross. A 4-cutset S of G is said to be *trivial* if there exists a vertex z of degree 4 such that $N_G(z)=S$; otherwise it is said to be *nontrivial.* For $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that e is 4-*contractible* or 4-*noncontractible* according as G/e is 4-connected or not. A 4-noncontractible edge $e=ab$ is said to be *trivially* 4-*noncontractible* if there exists a vertex z of degree 4 such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_{tn}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively. Note that if $|V(G)| \geq 6$, then $e \in E_n(G)$ if and only if there exists a 4-cutset S such that $V(e) \subseteq S$, and $e \in E_{tn}(G)$ if and only if there exists a trivial 4-cutset S such that $V(e) \subseteq S$.

The following theorem concerning the number of 4-contractible edges in a 4-connected graph was proved in [2].

Theorem A. *If G is a 4-connected graph, then $|E_c(G)| \geq (1/68)\sum_{u\in V(G)} (\deg_G(u)-4)$.*

The coefficient 1/68 in Theorem A seems far from best possible. The purpose of this paper is to prove two results which will be useful in refining Theorem A. Our results can be also seen as a "large-degree version" of the two structure theorems proved in [1] concerning edges not contained in triangles (see Theorems C and D below).

Throughout the rest of this paper, we let G be a 4-connected graph. Set

$$\begin{aligned}\mathcal{L} = \{(S,A) \mid\ & S \text{ is a 4-cutset, } A \text{ is the union of the vertex set of}\\ & \text{some components of } G-S,\ \emptyset \neq A \neq V(G)-S\},\\ \mathcal{L}_0 = \{(S,A) \in \mathcal{L} \mid\ & S \text{ is a nontrivial 4-cutset}\}.\end{aligned}$$

For $(S,A) \in \mathcal{L}$, we let $\overline{A} = V(G)-S-A$. Thus if $(S,A)\in\mathcal{L}$, then $(S,\overline{A})\in\mathcal{L}$ and $N_G(A)-A = N_G(\overline{A})-\overline{A} = S$.

Let F be a subset of $E_n(G)-E_{tn}(G)$. Let $\tilde{V}(G)$ denote the set of those vertices of G which are incident with an edge in F, and let $\tilde{G}$ denote the spanning subgraph of G with edge set F; that is to say, $\tilde{V}(G) = \bigcup_{e\in F} V(e)$ and $\tilde{G} = (V(G),F)$. Now take $(S_1,A_1),\dots,(S_k,A_k) \in \mathcal{L}$ so that for each $e \in F$, there exists S_i such that $V(e) \subseteq S_i$. We choose $(S_1,A_1),\dots,(S_k,A_k)$ so that k is minimum and so that $(|A_1|,\dots,|A_k|)$ is lexicographically minimum, subject to the condition that k is minimum (thus if $F=\emptyset$, then $k=0$). Note that the minimality of k implies that for each $1 \leq i \leq k$, we have $E(G[S_i]) \cap F \neq \emptyset$ and hence $(S_i, A_i) \in \mathcal{L}_0$. Set $\mathcal{S} = \{S_1, \ldots, S_k\}$. Further set

$$\begin{aligned}\mathcal{K} &= \big\{(u, S, A) \mid u \in \tilde{V}(G), S \in \mathcal{S}, (S, A) \in \mathcal{L}_0, \text{ there exists} \\ &\qquad e \in F \text{ such that } u \in V(e) \subseteq S\big\}, \\ \mathcal{K}^* &= \big\{(u, S, A) \in \mathcal{K} \mid \text{ there is no } (v, T, B) \in \mathcal{K} \text{ with} \\ &\qquad v = u \text{ and } (T, B) \neq (S, A) \text{ such that } B \subseteq A\big\}.\end{aligned}$$

Moreover let $\mathcal{K}_0$ be the set of those members $(u, S, A) \in \mathcal{K}^*$ which satisfy one of the following two conditions:

(1) $\deg_G(u) \geq 5$; or

(2) $\deg_G(u) = 4$, $|N_G(u) \cap A| = 1$ and, if we write $N_G(u) \cap A = \{a\}$, then $ua \in E_c(G)$.

We say that F is *admissible* if the following statement is true (note that this definition implies that if $F = \emptyset$, then F is admissible).

Statement B. *Let $uv \in F$, and let S be a 4-cutset with $u, v \in S$, and let A be the vertex set of a component of $G - S$. Then there exists $e \in E_c(G)$ such that either e is incident with u or there exists $a \in N_G(u) \cap (S \cup A) \cap V_4(G)$ such that e is incident with a.*

Now we let $\tilde{E}(G)$ denote the set of those edges of a 4-connected graph G which are not contained in a triangle. The following result appears as Theorem 1 in [1].

Theorem C. *The set $\tilde{E}(G) \cap E_n(G)$ is admissible.*

Let L be the set of edges e such that both endvertices of e have degree 4. In this paper, we prove the following theorem.

Theorem 1. *Let $F = E_n(G) - E_{tn}(G) - L$. Let $\mathcal{S}$ be as above, and suppose that $\mathcal{S}$ is cross free. Then F is admissible.*

Note that in the case where $F = \tilde{E}(G) \cap E_n(G)$, we can show that $\mathcal{S}$ is cross free (see Claim 4.1 in [1]), and this is why we do not need the assumption that $\mathcal{S}$ is cross free in Theorem C.

The following theorem appears as Theorem 2 in [1].

Theorem D. *Let $\mathcal{K}_0$ be as above with $F = \tilde{E}(G) \cap E_n(G)$. Then we can assign to each $(u, S, A) \in \mathcal{K}_0$ a 4-contractible edge $\varphi(u, S, A)$ having the property stated in Statement* B, *so that for each $e \in E_c(G)$ there are at most two members (u, S, A) of $\mathcal{K}_0$ such that $\varphi(u, S, A) = e$.*

The following theorem is our main result.

Theorem 2. *Let $\mathcal{S}$ and $\mathcal{K}_0$ be as above with $F = E_n(G) - E_{tn}(G) - L$, and suppose that $\mathcal{S}$ is cross free. Then we can assign to each $(u, S, A) \in \mathcal{K}_0$ a 4-contractible edge $\varphi(u, S, A)$ having the property stated in Statement* B, *so that for each $e \in E_c(G)$ there are at most four members (u, S, A) of $\mathcal{K}_0$ such that $\varphi(u, S, A) = e$.*

We remark that in Theorem 2, situations in which there are three or four members (u, S, A) of $\mathcal{K}_0$ such that $\varphi(u, S, A) = e$ are rather limited (see Claim 4.17).

Recall that Theorems 1 and 2 will be useful in refining Theorem A. The reasons are as follows. Let k be a maximum value with $|E_c(G)| \geq k \sum_{u \in V(G)}(\deg_G(u) -4)$ for a 4-connected graph G. Note that we know that $1/68 \leq k \leq 1/13$, and hence assume $1/68 \leq k \leq 1/13$ throughout the rest of this argument. If $|V_{\geq 5}(G)| = 0$, then the above inequality holds immediately. Thus we now assume that $|V_{\geq 5}(G)| \geq 1$. Let $\mathcal{S}$ be as above with $F = E_n(G) - E_{tn}(G) - L$. If $|E_c(G)| < k \sum_{u \in V(G)}(\deg_G(u) - 4)$, then we can show that $\mathcal{S}$ is cross free by Theorem 1 in [4]. Suppose that $|E_c(G)| < (1/28) \sum_{u \in V(G)}(\deg_G(u) - 4)$. Then $\mathcal{S}$ is cross free by the above argument. Hence we can use Theorem 2, and we can show that $|E_c(G)| \geq (1/28) \sum_{u \in V(G)}(\deg_G(u) - 4)$ by Theorem 2 (the verification of this statement involves lengthy calculations), which is a contradiction. Thus we have $|E_c(G)| \geq (1/28) \sum_{u \in V(G)}(\deg_G(u) - 4)$. However, it is likely that the coefficient 1/28 can further be improved in view of the fact that situations in which there are three or four members (u, S, A) of $\mathcal{K}_0$ such that $\varphi(u, S, A) = e$ are limited. Thus matters concerning refinements of Theorem A will be discussed in a separate paper.

Our notation is standard, and is mostly taken from Diestel [3]. The organization of this paper is as follows. In Section 2, we introduce known results proved in [1], and prove some preliminary results. We prove Theorem 1 in Section 3, and Theorem 2 in Section 4.

2. Preliminaries

Throughout the rest of this paper, we let G denote a 4-connected graph with $F = E_n(G) - E_{tn}(G) - L \neq \emptyset$ (note that in proving Theorems 1 and 2, we may clearly assume $F \neq \emptyset$). Thus $|V(G)| \geq 6$. Also let $\mathcal{L}$, $\mathcal{L}_0$ be as in the second paragraph following the statement of Theorem A.

In this section, we state several results which we use in the proof of Theorems 1 and 2.

2.1. Known results

In this subsection, we state results about the distribution of 4-contractible edges. The following lemmas follow from Lemmas 2.2 through 2.13, respectively, in [1].

Lemma 2.1. *Let* $(S, A), (T, B) \in \mathcal{L}_0$*, and suppose that* $S \cap T \neq \emptyset$*. Then either* $A \cap B \neq \emptyset$ *and* $\overline{A} \cap \overline{B} \neq \emptyset$*, or* $A \cap \overline{B} \neq \emptyset$ *and* $\overline{A} \cap B \neq \emptyset$*.*

Lemma 2.2. *Let* $(S, A), (T, B) \in \mathcal{L}$*, and suppose that* $A \cap B \neq \emptyset$ *and* $\overline{A} \cap \overline{B} \neq \emptyset$*. Then* $((S \cap T) \cup (S \cap B) \cup (A \cap T), A \cap B) \in \mathcal{L}$ *and* $((S \cap T) \cup (S \cap \overline{B}) \cup (\overline{A} \cap T), \overline{A} \cap \overline{B}) \in \mathcal{L}$*.*

Lemma 2.3. *Let* $(S, A) \in \mathcal{L}$*.*

(i) *If* $W \subseteq S$ *and* $|W| \leq |A|$*, then* $|N_G(W) \cap A| \geq |W|$*. Further if* $|W| < |A|$ *and* $|N_G(W) \cap A| = |W|$*, then* $((S - W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathcal{L}$*.*

(ii) *If* $x \in S$*, then* $N_G(x) \cap A \neq \emptyset$*. Further if* $(S, A) \in \mathcal{L}_0$ *and* $|N_G(x) \cap A| = 1$*, then* $((S - \{x\}) \cup (N_G(x) \cap A), A - (N_G(x) \cap A)) \in \mathcal{L}$*.*

Lemma 2.4. *Let* $ab \in E(G)$ *with* $\deg_G(a) = \deg_G(b) = 4$*. Then* $N_G(a) - \{b\} \neq N_G(b) - \{a\}$*.*

Lemma 2.5. *Let* u, a, b, w *be four distinct vertices with* $ua, ub, ab, aw, bw \in E(G)$ *and* $\deg_G(a) = \deg_G(b) = 4$*, and write* $N_G(a) = \{u, b, w, x\}$ *and* $N_G(b) = \{u, a, w, y\}$*. Then* $x \neq y$*, and we have* $ax, by \in E_c(G) \cup E_{tn}(G)$*.*

Lemma 2.6. *Under the notation of Lemma* 2.5*, suppose that* $\deg_G(u)$*,* $\deg_G(w) \geq 5$*. Then* $ax, by \in E_c(G)$*.*

Lemma 2.7. *Under the notation of Lemma* 2.5*, suppose that* $\deg_G(u) \geq 5$ *and* $\deg_G(w) = 4$*. Then one of the following holds:*

(i) $xw \notin E(G)$ *and* $ax \in E_c(G)$*; or*

(ii) $yw \notin E(G)$ *and* $by \in E_c(G)$*.*

Lemma 2.8. *Let* $(P, X) \in \mathcal{L}_0$ *and* $u \in P$*. Suppose that* X *is minimal, subject to the condition that* $u \in P$ *(i.e., there is no* $(R, Z) \in \mathcal{L}_0$ *with* $(P, X) \neq (R, Z)$ *such that* $u \in R$ *and* $Z \subseteq X$*). Then* $ua \in E_c(G) \cup E_{tn}(G)$ *for each* $a \in N_G(u) \cap X$*.*

Lemma 2.9. *Let* $(R, Z) \in \mathcal{L}_0$ *and* $a \in R$*. Suppose that* $|N_G(a) \cap Z| = 1$*, and write* $N_G(a) \cap Z = \{x\}$*. Then* $ax \in E_c(G) \cup E_{tn}(G)$*.*

Lemma 2.10. *Let* u, a, b *be three distinct vertices with* $ua, ub, ab \in E(G)$ *and* $\deg_G(a) = 4$*, and write* $N_G(a) = \{u, b, x, y\}$*. Suppose that there exists* $(R, Z) \in \mathcal{L}_0$ *such that* $u, a \in R$*,* $b, y \in Z$ *and* $x \in \overline{Z}$*. Suppose further that* Z *is minimal, subject to the condition that* $u, a \in R$ *and* $b \in Z$*. Then the following hold.*

(i) $xy \notin E(G)$.
(ii) $ax \in E_c(G) \cup E_{tn}(G)$.
(iii) $ay \in E_c(G) \cup E_{tn}(G)$.

Lemma 2.11. *Under the notation of Lemma* 2.10, *suppose that* $\deg_G(b) \geq 5$. *Then* $ax \in E_c(G)$ *or* $ay \in E_c(G)$.

Lemma 2.12. *Under the notation of Lemma* 2.10, *suppose that* $\deg_G(b)$, $\deg_G(u) \geq 5$. *Then* $ax, ay \in E_c(G)$.

2.2. Vertices not contained in $\tilde{V}(G)$

Recall that $F = E_n(G) - E_{tn}(G) - L$ and $\tilde{V}(G) = \bigcup_{e \in F} V(e)$. In this subsection, we prove results concerning conditions for a vertex not to belong to $\tilde{V}(G)$.

Lemma 2.13. *Under the notation of Lemma* 2.5, $a, b \notin \tilde{V}(G)$.

Proof. In view of the symmetry of the roles of a and b, it suffices to prove $a \notin \tilde{V}(G)$. Suppose that $a \in \tilde{V}(G)$. Then there exists $e \in F$ such that e is incident with a. Since $au, aw \in E_{tn}(G)$ and $ab \in L$, $e \neq au, ab, aw$. Hence $e = ax$. By Lemma 2.5, we get $e \in E_c(G) \cup E_{tn}(G)$, a contradiction. ■

Lemma 2.14. *Under the notation of Lemma* 2.10, *suppose that* $\deg_G(u) = 4$ *or* $\deg_G(b) = 4$. *Then* $a \notin \tilde{V}(G)$.

Proof. Suppose that $a \in \tilde{V}(G)$. Then there exists $e \in F$ such that e is incident with a. Since $au, ab \in E_{tn}(G) \cup L$, $e \neq au, ab$. Consequently $e = ax$ or ay, which contradicts Lemma 2.10(ii) or (iii). ■

3. Proof of Theorem 1

In the rest of this paper, we establish Theorems 1 and 2 by proving several claims. The proofs of most of the claims in this paper are quite similar to the proofs of the claims in [1] having virtually the same statements. However, considering that we are dealing with $E_n(G) - E_{tn}(G) - L$ instead of $\tilde{E}(G) \cap E_n(G)$, we have decided to include the details of the proofs in this paper. In this section, we prove Theorem 1.

3.1. Neighborhood of a vertex of degree 5

In this subsection, we prove that Statement B is true if $\deg_G(u) \geq 5$. Specifically, we prove the following proposition in a series of claims.

Proposition 3.1. *Let $(P, X) \in \mathcal{L}_0$ and $u \in P$, and suppose that* $\deg_G(u) \geq 5$. *Then one of the following holds*:

(1) *there exists $a \in N_G(u) \cap X$ such that $ua \in E_c(G)$; or*

(2) *there exists $a \in N_G(u) \cap (P \cup X) \cap V_4(G)$ for which there exists $e \in E_c(G)$ such that e is incident with a.*

Note that Proposition 3.1 implies that in Theorem 1, the assumption that $\mathcal{S}$ is cross free is not necessary for vertices u with $\deg_G(u) \geq 5$. Throughout this subsection, let $(P, X), u$ be as in Proposition 3.1. We may assume that X is minimal, subject to the condition that $u \in P$ (i.e., there is no $(R, Z) \in \mathcal{L}_0$ with $(R, Z) \neq (P, X)$ such that $u \in R$ and $Z \subseteq X$).

The following four claims are virtually the same as Claims 3.2 through 3.5 in [1].

Claim 3.2. *Suppose that there exists an edge e joining a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex $N_G(u) \cap (P \cup X) \cap V_4(G)$. Suppose that $e \in E_n(G)$, and write $e = ab$. Then a or b, say a, satisfies the following conditions.*

(i) *If we write $N_G(a) = \{u, b, x, y\}$, then $xy \notin E(G)$.*

(ii) $a \notin \tilde{V}(G)$.

(iii) *There exists $e' \in E_c(G)$ such that e' is incident with a.*

Proof. If $ab \in E_{tn}(G)$, then there exists $w \in V_4(G)$ such that $wa, wb \in E(G)$, and hence the desired conclusions follows from Lemmas 2.7 and 2.13. Thus we may assume that $ab \in E_n(G) - E_{tn}(G)$. Then there exists $(R, Z) \in \mathcal{L}_0$ with $a, b \in R$. We first show that $u \notin R$. Suppose that $u \in R$. Then by Lemma 2.1, we may assume $X \cap Z \neq \emptyset$ and $\overline{X} \cap \overline{Z} \neq \emptyset$. Since $a, b \in (P \cup X) \cap R$, it follows from Lemma 2.2 that $((P \cap R) \cup (P \cap Z) \cup (X \cap R), X \cap Z) \in \mathcal{L}_0$, which contradicts the minimality of X. Thus $u \notin R$. We may assume $u \in Z$. We may also assume that we have chosen (R, Z) so that Z is minimal, subject to the condition that $a, b \in R$ and $u \in Z$. By Lemma 2.3(i), we have $N_G(a) \cap Z \neq \{u\}$ or $N_G(b) \cap Z \neq \{u\}$. We may assume $N_G(a) \cap Z \neq \{u\}$. Since $N_G(a) \cap \overline{Z} \neq \emptyset$ by Lemma 2.3(ii), we have $|N_G(a) \cap Z| = 2$ and $|N_G(a) \cap \overline{Z}| = 1$. Write $N_G(a) \cap Z = \{u, y\}$ and $N_G(a) \cap \overline{Z} = \{x\}$. Then b, a, u, x, y satisfy the assumptions of Lemmas 2.10, 2.11 and 2.14 with the roles of b and u replaced by each other. Consequently the desired conclusions follow from (i) of Lemma 2.10 and Lemmas 2.11 and 2.14. ■

Claim 3.3. *Let $a \in X$, and suppose that $ua \in E_n(G)$. Then $ua \in E_{tn}(G)$.*

Proof. This follows from Lemma 2.8. ■

Claim 3.4. *Suppose that each edge joining u and a vertex in X is 4-noncontractible, and that there is no edge which joins a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$. Then $N_G(u) \cap X \cap V_4(G) = \emptyset$.*

Proof. Suppose that $N_G(u)\cap X\cap V_4(G)\neq\emptyset$, and take $a\in N_G(u)\cap X\cap V_4(G)$. We have $ua\in E_{tn}(G)$ by Claim 3.3. Hence there exists $b\in V_4(G)$ such that $ub, ab\in E(G)$. From $a\in X$ and $ab\in E(G)$, it follows that $b\in P\cup X$. Thus ab is an edge joining a vertex in $N_G(u)\cap X\cap V_4(G)$ and a vertex in $N_G(u)\cap(P\cup X)\cap V_4(G)$, a contradiction. ■

Claim 3.5. *Suppose that each edge joining u and a vertex in X is 4-noncontractible, and that there is no edge which joins a vertex in $N_G(u)\cap X\cap V_4(G)$ and a vertex in $N_G(u)\cap(P\cup X)\cap V_4(G)$. Then there exists $a\in N_G(u)\cap P\cap V_4(G)$ and $b\in N_G(u)\cap X$ such that $ab\in E(G)$, $|N_G(a)\cap X|=2$ and $|N_G(a)\cap\overline{X}|=1$.*

Proof. Take $z\in N_G(u)\cap X$. Then $uz\in E_{tn}(G)$ by Claim 3.3, and hence there exists $a_z\in V_4(G)$ such that $a_zu, a_zz\in E(G)$. Since $N_G(u)\cap X\cap V_4(G)=\emptyset$ by Claim 3.4, $a_z\in P$. Since $\deg_G(a_z)=4$ and $u\in N_G(a_z)\cap P$, $|N_G(a_z)\cap X|+|N_G(a_z)\cap\overline{X}|\le 3$, and hence it follows from Lemma 2.3(ii) that $1\le|N_G(a_z)\cap X|\le 2$. Now by way of contradiction, suppose that the claim is false. Then $|N_G(a_z)\cap X|=1$, i.e., $N_G(a_z)\cap X=\{z\}$. Since $z\in N_G(u)\cap X$ is arbitrary, this means that $a_y\neq a_z$ for any $y,z\in N_G(u)\cap X$ with $y\neq z$ and if we set $W=\{a_z\,|\,z\in N_G(u)\cap X\}$, then we have $|W|=|N_G(u)\cap X|$ and $N_G(\{u\}\cup W)\cap X=N_G(u)\cap X$, and hence $|N(\{u\}\cup W)\cap X|=|W|=|\{u\}\cup W|-1$. In view of Lemma 2.3(i), this implies $|\{u\}\cup W|\ge|X|+1$, i.e., $|W|\ge|X|$. Again fix $z\in N_G(u)\cap X$. Since $N_G(a_y)\cap X=\{y\}$ for each $y\in(N_G(u)\cap X)-\{z\}$, $N_G(z)\subseteq(P-(W-\{a_z\}))\cup(X-\{z\})$. Consequently $\deg_G(z)\le|P|-|W|+|X|\le|P|=4$, which implies $z\in N_G(u)\cap X\cap V_4(G)$. But this contradicts Claim 3.4, completing the proof. ■

The following claim corresponds to Claim 3.6 in [1].

Claim 3.6. *Suppose that each edge joining u and a vertex in X is 4-noncontractible, and that there is no edge which joins a vertex in $N_G(u)\cap X\cap V_4(G)$ and a vertex in $N_G(u)\cap(P\cup X)\cap V_4(G)$. Further let a, b be as in Claim 3.5, and write $N_G(a)\cap X=\{b,y\}$ and $N_G(a)\cap\overline{X}=\{x\}$. Then $xy\notin E(G)$, and $ax, ay\in E_c(G)$.*

Proof. Note that $\deg_G(b)\ge 5$ by Claim 3.4, and $\deg_G(u)\ge 5$ by the assumption of Proposition 3.1. Thus the desired conclusions follows from (i) of Lemma 2.10 and Lemma 2.12. ■

Proposition 3.1 now follows from Claims 3.2 and 3.6.

3.2. Non-crossing 4-cutsets

In this subsection, we complete the proof of Theorem 1. Throughout the rest of this paper, we let $\mathcal{S}$, $\mathcal{K}$, $\mathcal{K}^*$ and $\mathcal{K}_0$ be as in the paragraph preceding Statement B with $F=E_n(G)-E_{tn}(G)-L$, and suppose that $\mathcal{S}$ is cross free.

The following claim immediately follows from the definition of $\mathcal{K}^*$.

Claim 3.7. *Let $u \in \tilde{V}(G)$. Then for each $(u,S,A) \in \mathcal{K}$, there exists a member (v,T,B) of $\mathcal{K}^*$ with $v = u$ and $B \subseteq A$. In particular, there exist at least two members (v,T,B) of $\mathcal{K}^*$ with $v = u$.*

The following claim is virtually the same as Claim 4.3 in [1].

Claim 3.8. *Let (u,S,A), $(v,T,B) \in \mathcal{K}^*$ with $u = v$ and $(S,A) \neq (T,B)$. Then $(S \cup A) \cap B = A \cap (T \cup B) = \emptyset$.*

Proof. If $S = T$, the desired conclusion clearly holds. Thus we may assume that $S \neq T$. Since $\mathcal{S}$ is cross free, we have that $S \cap \overline{B} = T \cap \overline{A} = \emptyset$, $S \cap B = T \cap \overline{A} = \emptyset$, $S \cap \overline{B} = T \cap A = \emptyset$, or $S \cap B = T \cap A = \emptyset$. Suppose that $S \cap \overline{B} = T \cap \overline{A} = \emptyset$. Then since $S \neq T$, we have $A \cap T \neq \emptyset$ and $|(S \cap T) \cup (\overline{A} \cap T) \cup (S \cap \overline{B})| = |T| - |A \cap T| < 4$, and hence $\overline{A} \cap \overline{B} = \emptyset$. Since $S \cap \overline{B} = \emptyset$ and $A \cap T \neq \emptyset$, this implies $\overline{B}$ is a proper subset of A. But since $(u,T,\overline{B}) \in \mathcal{K}$ and $(u,S,A) \in \mathcal{K}^*$, this contradicts the definition of $\mathcal{K}^*$. If $S \cap B = T \cap \overline{A} = \emptyset$ or $S \cap \overline{B} = T \cap A = \emptyset$, then we obtain $B \subseteq A$ or $A \subseteq B$, respectively, and hence we similarly get a contradiction. Thus $S \cap B = T \cap A = \emptyset$. Since $S \neq T$, this also implies $A \cap B = \emptyset$, as desired. ■

Recall that $\tilde{G} = (V(G), F)$. The following claim corresponds to Claim 4.4 in [1].

Claim 3.9. *Let $u \in \tilde{V}(G)$. Then the following hold.*

(i) *There exists a member (v,T,B) of $\mathcal{K}_0$ with $v = u$.*

(ii) *Suppose that $\deg_G(u) \geq 5$, or $\deg_{\tilde{G}}(u) \geq 2$, or there exist three members (v,T,B) of $\mathcal{K}^*$ with $v = u$. Then for each $(u,S,A) \in \mathcal{K}^*$, we have $(u,S,A) \in \mathcal{K}_0$. In particular, if $\deg_G(u) = 4$ and $\deg_{\tilde{G}}(u) \geq 2$, then $\deg_{\tilde{G}}(u) = 2$ and there exist precisely two members (v,T,B) of $\mathcal{K}_0$ with $v = u$.*

Proof. If $\deg_G(u) \geq 5$, the desired conclusion immediately follows from Claim 3.7 and the definition of $\mathcal{K}_0$. Thus we may assume that $\deg_G(u) = 4$. We first prove (ii). Thus let u be as in (ii) with $\deg_G(u) = 4$. Then by Lemma 2.3(ii) and Claim 3.8, it follows that $|N_G(u) \cap A| = 1$ for each $(u,S,A) \in \mathcal{K}^*$, and that for each $a \in N_G(u) - N_{\tilde{G}}(u)$, there exists $(u,S,A) \in \mathcal{K}^*$ such that $a \in A$. Again by Claim 3.8, this implies that for each $(u,S,A) \in \mathcal{K}^*$, $N_G(u) \cap S = N_{\tilde{G}}(u) \cap S$. Note that this also implies that if $\deg_{\tilde{G}}(u) \geq 2$, then we have $\deg_{\tilde{G}}(u) = 2$ and there exist precisely two members (v,T,B) of $\mathcal{K}^*$ with $v = u$. Now let $(u,S,A) \in \mathcal{K}^*$, and write $N_G(u) \cap A = \{a\}$. To complete the proof of (ii), it suffices to show that $(u,S,A) \in \mathcal{K}_0$. Suppose that $(u,S,A) \notin \mathcal{K}_0$. Then $ua \in E_n(G)$, and hence $ua \in E_{tn}(G)$ by Lemma 2.9, which implies that there exists $c \in V_4(G)$ such that $cu, ca \in E(G)$. Since $N_G(u) \cap A = \{a\}$, this forces $c \in S$. But since $uc \in L$, $c \notin N_{\tilde{G}}(u)$, which contradicts the earlier assertion that $N_G(u) \cap S = N_{\tilde{G}}(u) \cap S$. Thus (ii) is proved.

We now prove (i). We may assume that there exists $(u,S,A) \in \mathcal{K}^*$ such that $(u,S,A) \notin \mathcal{K}_0$. Then arguing as above, we see that $|N_G(u) \cap (S \cup A)| \geq 3$ (note that if $|N_G(u) \cap A| \geq 2$, we clearly have $|N_G(u) \cap (S \cup A)| \geq 3$). Take $(u,T,B) \in \mathcal{K}^*$ with $B \subseteq \overline{A}$. Then $|N_G(u) \cap B| = 1$. Write $N_G(u) \cap B = \{b\}$. Suppose that $(u,T,B) \notin \mathcal{K}_0$. Then there exists $c' \in V_4(G)$ such that $c'u, c'b \in E(G)$. This in turn implies $|N_G(u) \cap A| = 1$. Write $N_G(u) \cap A = \{a\}$. Then there exists $c \in V_4(G)$ such that $cu, ca \in E(G)$. Since $\deg_G(u) = 4$, $\deg_{\tilde{G}}(u) \geq 1$ and $ab \notin E(G)$, this forces $c = c'$. But then applying Lemma 2.13 with a and b replaced by u and c, we obtain $u \notin \tilde{V}(G)$, which contradicts the assumption that $u \in \tilde{V}(G)$. Thus (i) is also proved. ■

We are now in a position to complete the proof of Theorem 1.

Let u, S, A be as in Statement B. Then $(S,A) \in \mathcal{L}_0$. Hence if $\deg_G(u) \geq 5$, then the desired conclusion follows from Proposition 3.1. Thus we may assume $\deg_G(u) = 4$. But then from Claim 3.9(i) and the definition of $\mathcal{K}_0$, we see that there exists $e \in E_c(G)$ such that e is incident with u. Consequently $F = E_n(G) - E_{tn}(G) - L$ is admissible, as desired. ■

4. Proof of Theorem 2

In this section, we prove Theorem 2. We continue with the notation of Subsection 3.2. In particular, we suppose that $\mathcal{S}$ is cross free, which is the assumption of Theorem 2.

4.1. Definition of $\lambda(u,S,A)$, $\alpha(u,S,A)$ and $\varphi(u,S,A)$

In this subsection, to each $(u,S,A) \in \mathcal{K}_0$, we assign an edge $\lambda(u,S,A)$, and an endvertex $\alpha(u,S,A)$ of $\lambda(u,S,A)$, and a 4-contractible edge $\varphi(u,S,A)$ incident with $\alpha(u,S,A)$. The following claim corresponds to Claim 5.1 in [1].

Claim 4.1. *Let $(u,S,A) \in \mathcal{K}_0$, and set $W = \{z \in S - \{u\} - N_{\tilde{G}}(u) \mid |N_G(z) \cap A| = 1\}$. Then $((S-W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathcal{L}_0$.*

Proof. By the definition of $\mathcal{K}$, there exists $e \in F$ such that $u \in V(e) \subseteq S$. Hence $W \subseteq S - V(e)$, which implies $|W| \leq 2$. On the other hand, since $(S,A) \in \mathcal{L}_0$, $|A| \geq 2$. Thus $|W| \leq |A|$. Suppose that $|W| = |A|$. Then $|W| = |A| = 2$. By Lemma 2.3(i), $N_G(\{x,z\}) \cap A = A$ for each $x \in V(e)$ and $z \in W$. Since we also have $N_G(W) \cap A = A$ by Lemma 2.3(i) and since $|N_G(z) \cap A| = 1$ for each $z \in W$, this means that $N_G(x) \cap A = A$ for each $x \in V(e)$. Consequently $\deg_G(a) = 4$ and $V(e) \subseteq N_G(a)$ for each $a \in A$, which implies $e \in E_{tn}(G)$, a contradiction. Thus $|W| < |A|$. Therefore it follows from Lemma 2.3(i) that $((S-W) \cup (N_G(W) \cap A), A - (N_G(W) \cap A)) \in \mathcal{L}$, which implies the desired conclusion because $V(e) \subseteq S - W$. ■

Now let $(u, S, A) \in \mathcal{K}_0$, and let W be as in Claim 4.1. We let $(P_{u,S,A}, X_{u,S,A})$ be a member of $\mathcal{L}_0$ with $u \in P_{u,S,A}$ and $X_{u,S,A} \subseteq A - (N_G(W) \cap A)$ such that $X_{u,S,A}$ is minimal, i.e., there is no $(R, Z) \in \mathcal{L}_0$ with $(R, Z) \neq (P_{u,S,A}, X_{u,S,A})$ such that $u \in R$ and $Z \subseteq X_{u,S,A}$. We remark that we do not require that there should exist an edge $e \in E_n(G)$ with $u \in V(e) \subseteq P_{u,S,A}$. The following claim immediately follows from the definition of $(P_{u,S,A}, X_{u,S,A})$.

Claim 4.2. *Let* $(u, S, A) \in \mathcal{K}_0$. *Let* $z \in S - \{u\} - N_{\tilde{G}}(u)$ *and suppose that* $|N_G(z) \cap A| = 1$. *Then* $z \notin P_{u,S,A}$.

Let again $(u, S, A) \in \mathcal{K}_0$, and let $(P, X) = (P_{u,S,A}, X_{u,S,A})$ be as above. We define the type of (u, S, A) as follows: (u, S, A) is of type 1 if there exists a 4-contractible edge joining u and a vertex in X; (u, S, A) is of type 2 if it is not of type 1 and there exists a 4-contractible edge joining a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$; (u, S, A) is of type 3 if it is not of type 1 or 2 but there exists an edge joining a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$; (u, S, A) is of type 4 if it is not of type i for any $i = 1, 2, 3$. We let $\mathcal{K}_i$ denote the set of those members of $\mathcal{K}_0$ which are the type i $(i = 1, 2, 3, 4)$. The following claim, which will be used implicitly throughout the rest of this paper, is virtually the same as Claim 5.3 in [1].

Claim 4.3. *Let* $(u, S, A) \in \mathcal{K}_0 - \mathcal{K}_1$. *Then* $\deg_G(u) \geq 5$.

Proof. Suppose that $\deg_G(u) = 4$. Then by the definition of $\mathcal{K}_0$, $|N_G(u) \cap A| = 1$ and, if we write $N_G(u) \cap A = \{a\}$, then $ua \in E_c(G)$. By Lemma 2.3(ii), $a \in X$. Consequently $(u, S, A) \in \mathcal{K}_1$ by definition, which contradicts the assumption that $(u, S, A) \in \mathcal{K}_0 - \mathcal{K}_1$. ■

We first define $\lambda(u, S, A)$. If $(u, S, A) \in \mathcal{K}_1$, let $\lambda(u, S, A)$ be a 4-contractible edge joining u and a vertex in X; if $(u, S, A) \in \mathcal{K}_2$, let $\lambda(u, S, A)$ be a 4-contractible edge joining a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$; if $(u, S, A) \in \mathcal{K}_3$, let $\lambda(u, S, A)$ be an edge joining a vertex in $N_G(u) \cap X \cap V_4(G)$ and a vertex in $N_G(u) \cap (P \cup X) \cap V_4(G)$; if $(u, S, A) \in \mathcal{K}_4$, let $\lambda(u, S, A) = ab$ where a, b are as in Claim 3.5. The following claim follows from the definition of $\lambda(u, S, A)$.

Claim 4.4. *Let* $2 \leq i, j \leq 4$ *with* $i \neq j$, *and let* $(u_1, S_1, A_1) \in \mathcal{K}_i$ *and* $(u_2, S_2, A_2) \in \mathcal{K}_j$. *Then* $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)$.

The following claims are virtually the same as Claims 5.5 and 5.6, respectively, in [1].

Claim 4.5. *Let* $(u_1, S_1, A_1), (u_2, S_2, A_2) \in \mathcal{K}_0$ *with* $u_1 = u_2$ *and* $(S_1, A_1) \neq (S_2, A_2)$. *Then* $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)$.

Proof. By Claim 3.8, $A_1 \cap A_2 = \emptyset$. Hence $X_{u_1,S_1,A_1} \cap X_{u_2,S_2,A_2} \subseteq A_1 \cap A_2 = \emptyset$. Since at least one of the endvertices of $\lambda(u_j, S_j, A_j)$ is in X_{u_j,S_j,A_j}, this implies $\lambda(u_1, S_1, A_1) \neq \lambda(u_2, S_2, A_2)$. ■

Claim 4.6. *Let e be an edge joining two vertices of degree 4. Then there exist at most two members (u, S, A) of $\mathcal{K}_2 \cup \mathcal{K}_3$ for which $\lambda(u, S, A) = e$.*

Proof. Suppose that there exist three members (u_j, S_j, A_j) $(1 \leq j \leq 3)$ of $\mathcal{K}_2 \cup \mathcal{K}_3$ such that $\lambda(u_j, S_j, A_j) = e$. By Claim 4.5, the u_j are all distinct. But this contradicts Lemma 2.4. ■

We prove two more claims concerning properties of $\lambda(u, S, A)$. The following claim corresponds to Claim 6.1 in [1].

Claim 4.7. *Let $(u, S, A), (v, T, B) \in \mathcal{K}_0 - \mathcal{K}_1$ with $u = v$ and $(S, A) \neq (T, B)$. Then $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_4(G) = \emptyset$.*

Proof. Suppose that $V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_4(G) \neq \emptyset$, and let $a \in V(\lambda(u, S, A)) \cap V(\lambda(v, T, B)) \cap V_4(G)$, and let $(P, X) = (P_{u,S,A}, X_{u,S,A})$. Then $a \in P \cup X \subseteq S \cup A$. Similarly $a \in T \cup B$. Hence $a \in (S \cup A) \cap (T \cap B) \subseteq S \cap T$ by Claim 3.8. Since $\deg_G(a) = 4$ and $u \in N_G(a) \cap S \cap T$, $|N_G(a) \cap (A \cup B)| \leq 3$. Since $A \cap B = \emptyset$ by Claim 3.8, this together with Lemma 2.3(ii) implies that we have $|N_G(a) \cap A| = 1$ or $|N_G(a) \cap B| = 1$. We may assume $|N_G(a) \cap A| = 1$. If $(u, S, A) \in \mathcal{K}_4$, then by the definition of $\lambda(u, S, A)$, a coincides with the vertex a in Claim 3.5, and hence $|N_G(a) \cap A| \geq |N_G(a) \cap X| = 2$ by Claim 3.5, a contradiction. Thus $(u, S, A) \in \mathcal{K}_2 \cup \mathcal{K}_3$. Consequently $ua \in E_{tn}(G)$ by the definition of types 2 and 3, and hence $a \notin N_{\tilde{G}}(u)$. By Claim 4.2, this implies $a \notin P$, which contradicts the fact that $a \in (P \cup X) \cap S \subseteq P$. ■

The following claim is virtually the same as Claim 6.2 in [1].

Claim 4.8. *Let $(u, S, A), (v, T, B) \in \mathcal{K}_4$ with $(u, S, A) \neq (v, T, B)$. Then $\lambda(u, S, A) \neq \lambda(v, T, B)$.*

Proof. Suppose that $\lambda(u, S, A) = \lambda(v, T, B)$. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$, and let a, b, x, y be as in Claims 3.5 and 3.6. Then $\lambda(u, S, A) = \lambda(v, T, B) = ab$, and hence $v \in N_G(a) \cap N_G(b)$. In particular, $v \in N_G(a) - \{b\} = \{u, x, y\}$. Since we get $xb \notin E(G)$ from $x \in \overline{X}$ and $b \in X$, $v \neq x$. We also have $v \neq u$ by Claim 4.5. Thus $v = y$, and hence $y, a \in P_{v,T,B}$. Consequently $ya \in E_n(G)$, which contradicts Claim 3.6. ■

We now define $\alpha(u, S, A)$. If $(u, S, A) \in \mathcal{K}_1$, let $\alpha(u, S, A) = u$. Now assume $(u, S, A) \in \mathcal{K}_2$. In this case, we let $\alpha(u, S, A)$ be an endvertex of $\lambda(u, S, A)$. If $\lambda(u, S, A)$ has an endvertex in P and there is no $(w, R, Z) \in \mathcal{K}_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$, then we let $\alpha(u, S, A)$ be the endvertex of $\lambda(u,S,A)$ in X. Next assume $(u,S,A)\in\mathcal{K}_3$. In this case, we let $\alpha(u,S,A)$ be an endvertex of $\lambda(u,S,A)$ which satisfies (ii) and (iii) of Claim 3.2. If there is no $(w,R,Z)\in\mathcal{K}_3$ with $(w,R,Z)\neq(u,S,A)$ such that $\lambda(w,R,Z)=\lambda(u,S,A)$, then we choose $\alpha(u,S,A)$ so that it also satisfies (i) of Claim 3.2. Finally, if $(u,S,A)\in\mathcal{K}_4$, let $\alpha(u,S,A)=a$, where a is as in Claim 3.5. Note that if $(u_1,S_1,A_1),(u_2,S_2,A_2)\in\mathcal{K}_3$ with $(u_1,S_1,A_1)\neq(u_2,S_2,A_2)$ and $\lambda(u_1,S_1,A_1)=\lambda(u_2,S_2,A_2)$, then $u_1\neq u_2$ by Claim 4.5, and hence it follows from Lemmas 2.6 and 2.13 that both endvertices of $\lambda(u_1,S_1,A_1)$ satisfy (ii) and (iii) of Claim 3.2. Thus in view of Claim 4.6, we can define $\alpha(u,S,A)$ so that the following claim holds.

Claim 4.9. *Let* $(u_1,S_1,A_1),(u_2,S_2,A_2)\in\mathcal{K}_2\cup\mathcal{K}_3$ *with* $(u_1,S_1,A_1)\neq(u_2,S_2,A_2)$ *and* $\lambda(u_1,S_1,A_1)=\lambda(u_2,S_2,A_2)$. *Then* $\alpha(u_1,S_1,A_1)\neq\alpha(u_2,S_2,A_2)$.

Finally we define $\varphi(u,S,A)$. If $(u,S,A)\in\mathcal{K}_1\cup\mathcal{K}_2$, simply let $\varphi(u,S,A)=\lambda(u,S,A)$; if $(u,S,A)\in\mathcal{K}_3$, let $\varphi(u,S,A)$ be a 4-contractible edge incident with $\alpha(u,S,A)$, whose existence is guaranteed by Claim 3.2(iii) or Lemma 2.6 (it is possible that the other endvertex of $\varphi(u,S,A)$ lies in $\overline{X}$); if $(u,S,A)\in\mathcal{K}_4$, let $\varphi(u,S,A)=ax$, where a,x are as in Claim 3.6.

4.2. Properties of $\boldsymbol{\varphi(u,S,A)}$

In this subsection, we complete the proof of Theorem 2 by showing that for any pair (e,a) of a 4-contractible edge e and an endvertex a of e, there are at most two members (u,S,A) of $\mathcal{K}_0$ for which $(\varphi(u,S,A),\alpha(u,S,A))=(e,a)$. The first two claims immediately follow from Claims 4.5 and 4.9, respectively.

Claim 4.10. *Let* $(u,S,A),(v,T,B)\in\mathcal{K}_1$ *with* $(u,S,A)\neq(v,T,B)$. *Then* $(\varphi(u,S,A),\alpha(u,S,A))\neq(\varphi(v,T,B),\alpha(v,T,B))$.

Claim 4.11. *Let* $(u,S,A),(v,T,B)\in\mathcal{K}_2$ *with* $(u,S,A)\neq(v,T,B)$. *Then* $(\varphi(u,S,A),\alpha(u,S,A))\neq(\varphi(v,T,B),\alpha(v,T,B))$.

The following claims are virtually the same as Claims 7.3 and 7.4, respectively, in [1].

Claim 4.12. *Let* $(u,S,A)\in\mathcal{K}_2$ *and* $(v,T,B)\in\mathcal{K}_1$, *and suppose that* $\varphi(u,S,A)=\varphi(v,T,B)$. *Then* $v\in P_{u,S,A}$, *and there is no* $(w,R,Z)\in\mathcal{K}_2$ *with* $(w,R,Z)\neq(u,S,A)$ *such that* $\varphi(w,R,Z)=\varphi(u,S,A)$.

Proof. Write $\varphi(u,S,A)=\varphi(v,T,B)=vb$. Also let vz be an edge in F such that $v,z\in T$. Let $(P,X)=(P_{u,S,A},X_{u,S,A})$. Suppose that $v\in X$. Then since $vz\in E(G)$, we have $z\in P\cup X$, and hence $z\in(P\cup X)\cap T$. Since $\deg_G(v)=4$, it follows from the definition of $\mathcal{K}_0$ that $N_G(v)\cap B=\{b\}$. Since $u\in N_G(v)\cap N_G(b)$, this implies $u\in T$, and hence $u\in P\cap T$. Thus by Lemmas 2.1 and 2.2, there exists a 4-cutset U with $U \supseteq (P \cup X) \cap T$ such that $G - U$ has a component H with $V(H) \subseteq X - (X \cap T) \subseteq X - \{v\}$. But then since $v \in X \cap T \subseteq U$, $z \in (P \cup X) \cap T \subseteq U$ and $vz \in F \subseteq E_n(G) - E_{tn}(G)$, U is a nontrivial 4-cutset, which contradicts the minimality of X because $u \in P \cap T \subseteq U$ (see the remark made in the paragraph preceding Claim 4.2). Thus $v \in P$. Now suppose that there exists $(w, R, Z) \in \mathcal{K}_2$ with $(w, R, Z) \neq (u, S, A)$ such that $\varphi(w, R, Z) = \varphi(u, S, A)$. Then $w \neq u$ by Claim 4.5. Hence applying Lemma 2.13 with $a = v$, we see that $v \notin \tilde{V}(G)$. But this contradicts the assumption that $(v, T, B) \in \mathcal{K}_1$. Thus there no such (w, R, Z). ■

Claim 4.13. *Let $(u, S, A) \in \mathcal{K}_2$ and $(v, T, B) \in \mathcal{K}_1$. Then $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.*

Proof. We may assume $\varphi(u, S, A) = \varphi(v, T, B)$. Write $\varphi(u, S, A) = vb$. We have $\alpha(v, T, B) = v$ by definition. On the other hand, in view of Claim 4.12, $\alpha(u, S, A) = b$ by the choice of $\alpha(u, S, A)$ described in Subsection 4.1. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$. ■

The following claim corresponds to Claim 7.5 in [1].

Claim 4.14. *Let $(u, S, A) \in \mathcal{K}_3$ and $(v, T, B) \in \mathcal{K}_1$. Then $\alpha(u, S, A) \neq \alpha(v, T, B)$.*

Proof. By Lemma 2.13 and Claim 3.2, $\alpha(u, S, A) \notin \tilde{V}(G)$. On the other hand, $\alpha(v, T, B) = v \in \tilde{V}(G)$. Thus $\alpha(u, S, A) \neq \alpha(v, T, B)$. ■

The following claims are virtually the same as Claims 7.6 and 7.7, respectively, in [1].

Claim 4.15. *Let $(u, S, A) \in \mathcal{K}_3 \cup \mathcal{K}_4$ and $(v, T, B) \in \mathcal{K}_2$. Then $\varphi(u, S, A) \neq \varphi(v, T, B)$.*

Proof. Suppose that $\varphi(u, S, A) = \varphi(v, T, B)$. Write $\lambda(u, S, A) = ab$ with $\alpha(u, S, A) = a$. Then $\deg_G(a) = 4$. Also write $\varphi(v, T, B) = ax$. Then $v \in N_G(a) \cap N_G(x)$. First assume that there exists $(w, R, Z) \in \mathcal{K}_3$ with $(w, R, Z) \neq (u, S, A)$ such that $\lambda(w, R, Z) = \lambda(u, S, A)$. Then $\deg_G(b) = 4$. By Claim 4.5, $w \neq u$. Thus $N_G(a) = \{u, b, w, x\}$. Since $\deg_G(v) \geq 5$ and $\deg_G(b) = 4$, $v \neq b$. Since $v \in N_G(a) \cap N_G(x) \subseteq N_G(a) - \{x\}$, this implies $v = u$ or w. On the other hand, $\deg_G(a) = 4$ and a is a common endvertex of $\varphi(v, T, B)$ and $\lambda(u, S, A) = \lambda(w, R, Z)$. Since $\varphi(v, T, B) = \lambda(v, T, B)$, this contradicts Claim 4.7. Next assume that there is no such (w, R, Z). Write $N_G(a) = \{u, b, x, y\}$. Suppose that $(u, S, A) \in \mathcal{K}_3$. Then $xy \notin E(G)$ by the choice of $\alpha(u, S, A)$, which implies $v \neq y$. Also we have $\deg_G(b) = 4$ by the definition of $\lambda(u, S, A)$, which implies $v \neq b$. Consequently, $v = u$, which contradicts Claim 4.7. Suppose that $(u, S, A) \in \mathcal{K}_4$. By Claim 3.6, $xy \notin E(G)$, which implies $v \neq y$. Again by

Claim 3.6, $xb \notin E(G)$, and hence $v \neq b$. Thus $v = u$, which again contradicts Claim 4.7. ∎

Claim 4.16. *Let* $(u, S, A), (v, T, B) \in \mathcal{K}_3$ *with* $(u, S, A) \neq (v, T, B)$. *Then* $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.

Proof. Suppose that $(\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))$. Write $\lambda(u, S, A) = ab$, $\varphi(u, S, A) = \varphi(v, T, B) = ax$, and $N_G(a) = \{u, b, x, y\}$. Then $\alpha(u, S, A) = \alpha(v, T, B) = a$, and $v \in N_G(a) - \{x\}$. Since $\deg_G(a) = 4$ and a is a common endvertex of $\lambda(u, S, A)$ and $\lambda(v, T, B)$, $v \neq u$ by Claim 4.7. Since $\deg_G(b) = 4$, $v \neq b$. Thus $v = y$, and hence $\lambda(v, T, B) = au$ or ab. On the other hand, since $\deg_G(u) \geq 5$, $\lambda(v, T, B) \neq au$. Consequently $\lambda(v, T, B) = ab$, which contradicts Claim 4.9. ∎

The following claim shows that in most cases, we have $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$ for $(u, S, A), (v, T, B) \in \mathcal{K}_0$ with $(u, S, A) \neq (v, T, B)$.

Claim 4.17. *The following hold.*

(i) *Let* $(u, S, A), (v, T, B) \in \mathcal{K}_0 - \mathcal{K}_4$ *with* $(u, S, A) \neq (v, T, B)$. *Then* $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.

(ii) *Let* $(u, S, A) \in \mathcal{K}_4$, $(v, T, B) \in \mathcal{K}_0 - \mathcal{K}_1$ *with* $(u, S, A) \neq (v, T, B)$. *Then* $(\varphi(u, S, A), \alpha(u, S, A)) \neq (\varphi(v, T, B), \alpha(v, T, B))$.

Proof. Statement (i) follows from Claims 4.10, 4.11 and 4.13 through 4.16. Thus we prove (ii). By Claim 4.15, we may assume that $(v, T, B) \in \mathcal{K}_3 \cup \mathcal{K}_4$. Suppose that $(\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))$. Let $(P, X) = (P_{u,S,A}, X_{u,S,A})$ and let a, b, x, y be as in Claims 3.5 and 3.6. Also let $(Q, Y) = (P_{v,T,B}, X_{v,T,B})$. Note that $N_G(a) = \{u, b, x, y\}$ and $v \in N_G(a) - \{x\}$. If $v = y$, then $a, y \in Q$, and hence $ay \in E_n(G)$, which contradicts Claim 3.6. Thus $v \neq y$. We also have $v \neq u$ by Claim 4.7. Consequently $v = b$, which implies $\lambda(v, T, B) = au$ or ay. Suppose that $(v, T, B) \in \mathcal{K}_3$. Then since $V(\lambda(v, T, B)) \subseteq V_4(G)$, $\lambda(v, T, B) = ay$. But then $ay \in E_n(G)$ by the definition of $\mathcal{K}_3$, which contradicts Claim 3.6. Thus we have $(v, T, B) \in \mathcal{K}_4$. Applying Claim 3.6 to (Q, Y), we now obtain $b, a \in Q$, $x \in \overline{Y}$ and $y, u \in Y$. In particular, $xu \notin E(G)$. Set $U = (P \cap Q) \cup (P \cap Y) \cup (X \cap Q)$. Since $y \in X \cap Y$ and $x \in \overline{X} \cap \overline{Y}$, it follows from Lemma 2.2 that $(U, X \cap Y) \in \mathcal{L}$. Since $u \in P \cap Y \subseteq U$, it follows from the minimality of X that $(U, X \cap Y) \notin \mathcal{L}_0$, i.e., U is a trivial 4-cutset. Hence there exists $c \in V_4(G)$ such that $N_G(c) = U$. Since $a, b, u \in U$, $c \in N_G(a) - \{b, u\} = \{x, y\}$. On the other hand, since $xu \notin E(G)$, $c \neq x$. Consequently $c = y$, which implies $y \in N_G(u) \cap X \cap V_4(G)$. But since $(u, S, A) \in \mathcal{K}_4$, this contradicts Claim 3.4. ∎

The following claim, together with Claim 4.17, shows that for each $e \in E_c(G)$ and for each endvertex a of e, there are at most two members (u, S, A) of $\mathcal{K}_0$ such that $(\varphi(u, S, A), \alpha(u, S, A)) = (e, a)$.

Claim 4.18. *Let* $(u, S, A) \in \mathcal{K}_4$, $(v, T, B) \in \mathcal{K}_1$ *with* $(\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(v, T, B), \alpha(v, T, B))$. *Then* $(\varphi(w, R, Z), \alpha(w, R, Z)) \neq (\varphi(u, S, A), \alpha(u, S, A))$ *for* $(w, R, Z) \in \mathcal{K}_0 - \{(u, S, A), (v, T, B)\}$.

Proof. Suppose that there exists $(w, R, Z) \in \mathcal{K}_0 - \{(u, S, A), (v, T, B)\}$ such that

$$(\varphi(w, R, Z), \alpha(w, R, Z)) = (\varphi(u, S, A), \alpha(u, S, A)).$$

By Claim 4.17(ii), we have $(w, R, Z) \in \mathcal{K}_1 - \{(v, T, B)\}$. On the other hand, since

$$(\varphi(v, T, B), \alpha(v, T, B)) = (\varphi(u, S, A), \alpha(u, S, A)) = (\varphi(w, R, Z), \alpha(w, R, Z)),$$

it follows from Claim 4.17(i) that $(w, R, Z) \in \mathcal{K}_4 - \{(u, S, A)\}$, which is a contradiction. ■

In view of the remark made before the statement of Claim 4.18, it follows from Claims 4.17 and 4.18 that for each $e \in E_c(G)$, there are at most four members (u, S, A) of $\mathcal{K}_0$ such that $\varphi(u, S, A) = e$. This completes the proof of Theorem 2. ■

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