

FAIR TOTAL DOMINATION NUMBER IN CACTUS GRAPHS

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Abstract

For $k \geq 1$, a k -fair total dominating set (or just kFTD-set) in a graph G is a total dominating set S such that $|N(v) \cap S| = k$ for every vertex $v \in V \setminus S$. The k -fair total domination number of G , denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A fair total dominating set, abbreviated FTD-set, is a kFTD-set for some integer $k \geq 1$. The fair total domination number of a nonempty graph G , denoted by $ftd(G)$, of G is the minimum cardinality of an FTD-set in G . In this paper, we present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds.

Keywords: fair total domination, cactus graph.

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1. INTRODUCTION

For notation and graph theory terminology not given here, we follow [12]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and let v be a vertex in V . The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from

the context, we simply write $N(v)$ rather than $N_G(v)$. The *degree* of a vertex v , is $\deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor a *support vertex*. We denote the set of leaves and support vertices of a graph G by $L(G)$ and $S(G)$, respectively. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \bigcup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. The *2-corona* $2\text{-cor}(G)$ of a graph G is a graph obtained from G by adding a path P_2 for every vertex v and joining v to a leaf of P_2 . Note that $2\text{-cor}(G)$ has order $3|V(G)|$. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the minimum number of edges of a path from u to v . For a subset S of vertices of a graph G , we denote by $G[S]$ the subgraph of G induced by S . A *cactus* graph is a graph such that no pair of cycles have a common edge.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . A dominating set S in a graph G with no isolated vertex, is a *total dominating set* of G if every vertex in S is adjacent to a vertex in S .

Caro *et al.* [1] studied the concept of fair domination in graphs. For $k \geq 1$, a *k-fair dominating set*, abbreviated kFD-set, in G is a dominating set S such that $|N(v) \cap D| = k$ for every vertex $v \in V \setminus D$. The *k-fair domination number* of G , denoted by $fd_k(G)$, is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality $fd_k(G)$ is called a $fd_k(G)$ -set. A *fair dominating set*, abbreviated FD-set, in G is a kFD-set for some integer $k \geq 1$. The *fair domination number*, denoted by $fd(G)$, of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G . An FD-set of G of cardinality $fd(G)$ is called a $fd(G)$ -set. A *perfect dominating set* in a graph G is a dominating set S such that every vertex in $V(G) \setminus S$ is adjacent to exactly one vertex in S . Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows *et al.* [7] with a different terminology which they called semiperfect domination. This concept was further studied in, for example, [2, 3, 5, 6, 8, 9, 11].

Maravilla *et al.* [13] introduced the concept of fair total domination in graphs. For an integer $k \geq 1$ and a graph G with no isolated vertex, a *k-fair total dominating set*, abbreviated kFTD-set, is a total dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. The *k-fair total domination number* of G , denoted by $ftd_k(G)$, is the minimum cardinality of a kFTD-set. A kFTD-set of G of cardinality $ftd_k(G)$ is called an $ftd_k(G)$ -set. A *fair total dominating set*, abbreviated FTD-set, in G is a kFTD-set for some integer $k \geq 1$. Thus, a fair total dominating set S of a graph G is a total dominating set S of G such that for every two distinct vertices u and v of $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$; that

is, S is both a fair dominating set and a total dominating set of G . The *fair total domination number* of G , denoted by $ftd(G)$, is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality $ftd(G)$ is called a minimum fair total dominating set or an *ftd-set* of G .

In [10], Volkmann and we studied fair total domination in trees and unicyclic graphs. In this paper, we study 1-fair total domination in cactus graphs. We present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds. The techniques used in this paper are similar to those presented in [9]. The following observations are easily verified.

Observation 1. *Any support vertex in a graph G with no isolated vertex belongs to every k FTD-set for each integer k .*

Observation 2. *Let S be a 1FTD-set in a graph G , and v be a vertex of degree at least two such that v is adjacent to a weak support vertex v' . If S contains a vertex $u \in N_G(v) \setminus \{v'\}$, then $v \in S$.*

2. UNICYCLIC GRAPHS

A vertex v of a graph is a *special vertex* if $\deg_G(v) = 2$ and v belongs to a cycle of G . Let $\mathcal{H}_1$ be the class of all graphs G that can be obtained from the 2-corona $2\text{-cor}(C)$ of a cycle C by removing precisely one support vertex v and the leaf adjacent to v . Let $\mathcal{G}_1$ be the class of all graphs G that can be obtained from a sequence $G_1, G_2, \dots, G_s = G$, where $G_1 \in \mathcal{H}_1$, and if $s \geq 2$, then G_{j+1} is obtained from G_j by one of the following Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, for $j = 1, 2, \dots, s-1$.

Operation $\mathcal{O}_1$. Let v be a vertex of G_j with $\deg(v) \geq 2$ such that v is not a special vertex. Then G_{j+1} is obtained from G_j by adding a path P_3 and joining v to a leaf of P_3 by means of an edge.

Operation $\mathcal{O}_2$. Let v be a support vertex of G_j and let u be a leaf adjacent to v . Then G_{j+1} is obtained from G_j by adding a vertex u' and a path P_2 , and joining u to u' and v to a leaf of P_2 .

Observation 3. *If $H \in \mathcal{H}_1$, then H has precisely one special vertex.*

Observation 4 [10]. *If $G \in \mathcal{G}_1$ has order n , and C is the cycle of G , then we have the following.*

- (1) G has precisely one special vertex.
- (2) G has $(n-1)/3$ leaves.
- (3) No vertex of C is a support vertex.

- (4) Any vertex of C is adjacent to at most one weak support vertex of degree two.

Lemma 5 [10]. If $G \in \mathcal{G}_1$, then every 1FTD-set in G contains every vertex of G of degree at least two.

Theorem 6 [10]. If G is a unicyclic graph of order n , then $ftd_1(G) \leq (2n+1)/3$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}_1$.

3. MAIN RESULT

Our aim in this paper is to give an upper bound for the fair total domination number of a cactus graph G in terms of the number of cycles of G , and then characterize all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let $\mathcal{H}_1$ and $\mathcal{G}_1$ be the families of unicyclic graphs described in Section 2. For $i = 2, \dots, k$, we construct a family $\mathcal{H}_i$ from $\mathcal{G}_{i-1}$, and a family $\mathcal{G}_i$ from $\mathcal{H}_i$ as follows.

- Family $\mathcal{H}_i$. Let $\mathcal{H}_i$ be the family of all graphs H_i such that H_i can be obtained from a graph $H_1 \in \mathcal{H}_1$ and a graph $G \in \mathcal{G}_{i-1}$, by the following procedure.

Procedure A. Let $w_0 \in V(H_1)$ be a vertex of degree at least two of H_1 such that w_0 is adjacent to a weak support vertex w'_0 , and $w \in V(G_{i-1})$ be a vertex of degree at least two of G_{i-1} such that w is adjacent to a weak support vertex w' of degree two. We remove w'_0 , the leaf adjacent to w'_0 , w' and the leaf adjacent to w' , and then identify the vertices w_0 and w .

- Family $\mathcal{G}_i$. Let $\mathcal{G}_i$ be the family of all graphs G that can be obtained from a sequence $G_1, G_2, \dots, G_s = G$, where $G_1 \in \mathcal{H}_i$, and if $s \geq 2$, then G_{j+1} is obtained from G_j by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, described in Section 2, for $j = 1, 2, \dots, s-1$.

Note that $\mathcal{H}_i \subseteq \mathcal{G}_i$, for $i = 1, 2, \dots, k$. Figure 1 demonstrates the construction of the family $\mathcal{G}_k$.

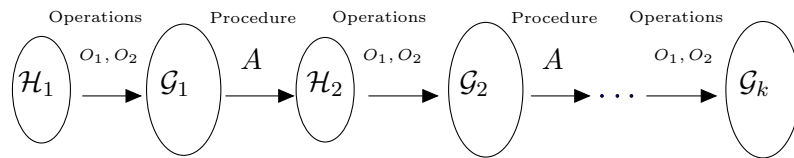


Figure 1. Construction of the family $\mathcal{G}_k$.

We will prove the following.

Theorem 7. If G is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $ftd_1(G) \leq (2(n+k)-1)/3$, with equality if and only if $G = C_7$ or $G \in \mathcal{G}_k$.

4. PRELIMINARY RESULTS AND OBSERVATIONS

4.1. Notation

We call a vertex w in a cycle C of a cactus graph G a *special cut-vertex* if w belongs to a shortest path from C to a cycle $C' \neq C$. We call a cycle C in G , a *leaf-cycle* if C contains exactly one special cut-vertex. In the cactus graph presented in Figure 2, v_i is a special cut-vertex, for $i = 1, 2, \dots, 8$. Moreover, C_j is a leaf-cycle for $j = 1, 2, 3$.

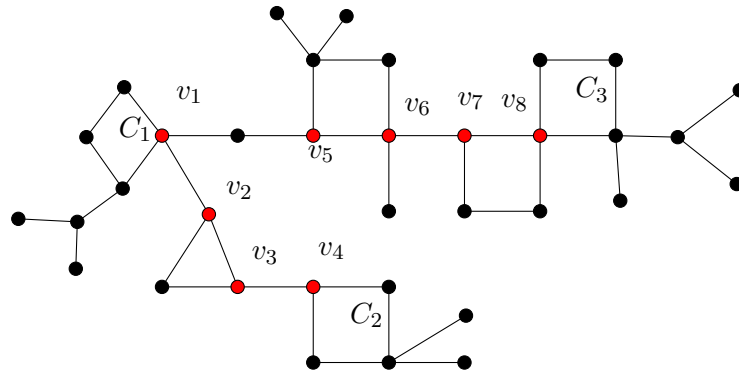


Figure 2. C_i is a leaf-cycle for $i = 1, 2, 3$ and v_j is a special cut-vertex for $j = 1, 2, \dots, 8$.

Observation 8. *Every cactus graph with at least two cycles contains at least two leaf-cycles.*

4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on k .

Observation 9. *If $G \in \mathcal{G}_k$ is a cactus graph of order n , then we have the following.*

- (1) *No cycle of G contains a support vertex. Furthermore, any cycle of G contains precisely one special vertex.*
- (2) *If a vertex v of G belongs to a cycle of G , then v is adjacent to at most one weak support vertex of degree two.*
- (3) $|L(G)| = (n + 1)/3 - 2k/3$.
- (4) *If a vertex v of G belongs to at least two cycles of G , then v is not adjacent to a weak support vertex, and v belongs to precisely two cycles of G .*

Proof. Let $G \in \mathcal{G}_k$ be a cactus graph of order n . To show (1), (2) or (3), we prove by an induction on k , that we call *first-induction*. For the base step, if $k = 1$, then $G \in \mathcal{G}_1$, and the result follows by Observation 4. Assume the result

holds for all graphs $G' \in \mathcal{G}_{k'}$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, G is obtained from a sequence $G_1, G_2, \dots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then G_{i+1} is obtained from G_i by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i = 1, 2, \dots, l-1$. We prove by an induction on l , that we call *second-induction*. For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that G is obtained from H and G' by Procedure A. It is easy to see that the base step of the second-induction holds. Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, G is obtained from G_{l-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$. It is easy to see that the result holds.

The proof for (4) is similarly verified. $\blacksquare$

Observation 10. Let $G \in \mathcal{G}_k$ be obtained from a sequence $G_1, G_2, \dots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and G_{j+1} is obtained from G_j by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$ or Procedure A, for $j = 1, 2, \dots, s-1$. If v is a vertex of G belonging to two cycles of G , then there is an integer $i \in \{2, 3, \dots, s\}$ such that G_i is obtained from G_{i-1} by applying Procedure A on the vertex v using a graph $H \in \mathcal{H}_1$, such that v belongs to a cycle of G_{i-1} .

Observation 11. Assume that $G \in \mathcal{G}_k$ and $v \in V(G)$ is a vertex of degree four belonging to two cycles. Let D_1 and D_2 be the components of $G - v$, G_1^* be the graph obtained from $G[D_1 \cup \{v\}]$ by joining v to a leaf of a path P_2 , and G_2^* be the graph obtained from $G[D_2 \cup \{v\}]$ by joining v to a leaf of a path P_2 . Then there exists an integer $k' < k$ such that $G_1^* \in \mathcal{G}_{k'}$ or $G_2^* \in \mathcal{G}_{k'}$.

Proof. Let $G \in \mathcal{G}_k$. Then G is obtained from a sequence $G_1, G_2, \dots, G_s = G$ ($s \geq 2$) such that $G_1 \in \mathcal{H}_1$ and G_{j+1} is obtained from G_j by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$ or procedure A, for $j = 1, 2, \dots, s-1$. Note that $s \geq k$. We define the j -th Procedure-Operation or just PO_j as one of the Operation $\mathcal{O}_1$, Operation $\mathcal{O}_2$, or Procedure A that can be applied to obtain G_{j+1} from G_j . Thus G is obtained from G_1 by Procedure-Operations $PO_1, PO_2, \dots, PO_{s-1}$.

Let v be a vertex of G of degree four belonging to two cycles of G , and D_1 and D_2 be the components of $G - v$. By Observation 10, there is an integer $i \in \{2, 3, \dots, s\}$ such that G_i is obtained from G_{i-1} by applying Procedure A on the vertex v using a graph $H \in \mathcal{H}_1$. Note that v is adjacent to a weak support vertex v' of G_{i-1} . Let v'' be the leaf of v' in G_{i-1} that is removed in Procedure A. Clearly, either $V(G_{i-1}) \cap D_1 \neq \emptyset$ or $V(G_{i-1}) \cap D_2 \neq \emptyset$. Without loss of generality, assume that $V(G_{i-1}) \cap D_1 \neq \emptyset$. Among $PO_i, PO_{i+1}, \dots, PO_{s-1}$, let $PO_{r_1}, PO_{r_2}, \dots, PO_{r_t}$ be those procedure-operations applied on a vertex of D_1 . Note that $i \leq t \leq s-1$. Let $G_{r_0} = G_{i-1}$ and $G_{r_{l+1}}$ be obtained from G_{r_l} by $PO_{r_{l+1}}$, for $l = 0, 1, 2, \dots, t-1$. Clearly, by an induction on t , we can deduce that there is an integer $k^* < k$ such that $G_{r_t} \in \mathcal{G}_{k^*}$. Note that $G_{r_t} = G_1^*$. $\blacksquare$

Lemma 12. *If $G \in \mathcal{G}_k$, then every 1FTD-set in G contains each vertex of G of degree at least two.*

Proof. Let $G \in \mathcal{G}_k$, and S be a 1FTD-set in G . We prove by an induction on k , that we call *first-induction*, that S contains every vertex of G of degree at least two. For the base step, if $k = 1$, then $G \in \mathcal{G}_1$, and the result follows by Lemma 5. Assume the result holds for all graphs $G' \in \mathcal{G}_{k'}$ with $k' < k$. Now consider the graph $G \in \mathcal{G}_k$, where $k > 1$. Clearly, G is obtained from a sequence $G_1, G_2, \dots, G_l = G$, of cactus graphs such that $G_1 \in \mathcal{H}_k$, and if $l \geq 2$, then G_{i+1} is obtained from G_i by one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i = 1, 2, \dots, l - 1$.

We prove by an induction on l , that we call *second-induction*, that S contains every vertex of G of degree at least two.

For the base step of the second-induction, let $l = 1$. Thus $G \in \mathcal{H}_k$. By the construction of graphs in the family $\mathcal{H}_k$, there are graphs $H \in \mathcal{H}_1$ and $G' \in \mathcal{G}_{k-1}$ such that G is obtained from H and G' by Procedure A. Clearly, H is obtained from the 2-corona $2\text{-cor}(C)$ of a cycle C , by removing precisely one support vertex v and the leaf adjacent to v of $2\text{-cor}(C)$.

Let $C = c_0c_1 \cdots c_rc_0$ be the cycle of H , where c_0 is a vertex of degree at least two of H that is adjacent to a weak support vertex c'_0 , and let c'_0 and its leaf (that we call c''_0) be removed according to Procedure A. By Observation 3, H has precisely one special vertex. Let c_t be the special vertex of H . Let $w \in V(G')$ be a vertex of degree at least two of G' that is adjacent to a weak support vertex w' , and let w' and its leaf (that we call w'') be removed according to Procedure A.

First we show that $\{c_1, c_r\} \cap S \neq \emptyset$. Clearly, $S \cap \{c_{t-1}, c_t, c_{t+1}\} \neq \emptyset$, since $\deg_G(c_t) = 2$. Assume that $c_t \in S$. Since at least one of c_{t-1} or c_{t+1} is adjacent to a weak support vertex, by Observation 2, $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$. By applying Observation 2, we obtain that $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \dots, c_r\} \setminus \{c_t\}$ is adjacent to a weak support vertex of G . Thus assume that $c_t \notin S$. Then $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$, and so $\{c_1, c_r\} \cap S \neq \emptyset$, since any vertex of $\{c_1, \dots, c_r\} \setminus \{c_t\}$ is adjacent to a weak support vertex of G . Hence, $\{c_1, c_r\} \cap S \neq \emptyset$. If $c_0 \notin S$, then $S \cup \{w', w''\}$ is a 1FTD-set for G' , and thus by the first-inductive hypothesis, S' contains $w = c_0$, a contradiction. Thus $c_0 \in S$. By Observation 2, $V(C) \subseteq S$, since any vertex of $\{c_1, \dots, c_r\} \setminus \{c_t\}$ is adjacent to a weak support vertex of G . Thus $S \cap V(G')$ is a 1FTD-set for G' . By the first-inductive hypothesis, $(S \cap V(G')) \cup \{w', w''\}$ contains every vertex of G' of degree at least two. Consequently, S contains every vertex of G of degree at least two. We conclude that the base step of the second-induction holds.

Assume that the result (for the second-induction) holds for $2 \leq l' < l$. Now let $G = G_l$. Clearly, G is obtained from G_{l-1} by applying one of the Operations $\mathcal{O}_1$ or $\mathcal{O}_2$.

Assume that G is obtained from G_{l-1} by applying Operation $\mathcal{O}_2$. Let x be a support vertex of G_{l-1} and let x' be a leaf adjacent to x . Let G be obtained from

G_{l-1} by adding a vertex u' and a path $P_2 = y_1y_2$, joining x' to u' and joining x to y_1 , according to Operation $\mathcal{O}_2$. By Observation 1, $x', y_1 \in S$ and so $x \in S$. Thus $S \setminus \{y_1\}$ is a 1FTD-set for G_{l-1} . By the second-inductive hypothesis, S contains all vertices of G_{l-1} of degree at least two. Consequently, S contains every vertex of G_k of degree at least two.

Next assume that G is obtained from G_{l-1} by applying Operation $\mathcal{O}_1$. Let $P_3 = x_1x_2x_3$ be a path and x_1 be joined to $y \in V(G_{l-1})$, where $\deg_{G_{l-1}}(y) \geq 2$ and y is not a special vertex of G_{l-1} , according to Operation $\mathcal{O}_2$. By Observation 1, $x_2 \in S$. Observe that $\{x_1, x_3\} \cap S \neq \emptyset$. If $x_1 \notin S$, then $x_3 \in S$ and $y \notin S$. Then $S \setminus \{x_2, x_3\}$ is a 1FTD-set for G_{l-1} that does not contain y , a contradiction by the second-inductive hypothesis. Thus assume that $x_1 \in S$. Suppose that $y \notin S$. Clearly, $N_{G_{l-1}}(y) \cap S = \emptyset$. Assume that there exists a component G'_1 of $G_{l-1} - y$ such that $|V(G'_1) \cap N_{G_{l-1}}(y)| = 1$. Then clearly $S' = (S \cap V(G_{l-1})) \cup V(G'_1)$ is a 1FTD-set for G_{l-1} , and by the second-inductive hypothesis, S' contains every vertex of G_{l-1} of degree at least two. Thus $y \in S'$, and so $y \in S$, a contradiction. Next assume that every component of $G_{l-1} - y$ has at least two vertices in $N_{G_{l-1}}(y)$. Since y is a non-special vertex of G_{l-1} , y belongs to at least two cycles of G_{l-1} . By Observation 9(4), y belongs to exactly two cycles of G_{l-1} . Thus $\deg_{G_{l-1}}(y) = 4$. By Observation 11, $G_{l-1} - y$ has exactly two components D_1 and D_2 . Let G^* be a graph obtained from $D_1 \cup \{y\}$ or $D_2 \cup \{y\}$ by adding a path $P_2 = y'y''$ to y . Then there exists $k' \leq k$ such that $G^* \in \mathcal{G}_{k'}$. Evidently, $S^* = (S \cap V(G^*)) \cup \{y', y''\}$ is a 1FTD-set for G^* , and so by the first-inductive hypothesis, S^* contains every vertex of G^* of degree at least two (since $G^* \in \mathcal{G}_{k'}$). Thus $y \in S^*$, and so $y \in S$, a contradiction. We conclude that $y \in S$. Observe that $S \cap V(G_{l-1})$ is a 1FTD-set for G_{l-1} , and so by the second-inductive hypothesis, $S \cap V(G_{l-1})$ contains every vertex of G_{l-1} of degree at least two. Consequently, S contains every vertex of G of degree at least two. ■

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

Corollary 13. *If $G \in \mathcal{G}_k$ is a cactus graph of order n , then $V(G) \setminus L(G)$ is the unique $ftd_1(G)$ -set.*

In what follows, we present an upper bound for the 1-fair domination number of a cactus graph in terms of the order and the number of cycles.

Theorem 14. *If G is a cactus graph of order $n \geq 4$ with $k \geq 1$ cycles, then $ftd_1(G) \leq (2(n+k) - 1)/3$.*

Proof. The result follows by Theorem 6 if $k = 1$. Thus assume that $k \geq 2$. Suppose to the contrary that $ftd_1(G) > (2(n(G) + k) - 1)/3$. Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is minimum. Let $C_1, C_2, \dots, C_k$ be the k cycles of G . Let C_i

be a leaf-cycle of G , where $i \in \{1, 2, \dots, k\}$. Let $C_i = c_0 c_1 \cdots c_r c_0$, where c_0 is the special cut-vertex of G . Suppose that G has a strong support vertex u , and u_1, u_2 are leaves adjacent to u . Let $G_0 = G - u_1$. By the choice of G , $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $u \in S'$. Clearly, S' is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. We deduce that every support vertex of G is adjacent to precisely one leaf.

Assume that $\deg_G(u_j) = 2$ for each $j = 1, 2, \dots, r$. Let $G' = G - c_2$. Then by the choice of G , $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then S' is a 1FTD-set for G cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Now $\{c_1\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. We deduce that $\deg_G(c_i) \geq 3$ for some $i \in \{1, 2, \dots, r\}$.

Let v_d be a leaf of G such that $d(v_d, C_i - c_0)$ is as maximum as possible, the shortest path from v_d to C_i does not contain c_0 and $\deg(v_{d-1})$ is as maximum as possible, where v_{d-1} is the neighbor of v_d on the shortest path from v_d to a vertex $v_0 \in C_i$.

Assume that $d \geq 3$. Observe that $\deg_G(v_{d-1}) = 2$, since G has no strong support vertex. Assume that $\deg_G(v_{d-2}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By the choice of G , $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$ -set. If $v_{d-3} \in S'$, then $\{v_{d-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. If $v_{d-3} \notin S'$, then $\{v_{d-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction. Thus assume that $\deg_G(v_{d-2}) \geq 3$. Assume that v_{d-2} is a support vertex. Let $G' = G - \{v_{d-1}, v_d\}$. By the choice of G , $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of G . Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of G such that $x \in N(v_{d-2})$. By the choice of the path $v_0 v_1 \cdots v_d$, (the part “ $\deg(v_{d-1})$ is as maximum as possible”), $\deg_G(x) = 2$. Let y be the leaf adjacent to x and $G' = G - \{v_d, v_{d-1}, y\}$. By the choice of G , $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of G' . Thus $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3$, a contradiction.

Next assume that $d = 2$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \dots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. Then by the choice of G , $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$,

then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ and so $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Let c_{j+1} be a support vertex of G , and $G' = G - c_j$. Then by the choice of G , $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 4/3$, a contradiction. Thus assume that $c_{j-1} \in S'$ and so $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus c_{j+1} is not a support vertex of G . Let $c'_{j+1} \in N(c_{j+1}) \setminus V(C_i)$. Clearly, c'_{j+1} is a support vertex, since $d = 2$. Observe that $\deg_G(c'_{j+1}) = 2$, since G has no strong support vertex. Let c''_{j+1} be the leaf of c'_{j+1} . Let $G' = G - c_j - c''_{j+1}$. By the choice of G , $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+1} \in S'$, since c_{j+1} is a support vertex in G' . If $c_{j-1} \notin S'$, then $S' \cup \{c'_{j+1}\}$ is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 1$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j, c'_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3$, a contradiction. Thus $\deg(c_i) \geq 3$ for $1 \leq i \leq r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G_1^* be the component of G^* containing c_r , and G_2^* be the component of G^* containing c_0 . Let $D = S(G_1^*) \setminus V(C_i)$. Clearly, $S' = D \cup \{c_1, c_2, \dots, c_r\}$ is a 1FTD-set for G_1^* of cardinality at most $2n(G_1^*)/3$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By the choice of G , $ftd_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$. Let S'' be an $ftd_1(G_3^*)$ -set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_3^*) + k - 1) - 1)/3 + 2n(G_1^*)/3 = (2(n+k) - 1)/3$, a contradiction.

Now assume that $d = 1$. Assume that $\deg_G(c_i) = 2$ for some $i \in \{1, 2, \dots, r\}$. Let $\deg_G(c_j) = 2$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By the choice of G , $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly, $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Thus c_{j+1} is a support vertex of G . Let $G' = G - c_j$. Then by the choice of G , $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then S' is a 1FTD-set for G , a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j\} \cup S'$ is a

1FTD-set in G of cardinality at most $(2(n+k)-1)/3-1/3$, a contradiction. We thus obtain that $\deg(c_i) \geq 3$ for $1 \leq i \leq r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G_1^* be the component of G^* containing c_r , and G_2^* be the component of G^* containing c_0 . Clearly, $S' = \{c_1, c_2, \dots, c_r\}$ is a 1FTD-set for G_1^* of cardinality at most $n(G_1^*)/2$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By the choice of G , $ftd_1(G_3^*) \leq (2(n(G_3^*)+k-1)-1)/3$. Let S'' be an $ftd_1(G_3^*)$ -set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_3^*)+k-1)-1)/3 + n(G_1^*)/2 < (2(n+k)-1)/3$, a contradiction. ■

It is evident that for the cycle C_7 the equality of the bound given in Theorem 14 holds.

Theorem 15. *If $G \neq C_7$ is a cactus graph of order $n \geq 5$ with $k \geq 1$ cycles, then $ftd_1(G) = (2(n+k)-1)/3$ if and only if $G \in \mathcal{G}_k$.*

Proof. We prove by an induction on k to show that any cactus graph $G \neq C_7$ of order $n \geq 5$ with $k \geq 1$ cycles and $ftd_1(G) = (2(n+k)-1)/3$ belongs to $\mathcal{G}_k$. The base step of the induction follows by Theorem 6. Assume the result holds for all cactus graphs $G' \neq C_7$ with $k' < k$ cycles. Now let $G \neq C_7$ be a cactus graph of order n with $k \geq 2$ cycles and $ftd_1(G) = (2(n+k)-1)/3$. Suppose to the contrary that $G \notin \mathcal{G}_k$. Assume that G has the minimum order, and among all such graphs, assume that the size of G is minimum.

Claim 1. *Every support vertex of G is weak support vertex.*

Proof. Suppose that G has a strong support vertex u , and assume that u_1 and u_2 are two leaves adjacent to u . Let $G' = G - u_1$, and S' be an $ftd_1(G')$ -set. By Observation 1, $u \in S'$. By Theorem 14, $|S'| \leq (2(n(G') + 2) - 1)/3 = (2(n+k)-1)/3 - 2/3$. Clearly, S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 2/3$, a contradiction. □

By Observation 8, G has at least two leaf-cycles. Let $C_1 = c_0c_1 \cdots c_rc_0$ be a leaf-cycle of G , where c_0 is a special cut-vertex of G . Let G'_1 be the component of $G - c_0c_1 - c_0c_r$ containing c_1 .

Claim 2. $V(G'_1) \neq \{c_1, \dots, c_r\}$.

Proof. Suppose that $V(G'_1) = \{c_1, \dots, c_r\}$. Then $\deg_G(c_i) = 2$, for each $i = 1, 2, \dots, r$. Let $G' = G - c_2$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k)-1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_0 \in S'$. If $|S' \cap \{c_1, c_3\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n+k)-1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 2$. Then $\{c_2\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_1, c_3\}| = 0$. Then $\{c_1\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n+k)-1)/3 - 1/3$, a contradiction. □

Let $v_d \in V(G'_1) \setminus \{c_1, \dots, c_r\}$ be a leaf of G'_1 at maximum distance from $\{c_1, \dots, c_r\}$, and assume that $\deg(v_{d-1})$ is as maximum as possible, $\deg_G(v_0)$ is as maximum as possible, and $\deg_G(v_1)$ is as maximum as possible, where $v_0 \in \{c_1, \dots, c_r\}$ and $v_0v_1 \cdots v_d$ is the shortest path from v_d to $\{c_1, \dots, c_r\}$.

Suppose that $d = 1$. Assume that $\deg_G(c_j) = 2$, for some $j \in \{1, 2, \dots, r\}$. Assume that $\deg_G(c_{j+1}) = 2$. Let $G' = G - c_j$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly, $\deg_G(c_{j-1}) \geq 3$. Clearly, $c_{j+1} \neq c_0$ or $c_{j-1} \neq c_0$. Assume, without loss of generality, that $c_{j+1} \neq c_0$. Then c_{j+1} is a support vertex of G . Let $G' = G - c_j$. Then by Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+1} \in S'$. If $c_{j-1} \notin S'$, then S' is a 1FTD-set for G of cardinality at most $(2(n + k) - 1)/3 - 4/3$, a contradiction. Thus assume that $c_{j-1} \in S'$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$, a contradiction. We thus obtain that $\deg(c_j) \geq 3$, for $1 \leq j \leq r$. Let $G^* = G - c_0c_1 - c_0c_r$. Let G_1^* be the component of G^* containing c_r , and G_2^* be the component of G^* containing c_0 . Clearly, $S' = \{c_1, c_2, \dots, c_r\}$ is a 1FTD-set for G_1^* of cardinality at most $n(G_1^*)/2$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. By Theorem 14, $ftd_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$. Let S'' be an $ftd_1(G_3^*)$ -set. By Observation 1, $c_0 \in S''$. Clearly, $S' \cup S''$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_3^*) + k - 1) - 1)/3 + n(G_1^*)/2 < (2(n + k) - 1)/3$, a contradiction.

Thus assume that $d \geq 2$.

Claim 3. If $d \geq 3$, then $G \in \mathcal{G}_k$.

Proof. Assume that $d \geq 3$. By Claim 1, $\deg_G(v_{d-1}) = 2$. Assume first that $\deg_G(v_{d-2}) \geq 3$. Assume that v_{d-2} is a support vertex. Let $G' = G - \{v_{d-1}, v_d\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $v_{d-2} \in S'$. Then $\{v_{d-1}\} \cup S'$ is a 1FTD-set in G , and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 1/3$, a contradiction. Thus assume that v_{d-2} is not a support vertex of G . Let $x \neq v_{d-1}, v_{d-3}$ be a support vertex of G such that $x \in N(v_{d-2})$. By the choice of the path $v_0v_1 \cdots v_d$, (the part “ $\deg(v_{d-1})$ is as maximum as possible”), $\deg_G(x) = 2$. Let y be the leaf adjacent to x , and $G' = G - \{v_d, v_{d-1}, y\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Assume that $ftd_1(G') < (2(n(G') + k) - 1)/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $v_{d-2} \in S'$, since v_{d-2} is a support vertex of G' . Then $\{v_{d-1}, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) < (2(n + k) - 1)/3$, a

contradiction. Thus $ftd_1(G') = (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. By the choice of G , $G' \in \mathcal{G}_k$. Thus G is obtained from G' by Operation $\mathcal{O}_2$, and so $G \in \mathcal{G}_k$.

Assume that $\deg_G(v_{d-2}) = 2$. We consider the following cases.

Case 1. $d \geq 4$. Suppose that $\deg_G(v_{d-3}) = 2$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}, v_{d-3}\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 8/3$. Let S' be an $ftd_1(G')$ -set. If $v_{d-4} \in S'$, then $\{v_{v-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. Thus $v_{d-4} \notin S'$. Then $\{v_{v-2}, v_{d-1}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$, a contradiction. We deduce that $\deg_G(v_{d-3}) \geq 3$. Let $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Assume that $ftd_1(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$ -set. If $v_{d-3} \in S'$, then $\{v_{v-1}, v_{d-2}\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $v_{d-3} \notin S'$. Then $\{v_{v-1}, v_d\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. We thus obtain that $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the choice of G , $G' \in \mathcal{G}_k$. Since $d \geq 4$, v_{d-3} is not a special vertex of G' . Thus G is obtained from G' by Operation $\mathcal{O}_1$, and so $G \in \mathcal{G}_k$.

Case 2. $d = 3$. Clearly, $\deg(v_0) \geq 3$. We show that $\deg(v_0) \geq 4$. Suppose that $\deg(v_0) = 3$. Let $G' = G - \{v_1, v_2, v_3\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Assume that $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the choice of G , $G' \in \mathcal{G}_k$. By Observation 9(1), v_0 is the unique special vertex of G' , since $\deg_{G'}(v_0) = 2$. We show that $\deg_{G'}(x) = 3$ for each $x \in \{c_1, \dots, c_r\} \setminus \{v_0\}$. Assume that $\deg_{G'}(c_j) \geq 4$ for some $c_j \in \{c_1, \dots, c_r\} \setminus \{v_0\}$. If there is a vertex $w \in V(G) \setminus V(C_1)$ such that $d(w, C_1) = d(w, c_j) = 3$, then w can play the same role of v_d , and thus $\deg(u_j) = 3$, a contradiction. Thus there is no vertex $w \in V(G) \setminus V(C_1)$ such that $d(w, C_1) = d(w, c_j) = 3$. Thus any vertex of $N(u_j) \setminus V(C_1)$ is a leaf or a weak support vertex. Assume that $N(c_j) \setminus V(C_1)$ contains t_1 leaves and t_2 support vertices, where $t_1 + t_2 \geq 2$. By Observation 9(1), $t_1 = 0$, since $G' \in \mathcal{G}_k$. Thus $t_2 \geq 2$. Let z_1 and z_2 be two weak support vertices in $N(c_j) \setminus V(C_1)$. Let z'_1 and z'_2 be the leaves adjacent to z_1 and z_2 , respectively. (We switch for a while to G). Let $G'' = G - \{z_1, z'_1, z'_2\}$. By Theorem 14, $ftd_1(G'') \leq (2(n(G'') + k) - 1)/3$. Suppose that $ftd_1(G'') = (2(n(G'') + k) - 1)/3$. By the choice of G , $G'' \in \mathcal{G}_k$. Clearly, $\deg_{G''}(c_i) \geq 3$, since v_0 is the unique special vertex of G' , a contradiction (by Observation 9(1)). Thus $ftd_1(G'') < (2(n(G'') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S'' be a 1FTD-set of G'' . By Observation 1, $c_j \in S''$. Then $S'' \cup \{z_1, z_2\}$ is a 1FTD-set of G . Thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. We deduce that $\deg_{G'}(c_i) = 3$ for each $c_i \in \{c_1, \dots, c_r\} \setminus \{v_0\}$. Thus $\deg_G(c_i) = 3$ for each $1 \leq i \leq r$. Note that by Observation 9(1), c_i is not a support vertex, for each i with $1 \leq i \leq r$ in G' , since $G' \in \mathcal{G}_k$. (We switch for a while to G). Let $F = \bigcup_{i=1}^r (N[c_i] \setminus \{c_0, \dots, c_r\})$. Clearly, $|F| = r$, since $\deg_{G'}(c_i) = 3$ for each $c_i \in \{c_1, \dots, c_r\} \setminus \{v_0\}$ and $\deg_G(v_0) = 3$. Let $F = \{u_1, u_2, \dots, u_r\}$. Clearly $\deg_G$

$(u_i) \geq 2$, for each i with $1 \leq i \leq r$, since c_i is not a support vertex for $1 \leq i \leq r$ in G' . By Claim 2, u_i is not a strong support vertex of G , for $1 \leq i \leq r$. If u_i is adjacent to a support vertex $u'_i \in V(G) \setminus V(C_1)$, for some integer i , then since the leaf of u'_i can play the role of v_3 , we obtain that $\deg(u_i) = 2$. Since $\deg_G(u_i) \geq 2$ for each i with $1 \leq i \leq r$, we find that $\deg_G(u_i) = 2$ for each i with $1 \leq i \leq r$.

Let $F' = \bigcup_{i=1}^r N(u_i) \setminus \{c_0, \dots, c_r\}$. Clearly, $|F'| = r$, since $\deg_G(u_i) = 2$, for each $u_i \in \{u_1, \dots, u_r\}$. Let $F' = \{u'_1, u'_2, \dots, u'_r\}$. By the choice of the path $v_0 v_1 \dots v_d$, (the part “ $\deg(v_{d-1})$ is as maximum as possible”), $\deg(u'_i) \leq 2$, for $1 \leq i \leq r$. Let $F'_1 = \{u'_i \in F' \mid \deg_G(u'_i) = 1\}$ and $F'_2 = F' - F'_1$. Then every vertex of F'_2 is a weak support vertex. Since $v_1 \in F'_2$, we have $|F'_2| \geq 1$. Let $G^* = G - c_0 c_1 - c_0 c_r$, and G_1^* and G_2^* be the components of G^* , where $c_1 \in V(G_1^*)$. By Theorem 14, $ftd_1(G_2^*) \leq (2(n(G_2^*) + k - 1) - 1)/3$. Clearly, $n(G_2^*) = n(G) - 3r - |F'_2|$. Let S_2^* be an $ftd_1(G_2^*)$ -set. If $c_0 \notin S_2^*$, then $S_2^* \cup F \cup F'$ is a 1FTD-set for G . Thus $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $c_0 \in S_2^*$. If $|F'_2| = 1$, then $S_2^* \cup V(C_1) \cup F \cup \{v_2\}$ is a 1FTD-set for G and thus $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r + 1 = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r + 1 < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|F'_2| \geq 2$. Let $\{u'_t, u'_{t'}\} \subseteq F'_2$ (assume without loss of generality that $t < t'$) such that $\deg_G(u'_i) = 1$, for $1 \leq i < t$ and $t' < i \leq r$. Let u''_t and $u''_{t'}$ be the leaves of u_t and $u_{t'}$, respectively. Clearly, $S_2^* \cup \{c_1, \dots, c_{t-1}\} \cup \{u_1, \dots, u_{t-1}\} \cup \{c_{t'+1}, \dots, c_r\} \cup \{u_{t'+1}, \dots, u_r\} \cup \{u_{t+1}, \dots, u_{t'-1}\} \cup \{u'_{t+1}, \dots, u'_{t'-1}\} \cup \{u'_t, u'_{t'}\} \cup \{u''_t, u''_{t'}\}$ is a 1FTD-set for G and thus $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n(G) - 3r - |F'_2| + k - 1) - 1)/3 + 2r + 1 < (2(n + k) - 1)/3$, a contradiction. We deduce that $ftd_1(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$ -set. If $v_0 \in S'$, then $S' \cup \{v_1, v_2\}$ is a 1FTD-set in G , and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S' \cup \{v_2, v_3\}$ is a 1FTD-set in G and thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\deg(v_0) \geq 4$. Let $G' = G - \{v_3, v_2, v_1\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Assume that $ftd_1(G') < (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$. Let S' be an $ftd_1(G')$ -set. If $v_0 \in S'$, then $S = S' \cup \{v_1, v_2\}$ is a 1FTD-set for G and thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $v_0 \notin S'$. Then $S = S' \cup \{v_2, v_3\}$ is a 1FTD-set for G and thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Hence, $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the inductive hypothesis, $G' \in \mathcal{G}_{k-1}$. Since $\deg(v_0) \geq 4$, v_0 is not a special vertex of G' . Thus G is obtained from G' by Operation $\mathcal{O}_1$ and so $G \in \mathcal{G}_k$. $\square$

By Claim 3, we assume that $d = 2$. We show that $\deg_G(v_0) = 3$. Suppose that $\deg_G(v_0) \geq 4$. Assume that v_0 is a support vertex. Let $G' = G - \{v_1, v_2\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $v_0 \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FTD-set in G , and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that v_0 is not a support vertex

of G . Let $x \neq v_1$ be a support vertex of G such that $x \in N(v_0) \setminus V(C_1)$. By the choice of the path $v_0 v_1 \cdots v_d$, (the part “ $\deg(v_{d-1})$ is as maximum as possible”), $\deg_G(x) = 2$. Let y be the leaf adjacent to x . Let $G' = G - \{v_2, v_1, y\}$. By Theorem 14, $ftd_1(G') \leq (2(n(G') + k) - 1)/3$. Let $ftd_1(G') < (2(n(G') + k) - 1)/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $v_0 \in S'$, since v_0 is a support vertex of G' . Then $\{v_1, x\} \cup S'$ is a 1FTD-set in G and so $ftd_1(G') < (2(n + k) - 1)/3$, a contradiction. Thus $ftd_1(G') = (2(n(G') + k) - 1)/3$. By the inductive hypothesis, $G' \in \mathcal{G}_k$, a contradiction by Observation 9(1), since v_0 is a support vertex of G' . Thus $\deg_G(v_0) = 3$. Observe that G has no strong support vertex. If c_i is adjacent to a support vertex c'_i of $N(c_i) \setminus V(C_1)$ for some i , then the leaf of c'_i can play the role of v_2 , and thus $\deg_G(c_i) = 3$. Thus we may assume that $\deg_G(c_i) \leq 3$ for each i with $i = 1, 2, \dots, r$. Assume that $\deg_G(c_i) = 3$ for each i with $1 \leq i \leq r$.

Let $F = \bigcup_{i=1}^r (N(c_i) \setminus \{c_0, \dots, c_r\})$. Clearly, $|F| = r$, since $\deg_G(c_i) = 3$, for each $c_i \in \{c_1, \dots, c_r\}$. Let $F = \{u_1, u_2, \dots, u_r\}$. Clearly, $\deg_G(u_i) \leq 2$, for $1 \leq i \leq r$, since G has no strong support vertex. Let $F' = \{u_i \mid \deg_G(u_i) = 2\}$. Clearly, $v_1 \in F'$. Let F'' be the set of leaves of F' . Clearly, $v_2 \in F''$. Let $G^* = G - c_0 c_1 - c_0 c_r$. Let G_1^* be the component of G^* containing c_r and G_2^* be the component of G^* containing c_0 . Assume that $F = F'$. Thus $n(G_1^*) = 3r$, since $d = 2$. Further, $n(G_2^*) = n - 3r$. By Theorem 14, $ftd_1(G_2^*) \leq (2(n(G_2^*) + k - 1) - 1)/3$. Let S'' be an $ftd_1(G_2^*)$ -set. If $c_0 \in S''$, then $S'' \cup V(C_1) \cup F$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n - 3r + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3$, a contradiction. Thus $c_0 \in S''$. Then $S'' \cup F'' \cup F$ is a 1FTD-set for G and so $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n - 3r + k - 1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3$, a contradiction. We conclude that $F \neq F'$. Let $|F'| = r'$. Clearly, $1 \leq r' < r$, since $v_1 \in F'$. Thus $n(G_1^*) = 2r + r'$. Then $n(G_2^*) = n - (2r + r')$. Let $G_3^* = G[V(G_2^*) \cup \{c_1\}]$. Then $n(G_3^*) = n - (2r + r') + 1$. By Theorem 14, $ftd_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$. Let S''' be an $ftd_1(G_3^*)$ -set. By Observation 1, $c_0 \in S'''$ and so $S''' \cup V(C_1) \cup F'$ is a 1FTD-set for G . Thus $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + r + r' = (2(n - (2r + r') + 1 + k - 1) - 1)/3 + r + r' = (2(n + k) - 1 + r' - r)/3 < (2(n + k) - 1)/3$, a contradiction. Therefore $\deg_G(c_t) = 2$ for some $1 \leq t \leq r$.

Claim 4. No vertex of $C_1 - c_0$ is a support vertex.

Proof. Let c_j be a support vertex of G . Assume that c_{j+1} is a special vertex. Let $G' = G - c_{j+1}$. Then by Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_j \in S'$. If $c_{j+2} \notin S'$, then S' is a 1FTD-set for G of cardinality at most $(2(n + k) - 1)/3 - 4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $c_{j+2} \in S'$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\deg_G(c_{j+1}) \neq 2$. Note that c_t is a special vertex of G . Assume without loss of generality that $j < t$. Let $c_{j'}$ be

a support vertex of G and $c_{t'}$ be a special vertex of G , where $j \leq j' < t' \leq t$, and among such vertices choose $c_{j'}$ and $c_{t'}$ such that c_i is neither a support vertex nor a special vertex of G for each i with $j' < i < t'$. Let $u_i \in N(c_i) \setminus V(C_1)$ for $j' < i < t'$. Clearly, $\deg_G(u_i) = 2$ for $j' < i < t'$, since G has no strong support vertex. Let $G^* = G - c_{j'}c_{j'+1} - c_{t'}c_{t'+1}$. Let G_1^* be the component of G^* containing $c_{j'}$ and G_2^* be the component of G^* containing $c_{t'}$. Clearly, $n(G_2^*) = 3(t' - j' - 1) + 1$. Thus $n(G_1^*) = n - (3(t' - j' - 1) + 1)$.

By Theorem 14, $ftd_1(G_1^*) \leq (2(n(G_1^*) + k - 1) - 1)/3$. Let S' be an $ftd_1(G_1^*)$ -set. By Observation 1, $c_{j'} \in S'$. Assume that $c_{t'+1} \notin S'$. Then $S' \cup \{c_{j'+1}, c_{j'+2}, \dots, c_{t'-1}\} \cup \{u_{j'+1}, u_{j'+2}, \dots, u_{t'-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t' - j' - 1) = (2(n - (3(t' - j' - 1) + 1) + k - 1) - 1)/3 + 2(t' - j' - 1) = (2(n + k) - 1)/3 - 4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $c_{t'+1} \in S'$. Then $S' \cup \{c_{j'+1}, c_{j'+2}, \dots, c_{t'}\} \cup \{u_{j'+1}, u_{j'+2}, \dots, u_{t'-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t' - j' - 1) + 1 = (2(n - (3(t' - j' - 1) + 1) + k - 1) - 1)/3 + 2(t' - j' - 1) = (2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. $\square$

Claim 5. If $\deg_G(c_j) = 2$ for some j with $1 \leq j \leq r$, then $\deg_G(c_{j+1}) = 3$ and $\deg_G(c_{j-1}) = 3$.

Proof. Assume that $\deg_G(c_j) = \deg_G(c_{j+1}) = 2$, for some j with $1 \leq j \leq r$, and among such vertices choose c_j such that $\deg_G(c_{j-1}) = 3$. Let $G' = G - c_j$. Then by Theorem 14, $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$. Let S' be an $ftd_1(G')$ -set. By Observation 1, $c_{j+2} \in S'$. If $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$, then S' is a 1FTD-set for G of cardinality at most $(2(n + k) - 1)/3 - 4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$. Then $\{c_j\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$. Then $\{c_{j+1}\} \cup S'$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\deg_G(c_{j+1}) \geq 3$. Similarly $\deg_G(c_{j-1}) \geq 3$. $\square$

Claim 6. C_1 has precisely one special vertex.

Proof. Let c_{t_1} and c_{t_2} be two special vertices of C_1 and among such vertices choose c_{t_1} and c_{t_2} such that c_i is not a special vertex of C_1 for $t_1 < i < t_2$. By Claim 5, $t_1 + 1 < t_2$. By Claim 4, c_i is not a support vertex for $t_1 < i < t_2$. Let $u_i \in N(c_i) \setminus V(C_1)$, for $t_1 < i < t_2$. Clearly, $\deg_G(u_i) = 2$, for $t_1 < i < t_2$. Let u'_i be the leaf adjacent to u_i , for $t_1 < i < t_2$, and $G^* = G - c_{t_1}c_{t_1+1} - c_{t_2}c_{t_2+1}$. Let G_1^* be the component of G^* containing c_{t_1} , and G_2^* be the component of G^* containing c_{t_2} . Clearly, $n(G_2^*) = 3(t_2 - t_1 - 1) + 1$. Then $n(G_1^*) = n - (3(t_2 - t_1 - 1) + 1)$. By Theorem 14, $ftd_1(G_1^*) \leq (2(n(G_1^*) + k - 1) - 1)/3$. Let S' be an $ftd_1(G_1^*)$ -set. By Observation 1, $c_{t_1-1} \in S'$. Assume that $\{c_{t_1}, c_{t_2+1}\} \cap S' = \emptyset$.

Then $S' \cup \{c_{t_1}, c_{t_1+1}, \dots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)/3 - 1/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction.

Thus $\{c_{t_1}, c_{t_2+1}\} \cap S' \neq \emptyset$. If $\{c_{t_1}, c_{t_2+1}\} \subseteq S'$, then $S' \cup \{c_{t_1+1}, c_{t_1+2}, \dots, c_{t_2}\} \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)/3 - 1/3$. Thus $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus $\{c_{t_1}, c_{t_2+1}\} \not\subseteq S'$. If $c_{t_1} \in S'$ and $c_{t_2+1} \notin S'$, then $S' \cup \{c_{t_1+1}, c_{t_1+2}, \dots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)/3 - 4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. Thus assume that $c_{t_2+1} \in S'$ and $c_{t_1} \notin S'$. Then $S' \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\} \cup \{u'_{t_1+1}, u'_{t_1+2}, \dots, u'_{t_2-1}\}$ is a 1FTD-set in G of cardinality at most $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)/3 - 4/3$ and so $ftd_1(G) < (2(n + k) - 1)/3$, a contradiction. $\square$

By Claims 4 and 6, c_i is not a support vertex and is not a special vertex, for $i \in \{1, 2, \dots, t-1, t+1, \dots, r\}$. Let $u_i \in N(c_i) \setminus V(C_1)$, for $i \in \{1, 2, \dots, t-1, t+1, \dots, r\}$. Clearly, $\deg_G(u_i) = 2$, for $i \in \{1, 2, \dots, t-1, t+1, \dots, r\}$.

Let G_1'' be the component of $G - c_0c_1 - c_0c_r$ that contains c_1 , G_2'' be the component of $G - c_0c_1 - c_0c_r$ that contains c_0 , and G^* be a graph obtained from G_2'' by adding a path $p_2 = x_1x_2$ and joining c_0 to x_1 . Clearly, $n(G^*) = n - (3r - 2) + 2$. By Theorem 14, $ftd_1(G^*) \leq (2(n(G^*) + k - 1) - 1)/3$. Suppose that $ftd_1(G^*) < (2(n(G^*) + k - 1) - 1)/3$. Let S^* be an $ftd_1(G^*)$ -set. By Observation 1, $x_1 \in S^*$. If $c_0 \in S^*$, then $S^* \cup \{c_1, c_2, \dots, c_r\} \cup \{u_1, u_2, \dots, u_{t-1}, u_{t+1}, \dots, u_r\} \setminus \{x_1\}$ is a 1FTD-set in G . Thus $ftd_1(G) < (2(n(G^*) + k - 1) - 1)/3 + 2r - 1 - 1 = (2(n - (3r - 2) + 2 + k - 1) - 1)/3 + 2r - 2 = (2(n + k) - 1)/3$, a contradiction. Thus $c_0 \notin S^*$. Then $x_2 \in S^*$. If $t > 1$, then $S^* \setminus \{x_1, x_2\} \cup \{c_1, \dots, c_{t-1}\} \cup \{u_1, \dots, u_{t-1}\} \cup \{u_{t+1}, \dots, u_r\} \cup \{u'_{t+1}, \dots, u'_r\}$ is a 1FTD-set in G . Thus $ftd_1(G) < (2(n(G^*) + k - 1) - 1)/3 + 2(r - 1) - 2 = (2(n - (3r - 2) + 2 + k - 1) - 1)/3 + 2r - 4 = (2(n + k) - 1)/3 - 2$, a contradiction. Thus assume that $t = 1$. Then $S^* \setminus \{x_1, x_2\} \cup \{c_2, \dots, c_r\} \cup \{u_2, \dots, u_r\}$ is a 1FTD-set in G of cardinality at most $(2(n + k) - 1)/3 - 2$ and so $ftd_1(G) < (2(n + k) - 1)/3 - 2$, a contradiction. Thus $ftd_1(G^*) = (2(n(G^*) + k - 1) - 1)/3$. By the inductive hypothesis, $G^* \in \mathcal{G}_{k-1}$. Let G_1^* be the graph obtained from $G[G_1'' \cup \{c_0\}]$ by adding a path $p_2 = x'_1x'_2$ and joining c_0 to x'_1 . Clearly, $G_1^* \in \mathcal{H}_1$. Thus G is obtained from $G^* \in \mathcal{G}_{k-1}$ and $G_1^* \in \mathcal{H}_1$ by Procedure A. Consequently, $G \in \mathcal{H}_k \subseteq \mathcal{G}_k$.

For the converse, by Corollary 13, $V(G) \setminus L(G)$ is the unique $ftd_1(G)$ -set. Now Observation 9 implies that $ftd_1(G) = (2(n + k) - 1)/3$. $\blacksquare$

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REFERENCES

- [1] Y. Caro, A. Hansberg and M.A. Henning, *Fair domination in graphs*, Discrete Math. **312** (2012) 2905–2914.
doi:10.1016/j.disc.2012.05.006
- [2] B. Chaluvvaraju, M. Chellali and K.A. Vidya, *Perfect k -domination in graphs*, Australas. J. Combin. **48** (2010) 175–184.
- [3] B. Chaluvvaraju and K. Vidya, *Perfect dominating set graph of a graph*, Adv. Appl. Discrete Math. **2** (2008) 49–57.
- [4] E.J. Cockayne, B.L. Hartnell, S.T. Hedetniemi and R. Laskar, *Perfect domination in graphs*, J. Combin. Inform. System Sci. **18** (1993) 136–148.
- [5] I.J. Dejter, *Perfect domination in regular grid graphs*, Australas. J. Combin. **42** (2008) 99–114.
- [6] I.J. Dejter and A.A. Delgado, *Perfect domination in rectangular grid graphs*, J. Combin. Math. Combin. Comput. **70** (2009) 177–196.
- [7] M.R. Fellows and M.N. Hoover, *Perfect domination*, Australas. J. Combin. **3** (1991) 141–150.
- [8] M. Hajian and N. Jafari Rad, *Trees and unicyclic graph with large fair domination number*, Util. Math., to appear.
- [9] M. Hajian and N. Jafari Rad, *Fair domination number in cactus graphs*, Discuss. Math. Graph Theory **39** (2019) 489–503.
doi:10.7151/dmgt.2088
- [10] M. Hajian, N. Jafari Rad and L. Volkmann, *Bounds on the fair total domination number in trees and unicyclic graphs*, Australas. J. Combin. **74** (2019) 460–475.
- [11] H. Hatami and P. Hatami, *Perfect dominating sets in the Cartesian products of prime cycles*, Electron. J. Combin. **14** (2007) #N8.
doi:10.37236/1009
- [12] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).
- [13] E.C. Maravilla, R.T. Isla and S.R. Canoy Jr., *Fair total domination in the join, corona, and composition of graphs*, Int. J. Math. Anal. **8** (2014) 2677–2685.
doi:10.12988/ijma.2014.49296

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