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RESTRAINED DOMINATION IN SELF-COMPLEMENTARY GRAPHS

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Abstract

A self-complementary graph is a graph isomorphic to its complement. A set S of vertices in a graph G is a restrained dominating set if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The restrained domination number of a graph G is the minimum cardinality of a restrained dominating set of G. In this paper, we study restrained domination in self-complementary graphs. In particular, we characterize the self-complementary graphs having equal domination and restrained domination numbers.

Keywords: domination, complement, restrained domination, self-complementary graph.

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1. Introduction

A self-complementary graph G is a graph isomorphic to its complement $\overline{G}$. The structure of self-complementary graphs has been well-studied in the literature, including [1, 9, 19, 20]. In this paper, we characterize the self-complementary

graphs having equal domination and restrained domination numbers.

A set S of vertices in a graph G is a *dominating set* of G if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ a γ-set of G. The notion of domination and its variations in graphs and has been studied a great deal. For a more thorough treatment of domination parameters, see the books [14, 15, 17].

A set S is a *restrained dominating set*, abbreviated RD-set, of G if every vertex of $V(G) \setminus S$ is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Equivalently, a restrained dominating set of G is a dominating set S of G such that the subgraph $G[V(G) \setminus S]$ induced by the complement of S in G has no isolated vertex. The *restrained domination number* $\gamma_r(G)$ is the minimum cardinality of a RD-set of G. We call a RD-set of cardinality $\gamma_r(G)$ a γ_r-set of G.

The concept of restrained domination was introduced by Telle [26] in his 1994 PhD thesis, albeit as a vertex partitioning problem, and the first paper on the concept was published by Telle and Proskurowski in their 1997 paper [27]. However, the parameter was formally defined by Domke, Hattingh, Hedetniemi, Laskar, and Markus in their 1999 paper [5] on restrained domination in graphs, and also studied by Henning in his 1999 paper [16]. Subsequently over the past twenty or so years, the restrained domination number has been extensively studied in the literature; a rough estimate says that it occurs in more than 100 papers to date. For a small sample of recent papers on the restrained domination we refer the reader to [12, 13, 21, 24, 28].

As explained in the introductory 1999 paper [5], the restrained domination is application driven. One application given is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. Note that each prisoner's position is observed by a guard's position (to effect security) while each prisoner's position is seen by at least one other prisoner's position (to protect the rights of prisoners). To be cost effective, it is desirable to place as few guards as possible (in the sense above). The associated optimal placement of guards corresponds to a restrained dominating set of minimum cardinality.

Since every RD-set of a graph G is a dominating set of G, it follows that $\gamma(G) \leq \gamma_r(G)$. Determining when equality is achieved in such an inequality is a frequently studied problem in graph theory. For examples, see [3, 7, 10, 11, 22]. Here we characterize the self-complementary graphs G having $\gamma(G) = \gamma_r(G)$. The result is stated formally in Section 3.

1.1. Terminology and notation

For notation and graph theory terminology, we in general follow [17]. Specifically,

let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n(G) = |V|$ and edge set $E = E(G)$ of size $m(G) = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. The minimum degree (respectively, maximum degree) among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$). An *end-vertex* in G is a vertex of degree 1 in G. The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we omit it in the above expressions. For example, we write n, m, $d(u)$, $N(v)$ and $N[v]$ rather than $n(G)$, $m(G)$, $d_G(u)$, $N_G(v)$ and $N_G[v]$, respectively.

Given a subset $S \subseteq V(G)$ and a vertex $v \in S$, the *S-external private neighborhood* of v in G is the set $\mathrm{epn}_G(v, S) = \{w \in V \setminus S \mid N(w) \cap S = \{v\}\}$. We call each vertex in $\mathrm{epn}_G(v, S)$ an *S-external private neighbor*, or just an *external private neighbor*, of v.

The term end-vertex (as in Theorem 7) is not standard.

The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.

2. Special Graph Families

In this section, we discuss some special graph families. Let P_n and C_n denote the path and cycle, respectively, on n vertices. A *bull graph* consists of a triangle with two disjoint pendant edges as illustrated in Figure 1.

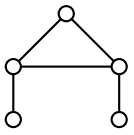


Figure 1. The bull graph.

2.1. The family $\mathcal{F}$

Let $\mathcal{F}$ be the family of self-complementary graphs G such that G is the cycle C_5 or G can be constructed from the disjoint union of a path P_4 and a self-complementary graph H, by adding all possible edges between the two support vertices of the path P_4 and the vertices of H. We note that if H is the trivial

graph K_1, then G is the bull graph. Hence, the two self-complementary graphs of order 5, namely, the cycle C_5 and the bull graph are in $\mathcal{F}$.

Suppose that $G \in \mathcal{F}$. If $G = C_5$, then $\gamma(G) = 2 < 3 = \gamma_r(G)$. If G is constructed from the union of a path $v_1v_2v_3v_4$ on four vertices and a self-complementary graph H, by adding all possible edges joining vertices in $\{v_2, v_3\}$ and $V(H)$, then $\{v_2, v_3\}$ is a dominating set of G and $\{v_1, v_2, v_4\}$ is a RD-set of G, implying that $\gamma(G) \leq 2$ and $\gamma_r(G) \leq 3$. In order to dominate the vertices v_1 and v_4, we note that at least one of the vertices v_1 and v_2 along with at least one of v_3 and v_4 are in every dominating set. It follows that $\gamma(G) \geq 2$ and every RD-set of G contains both v_1 and v_4. But $\{v_1, v_4\}$ is not a RD-set of G, so $\gamma_r(G) \geq 3$. Consequently, $\gamma(G) = 2 < 3 = \gamma_r(G)$. We state this formally as follows.

Observation 1. *If $G \in \mathcal{F}$, then G is a self-complementary graph and $\gamma(G) < \gamma_r(G)$.*

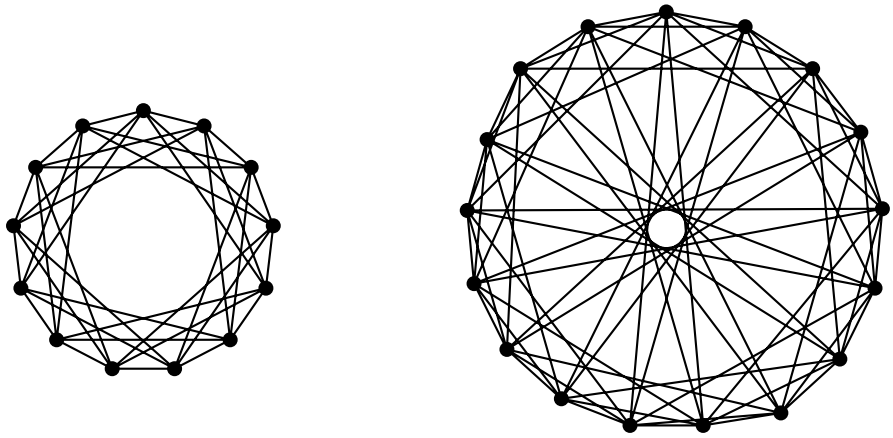
2.2. Paley graphs

Recall that a *prime power* is a positive integer power of a single prime number. Let q be a prime power such that $q \equiv 1 \pmod 4$; that is, q is either an arbitrary power of a Pythagorean prime (a prime congruent to $1 \pmod 4$) or an even power of an odd non-Pythagorean prime. This choice of q implies that in the unique finite field $\mathbb{F}_q$ of order q, the element -1 has a square root. The *Paley graph* G of order q is defined as follows. The vertex set of G is the field $\mathbb{F}_q$, while two vertices of G are adjacent if their difference is a square in the field $\mathbb{F}_q$. We note that Paley graphs exist for order 5, 9, 13, 25, 29, 37, 41, 49, 53, 61 (for a more comprehensive list, we refer the reader to http://oeis.org/A085759). Paley graphs were introduced independently by Sachs in his 1962 paper [23] and Erdős and Rényi in their 1963 paper [8]. Sachs [23] established the following fundamental property of Paley graphs.

Theorem 2 [23]**.** *The Paley graphs are self-complementary.*

The Paley graph G of order 5 is the 5-cycle which belongs to the family $\mathcal{F}$ and satisfies $\gamma(G) = 2 < 3 = \gamma_r(G)$. The Paley graph G of order 9 is the Cartesian product $K_3 \,\square\, K_3$ of two complete graphs K_3 (also known in the literature as a 3×3 rook's graph). The Paley graph of order 13 is the circulant graph $C_{13}\langle 1, 3, 4\rangle$ shown in Figure 2(a), while the Paley graph of order 17 is the circulant graph $C_{17}\langle 1, 2, 4, 8\rangle$ of order 17 shown in Figure 2(b) (used to show that the Ramsey number $R(4, 4) \geq 18$).

Recall that a strongly regular graph G with parameters (ν, k, λ, μ) is a k-regular graph of order ν, in which every two adjacent vertices in G have λ common neighbors and every two non-adjacent vertices have μ common neighbors. The Paley graph G of order q is a strongly regular graph with parameters $\left(q, \frac{1}{2}(q -\right.$



(a) The Paley graph of order 13. (b) The Paley graph of order 17.

Figure 2. The Paley graphs of order 13 and 17.

$1), \frac{1}{4}(q-5), \frac{1}{4}(q-1)\big)$. For example, the Paley graph of order $q = 13$ is a strongly regular graph with parameters $(13, 6, 2, 3)$. The Paley graphs G of order 9, 13, 17 and 25 all satisfy $\gamma(G) = 3 = \gamma_r(G)$. The Paley graphs G of orders between 29 and 81 (that is, of orders 29, 37, 41, 49, 53, 61, 73 and 81) all satisfy $\gamma(G) = 4 = \gamma_r(G)$. The Paley graphs G of orders between 89 and 373 all satisfy $\gamma(G) = 5 = \gamma_r(G)$ (see, for example, https://www.win.tue.nl/ aeb/graphs/Paley.html).

By using a greedy algorithm where one selects a vertex that dominates the maximum number of yet undominated vertices at each step one can show (or see [18]) that if G is a Paley graph of order q, then $\gamma(G) \leq 1 + \ln_2(q)$. Every γ-set in a Paley graph G of order q where $q \geq 9$ is therefore a RD-set of G noting that G is a $(\frac{q-1}{2})$-regular graph and therefore every vertex outside the γ-set has at least one neighbor outside the set. We state this formally as follows.

Theorem 3. *If G is a Paley graph different from the 5-cycle, then G is a self-complementary graph and $\gamma(G) = \gamma_r(G)$.*

3. Main Result

The simplest example of a self-complementary graphs G with $\gamma(G) = \gamma_r(G)$ is the path $G = P_4$ which satisfies $\gamma(G) = 2 = \gamma_r(G)$. As shown in Theorem 3, every Paley graph different from the 5-cycle has equal domination and restrained domination numbers. Hence, there are infinitely many self-complementary graphs G for which $\gamma(G) = \gamma_r(G)$. Our aim in this paper is to characterize the self-complementary graphs G having $\gamma(G) = \gamma_r(G)$. We shall prove the following result, a proof of which is given in Section 4.

Theorem 4. *Let G be a self-complementary graph. Then $\gamma(G) = \gamma_r(G)$ if and only if $G \notin \mathcal{F}$.*

3.1. Known results

We will use the following result due to Bollobás and Cockayne [2].

Theorem 5 [2]**.** *If G is a graph without isolated vertices, then G has a γ-set S such that for all $v \in S$, $\mathrm{epn}(v, S) \neq \emptyset$.*

Our next two results are well-known.

Theorem 6 [14]**.** *If G is a graph with $\mathrm{diam}(G) = 2$, then $\gamma(G) \leq \delta(G)$.*

Theorem 7 [14]**.** *If G is a graph with $\mathrm{diam}(G) = 3$, then $\gamma(\overline{G}) \leq 2$.*

Another useful result was proven in [4].

Theorem 8 [4]**.** *If a graph G has $\gamma(\overline{G}) \geq 4$, then every $\gamma(G)$-set is a $\gamma_r(G)$-set. In particular, $\gamma(G) = \gamma_r(G)$.*

The remaining results in this section pertain to self-complementary graphs.

Theorem 9 [23, 25]**.** *If G is a self-complementary graph, then $\mathrm{diam}(G) \in \{2, 3\}$.*

Theorem 10 [1]**.** *If G is a self-complementary graph with an end-vertex, then G contains exactly two end-vertices and two cut-vertices.*

It is well-known that if G is a self-complementary graph of order n, then $n \equiv 0, 1 \pmod 4$. We present next a straightforward observation concerning the order of self-complementary graphs in the literature. Let G be a self-complementary graph of order n, and let v be a vertex of G having maximum degree $\Delta(G)$. In the complement $\overline{G}$ of G, the vertex v has degree $\delta(\overline{G}) = n - 1 - \Delta(G)$. Since G is self-complementary, $\delta(\overline{G}) = \delta(G)$, and so, $n = \Delta(G) + \delta(G) + 1$. We state this formally as follows.

Observation 11. *If G is a self-complementary graph G of order n, then $n = 1 + \delta(G) + \Delta(G)$.*

4. Proof of Theorem 4

In this section, we prove Theorem 4. We begin with two lemmas and a corollary, which apply to general graphs.

Lemma 12. *Let G be a graph with $\gamma(G) \leq \gamma(\overline{G})$. If $\delta(G) \geq 4$, then $\gamma(G) = \gamma_r(G)$.*

Proof. Assume that $\gamma_r(G) > \gamma(G)$. Theorem 8 implies that $\gamma(G) \leq \gamma(\overline{G}) \leq 3$. Let S be any γ-set of G. Since $\delta(G) \geq 4$, it follows that every vertex in $V(G) \setminus S$ has a neighbor in $V(G) \setminus S$. Thus, S is a RD-set with cardinality $\gamma(G)$, a contradiction to our assumption. ■

Lemma 13. *Let G be a graph with no isolated vertices. If $\gamma_r(G) \geq \gamma(G) + 2$, then $\gamma_r(\overline{G}) \leq \gamma(G) + 1$.*

Proof. Suppose that $\gamma_r(G) \geq \gamma(G) + 2$. By Theorem 5, the graph G contains a γ-set S such that every vertex of S has an S-external private neighbor. Since $\gamma_r(G) \geq \gamma(G) + 2$, it follows there exist two vertices $u, v \in V(G) \setminus S$ such that $N_G(u) \subseteq S$ and $N_G(v) \subseteq S$. Suppose that $V(G) \setminus (S \cup \{u, v\}) = \emptyset$. In this case, our choice of S implies that $|S| = 2$. It follows that $G \in \{P_4, 2K_2\}$. If $G = P_4$, then $\gamma_r(G) = 2 = \gamma(G)$, a contradiction. Hence, $G = 2K_2$, in which case $\overline{G} = C_4$ and $\gamma_r(\overline{G}) = 2 = \gamma(G) < 3 = \gamma(G) + 1$. Hence, we may assume that $V(G) \setminus (S \cup \{u, v\}) \neq \emptyset$, for otherwise the desired result follows. In this case, each of u and v dominate $V(\overline{G}) \setminus S$ in $\overline{G}$. Hence, $S \cup \{u\}$ is a RD-set of $\overline{G}$, and so, $\gamma_r(\overline{G}) \leq \gamma(G) + 1$. ■

This results in the following corollary.

Corollary 14. *If G is a self-complementary graph, then $\gamma(G) \leq \gamma_r(G) \leq \gamma(G) + 1$.*

We are now ready to prove Theorem 4. Recall its statement.

Theorem 4. *Let G be a self-complementary graph. Then $\gamma(G) = \gamma_r(G)$ if and only if $G \notin \mathcal{F}$.*

Proof. First assume that $G \in \mathcal{F}$. Then G is self-complementary and $\gamma(G) = 2 < 3 = \gamma_r(G)$. This proves the necessity.

For the sufficiency, we again prove the contrapositive. Assume that G is a self-complementary graph with $\gamma_r(G) > \gamma(G)$. We show that $G \in \mathcal{F}$. From Theorem 9, we have that $\text{diam}(G) \in \{2, 3\}$. By Theorem 8, if $\gamma(G) \geq 4$, then $\gamma(G) = \gamma_r(G)$, a contradiction. Hence, we need only consider graphs G with $\gamma(G) \leq 3$. Since G is self-complementary, if $\gamma(G) = 1$, then G is the trivial graph K_1 and $\gamma(G) = \gamma_r(G) = 1$, a contradiction. Thus, $\gamma(G) \in \{2, 3\}$. Corollary 14 implies that $\gamma_r(G) = \gamma(G) + 1$, so $\gamma_r(G) \in \{3, 4\}$. Moreover, by Lemma 12, if $\delta(G) \geq 4$, then $\gamma(G) = \gamma_r(G)$, a contradiction. Hence, $\delta(G) \leq 3$. We proceed further by proving two claims.

Claim 15. $\gamma(G) = 2$.

Proof. By our previous discussion, $\gamma(G) \in \{2, 3\}$. Suppose, to the contrary, that $\gamma(G) = 3$. Since G is self-complementary, Theorem 7 implies that $\text{diam}(G) = 2$. It follows from Theorem 6 that $3 = \gamma(G) \leq \delta(G) \leq 3$, and so, $\delta(G) = 3$. Let $S = \{u, v, w\}$ be a γ-set of G. Since $\gamma_r(G) \neq \gamma(G)$, there exists some $x \in V(G) \setminus S$ such that $N_G(x) \subseteq S$. Since $\delta(G) = 3$, it follows that $d_G(x) = 3$ and $N_G(x) = S$. If there is an isolated vertex, say u, in $G[S]$, then $\{x, u\}$ is a dominating set of $\overline{G}$, and so, $\gamma(\overline{G}) \leq 2 < 3 = \gamma(G)$, a contradiction. Hence, there is no isolated vertex in $G[S]$. In other words, $G[S]$ is the path P_3 or the complete graph K_3. Since S is an arbitrary γ-set of G, every γ-set of G induces a path P_3 or a complete graph K_3. This in turn implies that if D is an arbitrary γ-set of G, then every vertex in D has a D-external private neighbor.

We now consider the γ-set $S = \{u, v, w\}$ of G. Let u', v' and w' be S-external private neighbors of u, v, and w, respectively. We note that $x \notin \{u', v', w'\}$ and $S' = \{x, u', w'\}$ dominates $\overline{G}$. Since G is self-complementary, $\gamma_r(G) \neq \gamma(G)$ and $\delta(G) = 3$, there exists some vertex $y \in V(\overline{G}) \setminus S'$ such that $N_{\overline{G}}(y) = S'$. But then $N_G(y) = V(G) \setminus S'$. We note that $y \neq x$ and $y \notin S$.

If u' and w' are adjacent in G, then $S' = \{y, u', v\}$ is a γ-set of G. However, the vertex u' is an isolated vertex in $G[S']$, a contradiction. Thus, u' and w' are not adjacent in G. Since $\text{diam}(G) = 2$, u' and w' have a common neighbor, say a, in G. We note that $a \notin S$ and $a \notin \{x, y\}$. Thus, $S'' = \{x, y, a\}$ is a γ-set of G. However, the vertex x is an isolated vertex in $G[S'']$, a contradiction. □

By Claim 15, $\gamma(G) = 2$, and so, $\gamma_r(G) = 3$. Let S be an arbitrary γ-set of G, and so $|S| = 2$. As observed earlier, $\delta(G) \leq 3$. If $\delta(G) = 3$, then every vertex in $V(G) \setminus S$ has a neighbor in both S and $V(G) \setminus S$, implying that S is a RD-set with cardinality $\gamma(G)$, a contradiction. Hence, $\delta(G) \in \{1, 2\}$.

Claim 16. *If* $\text{diam}(G) = 2$*, then* $G = C_5 \in \mathcal{F}$.

Proof. Assume that $\text{diam}(G) = 2$. We claim that every γ-set of G is independent. To see this, let $S = \{x, y\}$ be a γ-set of G, and suppose, to the contrary, that $xy \in E(G)$. Thus, x and y are not adjacent in $\overline{G}$. Since $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, the vertices x and y have a common neighbor, say z, in $\overline{G}$. But then the vertex z is not dominated by $S = \{x, y\}$ in G, contradicting the fact that S is a dominating set of G. Therefore, every γ-set of G is independent. Since G is a self-complementary, this implies that every γ-set of $\overline{G}$ is independent.

We note that since $\text{diam}(G) = 2$, the neighborhood of any vertex is a dominating set of G. Thus if $\delta(G) = 1$, then $\gamma(G) = 1$, contradicting our earlier observation that $\gamma(G) = 2$. Hence, $\delta(G) \geq 2$. As observed earlier, $\delta(G) \leq 2$. Consequently, $\delta(G) = 2$.

Let v be a vertex of degree 2 in G, and let $N_G(v) = \{v_1, v_2\}$. Since $\{v_1, v_2\}$ is a γ-set of G, the vertices v_1 and v_2 are not adjacent in G. We note that the set

$\{v, v_1\}$ is a γ-set of $\overline{G}$. Since $\{v, v_1\}$ is not a RD-set of $\overline{G}$, there exists a vertex u_2 that is adjacent only to v and v_1 in $\overline{G}$. We note that $u_2 \notin \{v, v_1, v_2\}$ and that u_2 is adjacent to every vertex of G except for v and v_1. Thus, the set $\{v, u_2\}$ is a γ-set of G. Since the vertex v_2 is the only common neighbor of v and u_2 in G and since $\{v, u_2\}$ is not a RD-set of $\overline{G}$, this implies that the vertex v_2 has degree 2 in G, and so $N_G(v_2) = \{v, u_2\}$. Analogously, the vertex v_1 has degree 2 in G. Let u_1 be the neighbor of v_1 different from v in G, and so $N_G(v_1) = \{v, u_1\}$. Since $\{v_1, v_2\}$ is a dominating set of G and since $\delta(G) = 2$, we deduce that G is the 5-cycle $vv_1u_1u_2v_2v$. Thus, $G = C_5 \in \mathcal{F}$. □

Recall that by Claim 15, we have $\gamma(G) = 2$, and so $\gamma_r(G) = 3$. By Claim 16, we may assume that $\text{diam}(G) = 3$, for otherwise the result holds. Since G is self-complementary, $\text{diam}(\overline{G}) = 3$. Let x and y be two vertices at distance 3 from each other in $\overline{G}$, and let x' and y' be two vertices at distance 3 from each other in G. We note that the vertices x and y are adjacent in G and the set $S = \{x, y\}$ is a γ-set of G, while the vertices x' and y' are adjacent in $\overline{G}$ and the set $S' = \{x', y'\}$ is a γ-set of $\overline{G}$. We note further that since x and y are adjacent vertices in G and the set $S = \{x, y\}$ is a dominating set of G, each of x and y is within distance 2 from every vertex in G. Since x' and y' are at distance 3 from each other in G, this implies that $x \notin \{x', y'\}$ and $y \notin \{x', y'\}$. Hence, the vertices x, x', y and y' are distinct vertices.

By our earlier observations, neither x' nor y' belongs to the γ-set S of G. If x' is adjacent to both x and y in G, then x or y would be a common neighbor in G of x' and y', implying that $d_G(x', y') = 2$, a contradiction. Hence, x' is adjacent to exactly one of x and y in G. Renaming vertices if necessary, we may assume that x' is adjacent to x in G, and so $x' \in \text{epn}_G(x, S)$. If y' is adjacent to x in G, then $d_G(x', y') = 2$, a contradiction. Hence, $y' \in \text{epn}_G(y, S)$. Since S is not a RD-set of G, there exists a vertex, say w, that does not belong to S and is adjacent to no vertex outside S; that is, $w \in V(G) \setminus S$ and $N_G(w) \subseteq S$.

Claim 17. *If $w \notin \{x', y'\}$, then G is the bull graph, and so $G \in \mathcal{F}$.*

Proof. Assume that $w \notin \{x', y'\}$. Thus, the vertices x, x', y, y' and w are distinct vertices. We show firstly that $N_G(w) = S$. Suppose, to the contrary, that w has degree 1 in G. Renaming vertices if necessary, we may assume that x and w are adjacent. We note that w is adjacent to every vertex in $\overline{G}$ except for the vertex x. Thus, $S^* = \{x, w\}$ is a γ-set of $\overline{G}$. Since $\gamma_r(\overline{G}) = 3$, the set S^* is not a RD-set of $\overline{G}$, implying that there exists a vertex $z \notin S^*$ that is only adjacent in $\overline{G}$ to x or w or to both x and w; that is, $N_{\overline{G}}(z) \subseteq S^*$. Since the vertices x' and y' are adjacent in $\overline{G}$, we note that $z \notin S' = \{x', y'\}$. The vertex z is therefore not dominated by S' in $\overline{G}$, contradicting the fact that S' is a γ-set of $\overline{G}$. Hence, $N_G(w) = S$.

Let $F = G[\{x', x, y, y', w\}]$ be the subgraph of G induced by the set $\{x', x, y, y', w\}$. By our earlier observations, the graph F is a bull graph, where $x'xyy'$ is an induced path in G and where $N_G(w) = \{x, y\}$. Further, recall that the set $S = \{x, y\}$ is a γ-set of G and the set $S' = \{x', y'\}$ is a γ-set of $\overline{G}$. Since G is a self-complementary, the graph $\overline{G}$ therefore contains a bull graph $H = \overline{G}[\{a', a, b, b', c\}]$ as an induced subgraph, where $a'abb'$ is an induced path in $\overline{G}$, $N_{\overline{G}}(c) = \{a, b\}$, the set $\{a, b\}$ is a γ-set of $\overline{G}$, and the set $\{a', b'\}$ is a γ-set of G.

We note that both vertices x' and y' have degree at least 3 in $\overline{G}$, while the vertex c has degree 2 in $\overline{G}$. Thus, $c \notin \{x', y'\}$. Since $S' = \{x', y'\}$ is a γ-set of $\overline{G}$ and the vertex c is only adjacent to a and b in $\overline{G}$, we note that $S' \cap \{a, b\} \neq \emptyset$. Analogous arguments show that $S \cap \{a', b'\} \neq \emptyset$. Renaming vertices if necessary, we may assume that $a = x'$. Since $\{a, b\}$ is a dominating set in $\overline{G}$ and the vertices a and b are adjacent in $\overline{G}$, this implies that $b \notin \{x, w, y\}$.

We show that $c = w$. Suppose, to the contrary, that $c \neq w$. Recall that $a = x'$. If $c \notin \{x, y\}$, then since the vertex w is adjacent in $\overline{G}$ to every vertex in $V(\overline{G}) \setminus \{x, y\}$ this would imply that the vertices c and w are adjacent in $\overline{G}$. By our earlier observations, w is distinct from a and b. Therefore, the vertex c has at least three neighbors in $\overline{G}$, namely a, b and w, and so c has degree at least 3 in $\overline{G}$, a contradiction. Hence, $c \in \{x, y\}$. Since x and x' are not adjacent in $\overline{G}$ and a and c are adjacent in $\overline{G}$, we note that $c \neq x$. Therefore, $c = y$. In particular, we note that $y \notin \{a', b'\}$. As observed earlier, $\{x, y\} \cap \{a', b'\} \neq \emptyset$. Therefore, we must have $x \in \{a', b'\}$. Since x and x' are not adjacent in $\overline{G}$, and since a and a' are adjacent in $\overline{G}$, we note that $x \neq a'$. Hence, $x = b'$. Thus the vertex b is a common neighbor of x and y in $\overline{G}$, and therefore the vertex b is not dominated by $\{x, y\}$ in G, contradicting the fact that $\{x, y\}$ is a dominating set of G. We deduce, therefore, that $c = w$.

Recall that $d_G(w) = 2$ and $d_{\overline{G}}(c) = 2$. Since $c = w$, we note that the vertex w has degree 2 in both G and $\overline{G}$. This implies that $F = G$; that is, G is the bull graph. □

By Claim 17, we may assume that $w \in \{x', y'\}$, for otherwise $G \in \mathcal{F}$ as desired. Renaming vertices if necessary, we may assume that $w = x'$, implying that $d_G(x') = 1$. By Theorem 10, the self-complementary G therefore has exactly two vertices of degree 1. Let $T = V(G) \setminus \{x', x, y', y\}$. If $T = \emptyset$, then $G = P_4$ and $\gamma_r(G) = \gamma(G) = 2$, a contradiction. Hence, $T \neq \emptyset$. If some vertex t in T is adjacent in G only to x or y or to both x and y, then as shown in Claim 17, G is the bull graph, and so $G \in \mathcal{F}$ as desired. Hence, we may assume that every vertex in T is adjacent in G to at least one vertex in $V(G) \setminus S$, implying that every vertex in T has degree at least 2 in G. This implies that the only possible vertex of degree 1 different from x' is the vertex y'. As observed earlier, the graph G has exactly two vertices of degree 1. Hence, x' and y' are the two vertices of degree 1 in G.

Since G is self-complementary of order n, the graph $\overline{G}$ contains exactly two vertices of degree 1 and therefore the graph G contains exactly two vertices of degree $n-2$. No vertex in T is adjacent in G to x' or y', implying that every vertex of T has degree at most $n-3$ in G. Hence the two vertices of degree $n-2$ in G are necessarily the vertices x and y. Therefore, every vertex of T is adjacent to the two support vertices of the P_4 induced by $\{y', y, x, x'\}$. Since G is self-complementary, it follows that $G[T]$ is also self-complementary, and so, $G \in \mathcal{F}$, as desired. This completes the proof of Theorem 4. ■

5. Closing Remark

As remarked in Section 3, there are infinitely many self-complementary graphs G for which $\gamma(G) = \gamma_r(G)$. In this paper, we characterize the self-complementary graphs G having $\gamma(G) = \gamma_r(G)$, and show that such graphs are precisely the class of self-complementary graphs that do not belong to the family $\mathcal{F}$ constructed in Section 2.

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