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FAIR TOTAL DOMINATION NUMBER IN CACTUS GRAPHS

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Abstract

For $k \geq 1$, a k-fair total dominating set (or just kFTD-set) in a graph G is a total dominating set S such that $|N(v) \cap S| = k$ for every vertex $v \in V \backslash S$. The k-fair total domination number of G, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A fair total dominating set, abbreviated FTD-set, is a kFTD-set for some integer $k \geq 1$. The fair total domination number of a nonempty graph G, denoted by $ftd(G)$, of G is the minimum cardinality of an FTD-set in G. In this paper, we present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds.

Keywords: fair total domination, cactus graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [12]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v, is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote the set of leaves and support vertices of a graph G by $L(G)$ and $S(G)$, respectively. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \bigcup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The 2-*corona* 2-*cor*(G) of a graph G is a graph obtained from G by adding a path P_2 for every vertex v and joining v to a leaf of P_2. Note that 2-*cor*(G) has order $3|V(G)|$. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. For a subset S of vertices of a graph G, we denote by $G[S]$ the subgraph of G induced by S. A *cactus* graph is a graph such that no pair of cycles have a common edge.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S in a graph G with no isolated vertex, is a *total dominating set* of G if every vertex in S is adjacent to a vertex in S.

Caro *et al.* [1] studied the concept of fair domination in graphs. For $k \geq 1$, a *k-fair dominating set*, abbreviated kFD-set, in G is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V\backslash D$. The *k-fair domination number* of G, denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$-set. A *fair dominating set*, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$-set. A *perfect dominating set* in a graph G is a dominating set S such that every vertex in $V(G)\backslash S$ is adjacent to exactly one vertex in S. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows *et al.* [7] with a different terminology which they called semiperfect domination. This concept was further studied in, for example, [2, 3, 5, 6, 8, 9, 11].

Maravilla *et al.* [13] introduced the concept of fair total domination in graphs. For an integer $k \geq 1$ and a graph G with no isolated vertex, a *k-fair total dominating set*, abbreviated kFTD-set, is a total dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G)\backslash S$. The k-fair total domination number of G, denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A kFTD-set of G of cardinality $ftd_k(G)$ is called an $ftd_k(G)$-set. A *fair total dominating set*, abbreviated FTD-set, in G is a kFTD-set for some integer $k \geq 1$. Thus, a fair total dominating set S of a graph G is a total dominating set S of G such that for every two distinct vertices u and v of $V(G)\backslash S$, $|N(u) \in S| = |N(v) \cap S|$; that is, S is both a fair dominating set and a total dominating set of G. The *fair total domination number* of G, denoted by $ftd(G)$, is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality $ftd(G)$ is called a minimum fair total dominating set or an ftd-set of G.

In [10], Volkmann and we studied fair total domination in trees and unicyclic graphs. In this paper, we study 1-fair total domination in cactus graphs. We present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds. The techniques used in this paper are similar to those presented in [9]. The following observations are easily verified.

Observation 1. *Any support vertex in a graph G with no isolated vertex belongs to every kFTD-set for each integer k.*

Observation 2. *Let S be a* 1FTD*-set in a graph G, and v be a vertex of degree at least two such that v is adjacent to a weak support vertex v'. If S contains a vertex $u \in N_G(v)\backslash\{v'\}$, then $v \in S$.*

2. Unicyclic Graphs

A vertex v of a graph is a *special vertex* if $\deg_G(v) = 2$ and v belongs to a cycle of G. Let $\mathcal{H}_1$ be the class of all graphs G that can be obtained from the 2-corona 2-*cor*(C) of a cycle C by removing precisely one support vertex v and the leaf adjacent to v. Let $\mathcal{G}_1$ be the class of all graphs G that can be obtained from a sequence $G_1, G_2, \ldots, G_s = G$, where $G_1 \in \mathcal{H}_1$, and if $s \geq 2$, then G_{j+1} is obtained from G_j by one of the following Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $j = 1, 2, \ldots, s-1$.

Operation $\mathcal{O}_1$. Let v be a vertex of G_j with $\deg(v) \geq 2$ such that v is not a special vertex. Then G_{j+1} is obtained from G_j by adding a path P_3 and joining v to a leaf of P_3 by means of an edge.

Operation $\mathcal{O}_2$. Let v be a support vertex of G_j and let u be a leaf adjacent to v. Then G_{j+1} is obtained from G_j by adding a vertex u' and a path P_2, and joining u to u' and v to a leaf of P_2.

Observation 3. *If $H \in \mathcal{H}_1$, then H has precisely one special vertex.*

Observation 4 [10]**.** *If $G \in \mathcal{G}_1$ has order n, and C is the cycle of G, then we have the following.*

(1) *G has precisely one special vertex.*

(2) *G has $(n-1)/3$ leaves.*

(3) *No vertex of C is a support vertex.*

(4) *Any vertex of C is adjacent to at most one weak support vertex of degree two.*

Lemma 5 [10]**.** *If $G \in \mathcal{G}_1$, then every* 1FTD*-set in G contains every vertex of G of degree at least two.*

Theorem 6 [10]**.** *If G is a unicyclic graph of order n, then $ftd_1(G) \leq (2n+1)/3$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}_1$.*

3. Main Result

Our aim in this paper is to give an upper bound for the fair total domination number of a cactus graph G in terms of the number of cycles of G, and then characterize all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $\mathcal{H}_1$ and $\mathcal{G}_1$ be the families of unicyclic graphs described in Section 2. For $i = 2, \dots, k$, we construct a family $\mathcal{H}_i$ from $\mathcal{G}_{i-1}$, and a family $\mathcal{G}_i$ from $\mathcal{H}_i$ as follows.

• Family $\mathcal{H}_i$. Let $\mathcal{H}_i$ be the family of all graphs H_i such that H_i can be obtained from a graph $H_1 \in \mathcal{H}_1$ and a graph $G \in \mathcal{G}_{i-1}$, by the following procedure.

Procedure A. Let $w_0 \in V(H_1)$ be a vertex of degree at least two of H_1 such that w_0 is adjacent to a weak support vertex w_0', and $w \in V(G_{i-1})$ be a vertex of degree at least two of G_{i-1} such that w is adjacent to a weak support vertex w' of degree two. We remove w_0', the leaf adjacent to w_0', w' and the leaf adjacent to w', and then identify the vertices w_0 and w.

• Family $\mathcal{G}_i$. Let $\mathcal{G}_i$ be the family of all graphs G that can be obtained from a sequence $G_1, G_2, \dots, G_s = G$, where $G_1 \in \mathcal{H}_i$, and if $s \geq 2$, then G_{j+1} is obtained from G_j by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, described in Section 2, for $j = 1, 2, \dots, s-1$.

Note that $\mathcal{H}_i \subseteq \mathcal{G}_i$, for $i = 1, 2, \dots, k$. Figure 1 demonstrates the construction of the family $\mathcal{G}_k$.

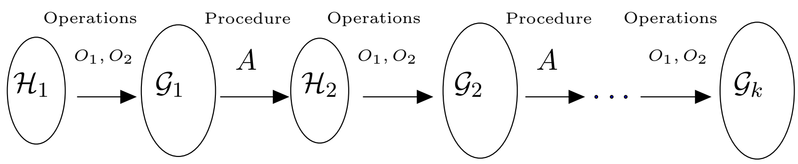

Figure 1. Construction of the family $\mathcal{G}_k$.

We will prove the following.

Theorem 7. *If G is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $ftd_1(G) \leq (2(n+k)-1)/3$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}_k$.*

4. Preliminary Results and Observations

4.1. Notation

We call a vertex w in a cycle C of a cactus graph G a *special cut-vertex* if w belongs to a shortest path from C to a cycle $C' \neq C$. We call a cycle C in G, a *leaf-cycle* if C contains exactly one special cut-vertex. In the cactus graph presented in Figure 2, v_i is a special cut-vertex, for $i = 1, 2, \ldots, 8$. Moreover, C_j is a leaf-cycle for $j = 1, 2, 3$.

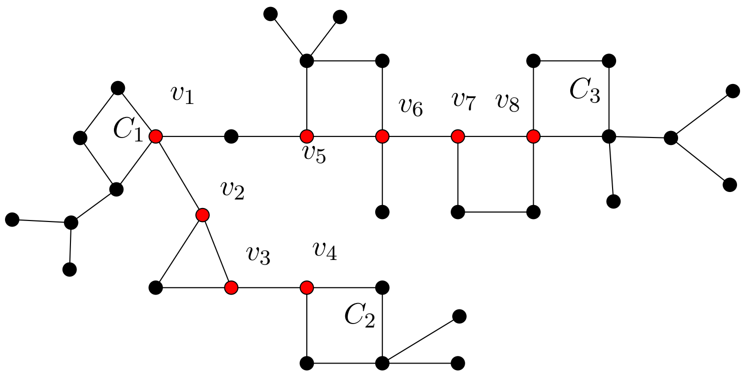

Figure 2. C_i is a leaf-cycle for $i = 1, 2, 3$ and v_j is a special cut-vertex for $j = 1, 2, \ldots, 8$.

Observation 8. *Every cactus graph with at least two cycles contains at least two leaf-cycles.*

4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on k.

Observation 9. *If $G \in \mathcal{G}_k$ is a cactus graph of order n, then we have the following.*

(1) *No cycle of G contains a support vertex. Furthermore, any cycle of G contains precisely one special vertex.*

(2) *If a vertex v of G belongs to a cycle of G, then v is adjacent to at most one weak support vertex of degree two.*

(3) $|L(G)| = (n + 1)/3 - 2k/3$.

(4) *If a vertex v of G belongs to at least two cycles of G, then v is not adjacent to a weak support vertex, and v belongs to precisely two cycles of G.*

Proof. Let $G \in \mathcal{G}_k$ be a cactus graph of order n. To show (1), (2) or (3), we prove by an induction on k, that we call *first-induction*. For the base step, if $k = 1$, then $G \in \mathcal{G}_1$, and the result follows by Observation 4. Assume the result holds for all graphs $G' \in \mathcal{G}_{k'}$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, G is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then G_{i+1} is obtained from G_i by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i = 1, 2, \ldots, l-1$. We prove by an induction on l, that we call *second-induction*. For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that G is obtained from H and G' by Procedure A. It is easy see that the base step of the second-induction holds. Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, G is obtained from G_{l-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$. It is easy see that the result holds.

The proof for (4) is similarly verified. ∎

Observation 10. *Let $G \in \mathcal{G}_k$ be obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and G_{j+1} is obtained from G_j by one of the Operations O_1 or O_2 or Procedure A, for $j = 1, 2, \ldots, s-1$. If v is a vertex of G belonging to two cycles of G, then there is an integer $i \in \{2, 3, \ldots, s\}$ such that G_i is obtained from G_{i-1} by applying Procedure A on the vertex v using a graph $H \in \mathcal{H}_1$, such that v belongs to a cycle of G_{i-1}.*

Observation 11. *Assume that $G \in \mathcal{G}_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let D_1 and D_2 be the components of $G - v$, G_1^* be the graph obtained from $G[D_1 \cup \{v\}]$ by joining v to a leaf of a path P_2, and G_2^* be the graph obtained from $G[D_2 \cup \{v\}]$ by joining v to a leaf of a path P_2. Then there exists an integer $k' < k$ such that $G_1^* \in \mathcal{G}_{k'}$ or $G_2^* \in \mathcal{G}_{k'}$.*

Proof. Let $G \in \mathcal{G}_k$. Then G is obtained from a sequence $G_1, G_2, \ldots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and G_{j+1} is obtained from G_j by one of the Operations O_1 or O_2 or procedure A, for $j = 1, 2, \ldots, s-1$. Note that $s \geq k$. We define the *j-th Procedure-Operation* or just PO_j as one of the Operation O_1, Operation O_2, or Procedure A that can be applied to obtain G_{j+1} from G_j. Thus G is obtained from G_1 by Procedure-Operations $PO_1, PO_2, \ldots, PO_{s-1}$.

Let v be a vertex of G of degree four belonging to two cycles of G, and D_1 and D_2 be the components of $G - v$. By Observation 10, there is an integer $i \in \{2, 3, \ldots, s\}$ such that G_i is obtained from G_{i-1} by applying Procedure A on the vertex v using a graph $H \in \mathcal{H}_1$. Note that v is adjacent to a weak support vertex v' of G_{i-1}. Let v'' be the leaf of v' in G_{i-1} that is removed in Procedure A. Clearly, either $V(G_{i-1}) \cap D_1 \neq \emptyset$ or $V(G_{i-1}) \cap D_2 \neq \emptyset$. Without loss of generality, assume that $V(G_{i-1}) \cap D_1 \neq \emptyset$. Among $PO_i, PO_{i+1}, \ldots, PO_{s-1}$, let $PO_{r_1}, PO_{r_2}, \ldots, PO_{r_t}$, be those procedure-operations applied on a vertex of D_1. Note that $i \leq t \leq s-1$. Let $G_{r_0} = G_{i-1}$ and $G_{r_{l+1}}$ be obtained from G_{r_l} by PO_{l+1}, for $l = 0, 1, 2, \ldots, t-1$. Clearly, by an induction on t, we can deduce that there is an integer $k^* < k$ such that $G_{r_t} \in \mathcal{G}_{k^*}$. Note that $G_{r_t} = G_1^*$. ∎

Lemma 12. *If $G \in \mathcal{G}_k$, then every* 1FTD*-set in G contains each vertex of G of degree at least two.*

Proof. Let $G \in \mathcal{G}_k$, and S be a 1FTD-set in G. We prove by an induction on k, that we call *first-induction*, that S contains every vertex of G of degree at least two. For the base step, if $k = 1$, then $G \in \mathcal{G}_1$, and the result follows by Lemma 5. Assume the result holds for all graphs $G' \in \mathcal{G}_{k'}$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, G is obtained from a sequence $G_1, G_2, \ldots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then G_{i+1} is obtained from G_i by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i = 1, 2, \ldots, l-1$.

We prove by an induction on l, that we call *second-induction*, that S contains every vertex of G of degree at least two.

For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that G is obtained from H and G' by Procedure A. Clearly, H is obtained from the 2-corona 2-$cor(C)$ of a cycle C, by removing precisely one support vertex v and the leaf adjacent to v of 2-$cor(C)$.

Let $C = c_0c_1 \cdots c_rc_0$ be the cycle of H, where c_0 is a vertex of degree at least two of H that is adjacent to a weak support vertex c_0', and let c_0' and its leaf (that we call c_0'') be removed according to Procedure A. By Observation 3, H has precisely one special vertex. Let c_t be the special vertex of H. Let $w \in V(G')$ be a vertex of degree at least two of G' that is adjacent to a weak support vertex w', and let w' and its leaf (that we call w'') be removed according to Procedure A.

First we show that $\{c_1, c_r\} \cap S \neq \emptyset$. Clearly, $S \cap \{c_{t-1}, c_t, c_{t+1}\} \neq \emptyset$, since $\deg_G(c_t) = 2$. Assume that $c_t \in S$. Since at least one of c_{t-1} or c_{t+1} is adjacent to a weak support vertex, by Observation 2, $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$. By applying Observation 2, we obtain that $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\} \setminus \{c_t\}$ is adjacent to a weak support vertex of G. Thus assume that $c_t \notin S$. Then $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$, and so $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \ldots, c_r\} \setminus \{c_t\}$ is adjacent to a weak support vertex of G. Hence, $\{c_1, c_r\} \cap S \neq \emptyset$. If $c_0 \notin S$, then $S \cup \{w', w''\}$ is a 1FTD-set for G', and thus by the first-inductive hypothesis, S' contains $w = c_0$, a contradiction. Thus $c_0 \in S$. By Observation 2, $V(C) \subseteq S$, since any vertex of $\{c_1, \ldots, c_r\} - \setminus \{c_t\}$ is adjacent to a weak support vertex of G. Thus $S \cap V(G')$ is a 1FTD-set for G'. By the first-inductive hypothesis, $(S \cap V(G')) \cup \{w', w''\}$ contains every vertex of G' of degree at least two. Consequently, S contains every vertex of G of degree at least two. We conclude that the base step of the second-induction holds.

Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, G is obtained from G_{l-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$.

Assume that G is obtained from G_{l-1} by applying Operation $\mathcal{O}_2$. Let x be a support vertex of G_{l-1} and let x' be a leaf adjacent to x. Let G be obtained from

G_{l-1} by adding a vertex u' and a path $P_2 = y_1y_2$, joining x' to u' and joining x to y_1, according to Operation $\mathcal{O}_2$. By Observation 1, $x', y_1 \in S$ and so $x \in S$. Thus $S\backslash\{y_1\}$ is a 1FTD-set for G_{l-1}. By the second-inductive hypothesis, S contains all vertices of G_{l-1} of degree at least two. Consequently, S contains every vertex of G_k of degree at least two.

Next assume that G is obtained from G_{l-1} by applying Operation $\mathcal{O}_1$. Let $P_3 = x_1x_2x_3$ be a path and x_1 be joined to $y \in V(G_{l-1})$, where $\deg_{G_{l-1}}(y) \geq 2$ and y is not a special vertex of G_{l-1}, according to Operation $\mathcal{O}_2$. By Observation 1, $x_2 \in S$. Observe that $\{x_1, x_3\} \cap S \neq \emptyset$. If $x_1 \notin S$, then $x_3 \in S$ and $y \notin S$. Then $S\backslash\{x_2, x_3\}$ is a 1FTD-set for G_{l-1} that does not contains y, a contradiction by the second-inductive hypothesis. Thus assume that $x_1 \in S$. Suppose that $y \notin S$. Clearly, $N_{G_{l-1}}(y) \cap S = \emptyset$. Assume that there exists a component G'_1 of $G_{l-1} - y$ such that $|V(G'_1) \cap N_{G_{l-1}}(y)| = 1$. Then clearly $S' = (S \cap V(G_{l-1})) \cup V(G'_1)$ is a 1FTD-set for G_{l-1}, and by the second-inductive hypothesis, S' contains every vertex of G_{l-1} of degree at least two. Thus $y \in S'$, and so $y \in S$, a contradiction. Next assume that every component of $G_{l-1} - y$ has at least two vertices in $N_{G_{l-1}}(y)$. Since y is a non-special vertex of G_{l-1}, y belongs to at least two cycles of G_{l-1}. By Observation 9(4), y belongs to exactly two cycles of G_{l-1}. Thus $\deg_{G_{l-1}}(y) = 4$. By Observation 11, $G_{l-1} - y$ has exactly two components D_1 and D_2. Let G^* be a graph obtained from $D_1 \cup \{y\}$ or $D_2 \cup \{y\}$ by adding a path $P_2 = y'y''$ to y. Then there exists $k' \leq k$ such that $G^* \in \mathcal{G}_{k'}$. Evidently, $S^* = (S \cap V(G^*)) \cup \{y', y''\}$ is a 1FTD-set for G^*, and so by the first-inductive hypothesis, S^* contains every vertex of G^* of degree at least two (since $G^* \in \mathcal{G}_{k'}$). Thus $y \in S^*$, and so $y \in S$, a contradiction. We conclude that $y \in S$. Observe that $S \cap V(G_{l-1})$ is a 1FTD-set for G_{l-1}, and so by the second-inductive hypothesis, $S \cap V(G_{l-1})$ contains every vertex of G_{l-1} of degree at least two. Consequently, S contains every vertex of G of degree at least two. ∎

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

Corollary 13. *If $G \in \mathcal{G}_k$ is a cactus graph of order n, then $V(G)\backslash L(G)$ is the unique $ftd_1(G)$-set.*

In what follows, we present an upper bound for the 1-fair domination number of a cactus graph in terms of the order and the number of cycles.

Theorem 14. *If G is a cactus graph of order $n \geq 4$ with $k \geq 1$ cycles, then $ftd_1(G) \leq (2(n + k) - 1)/3$.*

Proof. The result follows by Theorem 6 if $k = 1$. Thus assume that $k \geq 2$. Suppose to the contrary that $ftd_1(G) > (2(n(G) + k) - 1)/3$. Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is minimum. Let $C_1, C_2, \ldots, C_k$ be the k cycles of G. Let C_i

be a leaf-cycle of G, where $i \in \{1, 2, \ldots, k\}$. Let $C_i = c_0c_1 \cdots c_rc_0$, where c_0 is the special cut-vertex of G. Suppose that G has a strong support vertex u, and u_1, u_2 are leaves adjacent to u. Let $G_0 = G - u_1$. By the choice of G, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $u \in S'$. Clearly, S' is a 1FTD-set in G and so $ftd_1(G) \leq (2(n+k)-1)/3-2/3$, a contradiction. We deduce that every support vertex of G is adjacent to precisely one leaf.

Assume that $\deg_G(u_j) = 2$ for each $j = 1, 2, \ldots, r$. Let $G' = G - c_2$. Then by the choice of G, $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then S' is a 1FTD-set for G cardinality at most $(2(n+k)-1)/3-4/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Now $\{c_1\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. We deduce that $\deg_G(c_i) \geq 3$ for some $i \in \{1, 2, \ldots, r\}$.

Let v_d be a leaf of G such that $d(v_d, C_i - c_0)$ is as maximum as possible, the shortest path from v_d to C_i does not contain c_0 and $\deg(v_{d-1})$ is as maximum as possible, where v_{d-1} is the neighbor of v_d on the shortest path from v_d to a vertex $v_0 \in C_i$.

Assume that $d \geq 3$. Observe that $\deg_G(v_{d-1}) = 2$, since G has no strong support vertex. Assume that $\deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By the choice of G, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$-set. If $v_{d-3} \in S'$, then $\{v_{v-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. If $v_{d-3} \notin S'$, then $\{v_{v-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. Thus assume that $\deg_G(v_{d-2}) \geq 3$. Assume that v_{d-2} is a support vertex. Let $G' = G - \{v_{d-1}, v_d\}$. By the choice of G, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of G. Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of G such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1 \cdots v_d$, (the part "$\deg(v_{d-1})$ is as maximum as possible"), $\deg_G(x) = 2$. Let y be the leaf adjacent to x and $G' = G - \{v_d, v_{d-1}, y\}$. By the choice of G, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$-set. By Observation 1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of G'. Thus $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction.

Next assume that $d = 2$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \ldots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. Then by the choice of G, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ and so $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Let c_{j+1} be a support vertex of G, and $G' = G - c_j$. Then by the choice of G, $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-4/3$, a contradiction. Thus assume that $c_{j-1} \in S'$ and so $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus c_{j+1} is not a support vertex of G. Let $c'_{j+1} \in N(c_{j+1}) \backslash V(C_i)$. Clearly, c'_{j+1} is a support vertex, since $d = 2$. Observe that $\deg_G(c'_{j+1}) = 2$, since G has no strong support vertex. Let c''_{j+1} be the leaf of c'_{j+1}. Let $G' = G - c_j - c''_{j+1}$. By the choice of G, $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-2$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+1} \in S'$, since c_{j+1} is a support vertex in G'. If $c_{j-1} \notin S'$, then $S' \cup \{c'_{j+1}\}$ is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-1$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j, c'_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3$, a contradiction. Thus $\deg(c_i) \geq 3$ for $1 \leq i \leq r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G_1^* be the component of G^* containing c_r, and G_2^* be the component of G^* containing c_0. Let $D = S(G_1^*) \backslash V(C_i)$. Clearly, $S' = D \cup \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for G_1^* of cardinality at most $2n(G_1^*)/3$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By the choice of G, $ftd_1(G_3^*) \leq (2(n(G_3^*)+k-1)-1)/3$. Let S'' be an $ftd_1(G_3^*)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_3^*)+k-1)-1)/3 + 2n(G_1^*)/3 = (2(n+k)-1)/3$, a contradiction.

Now assume that $d = 1$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \ldots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By the choice of G, $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly, $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Thus c_{j+1} is a support vertex of G. Let $G' = G - c_j$. Then by the choice of G, $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then S' is a 1FTD-set for G, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j\} \cup S'$ is a

1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. We thus obtain that $\deg(c_i) \geq 3$ for $1 \leq i \leq r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G_1^* be the component of G^* containing c_r, and G_2^* be the component of G^* containing c_0. Clearly, $S' = \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for G_1^* of cardinality at most $n(G_1^*)/2$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By the choice of G, $ftd_1(G_3^*) \leq (2(n(G_3^*)+k-1)-1)/3$. Let S'' be an $ftd_1(G_3^*)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_3^*)+k-1)-1)/3 + n(G_1^*)/2 < (2(n+k)-1)/3$, a contradiction. ■

It is evident that for the cycle C_7 the equality of the bound given in Theorem 14 holds.

Theorem 15. *If $G \neq C_7$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $ftd_1(G) = (2(n+k)-1)/3$ if and only if $G \in \mathcal{G}_k$.*

Proof. We prove by an induction on k to show that any cactus graph $G \neq C_7$ of order $n \geq 5$ with $k \geq 1$ cycles and $ftd_1(G) = (2(n+k)-1)/3$ belongs to $\mathcal{G}_k$. The base step of the induction follows by Theorem 6. Assume the result holds for all cactus graphs $G' \neq C_7$ with $k' < k$ cycles. Now let $G \neq C_7$ be a cactus graph of order n with $k \geq 2$ cycles and $ftd_1(G) = (2(n+k)-1)/3$. Suppose to the contrary that $G \notin \mathcal{G}_k$. Assume that G has the minimum order, and among all such graphs, assume that the size of G is minimum.

Claim 1. *Every support vertex of G is weak support vertex.*

Proof. Suppose that G has a strong support vertex u, and assume that u_1 and u_2 are two leaves adjacent to u. Let $G' = G - u_1$, and S' be an $ftd_1(G')$-set. By Observation 1, $u \in S'$. By Theorem 14, $|S'| \leq (2(n(G')+2)-1)/3 = (2(n+k)-1)/3 - 2/3$. Clearly, S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 2/3$, a contradiction. □

By Observation 8, G has at least two leaf-cycles. Let $C_1 = c_0c_1 \cdots c_rc_0$ be a leaf-cycle of G, where c_0 is a special cut-vertex of G. Let G_1' be the component of $G - c_0c_1 - c_0c_r$ containing c_1.

Claim 2. $V(G_1') \neq \{c_1, \ldots, c_r\}$.

Proof. Suppose that $V(G_1') = \{c_1, \ldots, c_r\}$. Then $\deg_G(c_i) = 2$, for each $i = 1, 2, \ldots, r$. Let $G' = G - c_2$. By Theorem 14, $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3 - 4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Then $\{c_1\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. □

Let $v_d \in V(G_1')\backslash\{c_1,\ldots,c_r\}$ be a leaf of G_1' at maximum distance from $\{c_1,\ldots,c_r\}$, and assume that $\deg(v_{d-1})$ is as maximum as possible, $\deg_G(v_0)$ is as maximum as possible, and $\deg_G(v_1)$ is as maximum as possible, where $v_0 \in \{c_1,\ldots,c_r\}$ and $v_0v_1\cdots v_d$ is the shortest path from v_d to $\{c_1,\ldots,c_r\}$.

Suppose that $d=1$. Assume that $\deg_G(c_j)=2$, for some $j \in \{1,2,\ldots,r\}$. Assume that $\deg_G(c_{j+1})=2$. Let $G'=G-c_j$. By Theorem 14, $ftd_1(G') \le (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1},c_{j+1}\}|=1$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1},c_{j+1}\}|=2$. Then $\{c_j\}\cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1},c_{j+1}\}|=0$. Then $\{c_{j+1}\}\cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \ge 3$. Similarly, $\deg_G(c_{j-1}) \ge 3$. Clearly, $c_{j+1} \ne c_0$ or $c_{j-1} \ne c_0$. Assume, without loss of generality, that $c_{j+1} \ne c_0$. Then c_{j+1} is a support vertex of G. Let $G'=G-c_j$. Then by Theorem 14, $ftd_1(G') \le (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-4/3$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j\}\cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. We thus obtain that $\deg(c_j) \ge 3$, for $1 \le j \le r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G_1^* be the component of G^* containing c_r, and G_2^* be the component of G^* containing c_0. Clearly, $S'=\{c_1,c_2,\ldots,c_r\}$ is a 1FTD-set for G_1^* of cardinality at most $n(G_1^*)/2$. Let $G_3^* = G[V(G_2^*)\cup\{c_1\}]$. By Theorem 14, $ftd_1(G_3^*) \le (2(n(G_3^*)+k-1)-1)/3$. Let S'' be an $ftd_1(G_3^*)$-set. By Observation 1, $c_0 \in S''$. Clearly, $S'\cup S''$ is a 1FTD-set for G and so $ftd_1(G) \le (2(n(G_3^*)+k-1)-1)/3 + n(G_1^*)/2 < (2(n+k)-1)/3$, a contradiction.

Thus assume that $d \ge 2$.

Claim 3. *If $d \ge 3$, then $G \in \mathcal{G}_k$.*

Proof. Assume that $d \ge 3$. By Claim 1, $\deg_G(v_{d-1})=2$. Assume first that $\deg_G(v_{d-2}) \ge 3$. Assume that v_{d-2} is a support vertex. Let $G'=G-\{v_{d-1},v_d\}$. By Theorem 14, $ftd_1(G') \le (2(n(G')+k)-1)/3 = (2(n+k)-1)/3-4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $v_{d-2} \in S'$. Then $\{v_{d-1}\}\cup S'$ is a 1FTD-set in G, and so $ftd_1(G) \le (2(n+k)-1)/3-1/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of G. Let $x \ne v_{d-1}, v_{d-3}$ be a support vertex of G such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1\cdots v_d$, (the part "$\deg(v_{d-1})$ is as maximum as possible"), $\deg_G(x)=2$. Let y be the leaf adjacent to x, and $G'=G-\{v_d,v_{d-1},y\}$. By Theorem 14, $ftd_1(G') \le (2(n(G')+k)-1)/3 = (2(n+k)-1)/3-2$. Assume that $ftd_1(G') < (2(n(G')+k)-1)/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of G'. Then $\{v_{d-1},x\}\cup S'$ is a 1FTD-set in G and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction.Thus $ftd_1(G') = (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. By the choice of G, $G' \in \mathcal{G}_k$. Thus G is obtained from G' by Operation $\mathcal{O}_2$, and so $G \in \mathcal{G}_k$.

Assume that $\deg_G(v_{d-2}) = 2$. We consider the following cases.

Case 1. $d \geq 4$. Suppose that $\deg_G(v_{d-3}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}, v_{d-3}\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G')+k)-1)/3 = (2(n+k)-1)/3-8/3$. Let S' be an $ftd_1(G')$-set. If $v_{d-4} \in S'$, then $\{v_{v-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. Thus $v_{d-4} \notin S'$. Then $\{v_{v-2}, v_{d-1}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. We deduce that $\deg_G(v_{d-3}) \geq 3$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Assume that $ftd_1(G') < (2(n(G')+k)-1)/3 = (2(n+k)-1)/3-2$. Let S' be an $ftd_1(G')$-set. If $v_{d-3} \in S'$, then $\{v_{v-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $v_{d-3} \notin S'$. Then $\{v_{v-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. We thus obtain that $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the choice of G, $G' \in \mathcal{G}_k$. Since $d \geq 4$, v_{d-3} is not a special vertex of G'. Thus G is obtained from G' by Operation $\mathcal{O}_1$, and so $G \in \mathcal{G}_k$.

Case 2. $d = 3$. Clearly, $\deg(v_0) \geq 3$. We show that $\deg(v_0) \geq 4$. Suppose that $\deg(v_0) = 3$. Let $G' = G - \{v_1, v_2, v_3\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G')+ k) - 1)/3$. Assume that $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the choice of G, $G' \in \mathcal{G}_k$. By Observation 9(1), v_0 is the unique special vertex of G', since $\deg_{G'}(v_0) = 2$. We show that $\deg_{G'}(x) = 3$ for each $x \in \{c_1, \ldots, c_r\}\backslash\{v_0\}$. Assume that $\deg_{G'}(c_j) \geq 4$ for some $c_j \in \{c_1, \ldots, c_r\}\backslash\{v_0\}$. If there is a vertex $w \in V(G) \setminus V(C_1)$ such that $d(w, C_1) = d(w, c_j) = 3$, then w can plays the same role of v_d, and thus $\deg(u_j) = 3$, a contradiction. Thus there is no vertex $w \in V(G)\backslash V(C_1)$ such that $d(w, C_1) = d(w, c_j) = 3$. Thus any vertex of $N(u_j)\backslash V(C_1)$ is a leaf or a weak support vertex. Assume that $N(c_j)\backslash V(C_1)$ contains t_1 leaves and t_2 support vertices, where $t_1+t_2 \geq 2$. By Observation 9(1), $t_1 = 0$, since $G' \in \mathcal{G}_k$. Thus $t_2 \geq 2$. Let z_1 and z_2 be two weak support vertices in $N(c_j)\backslash V(C_1)$. Let z_1' and z_2' be the leaves adjacent to z_1 and z_2, respectively. (We switch for a while to G). Let $G'' = G-\{z_1, z_1', z_2'\}$. By Theorem 14, $ftd_1(G'') \leq (2(n(G'')+k)-1)/3$. Suppose that $ftd_1(G'') = (2(n(G') + k) - 1)/3$. By the choice of G, $G'' \in \mathcal{G}_k$. Clearly, $\deg_{G''}(c_i) \geq 3$, since v_0 is the unique special vertex of G', a contradiction (by Observation 9(1)). Thus $ftd_1(G'') < (2(n(G'')+k)-1)/3 = (2(n+k)-1)/3-2$. Let S'' be a 1FTD-set of G''. By Observation 1, $c_j \in S''$. Then $S'' \cup \{z_1, z_2\}$ is a 1FTD-set of G. Thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. We deduce that $\deg_{G'}(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\}\backslash\{v_0\}$. Thus $\deg_G(c_i) = 3$ for each $1 \leq i \leq r$. Note that by Observation 9(1), c_i is not a support vertex, for each i with $1 \leq i \leq r$ in G', since $G' \in \mathcal{G}_k$. (We switch for a while to G). Let $F = \bigcup_{i=1}^{r}(N[c_i])\backslash\{c_0, \ldots, c_r\}$. Clearly, $|F| = r$, since $\deg_{G'}(c_i) = 3$ for each $c_i \in \{c_1, \ldots, c_r\}\backslash\{v_0\}$ and $\deg_G(v_0) = 3$. Let $F = \{u_1, u_2, \ldots, u_r\}$. Clearly $\deg_G$

$(u_i) \geq 2$, for each i with $1 \leq i \leq r$, since c_i is not a support vertex for $1 \leq i \leq r$ in G'. By Claim 2, u_i is not a strong support vertex of G, for $1 \leq i \leq r$. If u_i is adjacent to a support vertex $u'_i \in V(G) \backslash V(C_1)$, for some integer i, then since the leaf of u'_i can play the role of v_3, we obtain that $\deg(u_i) = 2$. Since $\deg_G(u_i) \geq 2$ for each i with $1 \leq i \leq r$, we find that $\deg_G(u_i) = 2$ for each i with $1 \leq i \leq r$.

Let $F' = \bigcup_{i=1}^{r} N(u_i) \backslash \{c_0, \dots, c_r\}$. Clearly, $|F'| = r$, since $\deg_G(u_i) = 2$, for each $u_i \in \{u_1, \dots, u_r\}$. Let $F' = \{u'_1, u'_2, \dots, u'_r\}$. By the choice of the path $v_0 v_1 \cdots v_d$, (the part "$\deg(v_{d-1})$ is as maximum as possible"), $\deg(u'_i) \leq 2$, for $1 \leq i \leq r$. Let $F'_1 = \{u'_i \in F' | \deg_G(u'_i) = 1\}$ and $F'_2 = F' - F'_1$. Then every vertex of F'_2 is a weak support vertex. Since $v_1 \in F'_2$, we have $|F'_2| \geq 1$. Let $G^* = G - c_0 c_1 - c_0 c_r$, and G^*_1 and G^*_2 be the components of G^*, where $c_1 \in V(G^*_1)$. By Theorem 14, $ftd_1(G^*_2) \leq (2(n(G^*_2) + k - 1) - 1)/3$. Clearly, $n(G^*_2) = n(G) - 3r - |F'_2|$. Let S^*_2 be an $ftd_1(G^*_2)$-set. If $c_0 \notin S^*_2$, then $S^*_2 \cup F \cup F'$ is a 1FTD-set for G. Thus $ftd_1(G) \leq (2(n(G^*_2) + k - 1) - 1)/3 + 2r = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $c_0 \in S^*_2$. If $|F'_2| = 1$, then $S^*_2 \cup V(C_1) \cup F \cup \{v_2\}$ is a 1FTD-set for G and thus $ftd_1(G) \leq (2(n(G^*_2) + k - 1) - 1)/3 + 2r + 1 = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r + 1 < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|F'_2| \geq 2$. Let $\{u'_t, u'_{t'}\} \subseteq F'_2$ (assume without loss of generality that $t < t'$) such that $\deg_G(u'_i) = 1$, for $1 \leq i < t$ and $t' < i \leq r$. Let u''_t and $u''_{t'}$ be the leaves of u_t and $u_{t'}$, respectively. Clearly, $S^*_2 \cup \{c_1, \dots, c_{t-1}\} \cup \{u_1, \dots, u_{t-1}\} \cup \{c_{t'+1}, \dots, c_r\} \cup \{u_{t'+1}, \dots, u_r\} \cup \{u_{t+1}, \dots, u_{t'-1}\} \cup \{u'_{t+1}, \dots, u'_{t'-1}\} \cup \{u'_t, u'_{t'}\} \cup \{u''_t, u''_{t'}\}$ is a 1FTD-set for G and thus $ftd_1(G) \leq (2(n(G^*_2) + k - 1) - 1)/3 + 2r = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r + 1 < (2(n + k) - 1)/3$, a contradiction. We deduce that $ftd_1(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$-set. If $v_0 \in S'$, then $S' \cup \{v_1, v_2\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S' \cup \{v_2, v_3\}$ is a 1FTD-set in G and thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\deg(v_0) \geq 4$. Let $G' = G - \{v_3, v_2, v_1\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Assume that $ftd_1(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$-set. If $v_0 \in S'$, then $S = S' \cup \{v_1, v_2\}$ is a 1FTD-set for G and thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S = S' \cup \{v_2, v_3\}$ is a 1FTD-set for G and thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Hence, $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the inductive hypothesis, $G' \in \mathcal{G}_{k-1}$. Since $\deg(v_0) \geq 4$, v_0 is not a special vertex of G'. Thus G is obtained from G' by Operation $\mathcal{O}_1$ and so $G \in \mathcal{G}_k$. □

By Claim 3, we assume that $d = 2$. We show that $\deg_G(v_0) = 3$. Suppose that $\deg_G(v_0) \geq 4$. Assume that v_0 is a support vertex. Let $G' = G - \{v_1, v_2\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $v_0 \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FTD-set in G, and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that v_0 is not a support vertex of G. Let $x \neq v_1$ be a support vertex of G such that $x \in N(v_0)\backslash V(C_1)$. By the choice of the path $v_0v_1 \cdots v_d$, (the part "deg(v_{d-1}) is as maximum as possible"), $\deg_G(x) = 2$. Let y be the leaf adjacent to x. Let $G' = G - \{v_2, v_1, y\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G')+k)-1)/3$. Let $ftd_1(G') < (2(n(G')+k)-1)/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $v_0 \in S'$, since v_0 is a support vertex of G'. Then $\{v_1, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G') < (2(n+k)-1)/3$, a contradiction. Thus $ftd_1(G') = (2(n(G')+k)-1)/3$. By the inductive hypothesis, $G' \in \mathcal{G}_k$, a contradiction by Observation 9(1), since v_0 is a support vertex of G'. Thus $\deg_G(v_0) = 3$. Observe that G has no strong support vertex. If c_i is adjacent to a support vertex c'_i of $N(c_i)\backslash V(C_1)$ for some i, then the leaf of c'_i can play the role of v_2, and thus $\deg(c_i) = 3$. Thus we may assume that $\deg_G(c_i) \leq 3$ for each i with $i = 1, 2, \ldots, r$. Assume that $\deg_G(c_i) = 3$ for each i with $1 \leq i \leq r$.

Let $F = \bigcup_{i=1}^{r}(N(c_i)\backslash\{c_0, \ldots, c_r\}$. Clearly, $|F| = r$, since $\deg_G(c_i) = 3$, for each $c_i \in \{c_1, \ldots, c_r\}$. Let $F = \{u_1, u_2, \ldots, u_r\}$. Clearly, $\deg_G(u_i) \leq 2$, for $1 \leq i \leq r$, since G has no strong support vertex. Let $F' = \{u_i | \deg_G(u_i) = 2\}$. Clearly, $v_1 \in F'$. Let F'' be the set of leaves of F'. Clearly, $v_2 \in F''$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G^*_1 be the component of G^* containing c_r and G^*_2 be the component of G^* containing c_0. Assume that $F = F'$. Thus $n(G^*_1) = 3r$, since $d = 2$. Further, $n(G^*_2) = n - 3r$. By Theorem 14, $ftd_1(G^*_2) \leq (2(n(G^*_2)+k-1)-1)/3$. Let S'' be an $ftd_1(G^*_2)$-set. If $c_0 \in S''$, then $S'' \cup V(C_1) \cup F$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G^*_2) + k - 1) - 1)/3 + 2r = (2(n - 3r + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3$, a contradiction. Thus $c_0 \in S''$. Then $S'' \cup F'' \cup F$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G^*_2)+k-1)-1)/3+2r = (2(n - 3r + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3$, a contradiction. We conclude that $F \neq F'$. Let $|F'| = r'$. Clearly, $1 \leq r' < r$, since $v_1 \in F'$. Thus $n(G^*_1) = 2r + r'$. Then $n(G^*_2) = n - (2r + r')$. Let $G^*_3 = G[V(G^*_2) \cup \{c_1\}]$. Then $n(G^*_3) = n - (2r + r') + 1$. By Theorem 14, $ftd_1(G^*_3) \leq (2(n(G^*_3) + k - 1) - 1)/3$. Let S'' be an $ftd_1(G^*_3)$-set. By Observation 1, $c_0 \in S''$ and so $S'' \cup V(C_1) \cup F'$ is a 1FTD-set for G. Thus $ftd_1(G) \leq (2(n(G^*_2) + k - 1) - 1)/3 + r + r' = (2(n-(2r+r')+1+k-1)-1)/3+r+r' = (2(n+k)-1+r'-r)/3 < (2(n+k)-1)/3$, a contradiction. Therefore $\deg_G(c_t) = 2$ for some $1 \leq t \leq r$.

Claim 4. *No vertex of $C_1 - c_0$ is a support vertex.*

Proof. Let c_j be a support vertex of G. Assume that c_{j+1} is a special vertex. Let $G' = G - c_{j+1}$. Then by Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_j \in S'$. If $c_{j+2} \notin S'$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3-4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $c_{j+2} \in S'$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\deg_G(c_{j+1}) \neq 2$. Note that c_t is a special vertex of G. Assume without loss of generality that $j < t$. Let $c_{j'}$ be a support vertex of G and $c_{t'}$ be a special vertex of G, where $j \leq j' < t' \leq t$, and among such vertices choose $c_{j'}$ and $c_{t'}$ such that c_i is neither a support vertex nor a special vertex of G for each i with $j' < i < t'$. Let $u_i \in N(c_i)\backslash V(C_1)$ for $j' < i < t'$. Clearly, $\deg_G(u_i = 2$ for $j' < i < t'$, since G has no strong support vertex. Let $G^* = G - c_{j'}c_{j'+1} - c_{t'}c_{t'+1}$. Let G_1^* be the component of G^* containing $c_{j'}$ and G_2^* be the component of G^* containing $c_{t'}$. Clearly, $n(G_2^*) = 3(t' - j' - 1) + 1$. Thus $n(G_1^*) = n - (3(t' - j' - 1) + 1)$.

By Theorem 14, $ftd_1(G_1^*) \leq (2(n(G_1^*)+k-1)-1)/3$. Let S' be an $ftd_1(G_1^*)$-set. By Observation 1, $c_{j'} \in S'$. Assume that $c_{t'+1} \notin S'$. Then $S' \cup \{c_{j'+1}, c_{j'+2}, \ldots, c_{t'-1}\} \cup \{u_{j'+1}, u_{j'+2}, \ldots, u_{t'-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*)+k-1)-1)/3+2(t'-j'-1) = (2(n-(3(t'-j'-1)+1)+k-1)-1)/3+2(t'-j'-1) = (2(n+k)-1)/3-4/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus $c_{t'+1} \in S'$. Then $S' \cup \{c_{j'+1}, c_{j'+2}, \ldots, c_{t'}\} \cup \{u_{j'+1}, u_{j'+2}, \ldots, u_{t'-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*)+k-1)-1)/3+2(t'-j'-1)+1 = (2(n-(3(t'-j'-1)+1)+k-1)-1)/3+2(t'-j'-1) = (2(n+k)-1)/3-1/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. □

Claim 5. *If* $\deg_G(c_j) = 2$ *for some* j *with* $1 \leq j \leq r$*, then* $\deg_G(c_{j+1}) = 3$ *and* $\deg_G(c_{j-1}) = 3$.

Proof. Assume that $\deg_G(c_j) = \deg_G(c_{j+1}) = 2$, for some j with $1 \leq j \leq r$, and among such vertices choose c_j such that $\deg_G(c_{j-1}) = 3$. Let $G' = G - c_j$. Then by Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$-set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n + k) - 1)/3 - 4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly $\deg_G(c_{j-1}) \geq 3$. □

Claim 6. C_1 *has precisely one special vertex.*

Proof. Let c_{t_1} and c_{t_2} be two special vertices of C_1 and among such vertices choose c_{t_1} and c_{t_2} such that c_i is not a special vertex of C_1 for $t_1 < i < t_2$. By Claim 5, $t_1 + 1 < t_2$. By Claim 4, c_i is not a support vertex for $t_1 < i < t_2$. Let $u_i \in N(c_i)\backslash V(C_1)$, for $t_1 < i < t_2$. Clearly, $\deg_G(u_i) = 2$, for $t_1 < i < t_2$. Let u_i' be the leaf adjacent to u_i, for $t_1 < i < t_2$, and $G^* = G - c_{t_1}c_{t_1+1} - c_{t_2}c_{t_2+1}$. Let G_1^* be the component of G^* containing c_{t_1}, and G_2^* be the component of G^* containing c_{t_2}. Clearly, $n(G_2^*) = 3(t_2 - t_1 - 1) + 1$. Then $n(G_1^*) = n - (3(t_2 - t_1 - 1) + 1)$. By Theorem 14, $ftd_1(G_1^*) \leq (2(n(G_1^*) + k - 1) - 1)/3$. Let S' be an $ftd_1(G_1^*)$-set. By Observation 1, $c_{t_1-1} \in S'$. Assume that $\{c_{t_1}, c_{t_2+1}\} \cap S' = \emptyset$.

Then $S' \cup \{c_{t_1}, c_{t_1+1}, \ldots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*)+k-1)-1)/3+2(t_2-t_1-1)+1 = (2(n-(3(t_2-t_1-1)+1)+k-1)-1)/3+2(t_2-t_1-1)+1 = (2(n+k)-1)/3-1/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction.

Thus $\{c_{t_1}, c_{t_2+1}\} \cap S' \neq \emptyset$. If $\{c_{t_1}, c_{t_2+1}\} \subseteq S'$, then $S' \cup \{c_{t_1+1}, c_{t_1+2}, \ldots, c_{t_2}\} \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*)+k-1)-1)/3+2(t_2-t_1-1)+1 = (2(n-(3(t_2-t_1-1)+1)+k-1)-1)/3+2(t_2-t_1-1)+1 = (2(n+k)-1)/3-1/3$. Thus $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus $\{c_{t_1}, c_{t_2+1}\} \not\subseteq S'$. If $c_{t_1} \in S'$ and $c_{t_2+1} \notin S'$, then $S' \cup \{c_{t_1+1}, c_{t_1+2}, \ldots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*)+k-1)-1)/3+2(t_2-t_1-1) = (2(n-(3(t_2-t_1-1)+1)+k-1)-1)/3+2(t_2-t_1-1) = (2(n+k)-1)/3-4/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. Thus assume that $c_{t_2+1} \in S'$ and $c_{t_1} \notin S'$. Then $S' \cup \{u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2-1}\} \cup \{u'_{t_1+1}, u'_{t_1+2}, \ldots, u'_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*)+k-1)-1)/3+2(t_2-t_1-1) = (2(n-(3(t_2-t_1-1)+1)+k-1)-1)/3+2(t_2-t_1-1) = (2(n+k)-1)/3-4/3$ and so $ftd_1(G) < (2(n+k)-1)/3$, a contradiction. □

By Claims 4 and 6, c_i is not a support vertex and is not a special vertex, for $i \in \{1, 2, \ldots, t-1, t+1, \ldots, r\}$. Let $u_i \in N(c_i)\backslash V(C_1)$, for $i \in \{1, 2, \ldots, t-1, t+1, \ldots, r\}$. Clearly, $\deg_G(u_i) = 2$, for $i \in \{1, 2, \ldots, t-1, t+1, \ldots, r\}$.

Let G''_1 be the component of $G - c_0c_1 - c_0c_r$ that contains c_1, G''_2 be the component of $G - c_0c_1 - c_0c_r$ that contains c_0, and G^* be a graph obtained from G''_2 by adding a path $p_2 = x_1x_2$ and joining c_0 to x_1. Clearly, $n(G^*) = n-(3r-2)+2$. By Theorem 14, $ftd_1(G^*) \leq (2(n(G^*)+k-1)-1)/3$. Suppose that $ftd_1(G^*) < (2(n(G^*)+k-1)-1)/3$. Let S^* be an $ftd_1(G^*)$-set. By Observation 1, $x_1 \in S^*$. If $c_0 \in S^*$, then $S^* \cup \{c_1, c_2, \ldots, c_r\} \cup \{u_1, u_2, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\}\backslash\{x_1\}$ is a 1FTD-set in G. Thus $ftd_1(G) < (2(n(G^*)+k-1)-1)/3+2r-1-1 = (2(n-(3r-2)+2+k-1)-1)/3+2r-2 = (2(n+k)-1)/3$, a contradiction. Thus $c_0 \notin S^*$. Then $x_2 \in S^*$. If $t > 1$, then $S^*\backslash\{x_1, x_2\} \cup \{c_1, \ldots, c_{t-1}\} \cup \{u_1, \ldots, u_{t-1}\} \cup \{u_{t+1}, \ldots, u_r\} \cup \{u'_{t+1}, \ldots, u'_r\}$ is a 1FTD-set in G. Thus $ftd_1(G) < (2(n(G^*)+k-1)-1)/3+2(r-1)-2 = (2(n-(3r-2)+2+k-1)-1)/3+2r-4 = (2(n+k)-1)/3-2$, a contradiction. Thus assume that $t = 1$. Then $S^*\backslash\{x_1, x_2\} \cup \{c_2, \ldots, c_r\} \cup \{u_2, \ldots, u_r\}$, is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3-2$ and so $ftd_1(G) < (2(n+k)-1)/3-2$, a contradiction. Thus $ftd_1(G^*) = (2n(G^*+k-1)-1)/3$. By the inductive hypothesis, $G^* \in \mathcal{G}_{k-1}$. Let G_1^* be the graph obtained from $G[G''_1 \cup \{c_0\}]$ by adding a path $p_2 = x'_1x'_2$ and joining c_0 to x'_1. Clearly, $G_1^* \in \mathcal{H}_1$. Thus G is obtained from $G^* \in \mathcal{G}_{k-1}$ and $G_1^* \in \mathcal{H}_1$ by Procedure A. Consequently, $G \in \mathcal{H}_k \subseteq \mathcal{G}_k$.

For the converse, by Corollary 13, $V(G)\backslash L(G)$ is the unique $ftd_1(G)$-set. Now Observation 9 implies that $ftd_1(G) = (2(n+k)-1)/3$. ■

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