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ON EDGE *H*-IRREGULARITY STRENGTHS OF SOME GRAPHS

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Abstract

For a graph G an *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. In this case we say that G admits an $(H_1, H_2, \dots, H_t)$-(*edge*) *covering*. An H-*covering* of graph G is an $(H_1, H_2, \dots, H_t)$-(*edge*) *covering* in which every subgraph H_i is isomorphic to a given graph H.

Let G be a graph admitting H-covering. An edge k-labeling $\alpha : E(G) \to \{1, 2, \dots, k\}$ is called an *H-irregular edge k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H their weights

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$wt_\alpha(H')$ and $wt_\alpha(H'')$ are distinct. The weight of a subgraph H under an edge k-labeling α is the sum of labels of edges belonging to H. The *edge H-irregularity strength* of a graph G, denoted by $\mathrm{ehs}(G, H)$, is the smallest integer k such that G has an H-irregular edge k-labeling.

In this paper we determine the exact values of $\mathrm{ehs}(G, H)$ for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs. Moreover the subgraph H is isomorphic to only C_4, C_3 and K_4.

Keywords: H-irregular edge labeling, edge H-irregularity strength, prism, antiprism, triangular ladder, diagonal ladder, wheel, gear graph.

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1. Introduction

Consider a simple and finite graph $G = (V, E)$ of order at least 2. An edge k-labeling is a function $\alpha : E(G) \to \{1, 2, \ldots, k\}$, where k is a positive integer. Then the associated weight of a vertex $x \in V(G)$ is $w_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)$, where the sum is taken over all edges incident to x. Such a labeling α is called *irregular* if the obtained weights of all vertices are different. The smallest positive integer k for which there exists an irregular labeling of G is called the *irregularity strength* of G and is denoted by $\mathrm{s}(G)$. If it does not exist, then we write $\mathrm{s(G)} = \infty$. One can easily see that $\mathrm{s(G)} < \infty$ if and only if G contains no isolated edges and has at most one isolated vertex.

The notion of the irregularity strength was firstly introduced by Chartrand *et al.* in [7]. Some results on the irregularity strength can be found in [2, 3, 5, 6, 8, 9, 11–14].

A vertex k-labeling $\beta : V(G) \to \{1, 2, \ldots, k\}$ is called an *edge irregular k-labeling* of the graph G if the weights $w_\beta(xy) \neq w_\beta(x'y')$ for every two distinct edges xy and $x'y'$, where the weight of an edge $xy \in E(G)$ is $w_\beta(xy) = \beta(x)+\beta(y)$. The minimum k for which a graph G admits an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $\mathrm{es}(G)$. The notion of the edge irregularity strength was defined by Ahmad *et al.* in [1].

A family of subgraphs $H_1, H_2, \ldots, H_t$ is said to be an *edge-covering* of G if each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. In this case we say that G admits an $(H_1, H_2, \ldots, H_t)$-*(edge) covering.* If every subgraph H_i, $i = 1, 2, \ldots, t$, is isomorphic to a given graph H, then the graph G admits an *H-covering.*

Motivated by the irregularity strength and the edge irregularity strength of a graph G Ashraf *et al.* in [4] introduced a new parameter, edge H-irregularity strength, as a natural extension of the parameters $\mathrm{s}(G)$ and $\mathrm{es}(G)$. Let G be a graph admitting H-covering. An edge k-labeling α is called an *H-irregular* *edge k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H we have

$$wt_\alpha(H') = \sum_{e \in E(H')} \alpha(e) \neq \sum_{e \in E(H'')} \alpha(e) = wt_\alpha(H'').$$

The *edge H-irregularity strength* of a graph G, denoted by $\mathrm{ehs}(G, H)$, is the smallest integer k for which G has an H-irregular edge k-labeling.

Next theorem proved in [4] gives the lower bound of the edge H-irregularity strength of a graph G.

Theorem 1 [4]**.** *Let G be a graph admitting an H-covering and t is the number of all the subgraphs isomorphic to H. Then*

$$\mathrm{ehs}(G, H) \geq \left\lceil 1 + \frac{t-1}{|E(H)|} \right\rceil.$$

Note that the parameter t is the number of all subgraphs of G isomorphic to H. In this paper we determine exact values of the edge H-irregularity strength for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs for some H. Moreover the subgraph H is isomorphic to only C_4, C_3 and K_4.

2. Prism and Antiprism

The *prism* D_n can be defined as the Cartesian product $C_n \Box P_2$ of a cycle on n vertices with a path on 2 vertices. Let $V(C_n \Box P_2) = \{x_i, y_i : 1 \leq i \leq n\}$ be the vertex set and $E(C_n \Box P_2) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq n\}$ be the edge set, where the indices are taken modulo n. Hence, the graph D_n has $2n$ vertices and $3n$ edges.

Theorem 2. *Let $D_n = C_n \Box P_2$, $n \geq 3$, $n \neq 4$, be a prism. Then*

$$\mathrm{ehs}(D_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. The prism D_n, $n \geq 3$, $n \neq 4$, admits a C_4-covering with exactly n cycles C_4. We denote these 4-cycles by the symbols C_4^i, $i = 1, 2, \ldots, n$, such that the vertex set of C_4^i is $V(C_4^i) = \{x_i, x_{i+1}, y_i, y_{i+1}\}$ and the edge set is $E(C_4^i) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, x_{i+1} y_{i+1}\}$.

From Theorem 1 it follows that $\mathrm{ehs}(D_n, C_4) \geq \lceil \frac{n+3}{4} \rceil$. To show that $\lceil \frac{n+3}{4} \rceil$ is an upper bound for the edge C_4-irregularity strength of D_n we define a C_4-irregular edge labeling $\alpha_1 : E(D_n) \to \{1, 2, \ldots, \lceil \frac{n+3}{4} \rceil\}$, in the following way. We distinguish to cases according to the parity of n.

Case 1. When n is odd, then

$$\alpha_1(x_ix_{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \le i \le \frac{n+1}{2}, \\ \lceil \frac{n+1-i}{2} \rceil & \text{for } \frac{n+1}{2}+1 \le i \le n, \end{cases}$$

$$\alpha_1(y_iy_{i+1}) = \begin{cases} \lceil \frac{i+1}{2} \rceil & \text{for } 1 \le i \le \frac{n+1}{2}, \\ \lceil \frac{n+2-i}{2} \rceil & \text{for } \frac{n+1}{2}+1 \le i \le n, \end{cases}$$

$$\alpha_1(x_iy_i) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \le i \le \frac{n+1}{2}, \\ \lceil \frac{n+3-i}{2} \rceil & \text{for } \frac{n+1}{2}+1 \le i \le n. \end{cases}$$

Case 2. When n is even, then

$$\alpha_1(x_ix_{i+1}) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \le i \le \frac{n}{2}, \\ \lceil \frac{n+1-i}{2} \rceil & \text{for } \frac{n}{2}+1 \le i \le n, \end{cases}$$

$$\alpha_1(y_iy_{i+1}) = \begin{cases} \lceil \frac{i+1}{2} \rceil & \text{for } 1 \le i \le \frac{n}{2}, \\ \lceil \frac{n+3}{4} \rceil & \text{for } i = \frac{n}{2}+1, \\ \lceil \frac{n+2-i}{2} \rceil & \text{for } \frac{n}{2}+2 \le i \le n, \end{cases}$$

$$\alpha_1(x_iy_i) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \le i \le \frac{n}{2}, \\ \frac{n}{4}+1 & \text{for } i = \frac{n}{2}+1 \text{ and } n \equiv 0 \pmod 4, \\ \frac{n+2}{4} & \text{for } i = \frac{n}{2}+1 \text{ and } n \equiv 2 \pmod 4, \\ \lceil \frac{n+3-i}{2} \rceil & \text{for } \frac{n}{2}+2 \le i \le n. \end{cases}$$

It is easy to see that under the labeling α_1 all edge labels are at most $\lceil \frac{n+3}{4} \rceil$. The C_4-weights of the cycles C_4^i, $i = 1, 2, \dots, n$, under the edge labeling α_1, are given by

$$wt_{\alpha_1}(C_4^i) = \sum_{e \in E(C_4^i)} \alpha_1(e) = \alpha_1(x_ix_{i+1}) + \alpha_1(y_iy_{i+1}) + \alpha_1(x_iy_i) + \alpha_1(x_{i+1}y_{i+1}).$$

Case 1. When n is odd, then

$$\begin{aligned} wt_{\alpha_1}(C_4^i) &= \lceil \tfrac{i}{2} \rceil + \lceil \tfrac{i+1}{2} \rceil + \lceil \tfrac{i}{2} \rceil + \lceil \tfrac{i+1}{2} \rceil = 2i+2 \quad \text{for } 1 \le i \le \tfrac{n-1}{2}, \\ wt_{\alpha_1}\left(C_4^{\frac{n+1}{2}}\right) &= \lceil \tfrac{n+1}{4} \rceil + \lceil \tfrac{n+3}{4} \rceil + \lceil \tfrac{n+1}{4} \rceil + \lceil \tfrac{n+3}{4} \rceil = n+3, \\ wt_{\alpha_1}(C_4^i) &= \lceil \tfrac{n+1-i}{2} \rceil + \lceil \tfrac{n+2-i}{2} \rceil + \lceil \tfrac{n+3-i}{2} \rceil + \lceil \tfrac{n+2-i}{2} \rceil = 2n+5-2i \\ &\quad \text{for } \tfrac{n+1}{2}+1 \le i \le n-1, \\ wt_{\alpha_1}(C_4^n) &= \lceil \tfrac{1}{2} \rceil + \lceil \tfrac{2}{2} \rceil + \lceil \tfrac{3}{2} \rceil + \lceil \tfrac{1}{2} \rceil = 5. \end{aligned}$$

Case 2. When n is even, then

$$
\begin{aligned}
wt_{\alpha_1}(C_4^i) &= \left\lceil \tfrac{i}{2} \right\rceil + \left\lceil \tfrac{i+1}{2} \right\rceil + \left\lceil \tfrac{i}{2} \right\rceil + \left\lceil \tfrac{i+1}{2} \right\rceil = 2i+2 \quad \text{for } 1 \le i \le \tfrac{n}{2}-1,\\
wt_{\alpha_1}\left(C_4^{\frac{n}{2}}\right) &= \left\lceil \tfrac{n}{4} \right\rceil + \left\lceil \tfrac{n+2}{4} \right\rceil + \left\lceil \tfrac{n}{4} \right\rceil + \tfrac{n}{4} + 1 = n+2 \quad \text{for } n \equiv 0 \pmod 4,\\
wt_{\alpha_1}\left(C_4^{\frac{n}{2}}\right) &= \left\lceil \tfrac{n}{4} \right\rceil + \left\lceil \tfrac{n+2}{4} \right\rceil + \left\lceil \tfrac{n}{4} \right\rceil + \tfrac{n+2}{4} = n+2 \quad \text{for } n \equiv 2 \pmod 4,\\
wt_{\alpha_1}\left(C_4^{\frac{n}{2}+1}\right) &= \left\lceil \tfrac{n}{4} \right\rceil + \left\lceil \tfrac{n+3}{4} \right\rceil + \left\lceil \tfrac{n}{4} \right\rceil + 1 + \tfrac{n}{4} + 1 = n+3 \quad \text{for } n \equiv 0 \pmod 4,\\
wt_{\alpha_1}\left(C_4^{\frac{n}{2}+1}\right) &= \left\lceil \tfrac{n}{4} \right\rceil + \left\lceil \tfrac{n+3}{4} \right\rceil + \left\lceil \tfrac{n+2}{4} \right\rceil + \tfrac{n+2}{4} = n+3 \quad \text{for } n \equiv 2 \pmod 4,\\
wt_{\alpha_1}(C_4^i) &= \left\lceil \tfrac{n+1-i}{2} \right\rceil + \left\lceil \tfrac{n+2-i}{2} \right\rceil + \left\lceil \tfrac{n+3-i}{2} \right\rceil + \left\lceil \tfrac{n+2-i}{2} \right\rceil = 2n+5-2i\\
&\quad \text{for } \tfrac{n}{2}+2 \le i \le n-1,\\
wt_{\alpha_1}(C_4^n) &= \left\lceil \tfrac{1}{2} \right\rceil + \left\lceil \tfrac{2}{2} \right\rceil + \left\lceil \tfrac{3}{2} \right\rceil + \left\lceil \tfrac{1}{2} \right\rceil = 5.
\end{aligned}
$$

Combining the previous we get that

$$
wt_{\alpha_1}(C_4^i) = \begin{cases} 2(1+i) & \text{for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ 1+2(n+2-i) & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases}
$$

One can see that the weights of cycles C_4^i, for $i = 1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil$, are even and in increasing order, therefore $wt_{\alpha_1}\left(C_4^{i+1}\right) > wt_{\alpha_1}\left(C_4^i\right)$.

On the other hand, the weights of cycles C_4^i, for $i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n$, are odd and in decreasing order, therefore $wt_{\alpha_1}\left(C_4^{i+1}\right) < wt_{\alpha_1}\left(C_4^i\right)$.

Thus the edge weights are distinct numbers from the set $\{4, 5, \dots, n+3\}$. This shows that $\text{ehs}(D_n, C_4) \le \left\lceil \frac{n+3}{4} \right\rceil$. Hence the proof is concluded. ■

The *antiprism* A_n [10], $n \ge 3$, is a 4-regular graph (Archimedean convex polytope), consisting of $2n$ vertices and $4n$ edges. The vertex and edge set of A_n are defined as: $V(A_n) = \{x_i, y_i : 1 \le i \le n\}$, $E(A_n) = \{x_i y_i : 1 \le i \le n\} \cup \{x_i x_{i+1} : 1 \le i \le n\} \cup \{y_i x_{i+1} : 1 \le i \le n\} \cup \{y_i y_{i+1} : 1 \le i \le n\}$, with indices taken modulo n.

Theorem 3. *Let A_n, $n \ge 4$, be an antiprism. Then*

$$
\text{ehs}(A_n, C_3) = \left\lceil \frac{2n+2}{3} \right\rceil.
$$

Proof. The antiprism A_n, $n \ge 4$, admits a C_3-covering with exactly $2n$ cycles C_3. The first type of the cycle C_3 has the vertex set $V(C_3^i) = \{x_i, x_{i+1}, y_i : 1 \le i \le n\}$ and the edge set $E(C_3^i) = \{x_i x_{i+1}, x_i y_i, y_i x_{i+1} : 1 \le i \le n\}$. The second type of the cycle C_3 has the vertex set $V(\mathcal{C}_3^i) = \{y_i, y_{i+1}, x_{i+1} : 1 \le i \le n\}$ and the edge set $E(\mathcal{C}_3^i) = \{y_i y_{i+1}, y_i x_{i+1}, y_{i+1} x_{i+1} : 1 \le i \le n\}$. Note that the indices are taken modulo n.

From Theorem 1 it follows that $\mathrm{ehs}(A_n, C_3) \ge \left\lceil \frac{2n+2}{3} \right\rceil$. To show that $\left\lceil \frac{2n+2}{3} \right\rceil$ is an upper bound for the edge C_3-irregularity strength of A_n we define a C_3-irregular edge labeling $\alpha_2 : E(A_n) \to \left\{1, 2, \dots, \left\lceil \frac{2n+2}{3} \right\rceil\right\}$, in the following way. We distinguish two cases.

Case 1. When $n \equiv 0, 4, 5 \pmod 6$, then

$$\alpha_2(x_i x_{i+1}) = \begin{cases} i & \text{for } i = 1, 2, \\ i + \left\lfloor \frac{i}{3} \right\rfloor & \text{for } 3 \le i \le t-1, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = t, \\ n - i + 3 + \left\lfloor \frac{n-i-2}{3} \right\rfloor & \text{for } t+1 \le i \le n-2, \\ n - i + 2 & \text{for } i = n-1, n, \end{cases}$$

where $t = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 5 \pmod 6, \\ \frac{n}{2} + 1 & \text{if } n \equiv 0, 4 \pmod 6. \end{cases}$

$$\alpha_2(y_i y_{i+1}) = \begin{cases} i + 1 + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 5 \pmod 6, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 0, 4 \pmod 6, \\ n - i + 2 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n-1, \\ 1 & \text{for } i = n, \end{cases}$$

$$\alpha_2(x_i y_i) = \begin{cases} i + \left\lfloor \frac{i-1}{3} \right\rfloor & \text{for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 0, 5 \pmod 6, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ and } n \equiv 4 \pmod 6, \\ n - i + 3 + \left\lfloor \frac{n-i-1}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 2 \le i \le n-1, \\ 2 & \text{for } i = n, \end{cases}$$

$$\alpha_2(y_i x_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ i + 1 + \left\lfloor \frac{i-2}{3} \right\rfloor & \text{for } 2 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1, \\ \left\lceil \frac{2n+2}{3} \right\rceil - 1 & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 0, 4 \pmod 6, \\ \left\lceil \frac{2n+2}{3} \right\rceil & \text{for } i = \left\lceil \frac{n}{2} \right\rceil \text{ and } n \equiv 5 \pmod 6, \\ n - i + 2 + \left\lfloor \frac{n-i}{3} \right\rfloor & \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n. \end{cases}$$

Case 2. When $n \equiv 1, 2, 3 \pmod 6$, then

$$\alpha_2(x_i x_{i+1}) = \begin{cases} i & \text{for } i = 1, 2, \\ i + \lfloor \frac{i}{3} \rfloor & \text{for } 3 \le i \le t - 1, \\ \lceil \frac{2n+2}{3} \rceil & \text{for } i = t \text{ and } n \equiv 2 \pmod 6, \\ \lceil \frac{2n+2}{3} \rceil - 1 & \text{for } i = t \text{ and } n \equiv 1, 3 \pmod 6, \\ n - i + 3 + \lfloor \frac{n-i-2}{3} \rfloor & \text{for } t + 1 \le i \le n - 2, \\ n - i + 2 & \text{for } i = n - 1, n, \end{cases}$$

where $t = \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod 6, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 6. \end{cases}$

$$\alpha_2(y_i y_{i+1}) = \begin{cases} i + 1 + \lfloor \frac{i-1}{3} \rfloor & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 1, \\ \lceil \frac{2n+2}{3} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil, \\ n - i + 2 + \lfloor \frac{n-i-1}{3} \rfloor & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n - 1, \\ 1 & \text{for } i = n, \end{cases}$$

$$\alpha_2(x_i y_i) = \begin{cases} i + \lfloor \frac{i-1}{3} \rfloor & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil, \\ \lceil \frac{2n+2}{3} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \equiv 2 \pmod 6, \\ \lceil \frac{2n+2}{3} \rceil - 1 & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \equiv 1, 3 \pmod 6, \\ n - i + 3 + \lfloor \frac{n-i-1}{3} \rfloor & \text{for } \lceil \frac{n}{2} \rceil + 2 \le i \le n - 1, \\ 2 & \text{for } i = n, \end{cases}$$

$$\alpha_2(y_i x_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ i + 1 + \lfloor \frac{i-2}{3} \rfloor & \text{for } 2 \le i \le \lceil \frac{n}{2} \rceil - 1, \\ \lceil \frac{2n+2}{3} \rceil - 1 & \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } n \equiv 1, 2 \pmod 6, \\ \lceil \frac{2n+2}{3} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } n \equiv 3 \pmod 6, \\ n - i + 2 + \lfloor \frac{n-i}{3} \rfloor & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases}$$

Now we compute the C_3-weights under the edge labeling α_2 as follows. For the weights of 3-cycles of the first type we get

$$\begin{aligned} wt_{\alpha_2}(C_3^i) &= \sum_{e \in E(C_3^i)} \alpha_2(e) = \alpha_2(x_i x_{i+1}) + \alpha_2(x_i y_i) + \alpha_2(y_i x_{i+1}) \\ &= \begin{cases} 4i - 1 & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil, \\ 4n - 4i + 6 & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n, \end{cases} \end{aligned}$$

and for the weights of 3-cycles of the second type we have

$$wt_{\alpha_2}(\mathcal{C}_3^i) = \sum_{e \in E(\mathcal{C}_3^i)} \alpha_2(e) = \alpha_2(y_i y_{i+1}) + \alpha_2(y_i x_{i+1}) + \alpha_2(y_{i+1} x_{i+1})$$
$$= \begin{cases} 4i+1 & \text{for } 1 \le i \le \lceil \frac{n+1}{2} \rceil - 1, \\ 4n-4i+4 & \text{for } \lceil \frac{n+1}{2} \rceil \le i \le n. \end{cases}$$

Combining these two cases one can see that the weights of the cycles C_3^i are different to the weights of the cycles $\mathcal{C}_3^i$. This shows that α_2 is an edge C_3-irregular labeling of A_n. Therefore, $\text{ehs}(A_n, C_4) \le \lceil \frac{2n+2}{3} \rceil$ and we arrive at the desired result. ∎

3. Triangular Ladder and Diagonal Ladder

Let $L_n \cong P_n \square P_2$, $n \ge 2$, be a *ladder* with the vertex set $V(L_n) = \{x_i, y_i : i = 1, 2, \dots, n\}$ and the edge set $E(L_n) = \{x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \dots, n-1\} \cup \{x_i y_i : i = 1, 2, \dots, n\}$. The *triangular ladder* TL_n, $n \ge 2$, is obtained from a ladder L_n by adding the edges $y_i x_{i+1}$ for $i = 1, 2, \dots, n-1$.

Theorem 4. *Let TL_n, $n \ge 2$, be a triangular ladder. Then*

$$\text{ehs}(TL_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. The triangular ladder TL_n, $n \ge 2$, admits a C_3-covering with exactly $2(n-1)$ cycles C_3. There are two types of cycles C_3 that cover TL_n. The first type of cycles C_3 has the vertex set $V(C_3^i) = \{x_i, x_{i+1}, y_i : 1 \le i \le n-1\}$ and the edge set $E(C_3^i) = \{x_i x_{i+1}, x_i y_i, y_i x_{i+1} : 1 \le i \le n-1\}$. The second type of cycles C_3 has the vertex set $V(\mathcal{C}_3^i) = \{y_i, y_{i+1}, x_{i+1} : 1 \le i \le n-1\}$ and the edge set $E(\mathcal{C}_3^i) = \{y_i y_{i+1}, y_i x_{i+1}, y_{i+1} x_{i+1} : 1 \le i \le n-1\}$.

According to Theorem 1 it follows that $\text{ehs}(TL_n, C_3) \ge \lceil \frac{2n}{3} \rceil$. To show that $\lceil \frac{2n}{3} \rceil$ is an upper bound for the edge C_3-irregularity strength of TL_n we define a C_3-irregular edge labeling $\alpha_3 : E(TL_n) \to \{1, 2, \dots, \lceil \frac{2n}{3} \rceil\}$ as follows. Let us consider three cases.

Case 1. When $i \equiv 0 \pmod 3$, then

$$\begin{aligned} \alpha_3(x_i x_{i+1}) = \alpha_3(y_i y_{i+1}) &= \tfrac{2i}{3} && \text{for } i = 3, 6, \dots, n-1, \\ \alpha_3(y_i x_{i+1}) &= \tfrac{2i+3}{3} && \text{for } i = 3, 6, \dots, n-1, \\ \alpha_3(x_i y_i) &= \tfrac{2i}{3} && \text{for } i = 3, 6, \dots, n. \end{aligned}$$

Case 2. When $i \equiv 1 \pmod 3$, then

$$\begin{aligned} \alpha_3(x_ix_{i+1}) = \alpha_3(y_iy_{i+1}) = \alpha_3(y_ix_{i+1}) &= \tfrac{2i+1}{3} && \text{for } i = 1, 4, \ldots, n-1, \\ \alpha_3(x_iy_i) &= \tfrac{2i+1}{3} && \text{for } i = 1, 4, \ldots, n. \end{aligned}$$

Case 3. When $i \equiv 2 \pmod 3$, then

$$\begin{aligned} \alpha_3(x_ix_{i+1}) &= \tfrac{2i-1}{3} && \text{for } i = 2, 5, \ldots, n-1, \\ \alpha_3(y_iy_{i+1}) = \alpha_3(y_ix_{i+1}) &= \tfrac{2i+2}{3} && \text{for } i = 2, 5, \ldots, n-1, \\ \alpha_3(x_iy_i) &= \tfrac{2i+2}{3} && \text{for } i = 2, 5, \ldots, n. \end{aligned}$$

It is a routine matter to verify that under the labeling α_3 all edge labels are at most $\lceil \frac{2n}{3} \rceil$. It is not difficult to see that under the edge labeling α_3 the weights of the cycles C_3^i, $1 \le i \le n-1$, are of the form

$$wt_{\alpha_3}(C_3^i) = \alpha_3(x_ix_{i+1}) + \alpha_3(x_iy_i) + \alpha_3(y_ix_{i+1}) = 2i + 1.$$

The weights of the cycles $\mathcal{C}_3^i$, $1 \le i \le n-1$, are of the form

$$wt_{\alpha_3}(\mathcal{C}_3^i) = \alpha_3(y_iy_{i+1}) + \alpha_3(x_{i+1}y_{i+1}) + \alpha_3(y_ix_{i+1}) = 2(i + 1).$$

Combining these two cases we obtained that the weights are different for any two distinct cycles C_3. Thus $\text{ehs}(TL_n, C_3) \le \lceil \frac{2n}{3} \rceil$. This completes the proof. ■

The *diagonal ladder* DL_n is obtained from a ladder L_n by adding the edges $\{x_iy_{i+1}, x_{i+1}y_i : 1 \le i \le n-1\}$. So the diagonal ladder DL_n contains $2n$ vertices and $5n - 4$ edges.

Theorem 5. *Let DL_n, $n \ge 2$, be a diagonal ladder. Then*

$$\text{ehs}(DL_n, K_4) = \left\lceil \frac{n+4}{6} \right\rceil.$$

Proof. The diagonal ladder DL_n, $n \ge 2$, admits a K_4-covering with exactly $(n-1)$ complete graphs K_4. The K_4^i has the vertex set $V(K_4^i) = \{x_i, y_i, x_{i+1}, y_{i+1} : 1 \le i \le n-1\}$ and the edge set $E(K_4^i) = \{x_ix_{i+1}, x_iy_i, x_iy_{i+1}, y_iy_{i+1}, y_ix_{i+1}, x_{i+1}y_{i+1} : 1 \le i \le n-1\}$.

With respect to Theorem 1 it follows that $\text{ehs}(DL_n, K_4) \ge \lceil \frac{n+4}{6} \rceil$. To show that $\text{ehs}(DL_n, K_4) \le \lceil \frac{n+4}{6} \rceil$ we define a K_4-irregular edge labeling α_4 :

$E(DL_n) \to \{1, 2, \dots, \lceil \frac{n+4}{6} \rceil\}$, in the following way.

$$\alpha_4(y_i y_{i+1}) = \lceil \tfrac{i}{6} \rceil \quad \text{for } 1 \le i \le n-1,$$

$$\alpha_4(x_i x_{i+1}) = \begin{cases} \lceil \frac{i}{6} \rceil & \text{for } 1 \le i \le 5, \\ \lceil \frac{i+1}{6} \rceil & \text{for } 6 \le i \le n-1, \end{cases}$$

$$\alpha_4(x_i y_i) = \begin{cases} \lceil \frac{i}{6} \rceil & \text{for } 1 \le i \le 4, \\ \lceil \frac{i+2}{6} \rceil & \text{for } 5 \le i \le n, \end{cases}$$

$$\alpha_4(x_i y_{i+1}) = \begin{cases} 1 & \text{for } i = 1, \\ \lceil \frac{i+5}{6} \rceil & \text{for } 2 \le i \le n-1, \end{cases}$$

$$\alpha_4(x_{i+1} y_i) = \begin{cases} 1 & \text{for } i = 1, 2, \\ \lceil \frac{i+4}{6} \rceil & \text{for } 3 \le i \le n-1. \end{cases}$$

One can verify that under the labeling α_4 all edge labels are at least 1 and at most $\lceil \frac{n+4}{6} \rceil$. To show that α_4 is edge K_4-irregular labeling it will be enough to show that $wt_{\alpha_4}(K_4^i) < wt_{\alpha_4}(K_4^{i+1})$. It is a simple mathematical exercise that the weights of the subgraphs K_4^i, $i = 1, 2, 3, 4, 5$ are $wt_{\alpha_4}(K_4^i) = 5 + i$.

For $i = 6, 7, \dots, n-1$ we get

$$\begin{aligned} wt_{\alpha_4}(K_4^i) = \sum_{e \in E(K_4^i)} \alpha_4(e) &= \alpha_4(x_i x_{i+1}) + \alpha_4(y_i y_{i+1}) + \alpha_4(x_i y_i) + \alpha_4(x_{i+1} y_{i+1}) \\ &+ \alpha_4(x_i y_{i+1}) + \alpha_4(x_{i+1} y_i) = \lceil \tfrac{i+1}{6} \rceil + \lceil \tfrac{i}{6} \rceil + \lceil \tfrac{i+2}{6} \rceil + \lceil \tfrac{i+3}{6} \rceil + \lceil \tfrac{i+5}{6} \rceil \\ &+ \lceil \tfrac{i+4}{6} \rceil = 5 + i. \end{aligned}$$

This proves that $wt_{\alpha_4}(K_4^{i+1}) = wt_{\alpha_4}(K_4^i) + 1$ for $i = 1, 2, \dots, n-1$. Therefore, α_4 is an edge K_4-irregular labeling of DL_n. Thus $\text{ehs}(DL_n, K_4) \le \lceil \frac{n+4}{6} \rceil$. This concludes the proof. ∎

4. WHEEL AND GEAR GRAPH

A *wheel* W_n, $n \ge 3$, is a graph obtained by joining all vertices of cycle C_n to a further vertex c, called the *center*. Thus W_n contains $n+1$ vertices, say, $c, x_1, x_2, \dots, x_n$ and $2n$ edges, say, cx_i, $x_i x_{i+1}$, $1 \le i \le n$, where the indices are taken modulo n.

Theorem 6. *Let W_n, $n \ge 4$, be a wheel. Then*

$$\text{ehs}(W_n, C_3) = \left\lceil \frac{n+2}{3} \right\rceil.$$

Proof. The wheel W_n, $n \geq 4$, admits a C_3-covering with exactly n cycles C_3. Every cycle C_3 in W_n is of the form $C_3^i = cx_ix_{i+1}$, where $i = 1, 2, \ldots, n$ with indices taken modulo n.

According to Theorem 1 we have that $\text{ehs}(W_n, C_3) \geq \lceil \frac{n+2}{3} \rceil$. To show that $\lceil \frac{n+2}{3} \rceil$ is an upper bound for the edge C_3-irregularity strength of W_n we define a C_3-irregular edge labeling $\alpha_5 : E(W_n) \to \{1, 2, \ldots, \lceil \frac{n+2}{3} \rceil\}$ as follows.

$$\alpha_5(x_ix_{i+1}) = \begin{cases} i & \text{for } i = 1,\ 2, \\ i - \lfloor \frac{i}{3} \rfloor & \text{for } 3 \leq i \leq \lceil \frac{n+1}{2} \rceil - 1, \\ \lceil \frac{n+2}{3} \rceil & \text{for } i = \lceil \frac{n+1}{2} \rceil, \\ n - i + 1 - \lfloor \frac{n-i}{3} \rfloor & \text{for } \lceil \frac{n+1}{2} \rceil + 1 \leq i \leq n, \end{cases}$$

$$\alpha_5(cx_i) = \begin{cases} 1 & \text{for } i = 1, \\ i - 1 - \lfloor \frac{i-2}{3} \rfloor & \text{for } 2 \leq i \leq \lceil \frac{n}{2} \rceil, \\ \lceil \frac{n+2}{3} \rceil & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \equiv 0, 1 \pmod 3, \\ \lceil \frac{n+2}{3} \rceil - 1 & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \equiv 2 \pmod 3, \\ n - i + 1 - \lfloor \frac{n-i-1}{3} \rfloor & \text{for } \lceil \frac{n}{2} \rceil + 2 \leq i \leq n - 1, \\ 2 & \text{for } i = n. \end{cases}$$

It is a matter of routine checking that under the labeling α_5 all edge labels are at most $\lceil \frac{n+2}{3} \rceil$. For the C_3-weight of the cycle C_3^i we get

$$\begin{aligned} wt_{\alpha_5}(C_3^i) &= \alpha_5(cx_i) + \alpha_5(cx_{i+1}) + \alpha_5(x_ix_{i+1}) \\ &= \begin{cases} 2i + 1 & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil, \\ 2(n + 2 - i) & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n. \end{cases} \end{aligned}$$

Clearly, the weights of C_3^i for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ are odd and increasing. On the other hand the weights of C_3^i for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$ are even and decreasing. So, it concludes that all the weights of C_3^i are different. Thus α_5 is an edge C_3-irregular labeling of W_n and $\text{ehs}(W_n, C_3) \leq \lceil \frac{n+2}{3} \rceil$. This completes the proof. ∎

A *gear graph* G_n is obtained from W_n by inserting a vertex to each edge on the cycle C_n. Then the vertex set of G_n is $V(G_n) = \{c, x_i, y_i : 1 \leq i \leq n\}$ and the edge set is $E(W_n) = \{x_iy_i, y_ix_{i+1}, cx_i : 1 \leq i \leq n\}$ with indices taken modulo n.

Theorem 7. *Let G_n, $n \geq 3$, be a gear graph. Then*

$$\text{ehs}(G_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. The gear G_n, $n \geq 3$, admits a C_4-covering with exactly n cycles C_4. According to Theorem 1 we obtain that $\text{ehs}(G_n, C_4) \geq \lceil \frac{n+3}{4} \rceil$. To show that

$\lceil\frac{n+3}{4}\rceil$ is an upper bound for the edge C_4-irregularity strength of G_n we define a C_4-irregular edge labeling $\alpha_6 : E(G_n) \to \{1, 2, \dots, \lceil\frac{n+3}{4}\rceil\}$, in the following way.

$$\alpha_6(x_iy_i) = \begin{cases} \lceil\frac{i}{2}\rceil & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil - 1, \\ \lceil\frac{n}{4}\rceil & \text{for } i = \lceil\frac{n}{2}\rceil \text{ and } n \not\equiv 2 \pmod 4, \\ \frac{n-2}{4} & \text{for } i = \frac{n}{2} \text{ and } n \equiv 2 \pmod 4, \\ \lceil\frac{n-i+2}{2}\rceil & \text{for } \lceil\frac{n}{2}\rceil + 1 \le i \le n, \end{cases}$$

$$\alpha_6(y_ix_{i+1}) = \begin{cases} \lceil\frac{i+1}{2}\rceil & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil, \\ \lceil\frac{n-i+1}{2}\rceil & \text{for } \lceil\frac{n}{2}\rceil + 1 \le i \le n, \end{cases}$$

$$\alpha_6(cx_i) = \begin{cases} \lceil\frac{i}{2}\rceil & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil, \\ \lceil\frac{n-i+3}{2}\rceil & \text{for } \lceil\frac{n}{2}\rceil + 1 \le i \le n. \end{cases}$$

It is easy to verify that under the labeling α_6 all edge labels are at most $\lceil\frac{n+3}{4}\rceil$. For the C_4-weight of the cycle C_4^i, $i = 1, 2, \dots, n$, under the edge labeling α_6, we get

$$\begin{aligned} wt_{\alpha_6}(C_4^i) &= \sum_{e \in E(C_4^i)} \alpha_6(e) = \alpha_6(cx_i) + \alpha_6(cx_{i+1}) + \alpha_6(x_iy_i) + \alpha_6(y_ix_{i+1}) \\ &= \begin{cases} 2i + 2 & \text{for } 1 \le i \le \lceil\frac{n}{2}\rceil, \\ 2n + 5 - 2i & \text{for } \lceil\frac{n}{2}\rceil + 1 \le i \le n. \end{cases} \end{aligned}$$

Clearly, the weights of C_4^i for $1 \le i \le \lceil\frac{n}{2}\rceil$ are even and increasing. On the other hand the weights of C_4^i for $\lceil\frac{n}{2}\rceil + 1 \le i \le n$ are odd and decreasing. So, it concluded that all the weights of C_4^i are different. Thus α_6 is an edge C_4-irregular labeling of G_n. Hence $\text{ehs}(G_n, C_4) \le \lceil\frac{n+2}{3}\rceil$. This completes the proof of theorem. ∎

5. Conclusion

In this paper we have investigated the edge H-irregularity strength of some graphs. We have found the exact values of this parameter for several families of graphs namely, prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs.

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