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## FAIR TOTAL DOMINATION NUMBER IN CACTUS GRAPHS

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## Abstract

For  $k \geq 1$ , a k-fair total dominating set (or just kFTD-set) in a graph G is a total dominating set S such that  $|N(v) \cap S| = k$  for every vertex  $v \in V \setminus S$ . The k-fair total domination number of G, denoted by  $ftd_k(G)$ , is the minimum cardinality of a kFTD-set. A fair total dominating set, abbreviated FTD-set, is a kFTD-set for some integer  $k \geq 1$ . The fair total domination number of a nonempty graph G, denoted by ftd(G), of G is the minimum cardinality of an FTD-set in G. In this paper, we present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds.

Keywords: fair total domination, cactus graph.

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## 1. INTRODUCTION

For notation and graph theory terminology not given here, we follow [12]. Specifically, let G be a graph with vertex set V(G) = V of order |V| = n and let v be a vertex in V. The open neighborhood of v is  $N_G(v) = \{u \in V \mid uv \in E(G)\}$  and the closed neighborhood of v is  $N_G[v] = \{v\} \cup N_G(v)$ . If the graph G is clear from

the context, we simply write N(v) rather than  $N_G(v)$ . The degree of a vertex v, is deg(v) = |N(v)|. A vertex of degree one is called a *leaf* and its neighbor a support vertex. We denote the set of leaves and support vertices of a graph G by L(G) and S(G), respectively. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set  $S \subseteq V$ , its open neighborhood is the set  $N(S) = \bigcup_{v \in S} N(v)$ , and its closed neighborhood is the set  $N[S] = N(S) \cup S$ . The 2-corona 2-cor(G) of a graph G is a graph obtained from G by adding a path  $P_2$  for every vertex v and joining v to a leaf of  $P_2$ . Note that 2-cor(G) has order 3|V(G)|. The distance d(u, v) between two vertices u and v in a graph G is the minimum number of edges of a path from u to v. For a subset S of vertices of a graph is a graph such that no pair of cycles have a common edge.

A subset  $S \subseteq V$  is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by  $\gamma(G)$ , is the minimum cardinality of a dominating set of G. A dominating set S in a graph G with no isolated vertex, is a *total dominating set* of G if every vertex in S is adjacent to a vertex in S.

Caro *et al.* [1] studied the concept of fair domination in graphs. For  $k \ge 1$ , a k-fair dominating set, abbreviated kFD-set, in G is a dominating set S such that  $|N(v) \cap D| = k$  for every vertex  $v \in V \setminus D$ . The k-fair domination number of G, denoted by  $fd_k(G)$ , is the minimum cardinality of a kFD-set. A kFD-set of G of cardinality  $fd_k(G)$  is called a  $fd_k(G)$ -set. A fair dominating set, abbreviated FD-set, in G is a kFD-set for some integer  $k \ge 1$ . The fair domination number, denoted by fd(G), of a graph G that is not the empty graph is the minimum cardinality of an FD-set in G. An FD-set of G of cardinality fd(G) is called a fd(G)-set. A perfect dominating set in a graph G is a dominating set S such that every vertex in  $V(G) \setminus S$  is adjacent to exactly one vertex in S. Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne *et al.* in [4], and Fellows *et al.* [7] with a different terminology which they called semiperfect domination. This concept was further studied in, for example, [2,3,5,6,8,9,11].

Maravilla *et al.* [13] introduced the concept of fair total domination in graphs. For an integer  $k \geq 1$  and a graph G with no isolated vertex, a k-fair total dominating set, abbreviated kFTD-set, is a total dominating set  $S \subseteq V(G)$  such that  $|N(u) \cap S| = k$  for every  $u \in V(G) \setminus S$ . The k-fair total domination number of G, denoted by  $ftd_k(G)$ , is the minimum cardinality of a kFTD-set. A kFTD-set of G of cardinality  $ftd_k(G)$  is called an  $ftd_k(G)$ -set. A fair total dominating set, abbreviated FTD-set, in G is a kFTD-set for some integer  $k \geq 1$ . Thus, a fair total dominating set S of a graph G is a total dominating set S of G such that for every two distinct vertices u and v of  $V(G) \setminus S$ ,  $|N(u) \in S| = |N(v) \cap S|$ ; that is, S is both a fair dominating set and a total dominating set of G. The *fair total* domination number of G, denoted by ftd(G), is the minimum cardinality of an FTD-set. A fair total dominating set of cardinality ftd(G) is called a minimum fair total dominating set or an ftd-set of G.

In [10], Volkmann and we studied fair total domination in trees and unicyclic graphs. In this paper, we study 1-fair total domination in cactus graphs. We present upper bounds for the 1-fair total domination number of cactus graphs, and characterize cactus graphs achieving equality for the upper bounds. The techniques used in this paper are similar to those presented in [9]. The following observations are easily verified.

**Observation 1.** Any support vertex in a graph G with no isolated vertex belongs to every kFTD-set for each integer k.

**Observation 2.** Let S be a 1FTD-set in a graph G, and v be a vertex of degree at least two such that v is adjacent to a weak support vertex v'. If S contains a vertex  $u \in N_G(v) \setminus \{v'\}$ , then  $v \in S$ .

## 2. UNICYCLIC GRAPHS

A vertex v of a graph is a special vertex if  $\deg_G(v) = 2$  and v belongs to a cycle of G. Let  $\mathcal{H}_1$  be the class of all graphs G that can be obtained from the 2-corona 2-cor(C) of a cycle C by removing precisely one support vertex v and the leaf adjacent to v. Let  $\mathcal{G}_1$  be the class of all graphs G that can be obtained from a sequence  $G_1, G_2, \ldots, G_s = G$ , where  $G_1 \in \mathcal{H}_1$ , and if  $s \geq 2$ , then  $G_{j+1}$  is obtained from  $G_j$  by one of the following Operations  $\mathcal{O}_1$  or  $\mathcal{O}_2$ , for  $j = 1, 2, \ldots, s - 1$ .

**Operation**  $\mathcal{O}_1$ . Let v be a vertex of  $G_j$  with  $\deg(v) \geq 2$  such that v is not a special vertex. Then  $G_{j+1}$  is obtained from  $G_j$  by adding a path  $P_3$  and joining v to a leaf of  $P_3$  by means of an edge.

**Operation**  $\mathcal{O}_2$ . Let v be a support vertex of  $G_j$  and let u be a leaf adjacent to v. Then  $G_{j+1}$  is obtained from  $G_j$  by adding a vertex u' and a path  $P_2$ , and joining u to u' and v to a leaf of  $P_2$ .

**Observation 3.** If  $H \in \mathcal{H}_1$ , then H has precisely one special vertex.

**Observation 4** [10]. If  $G \in \mathcal{G}_1$  has order n, and C is the cycle of G, then we have the following.

- (1) G has precisely one special vertex.
- (2) G has (n-1)/3 leaves.
- (3) No vertex of C is a support vertex.

(4) Any vertex of C is adjacent to at most one weak support vertex of degree two.

**Lemma 5** [10]. If  $G \in \mathcal{G}_1$ , then every 1FTD-set in G contains every vertex of G of degree at least two.

**Theorem 6** [10]. If G is a unicyclic graph of order n, then  $ftd_1(G) \leq (2n+1)/3$ , with equality if and only if  $G = C_7$  or  $G \in \mathcal{G}_1$ .

## 3. Main Result

Our aim in this paper is to give an upper bound for the fair total domination number of a cactus graph G in terms of the number of cycles of G, and then characterize all cactus graphs achieving equality for the proposed bound. For this purpose we first introduce some families of graphs. Let  $\mathcal{H}_1$  and  $\mathcal{G}_1$  be the families of unicyclic graphs described in Section 2. For  $i = 2, \ldots, k$ , we construct a family  $\mathcal{H}_i$  from  $\mathcal{G}_{i-1}$ , and a family  $\mathcal{G}_i$  from  $\mathcal{H}_i$  as follows.

• Family  $\mathcal{H}_i$ . Let  $\mathcal{H}_i$  be the family of all graphs  $H_i$  such that  $H_i$  can be obtained from a graph  $H_1 \in \mathcal{H}_1$  and a graph  $G \in \mathcal{G}_{i-1}$ , by the following procedure.

**Procedure A.** Let  $w_0 \in V(H_1)$  be a vertex of degree at least two of  $H_1$  such that  $w_0$  is adjacent to a weak support vertex  $w'_0$ , and  $w \in V(G_{i-1})$  be a vertex of degree at least two of  $G_{i-1}$  such that w is adjacent to a weak support vertex w' of degree two. We remove  $w'_0$ , the leaf adjacent to  $w'_0$ , w' and the leaf adjacent to  $w'_0$ , and the leaf adjacent to  $w'_0$ , and the leaf adjacent to  $w'_0$ , and the leaf adjacent to  $w'_0$ .

• Family  $\mathcal{G}_i$ . Let  $\mathcal{G}_i$  be the family of all graphs G that can be obtained from a sequence  $G_1, G_2, \ldots, G_s = G$ , where  $G_1 \in \mathcal{H}_i$ , and if  $s \geq 2$ , then  $G_{j+1}$  is obtained from  $G_j$  by one of the Operations  $\mathcal{O}_1$  or  $\mathcal{O}_2$ , described in Section 2, for  $j = 1, 2, \ldots, s - 1$ .

Note that  $\mathcal{H}_i \subseteq \mathcal{G}_i$ , for i = 1, 2, ..., k. Figure 1 demonstrates the construction of the family  $\mathcal{G}_k$ .



Figure 1. Construction of the family  $\mathcal{G}_k$ .

We will prove the following.

**Theorem 7.** If G is a cactus graph of order  $n \ge 5$  with  $k \ge 1$  cycles, then  $ftd_1(G) \le (2(n+k)-1)/3$ , with equality if and only if  $G = C_7$  or  $G \in \mathcal{G}_k$ .

## 4. Preliminary Results and Observations

#### 4.1. Notation

We call a vertex w in a cycle C of a cactus graph G a special cut-vertex if w belongs to a shortest path from C to a cycle  $C' \neq C$ . We call a cycle C in G, a *leaf-cycle* if C contains exactly one special cut-vertex. In the cactus graph presented in Figure 2,  $v_i$  is a special cut-vertex, for i = 1, 2, ..., 8. Moreover,  $C_j$  is a leaf-cycle for j = 1, 2, 3.



Figure 2.  $C_i$  is a leaf-cycle for i = 1, 2, 3 and  $v_j$  is a special cut-vertex for  $j = 1, 2, \ldots, 8$ .

**Observation 8.** Every cactus graph with at least two cycles contains at least two leaf-cycles.

## 4.2. Properties of the family $\mathcal{G}_k$

The following observation can be proved by a simple induction on k.

**Observation 9.** If  $G \in \mathcal{G}_k$  is a cactus graph of order n, then we have the following.

- (1) No cycle of G contains a support vertex. Furthermore, any cycle of G contains precisely one special vertex.
- (2) If a vertex v of G belongs to a cycle of G, then v is adjacent to at most one weak support vertex of degree two.
- (3) |L(G)| = (n+1)/3 2k/3.
- (4) If a vertex v of G belongs to at least two cycles of G, then v is not adjacent to a weak support vertex, and v belongs to precisely two cycles of G.

**Proof.** Let  $G \in \mathcal{G}_k$  be a cactus graph of order n. To show (1), (2) or (3), we prove by an induction on k, that we call *first-induction*. For the base step, if k = 1, then  $G \in \mathcal{G}_1$ , and the result follows by Observation 4. Assume the result

holds for all graphs  $G' \in \mathcal{G}_{k'}$  with k' < k. Now consider the graph  $G \in \mathcal{G}_k$ , where k > 1. Clearly, G is obtained from a sequence  $G_1, G_2, \ldots, G_l = G$ , of cactus graphs such that  $G_1 \in \mathcal{H}_k$ , and if  $l \ge 2$ , then  $G_{i+1}$  is obtained from  $G_i$  by one of the Operations  $\mathcal{O}_1$  or  $\mathcal{O}_2$  for  $i = 1, 2, \ldots, l-1$ . We prove by an induction on l, that we call *second-induction*. For the base step of the second-induction, let l = 1. Thus  $G \in \mathcal{H}_k$ . By the construction of graphs in the family  $\mathcal{H}_k$ , there are graphs  $H \in \mathcal{H}_1$  and  $G' \in \mathcal{G}_{k-1}$  such that G is obtained from H and G' by Procedure A. It is easy see that the base step of the second-induction holds. Assume that the result (for the second-induction) holds for  $2 \le l' < l$ . Now let  $G = G_l$ . Clearly, G is obtained from  $G_{l-1}$  by applying one of the Operations  $\mathcal{O}_1$  or  $\mathcal{O}_2$ . It is easy see that the result holds.

The proof for (4) is similarly verified.

**Observation 10.** Let  $G \in \mathcal{G}_k$  be obtained from a sequence  $G_1, G_2, \ldots, G_s = G$  $(s \geq 2)$  such that  $G_1 \in \mathcal{H}_1$  and  $G_{j+1}$  is obtained from  $G_j$  by one of the Operations  $O_1$  or  $O_2$  or Procedure A, for  $j = 1, 2, \ldots, s-1$ . If v is a vertex of G belonging to two cycles of G, then there is an integer  $i \in \{2, 3, \ldots, s\}$  such that  $G_i$  is obtained from  $G_{i-1}$  by applying Procedure A on the vertex v using a graph  $H \in \mathcal{H}_1$ , such that v belongs to a cycle of  $G_{i-1}$ .

**Observation 11.** Assume that  $G \in \mathcal{G}_k$  and  $v \in V(G)$  is a vertex of degree four belonging to two cycles. Let  $D_1$  and  $D_2$  be the components of G - v,  $G_1^*$  be the graph obtained from  $G[D_1 \cup \{v\}]$  by joining v to a leaf of a path  $P_2$ , and  $G_2^*$  be the graph obtained from  $G[D_2 \cup \{v\}]$  by joining v to a leaf of a path  $P_2$ . Then there exists an integer k' < k such that  $G_1^* \in \mathcal{G}_{k'}$  or  $G_2^* \in \mathcal{G}_{k'}$ .

**Proof.** Let  $G \in \mathcal{G}_k$ . Then G is obtained from a sequence  $G_1, G_2, \ldots, G_s = G$  $(s \geq 2)$  such that  $G_1 \in \mathcal{H}_1$  and  $G_{j+1}$  is obtained from  $G_j$  by one of the Operations  $O_1$  or  $O_2$  or procedure A, for  $j = 1, 2, \ldots, s - 1$ . Note that  $s \geq k$ . We define the *j*-th Procedure-Operation or just  $PO_j$  as one of the Operation  $O_1$ , Operation  $O_2$ , or Procedure A that can be applied to obtain  $G_{j+1}$  from  $G_j$ . Thus G is obtained from  $G_1$  by Procedure-Operations  $PO_1, PO_2, \ldots, PO_{s-1}$ .

Let v be a vertex of G of degree four belonging to two cycles of G, and  $D_1$ and  $D_2$  be the components of G - v. By Observation 10, there is an integer  $i \in \{2, 3, \ldots, s\}$  such that  $G_i$  is obtained from  $G_{i-1}$  by applying Procedure A on the vertex v using a graph  $H \in \mathcal{H}_1$ . Note that v is adjacent to a weak support vertex v' of  $G_{i-1}$ . Let v'' be the leaf of v' in  $G_{i-1}$  that is removed in Procedure A. Clearly, either  $V(G_{i-1}) \cap D_1 \neq \emptyset$  or  $V(G_{i-1}) \cap D_2 \neq \emptyset$ . Without loss of generality, assume that  $V(G_{i-1}) \cap D_1 \neq \emptyset$ . Among  $PO_i, PO_{i+1}, \ldots, PO_{s-1}$ , let  $PO_{r_1}, PO_{r_2}, \ldots, PO_{r_t}$ , be those procedure-operations applied on a vertex of  $D_1$ . Note that  $i \leq t \leq s - 1$ . Let  $G_{r_0} = G_{i-1}$  and  $G_{r_{l+1}}$  be obtained from  $G_{r_l}$  by  $PO_{l+1}$ , for  $l = 0, 1, 2, \ldots, t - 1$ . Clearly, by an induction on t, we can deduce that there is an integer  $k^* < k$  such that  $G_{r_t} \in \mathcal{G}_{k^*}$ . Note that  $G_{r_t} = G_1^*$ . **Lemma 12.** If  $G \in \mathcal{G}_k$ , then every 1FTD-set in G contains each vertex of G of degree at least two.

**Proof.** Let  $G \in \mathcal{G}_k$ , and S be a 1FTD-set in G. We prove by an induction on k, that we call *first-induction*, that S contains every vertex of G of degree at least two. For the base step, if k = 1, then  $G \in \mathcal{G}_1$ , and the result follows by Lemma 5. Assume the result holds for all graphs  $G' \in \mathcal{G}_{k'}$  with k' < k. Now consider the graph  $G \in \mathcal{G}_k$ , where k > 1. Clearly, G is obtained from a sequence  $G_1, G_2, \ldots, G_l = G$ , of cactus graphs such that  $G_1 \in \mathcal{H}_k$ , and if  $l \geq 2$ , then  $G_{i+1}$  is obtained from  $G_i$  by one of the Operations  $\mathcal{O}_1$  or  $\mathcal{O}_2$  for  $i = 1, 2, \ldots, l-1$ .

We prove by an induction on l, that we call *second-induction*, that S contains every vertex of G of degree at least two.

For the base step of the second-induction, let l = 1. Thus  $G \in \mathcal{H}_k$ . By the construction of graphs in the family  $\mathcal{H}_k$ , there are graphs  $H \in \mathcal{H}_1$  and  $G' \in \mathcal{G}_{k-1}$  such that G is obtained from H and G' by Procedure A. Clearly, H is obtained from the 2-corona 2-cor(C) of a cycle C, by removing precisely one support vertex v and the leaf adjacent to v of 2-cor(C).

Let  $C = c_0c_1 \cdots c_rc_0$  be the cycle of H, where  $c_0$  is a vertex of degree at least two of H that is adjacent to a weak support vertex  $c'_0$ , and let  $c'_0$  and its leaf (that we call  $c''_0$ ) be removed according to Procedure A. By Observation 3, H has precisely one special vertex. Let  $c_t$  be the special vertex of H. Let  $w \in V(G')$  be a vertex of degree at least two of G' that is adjacent to a weak support vertex w', and let w' and its leaf (that we call w'') be removed according to Procedure A.

First we show that  $\{c_1, c_r\} \cap S \neq \emptyset$ . Clearly,  $S \cap \{c_{t-1}, c_t, c_{t+1}\} \neq \emptyset$ , since  $\deg_G(c_t) = 2$ . Assume that  $c_t \in S$ . Since at least one of  $c_{t-1}$  or  $c_{t+1}$  is adjacent to a weak support vertex, by Observation 2,  $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$ . By applying Observation 2, we obtain that  $\{c_1, c_r\} \cap S \neq \emptyset$ , since any vertex of  $\{c_1, \ldots, c_r\} \setminus \{c_t\}$  is adjacent to a weak support vertex of G. Thus assume that  $c_t \notin S$ . Then  $\{c_{t-1}, c_{t+1}\} \cap S \neq \emptyset$ , and so  $\{c_1, c_r\} \cap S \neq \emptyset$ , since any vertex of  $\{c_1, \ldots, c_r\} \setminus \{c_t\}$  is adjacent to a weak support vertex of G. Hence,  $\{c_1, c_r\} \cap S \neq \emptyset$ . If  $c_0 \notin S$ , then  $S \cup \{w', w''\}$  is a 1FTD-set for G', and thus by the first-inductive hypothesis, S' contains  $w = c_0$ , a contradiction. Thus  $c_0 \in S$ . By Observation 2,  $V(C) \subseteq S$ , since any vertex of  $\{c_1, \ldots, c_r\} - \setminus \{c_t\}$  is adjacent to a weak support vertex of G'. By the first-inductive hypothesis,  $(S \cap V(G')) \cup \{w', w''\}$  contains every vertex of G' of degree at least two. Consequently, S contains every vertex of G of degree at least two. We conclude that the base step of the second-induction holds.

Assume that the result (for the second-induction) holds for  $2 \leq l' < l$ . Now let  $G = G_l$ . Clearly, G is obtained from  $G_{l-1}$  by applying one of the Operations  $\mathcal{O}_1$  or  $\mathcal{O}_2$ .

Assume that G is obtained from  $G_{l-1}$  by applying Operation  $\mathcal{O}_2$ . Let x be a support vertex of  $G_{l-1}$  and let x' be a leaf adjacent to x. Let G be obtained from

 $G_{l-1}$  by adding a vertex u' and a path  $P_2 = y_1y_2$ , joining x' to u' and joining x to  $y_1$ , according to Operation  $\mathcal{O}_2$ . By Observation 1,  $x', y_1 \in S$  and so  $x \in S$ . Thus  $S \setminus \{y_1\}$  is a 1FTD-set for  $G_{l-1}$ . By the second-inductive hypothesis, S contains all vertices of  $G_{l-1}$  of degree at least two. Consequently, S contains every vertex of  $G_k$  of degree at least two.

Next assume that G is obtained from  $G_{l-1}$  by applying Operation  $\mathcal{O}_1$ . Let  $P_3 = x_1 x_2 x_3$  be a path and  $x_1$  be joined to  $y \in V(G_{l-1})$ , where  $\deg_{G_{l-1}}(y) \geq 2$ and y is not a special vertex of  $G_{l-1}$ , according to Operation  $\mathcal{O}_2$ . By Observation  $1, x_2 \in S$ . Observe that  $\{x_1, x_3\} \cap S \neq \emptyset$ . If  $x_1 \notin S$ , then  $x_3 \in S$  and  $y \notin S$ . Then  $S \setminus \{x_2, x_3\}$  is a 1FTD-set for  $G_{l-1}$  that does not contains y, a contradiction by the second-inductive hypothesis. Thus assume that  $x_1 \in S$ . Suppose that  $y \notin S$ . Clearly,  $N_{G_{l-1}}(y) \cap S = \emptyset$ . Assume that there exists a component  $G'_1$  of  $G_{l-1} - y$ such that  $|V(G'_1) \cap N_{G_{l-1}}(y)| = 1$ . Then clearly  $S' = (S \cap V(G_{l-1})) \cup V(G'_1)$ is a 1FTD-set for  $G_{l-1}$ , and by the second-inductive hypothesis, S' contains every vertex of  $G_{l-1}$  of degree at least two. Thus  $y \in S'$ , and so  $y \in S$ , a contradiction. Next assume that every component of  $G_{l-1} - y$  has at least two vertices in  $N_{G_{l-1}}(y)$ . Since y is a non-special vertex of  $G_{l-1}$ , y belongs to at least two cycles of  $G_{l-1}$ . By Observation 9(4), y belongs to exactly two cycles of  $G_{l-1}$ . Thus  $\deg_{G_{l-1}}(y) = 4$ . By Observation 11,  $G_{l-1} - y$  has exactly two components  $D_1$  and  $D_2$ . Let  $G^*$  be a graph obtained from  $D_1 \cup \{y\}$  or  $D_2 \cup \{y\}$ by adding a path  $P_2 = y'y''$  to y. Then there exists  $k' \leq k$  such that  $G^* \in \mathcal{G}_{k'}$ . Evidently,  $S^* = (S \cap V(G^*)) \cup \{y', y''\}$  is a 1FTD-set for  $G^*$ , and so by the firstinductive hypothesis,  $S^*$  contains every vertex of  $G^*$  of degree at least two (since  $G^* \in \mathcal{G}_{k'}$ ). Thus  $y \in S^*$ , and so  $y \in S$ , a contradiction. We conclude that  $y \in S$ . Observe that  $S \cap V(G_{l-1})$  is a 1FTD-set for  $G_{l-1}$ , and so by the second-inductive hypothesis,  $S \cap V(G_{l-1})$  contains every vertex of  $G_{l-1}$  of degree at least two. Consequently, S contains every vertex of G of degree at least two.

As a consequence of Observation 9(3) and Lemma 12, we obtain the following.

**Corollary 13.** If  $G \in \mathcal{G}_k$  is a cactus graph of order n, then  $V(G) \setminus L(G)$  is the unique  $ftd_1(G)$ -set.

In what follows, we present an upper bound for the 1-fair domination number of a cactus graph in terms of the order and the number of cycles.

**Theorem 14.** If G is a cactus graph of order  $n \ge 4$  with  $k \ge 1$  cycles, then  $ftd_1(G) \le (2(n+k)-1)/3$ .

**Proof.** The result follows by Theorem 6 if k = 1. Thus assume that  $k \ge 2$ . Suppose to the contrary that  $ftd_1(G) > (2(n(G) + k) - 1)/3$ . Assume that G has the minimum order, and among all such graphs, we may assume that the size of G is minimum. Let  $C_1, C_2, \ldots, C_k$  be the k cycles of G. Let  $C_i$  be a leaf-cycle of G, where  $i \in \{1, 2, ..., k\}$ . Let  $C_i = c_0c_1 \cdots c_rc_0$ , where  $c_0$  is the special cut-vertex of G. Suppose that G has a strong support vertex u, and  $u_1, u_2$  are leaves adjacent to u. Let  $G_0 = G - u_1$ . By the choice of G,  $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $u \in S'$ . Clearly, S' is a 1FTD-set in G and so  $ftd_1(G) \leq (2(n + k) - 1)/3 - 2/3$ , a contradiction. We deduce that every support vertex of G is adjacent to precisely one leaf.

Assume that  $\deg_G(u_j) = 2$  for each  $j = 1, 2, \ldots, r$ . Let  $G' = G - c_2$ . Then by the choice of G,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k) - 1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_0 \in S'$ . If  $|S' \cap \{c_1, c_3\}| = 1$ , then S' is a 1FTD-set for G cardinality at most (2(n+k)-1)/3-4/3, a contradiction. Thus assume that  $|S' \cap \{c_1, c_3\}| = 2$ . Then  $\{c_2\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3-1/3, a contradiction. Thus assume that  $|S' \cap \{c_1, c_3\}| = 0$ . Now  $\{c_1\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3-1/3, a contradiction. We deduce that  $\deg_G(c_i) \geq 3$  for some  $i \in \{1, 2, \ldots, r\}$ .

Let  $v_d$  be a leaf of G such that  $d(v_d, C_i - c_0)$  is as maximum as possible, the shortest path from  $v_d$  to  $C_i$  does not contain  $c_0$  and  $\deg(v_{d-1})$  is as maximum as possible, where  $v_{d-1}$  is the neighbor of  $v_d$  on the shortest path from  $v_d$  to a vertex  $v_0 \in C_i$ .

Assume that  $d \geq 3$ . Observe that  $\deg_G(v_{d-1}) = 2$ , since G has no strong support vertex. Assume that  $\deg_G(v_{d-2}) = 2$ . Let  $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$ . By the choice of G,  $ftd_1(G') \leq (2(n(G') + k) - 1)/3 = (2(n+k) - 1)/3 - 2.$ Let S' be an  $ftd_1(G')$ -set. If  $v_{d-3} \in S'$ , then  $\{v_{v-1}, v_{d-2}\} \cup S'$  is a 1FTDset in G and so  $ftd_1(G) \leq (2(n+k)-1)/3$ , a contradiction. If  $v_{d-3} \notin S'$ , then  $\{v_{v-1}, v_d\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) \leq (2(n+k)-1)/3$ , a contradiction. Thus assume that  $\deg_G(v_{d-2}) \geq 3$ . Assume that  $v_{d-2}$  is a support vertex. Let  $G' = G - \{v_{d-1}, v_d\}$ . By the choice of G,  $ftd_1(G') \leq v_{d-1}$ (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 4/3. Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $v_{d-2} \in S'$ . Then  $\{v_{d-1}\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) \leq (2(n+k)-1)/3 - 1/3)$ , a contradiction. Thus assume that  $v_{d-2}$  is not a support vertex of G. Let  $x \neq v_{d-1}, v_{d-3}$  be a support vertex of G such that  $x \in N(v_{d-2})$ . By the choice of the path  $v_0v_1 \cdots v_d$ , (the part "deg $(v_{d-1})$  is as maximum as possible"),  $\deg_G(x) = 2$ . Let y be the leaf adjacent to x and  $G' = G - \{v_d, v_{d-1}, y\}$ . By the choice of G,  $ftd_1(G') \leq (2(n(G') + k) - 1)/3 =$ (2(n+k)-1)/3-2. Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $v_{d-2} \in S'$ , since  $v_{d-2}$  is a support vertex of G'. Thus  $\{v_{d-1}, x\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) \leq (2(n+k)-1)/3$ , a contradiction.

Next assume that d = 2. Assume that  $\deg_G(c_i) = 2$  for some  $i \in \{1, 2, ..., r\}$ . Let  $\deg_G(c_j) = 2$ . Assume that  $\deg_G(c_{j+1}) = 2$ . Let  $G' = G - c_j$ . Then by the choice of G,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{j+2} \in S'$ . If  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$ ,

then S' is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3 - 4/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$ . Then  $\{c_j\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$  and so  $\{c_{j+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3-1/3, a contradiction. Thus deg<sub>G</sub>( $c_{j+1}$ )  $\geq 3$ . Similarly  $\deg_G(c_{i-1}) \geq 3$ . Clearly,  $c_{i+1} \neq c_0$  or  $c_{i-1} \neq c_0$ . Assume, without loss of generality, that  $c_{i+1} \neq c_0$ . Let  $c_{i+1}$  be a support vertex of G, and  $G' = G - c_i$ . Then by the choice of G,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3$ 1/3 - 4/3. Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{i+1} \in S'$ . If  $c_{i-1} \notin S'$ , then S' is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3 - 4/3, a contradiction. Thus assume that  $c_{i-1} \in S'$  and so  $\{c_i\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction. Thus  $c_{j+1}$  is not a support vertex of G. Let  $c'_{i+1} \in N(c_{i+1}) \setminus V(C_i)$ . Clearly,  $c'_{i+1}$  is a support vertex, since d = 2. Observe that  $\deg_G(c'_{j+1}) = 2$ , since G has no strong support vertex. Let  $c''_{j+1}$  be the leaf of  $c'_{j+1}$ . Let  $G' = G - c_j - c''_{j+1}$ . By the choice of G,  $ftd_1(G') \leq (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3 - 2$ . Let S' be an  $ftd_1(G')$ set. By Observation 1,  $c_{j+1} \in S'$ , since  $c_{j+1}$  is a support vertex in G'. If  $c_{j-1} \notin S'$ , then  $S' \cup \{c'_{i+1}\}$  is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3-1, a contradiction. Thus assume that  $c_{j-1} \in S'$ . Then  $\{c_j, c'_{j+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3, a contradiction. Thus deg $(c_i) \ge 3$  for  $1 \leq i \leq r$ . Let  $G^* = G - c_0 c_1 - c_0 c_r$ . Let  $G_1^*$  be the component of  $G^*$  containing  $c_r$ , and  $G_2^*$  be the component of  $G^*$  containing  $c_0$ . Let  $D = S(G_1^*) \setminus V(C_i)$ . Clearly,  $S' = D \cup \{c_1, c_2, \ldots, c_r\}$  is a 1FTD-set for  $G_1^*$  of cardinality at most  $2n(G_1^*)/3$ . Let  $G_3^* = G[V(G_2^*) \cup \{c_1\}]$ . By the choice of G,  $ftd_1(G_3^*) \le (2(n(G_3^*) + k - 1) - 1)/3$ . Let S'' be an  $ftd_1(G_3^*)$ -set. By Observation 1,  $c_0 \in S''$ . Clearly,  $S' \cup S''$  is a 1FTDset for G and so  $ftd_1(G) \le (2(n(G_3^*)+k-1)-1)/3+2n(G_1^*)/3 = (2(n+k)-1)/3,$ a contradiction.

Now assume that d = 1. Assume that  $\deg_G(c_i) = 2$  for some  $i \in \{1, 2, \ldots, r\}$ . Let  $\deg_G(c_j) = 2$ . Assume that  $\deg_G(c_{j+1}) = 2$ . Let  $G' = G - c_j$ . By the choice of G,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{j+2} \in S'$ . If  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$ , then S' is a 1FTD-set for G of cardinality at most (2(n + k) - 1)/3 - 4/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$ . Then  $\{c_j\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n + k) - 1)/3 - 4/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$ . Then  $\{c_j\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n + k) - 1)/3 - 1/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ . Then  $\{c_{j+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n + k) - 1)/3 - 1/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ . Then  $\{c_{j+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n + k) - 1)/3 - 1/3, a contradiction. Thus deg\_G(c\_{j+1}) \geq 3. Similarly,  $\deg_G(c_{j-1}) \geq 3$ . Clearly,  $c_{j+1} \neq c_0$  or  $c_{j-1} \neq c_0$ . Assume, without loss of generality, that  $c_{j+1} \neq c_0$ . Thus  $c_{j+1}$  is a support vertex of G. Let  $G' = G - c_j$ . Then by the choice of G,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{j+1} \in S'$ . If  $c_{j-1} \notin S'$ , then S' is a 1FTD-set for G, a contradiction. Thus assume that  $c_{j-1} \in S'$ . Then  $\{c_j\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3-1/3, a contradiction. We thus obtain that  $\deg(c_i) \geq 3$  for  $1 \leq i \leq r$ . Let  $G^* = G - c_0c_1 - c_0c_r$ . Let  $G_1^*$  be the component of  $G^*$  containing  $c_r$ , and  $G_2^*$  be the component of  $G^*$  containing  $c_0$ . Clearly,  $S' = \{c_1, c_2, \ldots, c_r\}$  is a 1FTD-set for  $G_1^*$  of cardinality at most  $n(G_1^*)/2$ . Let  $G_3^* = G[V(G_2^*) \cup \{c_1\}]$ . By the choice of G,  $ftd_1(G_3^*) \leq (2(n(G_3^*)+k-1)-1)/3$ . Let S" be an  $ftd_1(G_3^*)$ -set. By Observation 1,  $c_0 \in S''$ . Clearly,  $S' \cup S''$  is a 1FTD-set for G and so  $ftd_1(G) \leq (2(n(G_3^*)+k-1)-1)/3+n(G_1^*)/2 < (2(n+k)-1)/3$ , a contradiction.

It is evident that for the cycle  $C_7$  the equality of the bound given in Theorem 14 holds.

**Theorem 15.** If  $G \neq C_7$  is a cactus graph of order  $n \geq 5$  with  $k \geq 1$  cycles, then  $ftd_1(G) = (2(n+k)-1)/3$  if and only if  $G \in \mathcal{G}_k$ .

**Proof.** We prove by an induction on k to show that any cactus graph  $G \neq C_7$ of order  $n \geq 5$  with  $k \geq 1$  cycles and  $ftd_1(G) = (2(n+k)-1)/3$  belongs to  $\mathcal{G}_k$ . The base step of the induction follows by Theorem 6. Assume the result holds for all cactus graphs  $G' \neq C_7$  with k' < k cycles. Now let  $G \neq C_7$  be a cactus graph of order n with  $k \geq 2$  cycles and  $ftd_1(G) = (2(n+k)-1)/3$ . Suppose to the contrary that  $G \notin \mathcal{G}_k$ . Assume that G has the minimum order, and among all such graphs, assume that the size of G is minimum.

Claim 1. Every support vertex of G is weak support vertex.

**Proof.** Suppose that G has a strong support vertex u, and assume that  $u_1$  and  $u_2$  are two leaves adjacent to u. Let  $G' = G - u_1$ , and S' be an  $ftd_1(G')$ -set. By Observation 1,  $u \in S'$ . By Theorem 14,  $|S'| \leq (2(n(G') + 2) - 1)/3 = (2(n+k)-1)/3 - 2/3$ . Clearly, S' is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3 - 2/3, a contradiction.

By Observation 8, G has at least two leaf-cycles. Let  $C_1 = c_0 c_1 \cdots c_r c_0$  be a leaf-cycle of G, where  $c_0$  is a special cut-vertex of G. Let  $G'_1$  be the component of  $G - c_0 c_1 - c_0 c_r$  containing  $c_1$ .

# Claim 2. $V(G'_1) \neq \{c_1, \ldots, c_r\}.$

**Proof.** Suppose that  $V(G'_1) = \{c_1, \ldots, c_r\}$ . Then  $\deg_G(c_i) = 2$ , for each  $i = 1, 2, \ldots, r$ . Let  $G' = G - c_2$ . By Theorem 14,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k)-1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_0 \in S'$ . If  $|S' \cap \{c_1, c_3\}| = 1$ , then S' is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3 - 4/3, a contradiction. Thus assume that  $|S' \cap \{c_1, c_3\}| = 2$ . Then  $\{c_2\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction. Thus assume that  $|S' \cap \{c_1, c_3\}| = 2$ . Then  $\{c_2\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction. Thus assume that  $|S' \cap \{c_1, c_3\}| = 0$ . Then  $\{c_1\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction.  $\Box$ 

Let  $v_d \in V(G'_1) \setminus \{c_1, \ldots, c_r\}$  be a leaf of  $G'_1$  at maximum distance from  $\{c_1, \ldots, c_r\}$ , and assume that  $\deg(v_{d-1})$  is as maximum as possible,  $\deg_G(v_0)$  is as maximum as possible, and  $\deg_G(v_1)$  is as maximum as possible, where  $v_0 \in \{c_1, \ldots, c_r\}$  and  $v_0v_1 \cdots v_d$  is the shortest path from  $v_d$  to  $\{c_1, \ldots, c_r\}$ .

Suppose that d = 1. Assume that  $\deg_G(c_j) = 2$ , for some  $j \in \{1, 2, \dots, r\}$ . Assume that  $\deg_G(c_{j+1}) = 2$ . Let  $G' = G - c_j$ . By Theorem 14,  $ftd_1(G') \leq C_j$ (2(n(G')+k-1)-1)/3 = (2(n+k)-1)/3 - 4/3. Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{j+2} \in S'$ . If  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 1$ , then S' is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3 - 4/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$ . Then  $\{c_j\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ . Then  $\{c_{i+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3-1/3, a contradiction. Thus  $\deg_G(c_{j+1}) \geq 3$ . Similarly,  $\deg_G(c_{j-1}) \geq 3$ . Clearly,  $c_{i+1} \neq c_0$  or  $c_{i-1} \neq c_0$ . Assume, without loss of generality, that  $c_{i+1} \neq c_0$ . Then  $c_{j+1}$  is a support vertex of G. Let  $G' = G - c_j$ . Then by Theorem 14,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{j+1} \in S'$ . If  $c_{j-1} \notin S'$ , then S' is a 1FTDset for G of cardinality at most (2(n+k)-1)/3 - 4/3, a contradiction. Thus assume that  $c_{i-1} \in S'$ . Then  $\{c_i\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3, a contradiction. We thus obtain that deg $(c_i) \geq 3$ , for  $1 \leq j \leq r$ . Let  $G^* = G - c_0 c_1 - c_0 c_r$ . Let  $G_1^*$  be the component of  $G^*$  containing  $c_r$ , and  $G_2^*$  be the component of  $G^*$  containing  $c_0$ . Clearly,  $S' = \{c_1, c_2, \ldots, c_r\}$ is a 1FTD-set for  $G_1^*$  of cardinality at most  $n(G_1^*)/2$ . Let  $G_3^* = G[V(G_2^*) \cup \{c_1\}]$ . By Theorem 14,  $ftd_1(G_3^*) \leq (2(n(G_3^*) + k - 1) - 1)/3$ . Let S'' be an  $ftd_1(G_3^*)$ set. By Observation 1,  $c_0 \in S''$ . Clearly,  $S' \cup S''$  is a 1FTD-set for G and so  $ftd_1(G) \le (2(n(G_3^*)+k-1)-1)/3 + n(G_1^*)/2 < (2(n+k)-1)/3$ , a contradiction.

Thus assume that  $d \geq 2$ .

Claim 3. If  $d \geq 3$ , then  $G \in \mathcal{G}_k$ .

**Proof.** Assume that  $d \geq 3$ . By Claim 1,  $\deg_G(v_{d-1}) = 2$ . Assume first that  $\deg_G(v_{d-2}) \geq 3$ . Assume that  $v_{d-2}$  is a support vertex. Let  $G' = G - \{v_{d-1}, v_d\}$ . By Theorem 14,  $ftd_1(G') \leq (2(n(G')+k)-1)/3 = (2(n+k)-1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $v_{d-2} \in S'$ . Then  $\{v_{d-1}\} \cup S'$  is a 1FTD-set in G, and so  $ftd_1(G) \leq (2(n+k)-1)/3 - 1/3$ , a contradiction. Thus assume that  $v_{d-2}$  is not a support vertex of G. Let  $x \neq v_{d-1}, v_{d-3}$  be a support vertex of G such that  $x \in N(v_{d-2})$ . By the choice of the path  $v_0v_1 \cdots v_d$ , (the part " $\deg(v_{d-1})$ ) is as maximum as possible"),  $\deg_G(x) = 2$ . Let y be the leaf adjacent to x, and  $G' = G - \{v_d, v_{d-1}, y\}$ . By Theorem 14,  $ftd_1(G') \leq (2(n(G')+k)-1)/3 = (2(n+k)-1)/3 - 2$ . Assume that  $ftd_1(G') < (2(n(G')+k)-1)/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $v_{d-2} \in S'$ , since  $v_{d-2}$  is a support vertex of G'. Then  $\{v_{d-1}, x\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) < (2(n+k)-1)/3$ , a

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contradiction. Thus  $ftd_1(G') = (2(n(G') + k) - 1)/3 = (2(n + k) - 1)/3 - 2$ . By the choice of  $G, G' \in \mathcal{G}_k$ . Thus G is obtained from G' by Operation  $\mathcal{O}_2$ , and so  $G \in \mathcal{G}_k$ .

Assume that  $\deg_G(v_{d-2}) = 2$ . We consider the following cases.

Case 1.  $d \ge 4$ . Suppose that  $\deg_G(v_{d-3}) = 2$ . Let  $G' = G - \{v_d, v_{d-1}, v_{d-2}, v_{d-3}\}$ . By Theorem 14,  $ftd_1(G') \le (2(n(G')+k)-1)/3 = (2(n+k)-1)/3-8/3$ . Let S' be an  $ftd_1(G')$ -set. If  $v_{d-4} \in S'$ , then  $\{v_{v-1}, v_d\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) \le (2(n+k)-1)/3-2/3$ , a contradiction. Thus  $v_{d-4} \notin S'$ . Then  $\{v_{v-2}, v_{d-1}\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) \le (2(n+k)-1)/3-2/3$ , a contradiction. We deduce that  $\deg_G(v_{d-3}) \ge 3$ . Let  $G' = G - \{v_d, v_{d-1}, v_{d-2}\}$ . By Theorem 14,  $ftd_1(G') \le (2(n(G')+k)-1)/3$ . Assume that  $ftd_1(G') < (2(n(G')+k)-1)/3 = (2(n+k)-1)/3-2$ . Let S' be an  $ftd_1(G) < (2(n+k)-1)/3$ , then  $\{v_{v-1}, v_{d-2}\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus  $v_{d-3} \notin S'$ . Then  $\{v_{v-1}, v_d\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. We thus obtain that  $ftd_1(G') = (2(n(G')+k)-1)/3$ . By the choice of G,  $G' \in \mathcal{G}_k$ . Since  $d \ge 4$ ,  $v_{d-3}$  is not a special vertex of G'. Thus G is obtained from G' by Operation  $\mathcal{O}_1$ , and so  $G \in \mathcal{G}_k$ .

Case 2. d = 3. Clearly,  $\deg(v_0) \ge 3$ . We show that  $\deg(v_0) \ge 4$ . Suppose that deg $(v_0) = 3$ . Let  $G' = G - \{v_1, v_2, v_3\}$ . By Theorem 14,  $ftd_1(G') \leq 1$ (2(n(G')+k)-1)/3. Assume that  $ftd_1(G') = (2(n(G')+k)-1)/3$ . By the choice of  $G, G' \in \mathcal{G}_k$ . By Observation 9(1),  $v_0$  is the unique special vertex of G', since  $\deg_{G'}(v_0) = 2$ . We show that  $\deg_{G'}(x) = 3$  for each  $x \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$ . Assume that  $\deg_{G'}(c_j) \ge 4$  for some  $c_j \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$ . If there is a vertex  $w \in V(G) \setminus V(C_1)$  such that  $d(w, C_1) = d(w, c_j) = 3$ , then w can plays the same role of  $v_d$ , and thus deg $(u_i) = 3$ , a contradiction. Thus there is no vertex  $w \in$  $V(G)\setminus V(C_1)$  such that  $d(w, C_1) = d(w, c_j) = 3$ . Thus any vertex of  $N(u_j)\setminus V(C_1)$ is a leaf or a weak support vertex. Assume that  $N(c_i) \setminus V(C_1)$  contains  $t_1$  leaves and  $t_2$  support vertices, where  $t_1 + t_2 \ge 2$ . By Observation 9(1),  $t_1 = 0$ , since  $G' \in$  $\mathcal{G}_k$ . Thus  $t_2 \geq 2$ . Let  $z_1$  and  $z_2$  be two weak support vertices in  $N(c_j) \setminus V(C_1)$ . Let  $z'_1$  and  $z'_2$  be the leaves adjacent to  $z_1$  and  $z_2$ , respectively. (We switch for a while to G). Let  $G'' = G - \{z_1, z'_1, z'_2\}$ . By Theorem 14,  $ftd_1(G'') \le (2(n(G'')+k)-1)/3)$ . Suppose that  $ftd_1(G'') = (2(n(G')+k)-1)/3$ . By the choice of  $G, G'' \in \mathcal{G}_k$ . Clearly,  $\deg_{G''}(c_i) \geq 3$ , since  $v_0$  is the unique special vertex of G', a contradiction (by Observation 9(1)). Thus  $ftd_1(G'') < (2(n(G'')+k)-1)/3 = (2(n+k)-1)/3 - (2(n+k)-1)/3 - (2(n+k)-1)/3) - (2(n+k)-1)/3 - (2(n+k)-1)/3) - (2(n+k)-1)/3 - (2(n+k)-1)/3 - (2(n+k)-1)/3 - (2(n+k)-1)/3 - (2(n+k)-1)/3) - (2(n+k)-1)/3 - (2(n+k)-1)/3$ 2. Let S'' be a 1FTD-set of G''. By Observation 1,  $c_i \in S''$ . Then  $S'' \cup \{z_1, z_2\}$ is a 1FTD-set of G. Thus  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. We deduce that  $\deg_{G'}(c_i) = 3$  for each  $c_i \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$ . Thus  $\deg_G(c_i) = 3$ for each  $1 \leq i \leq r$ . Note that by Observation 9(1),  $c_i$  is not a support vertex, for each i with  $1 \leq i \leq r$  in G', since  $G' \in \mathcal{G}_k$ . (We switch for a while to G). Let  $F = \bigcup_{i=1}^{r} (N[c_i]) \setminus \{c_0, \ldots, c_r\}$ . Clearly, |F| = r, since  $\deg_{G'}(c_i) = 3$  for each  $c_i \in \{c_1, \ldots, c_r\} \setminus \{v_0\}$  and  $\deg_G(v_0) = 3$ . Let  $F = \{u_1, u_2, \ldots, u_r\}$ . Clearly  $\deg_G(v_0) = 1$ .  $(u_i) \geq 2$ , for each i with  $1 \leq i \leq r$ , since  $c_i$  is not a support vertex for  $1 \leq i \leq r$ in G'. By Claim 2,  $u_i$  is not a strong support vertex of G, for  $1 \leq i \leq r$ . If  $u_i$  is adjacent to a support vertex  $u'_i \in V(G) \setminus V(C_1)$ , for some integer i, then since the leaf of  $u'_i$  can play the role of  $v_3$ , we obtain that  $\deg(u_i) = 2$ . Since  $\deg_G(u_i) \geq 2$ for each i with  $1 \leq i \leq r$ , we find that  $\deg_G(u_i) = 2$  for each i with  $1 \leq i \leq r$ .

Let  $F' = \bigcup_{i=1}^r N(u_i) \setminus \{c_0, \ldots, c_r\}$ . Clearly, |F'| = r, since  $\deg_G(u_i) = 2$ , for each  $u_i \in \{u_1, \ldots, u_r\}$ . Let  $F' = \{u'_1, u'_2, \ldots, u'_r\}$ . By the choice of the path  $v_0v_1\cdots v_d$ , (the part "deg $(v_{d-1})$  is as maximum as possible"), deg $(u'_i) \leq 2$ , for  $1 \leq i \leq r$ . Let  $F'_1 = \{u'_i \in F' | \deg_G(u'_i) = 1\}$  and  $F'_2 = F' - F'_1$ . Then every vertex of  $F'_2$  is a weak support vertex. Since  $v_1 \in F'_2$ , we have  $|F'_2| \ge 1$ . Let  $G^* = G - c_0 c_1 - c_0 c_r$ , and  $G_1^*$  and  $G_2^*$  be the components of  $G^*$ , where  $c_1 \in V(G_1^*)$ . By Theorem 14,  $ftd_1(G_2^*) \leq (2(n(G_2^*) + k - 1) - 1)/3$ . Clearly,  $n(G_2^*) = n(G) - 3r - |F_2'|$ . Let  $S_2^*$  be an  $ftd_1(G_2^*)$ -set. If  $c_0 \notin S_2^*$ , then  $S_2^* \cup F \cup F'$ is a 1FTD-set for G. Thus  $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n(G) - 1)/3)/3 + 2r$  $3r - |F'_2| + k - 1) - 1/3 + 2r$  and so  $ftd_1(G) < (2(n+k) - 1)/3$ , a contradiction. Thus  $c_0 \in S_2^*$ . If  $|F_2| = 1$ , then  $S_2^* \cup V(C_1) \cup F \cup \{v_2\}$  is a 1FTD-set for G and 1/3 + 2r + 1 < (2(n+k) - 1)/3, a contradiction. Thus assume that  $|F'_2| \ge 2$ . Let  $\{u'_t, u'_{t'}\} \subseteq F'_2$  (assume without loss of generality that t < t') such that  $\deg_G(u'_i) = 1$ , for  $1 \le i < t$  and  $t' < i \le r$ . Let  $u''_t$  and  $u''_{t'}$  be the leaves of  $u_t$  and  $u_{t'}$ , respectively. Clearly,  $S_2^* \cup \{c_1, \ldots, c_{t-1}\} \cup \{u_1, \ldots, u_{t-1}\} \cup \{c_{t'+1}, \ldots, c_r\} \cup$  $\{u_{t'+1},\ldots,u_r\} \cup \{u_{t+1},\ldots,u_{t'-1}\} \cup \{u'_{t+1},\ldots,u'_{t'-1}\} \cup \{u'_t,u'_{t'}\} \cup \{u''_t,u''_{t'}\}$  is a 1FTD-set for G and thus  $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n(G) - 1)/3)/3 + 2r =$  $3r - |F'_2| + k - 1) - 1/3 + 2r + 1 < (2(n+k) - 1)/3$ , a contradiction. We deduce that  $ftd_1(G') < (2(n(G')+k)-1)/3 = (2(n+k)-1)/3 - 2$ . Let S' be an  $ftd_1(G')$ -set. If  $v_0 \in S'$ , then  $S' \cup \{v_1, v_2\}$  is a 1FTD-set in G, and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus assume that  $v_0 \notin S'$ . Then  $S' \cup \{v_2, v_3\}$  is a 1FTD-set in G and thus  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus  $\deg(v_0) \ge 4$ . Let  $G' = G - \{v_3, v_2, v_1\}$ . By Theorem 14,  $ftd_1(G') \leq (2(n(G') + k) - 1)/3$ . Assume that  $ftd_1(G') < (2(n(G') + k) - 1)/3 = (2(n+k) - 1)/3 - 2$ . Let S' be an  $ftd_1(G')$ -set. If  $v_0 \in S'$ , then  $S = S' \cup \{v_1, v_2\}$  is a 1FTD-set for G and thus  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus assume that  $v_0 \notin S'$ . Then  $S = S' \cup \{v_2, v_3\}$  is a 1FTD-set for G and thus  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Hence,  $ftd_1(G') = (2(n(G') + k) - 1)/3$ . By the inductive hypothesis,  $G' \in \mathcal{G}_{k-1}$ . Since deg $(v_0) \geq 4$ ,  $v_0$  is not a special vertex of G'. Thus G is obtained from G' by Operation  $\mathcal{O}_1$  and so  $G \in \mathcal{G}_k$ . 

By Claim 3, we assume that d = 2. We show that  $\deg_G(v_0) = 3$ . Suppose that  $\deg_G(v_0) \ge 4$ . Assume that  $v_0$  is a support vertex. Let  $G' = G - \{v_1, v_2\}$ . By Theorem 14,  $ftd_1(G') \le (2(n(G') + k) - 1)/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $v_0 \in S'$ . Then  $S' \cup \{v_{d-1}\}$  is a 1FTD-set in G, and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus assume that  $v_0$  is not a support vertex of G. Let  $x \neq v_1$  be a support vertex of G such that  $x \in N(v_0) \setminus V(C_1)$ . By the choice of the path  $v_0v_1 \cdots v_d$ , (the part "deg $(v_{d-1})$  is as maximum as possible"), deg<sub>G</sub>(x) = 2. Let y be the leaf adjacent to x. Let  $G' = G - \{v_2, v_1, y\}$ . By Theorem 14,  $ftd_1(G') \leq (2(n(G')+k)-1)/3$ . Let  $ftd_1(G') < (2(n(G')+k)-1)/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $v_0 \in S'$ , since  $v_0$  is a support vertex of G'. Then  $\{v_1, x\} \cup S'$  is a 1FTD-set in G and so  $ftd_1(G') < (2(n+k)-1)/3$ , a contradiction. Thus  $ftd_1(G') = (2(n(G')+k)-1)/3$ . By the inductive hypothesis,  $G' \in \mathcal{G}_k$ , a contradiction by Observation 9(1), since  $v_0$  is a support vertex of G'. Thus deg<sub>G</sub> $(v_0) = 3$ . Observe that G has no strong support vertex. If  $c_i$  is adjacent to a support vertex  $c'_i$  of  $N(c_i) \setminus V(C_1)$  for some i, then the leaf of  $c'_i$  can play the role of  $v_2$ , and thus deg $(c_i) = 3$ . Thus we may assume that deg<sub>G</sub> $(c_i) \leq 3$  for each i with  $i = 1, 2, \ldots, r$ . Assume that deg<sub>G</sub> $(c_i) = 3$  for each i with  $1 \leq i \leq r$ .

Let  $F = \bigcup_{i=1}^{r} (N(c_i) \setminus \{c_0, \ldots, c_r\})$ . Clearly, |F| = r, since  $\deg_G(c_i) = 3$ , for each  $c_i \in \{c_1, ..., c_r\}$ . Let  $F = \{u_1, u_2, ..., u_r\}$ . Clearly,  $\deg_G(u_i) \leq 2$ , for  $1 \leq i \leq r$ , since G has no strong support vertex. Let  $F' = \{u_i | \deg_G(u_i) = 2\}$ . Clearly,  $v_1 \in F'$ . Let F'' be the set of leaves of F'. Clearly,  $v_2 \in F''$ . Let  $G^* = G - c_0 c_1 - c_0 c_r$ . Let  $G_1^*$  be the component of  $G^*$  containing  $c_r$  and  $G_2^*$  be the component of  $G^*$  containing  $c_0$ . Assume that F = F'. Thus  $n(G_1^*) = 3r$ , since d =2. Further,  $n(G_2^*) = n - 3r$ . By Theorem 14,  $ftd_1(G_2^*) \le (2(n(G_2^*) + k - 1) - 1)/3$ . Let S'' be an  $ftd_1(G_2^*)$ -set. If  $c_0 \in S''$ , then  $S'' \cup V(C_1) \cup F$  is a 1FTD-set for G and so  $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r = (2(n - 3r + k - 1))/3 + 2r = (2(n - 3r + 1))/3 + 2r = ($ (1) - 1)/3 + 2r = (2(n + k - 1) - 1)/3, a contradiction. Thus  $c_0 \in S''$ . Then  $S'' \cup F'' \cup F$  is a 1FTD-set for G and so  $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + 2r =$ (2(n-3r+k-1)-1)/3 + 2r = (2(n+k-1)-1)/3, a contradiction. We conclude that  $F \neq F'$ . Let |F'| = r'. Clearly,  $1 \leq r' < r$ , since  $v_1 \in F'$ . Thus  $n(G_1^*) = 2r + r'$ . Then  $n(G_2^*) = n - (2r + r')$ . Let  $G_3^* = G[V(G_2^*) \cup \{c_1\}]$ . Then  $n(G_3^*) = n - (2r + r') + 1$ . By Theorem 14,  $ftd_1(G_3^*) \le (2(n(G_3^*) + k - 1) - 1)/3$ . Let S'' be an  $ftd_1(G_3^*)$ -set. By Observation 1,  $c_0 \in S''$  and so  $S'' \cup V(C_1) \cup F'$ is a 1FTD-set for G. Thus  $ftd_1(G) \leq (2(n(G_2^*) + k - 1) - 1)/3 + r + r' =$ (2(n-(2r+r')+1+k-1)-1)/3+r+r' = (2(n+k)-1+r'-r)/3 < (2(n+k)-1)/3,a contradiction. Therefore  $\deg_G(c_t) = 2$  for some  $1 \le t \le r$ .

## **Claim 4.** No vertex of $C_1 - c_0$ is a support vertex.

**Proof.** Let  $c_j$  be a support vertex of G. Assume that  $c_{j+1}$  is a special vertex. Let  $G' = G - c_{j+1}$ . Then by Theorem 14,  $ftd_1(G') \leq (2(n(G') + k - 1) - 1)/3 = (2(n+k)-1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_j \in S'$ . If  $c_{j+2} \notin S'$ , then S' is a 1FTD-set for G of cardinality at most (2(n+k)-1)/3-4/3 and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus  $c_{j+2} \in S'$ . Then  $\{c_{j+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 1/3 and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus  $\deg_G(c_{j+1}) \neq 2$ . Note that  $c_t$  is a special vertex of G. Assume without loss of generality that j < t. Let  $c_{j'}$  be a support vertex of G and  $c_{t'}$  be a special vertex of G, where  $j \leq j' < t' \leq t$ , and among such vertices choose  $c_{j'}$  and  $c_{t'}$  such that  $c_i$  is neither a support vertex nor a special vertex of G for each i with j' < i < t'. Let  $u_i \in N(c_i) \setminus V(C_1)$ for j' < i < t'. Clearly,  $\deg_G(u_i = 2$  for j' < i < t', since G has no strong support vertex. Let  $G^* = G - c_{j'}c_{j'+1} - c_{t'}c_{t'+1}$ . Let  $G_1^*$  be the component of  $G^*$  containing  $c_{j'}$  and  $G_2^*$  be the component of  $G^*$  containing  $c_{t'}$ . Clearly,  $n(G_2^*) = 3(t' - j' - 1) + 1$ . Thus  $n(G_1^*) = n - (3(t' - j' - 1) + 1)$ .

By Theorem 14,  $ftd_1(G_1^*) \leq (2(n(G_1^*)+k-1)-1)/3$ . Let S' be an  $ftd_1(G_1^*)$ -set. By Observation 1,  $c_{j'} \in S'$ . Assume that  $c_{t'+1} \notin S'$ . Then  $S' \cup \{c_{j'+1}, c_{j'+2}, \ldots, c_{t'-1}\} \cup \{u_{j'+1}, u_{j'+2}, \ldots, u_{t'-1}\}$  is a 1FTD-set in G of cardinality at most  $(2(n(G_1^*)+k-1)-1)/3+2(t'-j'-1)=(2(n-(3(t'-j'-1)+1)+k-1)-1)/3+2(t'-j'-1)=(2(n+k)-1)/3-4/3$  and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction. Thus  $c_{t'+1} \in S'$ . Then  $S' \cup \{c_{j'+1}, c_{j'+2}, \ldots, c_{t'}\} \cup \{u_{j'+1}, u_{j'+2}, \ldots, u_{t'-1}\}$  is a 1FTD-set in G of cardinality at most  $(2(n(G_1^*)+k-1)-1)/3+2(t'-j'-1)+1=(2(n-(3(t'-j'-1)+1)+k-1)-1)/3+2(t'-j'-1))=(2(n+k)-1)/3-1/3$  and so  $ftd_1(G) < (2(n+k)-1)/3$ , a contradiction.  $\Box$ 

**Claim 5.** If  $\deg_G(c_j) = 2$  for some j with  $1 \le j \le r$ , then  $\deg_G(c_{j+1}) = 3$  and  $\deg_G(c_{j-1}) = 3$ .

**Proof.** Assume that  $\deg_G(c_j) = \deg_G(c_{j+1}) = 2$ , for some j with  $1 \le j \le r$ , and among such vertices choose  $c_j$  such that  $\deg_G(c_{j-1}) = 3$ . Let  $G' = G - c_j$ . Then by Theorem 14,  $ftd_1(G') \le (2(n(G') + k - 1) - 1)/3 = (2(n + k) - 1)/3 - 4/3$ . Let S' be an  $ftd_1(G')$ -set. By Observation 1,  $c_{j+2} \in S'$ . If  $|S' \cap \{c_{j-1}, c_{j+1}\}| =$ 1, then S' is a 1FTD-set for G of cardinality at most (2(n + k) - 1)/3 - 4/3and so  $ftd_1(G) < (2(n + k) - 1)/3$ , a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 2$ . Then  $\{c_j\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n + k) - 1)/3 - 1/3 and so  $ftd_1(G) < (2(n + k) - 1)/3$ , a contradiction. Thus assume that  $|S' \cap \{c_{j-1}, c_{j+1}\}| = 0$ . Then  $\{c_{j+1}\} \cup S'$  is a 1FTD-set in G of cardinality at most (2(n + k) - 1)/3 - 1/3 and so  $ftd_1(G) < (2(n + k) - 1)/3$ , a contradiction. Thus  $\deg_G(c_{j+1}) \ge 3$ . Similarly  $\deg_G(c_{j-1}) \ge 3$ .

## Claim 6. $C_1$ has precisely one special vertex.

**Proof.** Let  $c_{t_1}$  and  $c_{t_2}$  be two special vertices of  $C_1$  and among such vertices choose  $c_{t_1}$  and  $c_{t_2}$  such that  $c_i$  is not a special vertex of  $C_1$  for  $t_1 < i < t_2$ . By Claim 5,  $t_1 + 1 < t_2$ . By Claim 4,  $c_i$  is not a support vertex for  $t_1 < i < t_2$ . Let  $u_i \in N(c_i) \setminus V(C_1)$ , for  $t_1 < i < t_2$ . Clearly,  $\deg_G(u_i) = 2$ , for  $t_1 < i < t_2$ . Let  $u'_i$  be the leaf adjacent to  $u_i$ , for  $t_1 < i < t_2$ , and  $G^* = G - c_{t_1}c_{t_1+1} - c_{t_2}c_{t_2+1}$ . Let  $G_1^*$  be the component of  $G^*$  containing  $c_{t_1}$ , and  $G_2^*$  be the component of  $G^*$  containing  $c_{t_2}$ . Clearly,  $n(G_2^*) = 3(t_2 - t_1 - 1) + 1$ . Then  $n(G_1^*) = n - (3(t_2 - t_1 - 1) + 1)$ . By Theorem 14,  $ftd_1(G_1^*) \leq (2(n(G_1^*) + k - 1) - 1)/3$ . Let S' be an  $ftd_1(G_1^*)$ -set. By Observation 1,  $c_{t_1-1} \in S'$ . Assume that  $\{c_{t_1}, c_{t_2+1}\} \cap S' = \emptyset$ .

Then  $S' \cup \{c_{t_1}, c_{t_1+1}, \dots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\}$  is a 1FTD-set in G of cardinality at most  $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)/3 - 1/3$  and so  $ftd_1(G) < (2(n + k) - 1)/3$ , a contradiction.

Thus  $\{c_{t_1}, c_{t_2+1}\} \cap S' \neq \emptyset$ . If  $\{c_{t_1}, c_{t_2+1}\} \subseteq S'$ , then  $S' \cup \{c_{t_1+1}, c_{t_1+2}, \dots, c_{t_2}\} \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\}$  is a 1FTD-set in G of cardinality at most  $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) + 1 = (2(n + k) - 1)/3 - 1/3$ . Thus  $ftd_1(G) < (2(n + k) - 1)/3$ , a contradiction. Thus  $\{c_{t_1}, c_{t_2+1}\} \not\subseteq S'$ . If  $c_{t_1} \in S'$  and  $c_{t_2+1} \notin S'$ , then  $S' \cup \{c_{t_1+1}, c_{t_1+2}, \dots, c_{t_2-1}\} \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\}$  is a 1FTD-set in G of cardinality at most  $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1)) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3$ , a contradiction. Thus assume that  $c_{t_2+1} \in S'$  and  $c_{t_1} \notin S'$ . Then  $S' \cup \{u_{t_1+1}, u_{t_1+2}, \dots, u_{t_2-1}\} \cup \{u'_{t_1+1}, u'_{t_1+2}, \dots, u'_{t_2-1}\}$  is a 1FTD-set in G of cardinality at most  $(2(n(G_1^*) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1)) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1)) = (2(n - (3(t_2 - t_1 - 1) + 1) + k - 1) - 1)/3 + 2(t_2 - t_1 - 1)) = (2(n + k) - 1)/3 - 4/3$  and so  $ftd_1(G) < (2(n + k) - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)/3 - 4/3$  and so  $ftd_1(G) < (2(n + k) - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)/3 - 4/3$  and so  $ftd_1(G) < (2(n + k) - 1) - 1)/3 + 2(t_2 - t_1 - 1) = (2(n + k) - 1)/3 - 4/3$  and so  $ftd_1(G) < (2(n + k) - 1)/3$ , a contradiction.  $\Box$ 

By Claims 4 and 6,  $c_i$  is not a support vertex and is not a special vertex, for  $i \in \{1, 2, ..., t - 1, t + 1, ..., r\}$ . Let  $u_i \in N(c_i) \setminus V(C_1)$ , for  $i \in \{1, 2, ..., t - 1, t + 1, ..., r\}$ . Clearly,  $\deg_G(u_i) = 2$ , for  $i \in \{1, 2, ..., t - 1, t + 1, ..., r\}$ .

Let  $G''_1$  be the component of  $G - c_0c_1 - c_0c_r$  that contains  $c_1, G''_2$  be the component of  $G-c_0c_1-c_0c_r$  that contains  $c_0$ , and  $G^*$  be a graph obtained from  $G_2''$ by adding a path  $p_2 = x_1 x_2$  and joining  $c_0$  to  $x_1$ . Clearly,  $n(G^*) = n - (3r - 2) + 2$ . By Theorem 14,  $ftd_1(G^*) \leq (2(n(G^*) + k - 1) - 1)/3$ . Suppose that  $ftd_1(G^*) < 1$  $(2(n(G^*)+k-1)-1)/3$ . Let  $S^*$  be an  $ftd_1(G^*)$ -set. By Observation 1,  $x_1 \in S^*$ . If  $c_0 \in S^*$ , then  $S^* \cup \{c_1, c_2, \ldots, c_r\} \cup \{u_1, u_2, \ldots, u_{t-1}, u_{t+1}, \ldots, u_r\} \setminus \{x_1\}$  is a 1FTD-set in G. Thus  $ftd_1(G) < (2(n(G^*)+k-1)-1)/3+2r-1-1 = (2(n-(3r-1)/3+2r-1)/3+2r-1)/3+2r-1-1 = (2(n-(3r-1)/3+2r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1)/3 + (2(n-(3r-1)/3+2r-1))/3 +$ (2) + 2 + k - 1) - 1)/3 + 2r - 2 = (2(n+k) - 1)/3, a contradiction. Thus  $c_0 \notin S^*$ . Then  $x_2 \in S^*$ . If t > 1, then  $S^* \setminus \{x_1, x_2\} \cup \{c_1, \ldots, c_{t-1}\} \cup \{u_1, \ldots, u_{t-1}\} \cup \{u_1, \ldots, u$  $\{u_{t+1}, \ldots, u_r\} \cup \{u'_{t+1}, \ldots, u'_r\}$  is a 1FTD-set in G. Thus  $ftd_1(G) < (2(n(G^*) + u_{t+1})) < (2(n(G^*) + u_{t+1}))$ 4 = (2(n+k)-1)/3 - 2, a contradiction. Thus assume that t = 1. Then  $S^* \setminus \{x_1, x_2\} \cup \{c_2, \ldots, c_r\} \cup \{u_2, \ldots, u_r\}$ , is a 1FTD-set in G of cardinality at most (2(n+k)-1)/3 - 2 and so  $ftd_1(G) < (2(n+k)-1)/3 - 2$ , a contradiction. Thus  $ftd_1(G^*) = (2n(G^*+k-1)-1)/3$ . By the inductive hypothesis,  $G^* \in \mathcal{G}_{k-1}$ . Let  $G_1^*$  be the graph obtained from  $G[G_1'' \cup \{c_0\}]$  by adding a path  $p_2 = x_1' x_2'$ and joining  $c_0$  to  $x'_1$ . Clearly,  $G_1^* \in \mathcal{H}_1$ . Thus G is obtained from  $G^* \in \mathcal{G}_{k-1}$  and  $G_1^* \in \mathcal{H}_1$  by Procedure A. Consequently,  $G \in \mathcal{H}_k \subseteq \mathcal{G}_k$ .

For the converse, by Corollary 13,  $V(G) \setminus L(G)$  is the unique  $ftd_1(G)$ -set. Now Observation 9 implies that  $ftd_1(G) = (2(n+k)-1)/3$ .

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