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NEIGHBOR PRODUCT DISTINGUISHING TOTAL COLORINGS OF PLANAR GRAPHS WITH MAXIMUM DEGREE AT LEAST TEN [1]

Aijun Dong

School of Data and Computer Science
Shandong Women's University
Jinan, 250300, P.R. China

e-mail: dongaijun@mail.sdu.edu.cn

and

Tong Li

School of Mathematics
Shandong University
Jinan 250100, P.R. China

e-mail: tli@sdu.edu.cn

Abstract

A proper $[k]$-total coloring c of a graph G is a proper total coloring c of G using colors of the set $[k] = \{1, 2, \ldots, k\}$. Let $p(u)$ denote the product of the color on a vertex u and colors on all the edges incident with u. For each edge $uv \in E(G)$, if $p(u) \neq p(v)$, then we say the coloring c distinguishes adjacent vertices by product and call it a neighbor product distinguishing k-total coloring of G. By $\chi''_{\prod}(G)$, we denote the smallest value of k in such a coloring of G. It has been conjectured by Li *et al.* that $\Delta(G) + 3$ colors enable the existence of a neighbor product distinguishing total coloring. In this paper, by applying the Combinatorial Nullstellensatz, we obtain that the conjecture holds for planar graph with $\Delta(G) \geq 10$. Moreover, for planar graph G with $\Delta(G) \geq 11$, it is neighbor product distinguishing $(\Delta(G) + 2)$-total colorable, and the upper bound $\Delta(G) + 2$ is tight.

Keywords: total coloring, neighbor product distinguishing coloring, planar graph.

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1. Introduction

The terminology and notation used but undefined in this paper can be found in [2]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. Let $d_G(v)$ or simply $d(v)$ denote the degree of a vertex v in G. If $d(x) = k$, $d(x) \geq k$ and $d(x) \leq k$, then the vertex x is called a *k-vertex*, *k^+-vertex* and *k^--vertex*, respectively. Let $N_G(u)$ be the set of neighbors of u in the graph G. We use $n_i(u)$ to denote the number of i-neighbors of u.

Let $[k]$ be a set of colors where $[k] = \{1, 2, \ldots, k\}$ and let c be a total coloring of G for which $c : E(G) \cup V(G) \to [k]$. By $p(v)$ (respectively, $s(v)$), we denote the product (respectively, set) of colors taken on the edges incident to v and the color on the vertex v, i.e., $p(v) = c(v)\prod_{uv\in E(G)} c(uv)$ (respectively, $s(v) = \{c(uv)|uv \in E(G)\} \cup \{c(v)\}$). If the coloring c is proper, then we call the coloring c such that $p(v) \neq p(u)$ (respectively, $s(u) \neq s(v)$) for each edge $uv \in E(G)$ a *neighbor product distinguishing $[k]$-total coloring* (respectively, *adjacent vertex distinguishing $[k]$-total coloring*) of G, or a *tnpd-k*-coloring (respectively, *tndi-k*-coloring) for simplicity. By $\chi''_{\prod}(G)$ (respectively, $(tndi(G))$), we denote the smallest value k such that G has a neighbor product (respectively, vertex) distinguishing $[k]$-total coloring of G. It is easy to observe that if two vertices are distinguished by product, then they are also distinguished by sets, but not necessarily conversely. That is to say $tndi(G) \leq \chi''_{\prod}(G)$.

In 2005, Zhang *et al.* introduced the notion of adjacent vertex distinguishing k-total coloring and brought forward the following conjecture.

Conjecture 1.1 [25]**.** *Let G be a connected graph with at least two vertices, then $tndi(G) \leq \Delta(G) + 3$.*

Zhang *et al.* proved the conjecture for graphs which are cliques, paths, cycles, fans, wheels, stars, complete graphs, bipartite complete graphs and trees. Wang and Chen confirmed the conjecture for graphs with $\Delta(G) = 3$ [3, 18]. Recently, Lu *et al.* verified the conjecture for all graphs with maximum degree 4 [13]. Wang proved that if G is 1-tree, then $tndi(G) \leq \Delta(G)+2$ [19]. Wang *et al.* investigated some planar graphs such as outerplanars and series-parallel graphs and confirmed the conjecture [20, 21]. In 2008, Wang *et al.* showed that if G is a graph with $mad(G) < 3$, then $tndi(G) \leq \Delta(G) + 2$ [22]. In 2012, Huang *et al.* proved that if G is a planar graph with $\Delta(G) \geq 11$, then $tndi(G) \leq \Delta(G) + 3$ [9]. Recently, Cheng *et al.* verified the conjecture for planar graphs with $\Delta(G) \geq 10$ [4]. In 2014, Wang *et al.* obtained that if G is a planar graph with $\Delta(G) \geq 14$, then $\Delta(G) + 1 \leq tndi(G) \leq \Delta(G) + 2$ [23]. Recently, Sun *et al.* confirmed the Conjecture 1.1 for the planar graph with $\Delta(G) \geq 8$ and without adjacent 4-cycles [17]. More related results can be seen in [6–8, 11, 12, 14–16, 24].

Recently, Li *et al.* completely determined the neighbor product distinguishing total coloring index for complete graphs, trees, cycles, bipartite graphs, subcubic graphs and K_4-minor free graphs. Based on these examples, they proposed the following conjecture.

Conjecture 1.2 [10]. *If G is a graph with at least two vertices, then $\chi''_{\prod}(G) \leq \Delta(G) + 3$.*

As for the sparse graph G with $\Delta(G) \leq 3$, Li *et al.* proved that $\chi''_{\prod}(G) = 5$ if G is an odd cycle, $\chi''_{\prod}(G) = 4$ if G is an even cycle and $\chi''_{\prod}(G) \leq \Delta(G)+3$ if G is a subcubic graph [10]. In 2017, Ding *et al.* confirmed the Conjecture 1.2 for sparse graph G with bounded maximum degree [6]. In this paper, we consider the planar graph G with $\Delta(G) \geq 10$ and obtain the following result.

Theorem 1.3. *Let G be a planar graph such that $\Delta(G) \geq 10$. Then $\chi''_{\prod}(G) \leq \Delta(G) + 3$.*

For $\Delta(G) \geq 11$, we prove the following tight upper bound.

Theorem 1.4. *If G is a planar graph G with $\Delta(G) \geq 11$, then $\chi''_{\prod}(G) \leq \Delta(G) + 2$.*

Since $tndi(G) \leq \chi''_{\prod}(G)$, Theorem 1.4 implies the following result in [23].

Theorem 1.5. *If G is a planar graph G with $\Delta(G) \geq 11$, then $tndi(G) \leq \Delta(G) + 2$.*

2. Some Important Lemmas

Lemma 2.1 [1]. *Let L_i be the set of real numbers, where $|L_i| = l_i$ for $1 \leq i \leq t$, and $l_1 \geq l_2 \geq \cdots \geq l_t$. Let $L = \left\{\sum_{i=1}^{t} x_i \,|\, x_i \in L_i, \prod_{1\leq i<j\leq t}(x_i - x_j) \neq 0\right\}$. Define $l'_1, l'_2, \ldots, l'_t$ by $l'_1 = l_1$ and $l'_i = \min\left\{l'_{i-1} - 1, l_i\right\}$ for $2 \leq i \leq t$. If $l'_t > 0$, then $|L| \geq \sum_{i=1}^{t} l'_i - \frac{1}{2}t(t+1) + 1$.*

From Lemma 2.1, it is easy to get the following lemma.

Lemma 2.2. *Let S_i be the set of positive real numbers, where $|S_i| = s_i$ for $1 \leq i \leq t$, and $s_1 \geq s_2 \geq \cdots \geq s_t$. Let $S = \left\{\prod_{i=1}^{t} x_i \,|\, x_i \in S_i, \prod_{1\leq i<j\leq t}(x_i - x_j) \neq 0\right\}$. Define $s'_1, s'_2, \ldots, s'_t$ by $s'_1 = s_1$ and $s'_i = \min\left\{s'_{i-1} - 1, s_i\right\}$ for $2 \leq i \leq t$. If $s'_t > 0$, then $|S| \geq \sum_{i=1}^{t} s'_i - \frac{1}{2}t(t+1) + 1$.*

Proof. For convenience, let $S_i = \left\{x_{i_1}, x_{i_2}, \ldots, x_{i_{s_i}}\right\}$, $L_i = \left\{\ln x_{i_1}, \ln x_{i_2}, \ldots, \ln x_{i_{s_i}}\right\}$ for $1 \leq i \leq t$. Let $L = \left\{\sum_{i=1}^{t} \ln x_i \mid \ln x_i \in L_i, \prod_{1\leq i<j\leq t}(\ln x_i - \ln x_j) \neq 0\right\}$. From Lemma 2.1, we have $|L| \geq \sum_{i=1}^{t} s'_i - \frac{1}{2}t(t+1) + 1$. Clearly, $|S| = |L|$. Thus we have $|S| \geq \sum_{i=1}^{t} s'_i - \frac{1}{2}t(t+1) + 1$. ■

Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables. By $c_P\left(x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}\right)$, we denote the coefficient of the monomial $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in the expansion of $P(x_1, x_2, \ldots, x_n)$, where k_i is a non-negative integer for $1 \le i \le n$.

Lemma 2.3 (Combinatorial Nullstellensatz [1])**.** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose the degree $deg(P)$ of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a nonnegative integer, and suppose the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in P is non-zero. If $S_1, S_2, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$ for $1 \le i \le n$, then there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.*

In the following, we will prove the main theorems. For convenience, for the coloring c of G, we use $p_c(v)$ to denote the product of the color on the vertex v and the colors taken on edges which are incident with v, i.e., $p_c(v) = \prod_{v \in e} c(e)c(v)$. We use $S_c(x)$ to denote the set of colors available for each element $x \in E(G) \cup V(G)$ in the coloring c. In addition, the following configurations in Figure 1 will be used in the proof of the theorems.

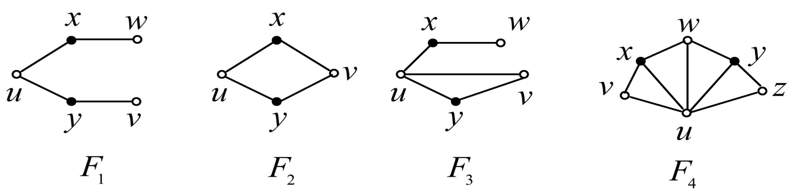

Figure 1

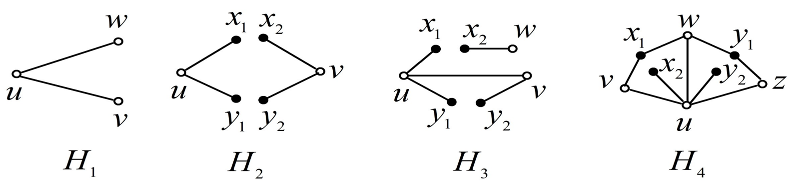

Figure 2

3. PROOF OF THEOREM 1.3

For any graph G, let $n_i(G) = |\{v \,|\, d_G(v) = i\}|$ for $i = 1, 2, \ldots, \Delta(G)$. A graph G' is smaller than the graph G if any of the following is true.

- $|E(G')| < |E(G)|$;
- $|E(G')| = |E(G)|$ and $(n_t(G'), n_{t-1}(G'), \ldots, n_2(G'), n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \ldots, n_2(G), n_1(G))$ with respect to the lexicographic order, where $t = \max\{\Delta(G), \Delta(G')\}$.

A graph G is minimal for the front property when no smaller graph satisfies it.

Suppose G is a minimal counterexample to Theorem 1.3. That is, the graph G does not admit any tnpd-k-coloring, and its smaller graph G' constructed from G by deleting edge, contracting edge or splitting vertex which is shown in the following discussion admits a tnpd-k-coloring c'.

Let H be the graph obtained by removing all the 2^--vertices of G. In the following, we will discuss the structural property of G and H by extending the coloring c' of G' to the desired coloring c of G. And then apply the discharging method to obtain a contradiction to the planarity of graph G.

For each $v \in V(G)$ and each coloring c of G, if $d(v) \leq 4$, then it has at most 12 forbidden colors since v has at most four adjacent vertices, four incident edges, and we have to guarantee that $p_c(v) \neq p_c(u)$ for each edge $uv \in E(G)$. Since $k \geq 13$, we can recolor v if necessary to get a coloring as desired. So in the following discussion, we will omit the coloring of all 4^--vertices.

For convenience, a 4-face f is good if it is incident with at most one 5^--vertex, otherwise, f is bad. A k-vertex v is called a *bad k-neighbor* of u if the edge uv is incident with two 3-faces. And v is called a *special k-neighbor* of u if the edge uv is incident with a 3-face and a bad 4-face. We use $n^H_{kb}(u)$ and $n^H_{3s}(u)$ to denote the number of bad k-neighbors and special 3-neighbors of u in H, respectively.

Now, we give some structural properties of G.

Property 1. *Every 6^--vertex is not adjacent to any 4^--vertex.*

Proof. Suppose to the contrary that there exists a 6^--vertex u which is adjacent to a 4^--vertex v. We consider the smaller graph $G' = G - uv$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the color of u. Let S_1, S_2 be the sets of available colors for u, uv, respectively. It is easy to know that $|S_1| \geq 13 - 5 - 5 = 3$, $|S_2| \geq 13 - 5 - 3 = 5$. Let $B = \{x_1x_2 | x_1 \in S_1, x_2 \in S_2, x_1 \neq x_2\}$. By Lemma 2.2, we have $|B| \geq 3 + 5 - 3 + 1 = 6 > 5$. Thus there exist $x_1 \in S_1$, $x_2 \in S_2$ for u and uv such that u does not conflict with any adjacent vertex. Then we can color u and uv with x_1 and x_2, respectively, to get a tnpd-k-coloring, a contradiction. ■

Property 2. *For any vertex $u \in V(G)$, we have $n_2(u) \leq 1$.*

Proof. Suppose to the contrary that $n_2(u) \geq 2$, and let x, y be two 2-neighbors of u. It is clear that x is not adjacent to y by Proposition 1. By analyzing whether the multiple edges appear or not when contracting the edges ux and uy, G must contain one of configurations F_1, F_2 and F_3. We divide the proof into the following three cases:

Case 1. There is a structure isomorphic to the configuration F_1 in G, i.e., x and y are not incident with any 3-face, and u, x and y are not incident with one and the same 4-face. Now we contract the edges ux and uy to get a smaller graph G' (see H_1 in Figure 2). By the minimality of G, we have G' admits a tnpd-k-coloring c'. For convenience, let $c'(uw) = a$ and $c'(uv) = b$. In the following, we subdivide uw, uv with x and y respectively. By coloring ux, yv with b and uy, xw with a, we get a tnpd-k-coloring of G, a contradiction.

Case 2. There is a structure isomorphic to the configuration F_2 in G, i.e., u, x and y are incident with one and the same 4-face. We split x and y into x_1, x_2 and y_1, y_2, respectively, to obtain a smaller graph G' (see H_2 in Figure 2). By the minimality of G, we have G' admits a tnpd-k-coloring c'. For convenience, let $c'(ux_1) = a$, $c'(uy_1) = b$, $c'(x_2v) = c$ and $c'(y_2v) = d$. In the following, we can stick x_1 and x_2 together, and stick y_1 and y_2 together (if necessary exchange the colors of ux_1 and uy_1 to guarantee a proper total coloring) to get a tnpd-k-coloring of G, a contradiction.

Case 3. There is a structure isomorphic to the configuration F_3 in G, i.e., at least one 2-neighbor of u is incident with some 3-face. We split x and y into x_1, x_2 and y_1, y_2, respectively, to obtain a smaller graph G' (see H_3 in Figure 2). By the minimality of G, we have G' admits a tnpd-k-coloring c'. For convenience, let $c'(ux_1) = a$, $c'(uv) = b$, $c'(uy_1) = c$, $c'(x_2w) = d$ and $c'(y_2v) = e$.

If $e = d \notin \{a, c\}$ or $d \neq e$, then we can stick x_1 and x_2, y_1 and y_2 together (if necessary exchange the colors of ux_1 and uy_1 to guarantee a proper total coloring) to get a tnpd-k-coloring of G, a contradiction.

If $d = e = a$, then we recolor ux_1 with b, recolor uv with a and recolor vy_2 with b. Now we can stick x_1 and x_2, y_1 and y_2 together, a contradiction.

If $d = e = c$, then we recolor ux_1 with b, recolor uv with c, recolor vy_2 with b and recolor uy_1 with a. Now we can stick x_1 and x_2, y_1 and y_2 together, a contradiction. ■

Property 3. *For any vertex $u \in V(G)$, let x, y be bad 3-neighbors of u, then any 3-face which is incident with x is not adjacent to any 3-face which is incident with y.*

Proof. By the contrary, G contains a structure isomorphic to the configuration F_4. We split the bad 3-neighbors x and y of u into x_1, x_2, y_1 and y_2, respectively to get a smaller graph G' (see H_4 in Figure 2). By the minimality of G, we have G' admits a tnpd-k-coloring c'. For convenience, let $c'(vx_1) = a$, $c'(wx_1) = b$, $c'(uv) = c$, $c'(uz) = d$, $c'(uw) = e$, $c'(uy_2) = f$, $c'(wy_1) = g$, $c'(zy_1) = h$ and $c'(ux_2) = i$. Next, we try to stick x_1, y_1 with x_2, y_2 together, respectively. If x_1, y_1 can be stuck with x_2, y_2 to get a proper total coloring, then we get a tnpd-k-coloring of G, a contradiction. Otherwise, we consider the following cases.

Case 1. If only one pair of the vertices x_i and y_i for $i = 1, 2$ cannot be stuck properly, without loss of generality, we say x_2 cannot be stuck properly with x_1.

Then $i \in \{a, b\}$. Without loss of generality, let $i = a$.

If $a \notin \{g, h\}$, then $f = b$ (otherwise, we exchange the colors of ux_2 and uy_2 so that we can stick y_1, y_2 and x_1, x_2 together to get a tnpd-k-coloring of G, a contradiction). We exchange the colors of ux_2 and uy_2, and meanwhile exchange the colors of wx_1 and wy_1. Now, we can stick x_1, x_2 with y_1, y_2 together, respectively, to get a tnpd-k-coloring of G, a contradiction.

If $g = a$, then we consider the following subcases.

- If $h \neq e$, then exchange the colors ux_2 and uw, recolor wy_1 with e. Now we can stick x_1, x_2 with y_1, y_2 together, respectively, a contradiction.
- If $h = e$ and $d \neq b$, then first, exchange the colors of zu and zy_1, and meanwhile exchange the colors of wu and wy_1. Then recolor ux_2 with d. Now we can stick x_1, x_2 with y_1, y_2 together, respectively, a contradiction.
- If $h = e$ and $d = b$, then first, exchange the colors of vu and vx_1. And then recolor ux_2 with f, recolor uy_2 with c. Now we can stick x_1, x_2 with y_1, y_2 together, respectively, a contradiction.

If $h = a$, then we consider the following subcases.

- If $g \neq d$ and $d \neq b$, then exchange the colors ux_2 and uz, recolor zy_1 with d. Now we can stick y_1, y_2 and x_1, x_2 together properly, a contradiction.
- If $g \neq d$ and $d = b$, then first, exchange the colors of zu and zy_1, and meanwhile exchange the colors of wu and wx_1. Then recolor ux_2 with f and recolor uy_2 with e. Now we can stick x_1, x_2 and y_1, y_2 together properly, a contradiction.
- If $g = d$, then first, exchange the colors of zu and zy_1 and meanwhile exchange the colors of wy_1 and wu. And then recolor ux_2 with e. Now we can stick x_1, x_2 and y_1, y_2 together properly, a contradiction.

Case 2. x_2 and y_2 cannot be properly stuck with x_1 and y_1, respectively. Without loss of generality, let $c'(ux_2) = g = a$ and $h = f = b$. Now we exchange the colors of uv and vx_1, and exchange the colors of uz and y_1z. Next, recolor ux_2 with d and recolor uy_2 with c. Now, we can stick x_1, x_2 and y_1, y_2 together properly, a contradiction. ■

Property 4. *Every* 5^-*-vertex is not adjacent to any* 5^-*-vertex.*

Proof. Suppose to the contrary that there exists a 5^--vertex u which is adjacent to a 5^--vertex v. Without loss of generality, we assume $d_G(u) = d_G(v) = 5$, and let $N_G(u) = \{u_1, u_2, u_3, u_4, v\}$, and $N_G(v) = \{v_1, v_2, v_3, v_4, u\}$. We consider the smaller graph $G' = G - uv$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the colors of u and v. For convenience, we use φ to denote the current coloring of G'. Let S_1, S_2 and S_3 be the sets of available colors for u, uv and v, respectively. It is easy to know that $|S_i| \geq 5$ for $1 \leq i \leq 3$. We associate u, uv, v with the variables x_1, x_2 and x_3, respectively, and let $\ln x_i = y_i$ for $1 \le i \le 3$. For convenience, let $S'_i = \{y_i | \ln x_i = y_i, x_i \in S_i\}$ for $1 \le i \le 3$. Obviously, $|S'_i| \ge 5$ for $1 \le i \le 3$. Now we consider the following polynomial.

$$\begin{aligned} P(y_1, y_2, y_3) = & \prod_{1 \le k < l \le 3} (y_k - y_l) \prod_{i=1}^{4} \big(y_1 + y_2 + \ln P_\varphi(u) - \ln P_\varphi(u_i)\big) \\ & \prod_{j=1}^{4} \big(y_2 + y_3 + \ln P_\varphi(v) - \ln P_\varphi(v_j)\big) \\ & \big(y_1 + y_2 + \ln P_\varphi(u) - (y_2 + y_3 + \ln P_\varphi(v))\big), \end{aligned}$$

where $P_\varphi(x)$ denotes the product of colors which are used for x and the elements which are incident with x in G' in the coloring φ. It is not difficult to obtain that $c_P\big(y_1^4 y_2^4 y_3^4\big) = 20 \ne 0$ by MATLAB. Since $deg(P) = 12 = 4+4+4$, by Lemma 2.3, there is $s_i \in S'_i$ for $1 \le i \le 3$ such that $P(s_1, s_2, s_3) \ne 0$. Finally, from the above discussion, we can color u, uv and v with e^{s_1}, e^{s_2} and e^{s_3}, respectively, to obtain a tnpd-k-coloring of G, a contradiction. ■

Note that the coefficient of the monomial $y_1^4 y_2^4 y_3^4$ in the expansion of $P(y_1 y_2 y_3)$ is equal to that of the same monomial in the polynomial $\prod_{1 \le k < l \le 3}(y_k - y_l)(y_1 + y_2)^4(y_2 + y_3)^4(y_1 - y_3)$ in Property 4. Thus in the following proofs when discussing the coefficient of some monomial in the expansion of the polynomial, if its degree is equal to the degree of the polynomial, we will omit the constant term in the polynomial.

Property 5. *There exists no $(5^-, 6^-, 6^-)$-cycle.*

Proof. Suppose to the contrary that there exists a $(5^-, 6^-, 6^-)$-cycle uvw. Without loss of generality, we assume $d_G(u) = 5$, $d_G(v) = d_G(w) = 6$. We consider the smaller graph $G' = G - uv - uw - vw$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the colors of u, v and w. Let S_1, S_2, S_3, S_4, S_5 and S_6 be the sets of available colors for v, w, u, vw, vu and uw, respectively. It is easy to know that $|S_1| \ge 13 - 8 = 5 > 3$, $|S_2| \ge 13 - 8 = 5 > 4$, $|S_3| \ge 13 - 6 = 7 > 6$, $|S_4| \ge 13 - 8 = 5 > 4$, $|S_5| \ge 13 - 7 = 6 > 5$ and $|S_6| \ge 13 - 7 = 6 > 4$. We associate v, w, u, vw, vu and uw with the variables $x_1, x_2, \ldots, x_6$, respectively, and let $\ln x_i = y_i$ for $1 \le i \le 6$. For convenience, let $S'_i = \{y_i \mid \ln x_i = y_i, x_i \in S_i\}$ for $1 \le i \le 6$. Obviously, $|S'_1| \ge 5 > 3$, $|S'_2| \ge 5 > 4$, $|S'_3| \ge 7 > 6$, $|S'_4| \ge 5 > 4$, $|S'_5| \ge 6 > 5$ and $|S'_6| \ge 6 > 4$. Now we consider the following polynomial.

$$\begin{aligned} P(y_1, y_2, \ldots, y_6) = & \prod_{1 \le k < l \le 3} (y_k - y_l) \prod_{4 \le i < j \le 6} (y_i - y_j)(y_1 - y_4)(y_2 - y_4) \\ & (y_1 - y_5)(y_3 - y_5)(y_2 - y_6)(y_3 - y_6) \end{aligned}$$

$$(y_1 + y_5 - y_2 - y_6)(y_2 + y_4 - y_3 - y_5)$$
$$(y_1 + y_4 - y_3 - y_6)(y_2 + y_4 + y_6)^4(y_3 + y_5 + y_6)^3$$
$$(y_1 + y_4 + y_5)^4.$$

It is not difficult to obtain that $c_P\left(y_1^3y_2^4y_3^6y_4^4y_5^5y_6^4\right) = 346 \neq 0$ by MATLAB. Since $deg(P) = 26 = 3 + 4 + 6 + 4 + 5 + 4$, by Lemma 2.3, there is $s_i \in S_i'$ for $1 \leq i \leq 6$ such that $P(s_1, s_2, \ldots, s_6) \neq 0$. Finally, we can color v, w, u, vw, vu and uw with $e^{s_1}, e^{s_2}, \ldots, e^{s_6}$, respectively, to obtain a tnpd-k-coloring of G, a contradiction. ■

Property 6. *For any non-zero integer t, if each $(t+1)$-vertex in G can be recolored, then for any vertex $u \in V(G)$ with $n_1(u) \geq t$, we have $n_d(u) = 0$ where $2 \leq d \leq t+1$.*

Proof. Suppose to the contrary that there exists a vertex u with $n_1(u) \geq t$ and $n_d(u) \neq 0$. Let $v_1, v_2, \ldots, v_t$ be some 1-neighbors of u, and v_0 be a d-neighbor of u where $2 \leq d \leq t+1$. Since v_0 can be recolored, then we split the vertex v_0 into v_{00} and v_{01} to obtain a smaller graph G' where $d_{G'}(v_{00}) = 1$ and $d_{G'}(v_{01}) = d-1$. By the minimality of G, G' has a tnpd-k-coloring. Now, we stick v_{00} and v_{01} together properly (if necessary, we can exchange the color of uv_{00} with some uv_i for $1 \leq i \leq t$) to obtain a tnpd-k-coloring of G, a contradiction. ■

In the following, we give some structural properties of H.

Fact 1. *For each $u \in V(H)$, if $d_H(u) \leq 5$, then $d_H(u) = d_G(u)$.*

Proof. Suppose to the contrary that there exists a vertex $u \in V(H)$ such that $d_H(u) \leq 5$ and $n_{2^-}^G(u) \geq 1$. Without loss of generality, we assume that $d_H(u) = 5$.

First, we assume that $n_1^G(u) \geq 1$. By Property 6, we have $n_2^G(u) = 0$. Clearly, $n_1^G(u) = d_G(u) - 5$. If $n_1^G(u) = 1$, then $d_G(u) = 6$, a contradiction by Property 1. So we have $n_1^G(u) \geq 2$. Let $uu_1, uu_2, \ldots, uu_{d-5}$ be the 1-neighbors of u where $d = d_G(u)$. Now, we consider the smaller graph $G' = G - \{uu_1, uu_2, \ldots, uu_{d-5}\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Let $S_1, S_2, \ldots, S_{d-5}$ be the sets of available colors for $uu_1, uu_2, \ldots, uu_{d-5}$, respectively. It is easy to know that $|S_i| \geq (\Delta(G) + 2) - 6 = \Delta(G) - 4$ for $1 \leq i \leq d-5$. Let $B = \Big\{x_1 x_2 \cdots x_{d-5} \,|\, x_k \in S_k, 1 \leq k \leq d-5, \prod_{1 \leq i < j \leq d-5}(x_i - x_j) \neq 0\Big\}$. By Lemma 2.2, we have $|B| \geq (\Delta(G)-4) + (\Delta(G)-5) + \cdots + (\Delta(G)+2-d) - \frac{1}{2}(d-5)(d-4) + 1 = \frac{1}{2}(2\Delta(G) - d - 2)(d-5) - \frac{1}{2}(d-5)(d-4) + 1 = \frac{1}{2}(d-5)(2\Delta(G) - 2d + 2) + 1 = (d-5)(\Delta(G) - d + 1) + 1$.

Clearly, if $d = 7$, 8, 9 and 10, since $\Delta(G) \geq 11$, we have $|B| \geq (7-5)(11-7+1)+1 = 11 > 5$, $|B| \geq (8-5)(11-8+1)+1 = 13 > 5$, $|B| \geq (9-5)(11-9+1)+1 = 13 > 5$ and $|B| \geq (10-5)(11-10+1)+1 = 11 > 5$, respectively. In each of the above-mentioned three situations, we can choose $\alpha_i \in S_i$ to color uu_i for $1 \le i \le d-5$ to obtain a tnpd-k-coloring of G, a contradiction. If $d \ge 11$, then we have $|B| \ge (11-5)(\Delta(G)-d+1)+1 \ge 6+1 = 7 > 5$. Now, we can choose $\alpha_i \in S_i$ to color uu_i for $1 \le i \le d-5$ to obtain a tnpd-k-coloring of G, a contradiction.

Now, we assume that $n_1^G(u) = 0$. Then by Property 2, $n_2^G(u) = 1$. Thus $d_G(u) = 6$, a contradiction by Property 1. ■

By Fact 1, it is easy to obtain the following fact.

Fact 2. $\delta(H) \ge 3$.

Fact 3. *For each $u \in V(H)$ with $d_H(u) = 6$, if $d_H(u) < d_G(u)$, then u is not adjacent to any 5^--vertex in H.*

Proof. Suppose to the contrary that there exists a vertex $u \in V(H)$ with $d_H(u) = 6$ and $n_{5^-}^H(u) \ge 1$. By Fact 1, we have $n_{5^-}^G(u) \ge 1$.

First, we assume $d_G(u) = 7$. Let w be the 2^--neighbor and v be some 5^--neighbor of u, respectively. Without loss of generality, we assume $d_G(w) = 2$ and $d_G(v) = 5$. Now, we consider the smaller graph $G' = G - \{uv, uw\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. We delete the colors of u and v. Let S_1, S_2, S_3 and S_4 be the sets of available colors for u, uv, uw and v, respectively. It is easy to know that $|S_1| \ge 13-10 = 3 > 2$, $|S_2| \ge 13-5-4 = 4 > 3$, $|S_3| \ge 13-6 = 7 > 6$ and $|S_4| \ge 13-8 = 5 > 4$. We associate u, uv, uw and v with the variables x_1, x_2, x_3 and x_4, respectively, and let $\ln x_i = y_i$ for $1 \le i \le 4$. For convenience, let $S_i' = \{y_i \mid \ln x_i = y_i, x_i \in S_i\}$ for $1 \le i \le 4$. Obviously, $|S_1'| \ge 3 > 2$, $|S_2'| \ge 4 > 3$, $|S_3'| \ge 7 > 6$ and $|S_4'| \ge 5 > 4$. Now we consider the following polynomial.

$$P(y_1, y_2, y_3, y_4) = \prod_{1 \le k < j \le 3} (y_k - y_j)(y_1 - y_4)(y_2 - y_4)$$
$$(y_1 + y_3 - y_4)(y_1 + y_2 + y_3)^5 (y_2 + y_4)^4.$$

It is not difficult to obtain that $c_P\big(y_1^2 y_2^3 y_3^6 y_4^4\big) = 50 \ne 0$ by MATLAB. Since $deg(P) = 15 = 2+3+6+4$, by Lemma 2.3, there is $s_i \in S_i'$ for $1 \le i \le 4$ such that $P(s_1, s_2, s_3, s_4) \ne 0$. Finally, we can color u, uv, uw and v with e^{s_1}, $e^{s_2}, \ldots, e^{s_4}$, respectively, to obtain a tnpd-k-coloring of G, a contradiction.

Now, we assume $d_G(u) \ge 8$. Then $n_{2^-}^G(u) \ge 2$. By Property 2 and Property 6, we have $n_1^G(u) = d_G(u) - d_H(u)$. Let $uu_1, uu_2, \ldots, uu_{d-6}$ be the 1-neighbors of u where $d = d_G(u)$. Now, we consider the smaller graph $G' = G - \{uu_1, uu_2, \ldots, uu_{d-6}\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Let $S_1, S_2, \ldots, S_{d-6}$ be the sets of available colors for $uu_1, uu_2, \ldots,$ uu_{d-6}, respectively. It is easy to know that $|S_i| \ge (\Delta(G)+2) - 7 = \Delta(G) - 5$ for

$1 \le i \le d-6$. Let $B = \{x_1x_2\cdots x_{d-6} \mid x_k \in S_k, 1 \le k \le d-6, \prod_{1\le i<j\le d-6}(x_i - x_j) \ne 0\}$. By Lemma 2.2, we have $|B| \ge (\Delta(G)-5)+(\Delta(G)-6)+\cdots+(\Delta(G)+2-d) - \frac{1}{2}(d-6)(d-5)+1 = \frac{1}{2}(2\Delta(G)-d-3)(d-6) - \frac{1}{2}(d-6)(d-5)+1 = \frac{1}{2}(d-6)(2\Delta(G)-2d+2)+1 = (d-6)(\Delta(G)-d+1)+1$.

Clearly, if $d = 8$, 9, and 10, since $\Delta(G) \ge 11$, we have $|B| \ge (8-6)(11-8+1)+1 = 9 > 6$, $|B| \ge (9-6)(11-9+1)+1 = 10 > 6$, and $|B| \ge (10-6)(11-10+1)+1 = 9 > 6$, respectively. In each of the above-mentioned three situations, we can choose $\alpha_i \in S_i$ to color uu_i for $1 \le i \le d-6$ to obtain a tnpd-k-coloring of G, a contradiction.

If $d = 11$, then we consider the following subcases.

- If $\Delta(G) = 11$, then the color set which is used in the coloring is $\{1, 2, \ldots, 13\}$. Let v be some 5^--neighbor of u. Without loss of generality, we assume $d_G(v) = 5$. Clearly, in any coloring of G, $p(u) \ge 1 \times 2 \times \cdots \times 12$, $p(v) \le 13 \times 12 \times \cdots \times 8$. Thus $\frac{p(u)}{p(v)} = \frac{7!}{13} > 1$. So we have $p(u) \ne p(v)$ in any coloring. Since $|B| \ge (11-6)(11-11+1)+1 = 6 > 5$, we can choose $\alpha_i \in S_i$ to color uu_i for $1 \le i \le 5$ to obtain a tnpd-k-coloring of G, a contradiction.
- If $\Delta(G) \ge 12$, then we have $|B| \ge (11-6)(12-11+1)+1 = 11 > 6$. Now, we can choose $\alpha_i \in S_i$ to color uu_i for $1 \le i \le d-6$ to obtain a tnpd-k-coloring of G, a contradiction.

If $d \ge 12$, then we have $|B| \ge (12-6)(\Delta(G)-d+1)+1 \ge 6+1 = 7 > 6$. Now, we can choose $\alpha_i \in S_i$ to color uu_i for $1 \le i \le d-6$ to obtain a tnpd-k-coloring of G, a contradiction. ∎

Fact 4. *For each $u \in V(H)$ with $d_H(u) = 7$, if $d_H(u) < d_G(u)$, then u is not adjacent to any 4^--vertex in H.*

Proof. Suppose to the contrary that there exists a vertex $u \in V(H)$ with $d_H(u) = 7$ and $n^H_{4^-}(u) \ge 1$. By Fact 1, we have $n^G_{4^-}(u) \ge 1$.

If $d_G(u) = 8$, then let v be the 2^--neighbor and w be some 4^--neighbor of u, respectively. Now, we consider the smaller graph $G' = G - \{uv, uw\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Let S_1, S_2 be the sets of available colors for uv and uw, respectively. It is easy to know that $|S_1| \ge 13-8 = 5$, $|S_2| \ge 13-10 = 3$. Let $B = \{x_1x_2 \mid x_k \in S_k, k = 1, 2, x_1 \ne x_2\}$. By Lemma 2.2, we have $|B| \ge 5+3-3+1 = 6 > 5$. Thus there exist $x_1 \in S_1$, $x_2 \in S_2$ for uv and uw such that u does not conflict with any adjacent vertex. Now we can color uv and uw with x_1 and x_2, respectively, to get a tnpd-k-coloring, a contradiction.

If $d_G(u) = 9$, then $n^G_{2^-}(u) = 2$. By Property 2 and Property 6, we have $n^G_1(u) = 2$. Let v, w be the 1-neighbors. Now, we consider the smaller graph $G' = G - \{uv, uw\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Let S_1, S_2 be the sets of available colors for uv and uw, respectively. It is easy to know that $|S_1| \geq 13 - 8 = 5$, $|S_2| \geq 13 - 8 = 5$. Let $B = \{x_1x_2 \,|\, x_k \in S_k, k = 1, 2, x_1 \neq x_2\}$. By Lemma 2.2, we have $|B| \geq 5 + 4 - 3 + 1 = 7 > 5$. Thus there exist $x_1 \in S_1$, $x_2 \in S_2$ for uv and uw such that u does not conflict with any adjacent vertex. Now we can color uv and uw with x_1 and x_2, respectively, to get a tnpd-k-coloring, a contradiction.

If $d_G(u) \geq 10$, then $n^G_{2^-}(u) \geq 3$. By Property 2 and Property 6, we have $n^G_1(u) \geq 3$. By Property 6, we have $n^G_d(u) = 0$ for $2 \leq d \leq 4$. Thus $n^H_{4^-}(u) = 0$ by Fact 1. ■

Fact 5. *For each $u \in V(H)$ with $d_H(u) = 7$, u is adjacent to at most one 4^--vertex in H.*

Proof. Suppose to the contrary that there exists a vertex $u \in V(H)$ with $d_H(u) = 7$ and $n^H_{4^-}(u) \geq 2$. By Fact 1, we have $n^G_{4^-}(u) \geq 2$.

If $d_G(u) = d_H(u) = 7$, then let v, w be the 4^--neighbors of u. Without loss of generality, we assume that $d_G(v) = d_G(w) = 4$. Now, we consider the smaller graph $G' = G - \{uv, uw\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. We delete the color of u. Let S_1, S_2 and S_3 be the sets of available colors for u, uv and uw, respectively. It is easy to know that $|S_1| \geq 13 - 10 = 3$, $|S_2| \geq 13 - 8 = 5$ and $|S_3| \geq 13 - 8 = 5$. Let $B = \big\{x_1x_2x_3 \,|\, x_k \in S_k, 1 \leq k \leq 3, \prod_{1 \leq i < j \leq 3}(x_i - x_j) \neq 0\big\}$. By Lemma 2.2, we have $|B| \geq 3 + 4 + 5 - 6 + 1 = 7 > 5$. Thus there exist $x_1 \in S_1$, $x_2 \in S_2$ and $x_3 \in S_3$ for u, uv and uw such that u does not conflict with any adjacent vertex. Now we can color u, uv and uw with x_1, x_2 and x_3, respectively, to get a tnpd-k-coloring, a contradiction.

If $d_G(u) \geq 8$, then $n_{4^-}(u) = 0$ by Fact 4. ■

For convenience, for each $u \in V(H)$ with $d_H(u) = 7$, if $n^H_3(u) = 1$, then we call it a *bad* 7-vertex. Otherwise, it is called a *good* 7-vertex.

Fact 6. *There is no $(5, 6, 7)$-cycle or $(5, 7, 7)$-cycle such that the 7-vertices are bad 7-vertices in H.*

Proof. Without loss of generality, suppose to the contrary that there exists a (u, v, w)-cycle such that $d_H(u) = d_H(w) = 7$, $d_H(v) = 5$ and u, w are bad 7-vertices, or $d_H(u) = 7$, $d_H(w) = 6$, $d_H(v) = 5$ and u is a bad 7-vertex. By Fact 1, Fact 3 and Fact 4, we have $d_H(x) = d_G(x)$ where x is u, w or v. For convenience, let $uu_1 \in E(H)$ with $d_H(u_1) = 3$. Now, we consider the smaller graph $G' = G - \{uu_1, uv, uw, vw\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the colors of u, v and w. Let S_1, S_2, S_3, S_4, S_5, S_6 and S_7 be the sets of available colors for u, v, w, uv, uw, wv and uu_1, respectively. It is easy to know that $|S_1| \geq 13 - 8 = 5 > 4$, $|S_2| \geq 13 - 6 = 7 > 6$, $|S_3| \geq 13 - 10 = 3 > 2$, $|S_4| \geq 13 - 7 = 6 > 5$, $|S_5| \geq 13 - 9 = 4 > 3$, $|S_6| \geq 13 - 8 = 5 > 4$ and $|S_7| \geq 13 - 6 = 7 > 6$. We associate u, v, w, uv,

uw, wv and uu_1 with the variables $x_1, x_2, \ldots, x_7$, respectively, and let $\ln x_i = y_i$ for $1 \le i \le 7$. For convenience, let $S'_i = \{y_i \mid \ln x_i = y_i, x_i \in S_i\}$ for $1 \le i \le 7$. Obviously, $|S'_1| \ge 5 > 4$, $|S'_2| \ge 7 > 6$, $|S'_3| \ge 3 > 2$, $|S'_4| \ge 6 > 5$, $|S'_5| \ge 4 > 3$, $|S'_6| \ge 5 > 4$ and $|S'_7| \ge 7 > 6$. Now we consider the following polynomial.

$$\begin{aligned} P(y_1, y_2, \ldots, y_7) = & \prod_{1 \le k < j \le 3} (y_k - y_j) \prod_{4 \le i < j \le 6} (y_i - y_j)(y_1 - y_4)(y_2 - y_4) \\ & (y_1 - y_5)(y_3 - y_5)(y_2 - y_6)(y_3 - y_6)(y_1 - y_7)(y_4 - y_7) \\ & (y_5 - y_7)(y_1 + y_5 + y_7 - y_2 - y_6)(y_2 + y_4 - y_3 - y_5) \\ & (y_1 + y_4 + y_7 - y_3 - y_6)(y_2 + y_4 + y_6)^3 (y_3 + y_5 + y_6)^5 \\ & (y_1 + y_4 + y_5 + y_7)^4. \end{aligned}$$

It is not difficult to obtain that $c_P\left(y_1^4 y_2^6 y_3^2 y_4^5 y_5^3 y_6^4 y_7^6\right) = 200 \ne 0$ by MATLAB. Since $deg(P) = 30 = 4 + 6 + 2 + 5 + 3 + 4 + 6$, by Lemma 2.3, there is $s_i \in S'_i$ for $1 \le i \le 7$ such that $P(s_1, s_2, \ldots, s_7) \ne 0$. Finally, we can color u, v, w, uv, uw, wv and uu_1 with $e^{s_1}, e^{s_2}, \ldots, e^{s_7}$, respectively, to obtain a tnpd-k-coloring of G, a contradiction. ■

Fact 7. *For each $u \in V(H)$ with $d_H(u) = 7$, if $n^H_{4^-}(u) = 1$, then u is adjacent to at most one 5-vertex in H.*

Proof. Suppose to the contrary that there exists a vertex $u \in V(H)$ with $d_H(u) = 7$, $n^H_{4^-}(u) = 1$ and $n^H_5(u) \ge 2$. By Fact 1, we have $n^G_{4^-}(u) = 1$ and $n^G_5(u) \ge 2$. Let u_1 be the 4^--neighbor of u, u_2 and u_3 be the 5-neighbors of u.

By Fact 4, we only consider the situation $d_G(u) = d_H(u)$. Without loss of generality, we assume $d_G(u_1) = 4$. Now, we consider the smaller graph $G' = G - \{uu_1, uu_2, uu_3\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the colors of u, u_2 and u_3. Let S_1, S_2, S_3, S_4, S_5 and S_6 be the sets of available colors for u, uu_1, uu_2, uu_3, u_2 and u_3, respectively. It is easy to know that $|S_1| \ge 13 - 8 = 5 > 4$, $|S_2| \ge 13 - 4 - 3 = 6 > 5$, $|S_3| \ge 13 - 8 = 5 > 3$, $|S_4| \ge 13 - 8 = 5 > 4$, $|S_5| \ge 13 - 8 = 5 > 4$ and $|S_6| \ge 13 - 8 = 5 > 4$. We associate u, uu_1, uu_2, uu_3, u_2 and u_3 with the variables $x_1, x_2, \ldots, x_6$, respectively, and let $\ln x_i = y_i$ for $1 \le i \le 6$. For convenience, let $S'_i = \{y_i \mid \ln x_i = y_i, x_i \in S_i\}$ for $1 \le i \le 6$. Obviously, $|S'_1| \ge 5 > 4$, $|S'_2| \ge 6 > 5$, $|S'_3| \ge 5 > 3$, $|S'_4| \ge 5 > 4$, $|S'_5| \ge 5 > 4$ and $|S'_6| \ge 5 > 4$. Now we consider the following polynomial.

$$\begin{aligned} P(y_1, y_2, \ldots, y_6) = & \prod_{2 \le k \le 6} (y_1 - y_k) \prod_{2 \le i < j \le 4} (y_i - y_j)(y_3 - y_5)(y_4 - y_6) \\ & (y_1 + y_2 + y_4 - y_5)(y_1 + y_2 + y_3 - y_6)(y_3 + y_5)^4 \\ & (y_4 + y_6)^4 (y_1 + y_2 + y_3 + y_4)^4. \end{aligned}$$

It is not difficult to obtain that $c_P\big(y_1^4y_2^5y_3^3y_4^4y_5^4y_6^4\big) = 176 \neq 0$ by MATLAB. Since $deg(P) = 24 = 4+5+3+4+4+4$, by Lemma 2.3, there is $s_i \in S_i'$ for $1 \le i \le 6$ such that $P(s_1, s_2, \ldots, s_6) \neq 0$. Finally, we can color u, uu_1, uu_2, uu_3, u_2 and u_3 with $e^{s_1}, e^{s_2}, \ldots, e^{s_6}$, respectively, to obtain a tnpd-k-coloring of G, a contradiction. ∎

Fact 8. *For each $u \in V(H)$ with $d_H(u) = 8$, if $n_3^H(u) \ge 1$ and $n_{4^-}^H(u) \ge 2$, then $n_{5^-}^H(u) = 2$.*

Proof. Suppose to the contrary that there exists a vertex $u \in V(H)$ with $d_H(u) = 8$ such that $n_3^H(u) \ge 1$, $n_{4^-}^H(u) \ge 2$ and $n_{5^-}^H(u) \ge 3$. By Fact 1, we have $n_3^G(u) \ge 1$, $n_{4^-}^G(u) \ge 2$ and $n_{5^-}^G(u) \ge 3$.

If $d_G(u) = d_H(u) = 8$, then let u_1, u_2 and u_3 be the 5^--neighbor of u. Without loss of generality, we assume $d_G(u_1) = 3$, $d_G(u_2) = 4$ and $d_G(u_3) = 5$. Now, we consider the smaller graph $G' = G - \{uu_1, uu_2, uu_3\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the colors of u and u_3. Let S_1, S_2, S_3, S_4 and S_5 be the sets of available colors for u, uu_1, uu_2, uu_3 and u_3, respectively. It is easy to know that $|S_1| \ge 13 - 10 = 3 > 2$, $|S_2| \ge 13 - 5 - 2 = 6 > 5$, $|S_3| \ge 13 - 8 = 5 > 4$, $|S_4| \ge 13 - 9 = 4 > 3$ and $|S_5| \ge 13 - 8 = 5 > 4$. We associate u, uu_1, uu_2, uu_3 and u_3 with the variables $x_1, x_2, \ldots, x_5$, respectively, and let $\ln x_i = y_i$ for $1 \le i \le 5$. For convenience, let $S_i' = \{y_i \mid \ln x_i = y_i, x_i \in S_i\}$ for $1 \le i \le 5$. Obviously, $|S_1'| \ge 3 > 2$, $|S_2'| \ge 6 > 5$, $|S_3'| \ge 5 > 4$, $|S_4'| \ge 4 > 3$ and $|S_5'| \ge 5 > 4$. Now we consider the following polynomial.

$$\begin{aligned} P(y_1, y_2, \ldots, y_5) = & \prod_{1 \le k < l \le 4} (y_k - y_l)(y_1 - y_5)(y_4 - y_5)(y_1 + y_2 + y_3 - y_5) \\ & (y_4 + y_5)^4 (y_1 + y_2 + y_3 + y_4)^5. \end{aligned}$$

It is not difficult to obtain that $c_P\big(y_1^2y_2^5y_3^4y_4^3y_5^5\big) = 45 \neq 0$ by MATLAB. Since $deg(P) = 18 = 2+5+4+3+4$, by Lemma 2.3, there is $s_i \in S_i'$ for $1 \le i \le 5$ such that $P(s_1, s_2, \ldots, s_5) \neq 0$. Finally, we can color u, uu_1, uu_2, uu_3 and u_3 with $e^{s_1}, e^{s_2}, \ldots, e^{s_5}$, respectively, to obtain a tnpd-k-coloring of G, a contradiction.

If $d_G(u) = 9$, then $n_{2^-}(u) = 1$. Let u_1, u_2 and u_3 be the 4^--neighbor of u. Without loss of generality, we assume $d_G(u_1) = 2$, $d_G(u_2) = 3$ and $d_G(u_3) = 4$. Now, we consider the smaller graph $G' = G - \{uu_1, uu_2, uu_3\}$. By the minimality of G, we have G' admits a tnpd-k-coloring c'. Now, we delete the colors of u. Let S_1, S_2, S_3 and S_4 be the sets of available colors for u, uu_1, uu_2 and uu_3, respectively. It is easy to know that $|S_1| \ge 13 - 12 = 1$, $|S_2| \ge 13 - 7 = 6$, $|S_3| \ge 13 - 8 = 5$ and $|S_4| \ge 13 - 9 = 4$. Let $B = \big\{x_1x_2x_3x_4 \mid x_k \in S_k, 1 \le k \le 4, \prod_{1 \le i < j \le 4}(x_i - x_j) \neq 0\big\}$. By Lemma 2.2, we have $|B| \ge 1+4+5+6-10+1 = 7 > 6$. Thus there exist $x_1 \in S_1$, $x_2 \in S_2$, $x_3 \in S_3$ and $x_4 \in S_4$ for u, uu_1, uu_2 and uu_3

such that u does not conflict with any adjacent vertex. Now we can color u, uu_1, uu_2 and uu_3 with x_1, x_2, x_3 and x_4, respectively, to get a tnpd-k-coloring, a contradiction.

If $d_G(u) \geq 10$, then $n^G_{2^-}(u) \geq 2$. By Property 2 and Property 6, we have $n^G_1(u) \geq 2$. By Property 6, we have $n^G_d(u) = 0$ for $2 \leq d \leq 3$. Thus $n^H_3(u) = 0$ by Fact 1. A contradiction to $n^H_3(u) \geq 1$. ■

In order to complete the proof, we use the discharging method. By Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$ and $\sum_{v\in V(H)} d_H(v) = \sum_{f\in F(H)} d_H(f) = 2|E(H)|$, thus

$$\sum_{v\in V(H)} (d_H(v) - 6) + \sum_{f\in F(H)} (2d_H(f) - 6) = -6(|V(H)| - |E(H)| + |F(H)|) = -12.$$

Define an initial charge function w on $V(H) \cup F(H)$ by setting $w(v) = d_H(v) - 6$ if $v \in V(H)$ and $w(f) = 2d_H(f) - 6$ if $f \in F(H)$. Clearly, we have $\sum_{x\in V(H)\cup F(H)} w(x) = -12$.

Now redistribute the charges according to the following discharging rules.

$D1$. If v is a bad 3-neighbor of u, then u gives 1 to v.

$D2$. Assume that v is a special 3-neighbor of u, then u gives $\frac{1}{2}$ to v.

$D3$. If v is a bad 4-neighbor of u, then u gives $\frac{1}{2}$ to v.

$D4$. For each $u \in V(H)$, if u is a good 7-vertex and v is a bad 5-neighbor of u, then u gives $\frac{1}{3}$ to v.

$D5$. If v is a bad 5-neighbor of u with $d_H(u) \geq 8$, then u gives $\frac{1}{3}$ to v.

$D6$. Assume that f is a 4-face. If f is bad, then f gives 1 to each of its incident 5^--vertices. If f is good, then f gives 2 to each of its incident 5^--vertices.

$D7$. If f is a 5^+-face, then f gives 2 to each of its incident 5^--vertices.

Let the new charge of each element $x \in V(H)\cup F(H)$ be $w'(x)$. In the following, we will show that $\sum_{x\in V(H)\cup F(H)} w'(x) \geq 0$, a contradiction to $\sum_{x\in V(H)\cup F(H)} w(x) = -12$. This will complete the proof.

Consider any vertex $v \in V(H)$, suppose $d_H(v) = 3$. Then $w(v) = -3$.

If v is incident with three 3-faces, then $w'(v) = w(v) + 3 \times 1 = 0$ by $D1$.

If v is incident with two 3-faces and one bad 4-face, then $w'(v) = w(v) + 1 + \frac{1}{2} \times 2 + 1 = 0$ by $D1$, $D2$ and $D6$.

If v is incident with two 3-faces and one good 4-face (or 5^+-face), then $w'(v) = w(v) + 1 + 2 = 0$ by $D1$ and $D6$ (or $D7$).

If v is incident with one 3-face and two bad 4-face, then $w'(v) = w(v) + \frac{1}{2} \times 2 + 1 \times 2 = 0$ by $D2$ and $D6$.

If v is incident with one 3-face and at least one good 4-face (or 5^+-face), then $w'(v) \geq w(v) + \frac{1}{2} + 1 + 2 = \frac{1}{2} > 0$ by $D2$ and $D6$ (or $D7$).

Otherwise, v is incident with three 4^+-faces. Then $w'(v) \geq w(v) + 1 \times 3 = 0$ by $D6$ and $D7$.

Suppose $d_H(v) = 4$. Then $w(v) = -2$.

If v is incident with four 3-faces, then $w'(v) = w(v) + \frac{1}{2} \times 4 = 0$ by $D3$.

Otherwise, v is incident with at least one 4^+-face. Then $w'(v) \geq w(v) + 1 + \frac{1}{2} \times 2 = 0$ by $D6$, $D7$ and $D3$.

Suppose $d_H(v) = 5$. Then $w(v) = -1$.

If v is incident with at least one 4^+-face, then $w'(v) \geq w(v) + 1 = 0$ by $D6$. Otherwise, v is incident with five 3-faces. By Property 5 and Fact 3, v has at least three 7^+-neighbors in H. Furthermore, v is adjacent to at most two bad 7-vertices by Fact 4 and Fact 6. For convenience, we divide the proof into the following cases.

- If v is not adjacent to any bad 7-vertex, then clearly we have $w'(v) \geq w(v) + \frac{1}{3} \times 3 = 0$ by $D5$.
- If v is adjacent to one bad 7-vertex, then we have $n^H_{7^+}(v) \geq 4$ by Fact 6. We have $w'(v) \geq w(v) + \frac{1}{3} \times 3 = 0$ by $D4$ and $D5$.
- If v is adjacent to two bad 7-vertices, then we have $n^H_{7^+}(v) = 5$ by Fact 6. We have $w'(v) \geq w(v) + \frac{1}{3} \times 3 = 0$ by $D4$ and $D5$.

Suppose $d_H(v) = 6$. Then $w(v) = 0$. By Property 1 and Fact 3, we have v is not adjacent to any 4^--vertex in H. Thus we have $w'(v) = w(v) = 0$.

Suppose $d_H(v) = 7$. Then $w(v) = 1$. By Fact 5, we have $n^H_{4^-}(v) \leq 1$. If $n^H_3(v) = 1$, i.e., v is a bad 7-vertex, then $n^H_4(v) = 0$. We have $w'(v) \geq w(v) - 1 = 0$ by $D1$, $D2$ and $D4$. If $n^H_4(v) = 1$, then $n^H_3(v) = 0$ and v is adjacent to at most one 5-vertex by Fact 7. We have $w'(v) \geq w(v) - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$ by $D3$ and $D4$. Otherwise, $n_{4^-}(v) = 0$, since v has at most three bad 5-neighbors by Proposition 4 and Fact 1, we have $w'(v) \geq w(v) - \frac{1}{3} \times 3 = 0$ by $D4$.

Suppose $d_H(v) = 8$. Then $w(v) = 2$. By Fact 8, we have $n^H_3(v) \leq 2$.

If $n^H_3(v) = 2$, then $n^H_4(v) = 0$ and $n^H_5(v) = 0$. Thus we have $w'(v) \geq w(v) - 1 \times 2 = 0$ by $D1$.

If $n^H_3(v) = 1$, then $n^H_4(v) \leq 1$ by Fact 8.

- If $n^H_4(v) = 1$, then $n^H_5(v) = 0$ by Fact 8. We have $w'(v) \geq w(v) - 1 - \frac{1}{2} = \frac{1}{2} > 0$ by $D1$ and $D2$.
- If $n^H_4(v) = 0$, then $n^H_{5b}(v) \leq 3$ by Property 4 and Fact 1. We have $w'(v) \geq w(v) - 1 - \frac{1}{3} \times 3 = 0$ by $D1$ and $D5$.

Otherwise, $n^H_3(v) = 0$. Then $n^H_{4b}(v) + n^H_{5b}(v) \leq 4$ by Property 4 and Fact 1. We have $w'(v) \geq w(v) - \frac{1}{2} \times 4 = 0$ by $D3$ and $D5$.

Suppose $d_H(v) = 9$. Then $w(v) = 3$. We have $n^H_{3b}(v) \leq 3$ by Property 3.

If $n_{3b}^H(v) = 3$, then $n_4^H(v) = 0$ and $n_5^H(v) = 0$ by Property 4 and Fact 1. Thus we have $w'(v) = w(v) - 1 \times 3 = 0$ by $D1$.

If $n_{3b}^H(v) = 2$, then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 2$ by Property 4 and Fact 1. Thus we have $w'(v) \geq w(v) - 1 \times 2 - \frac{1}{2} \times 2 = 0$ by $D1$, $D2$, $D3$ and $D5$.

If $n_{3b}^H(v) = 1$, then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 4$ by Property 4 and Fact 1. Thus we have $w'(v) \geq w(v) - 1 - \frac{1}{2} \times 3 = \frac{1}{2} > 0$ by $D1$, $D2$, $D3$ and $D5$.

Otherwise $n_{3b}^H(v) = 0$. Then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 5$ by Property 4 and Fact 1. Thus we have $w'(v) \geq w(v) - \frac{1}{2} \times 5 = \frac{1}{2} > 0$ by $D2$, $D3$ and $D5$.

Suppose $d_H(v) = 10$. Then $w(v) = 4$. We have $n_{3b}^H(v) \leq 3$ by Property 3.

If $n_{3b}^H(v) = 3$, then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 1$ by Property 4 and Fact 1. Thus we have $w'(v) = w(v) - 1 \times 3 - \frac{1}{2} = \frac{1}{2} > 0$ by $D1$, $D2$, $D3$ and $D5$.

If $n_{3b}^H(v) = 2$, then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 4$ by Property 4 and Fact 1. Thus we have $w'(v) \geq w(v) - 1 \times 2 - \frac{1}{2} \times 4 = 0$ by $D1$, $D2$, $D3$ and $D5$.

If $n_{3b}^H(v) = 1$, then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 5$ by Property 4 and Fact 1. Thus we have $w'(v) \geq w(v) - 1 - \frac{1}{2} \times 5 = \frac{1}{2} > 0$ by $D1$, $D2$, $D3$ and $D5$.

Otherwise $n_{3b}^H(v) = 0$. Then $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq 6$ by Property 4 and Fact 1. Thus we have $w'(v) \geq w(v) - \frac{1}{2} \times 6 = 1 > 0$ by $D2$, $D3$ and $D5$.

Suppose $d_H(v) \geq 11$. Then $w(v) = d_H(v) - 6$. Since two special 3-neighbors may be incident with one and the same bad 4-face, it is not difficult to obtain that $n_{3s}^H(v) \leq \frac{2}{3}\left(d_H(v) - 2n_{3b}^H(v)\right)$. Clearly, $n_{3s}^H(v) + n_{4b}^H(v) + n_{5b}^H(v) \leq \frac{2}{3}\left(d_H(v) - 2n_{3b}^H(v)\right)$ by Property 4 and Fact 1.

Thus we have $w'(v) \geq w(v) - n_{3b}^H(v) - \left[\frac{2}{3}\left(d_H(v) - 2n_{3b}^H(v)\right)\right] \times \frac{1}{2} = d_H(v) - 6 - n_{3b}^H(v) - \left(d_H(v) - 2n_{3b}^H(v)\right) \times \frac{1}{3} = d_H(v) - 6 - n_{3b}^H(v) - \frac{1}{3}d_H(v) + \frac{2}{3}n_{3b}^H(v) = \frac{2}{3}d_H(v) - 6 - \frac{1}{3}n_{3b}^H(v)$ by $D1$, $D2$, $D3$ and $D5$.

Since $n_{3b}^H(v) \leq \frac{1}{3}d_H(v)$ by Property 3, we have $w'(v) \geq \frac{2}{3}d_H(v) - 6 - \frac{1}{3} \times \frac{1}{3}d_H(v) = \frac{5}{9}d_H(v) - 6 \geq \frac{1}{9} > 0$.

For each $f \in F(H)$, suppose $d_H(f) = 3$. Then $w'(v) = w(v) = 0$.

Suppose $d_H(f) = 4$. Then $w(f) = 2$. By Property 4 and Fact 1, f is incident with at most two 5^--vertices. If f is bad, then $w'(f) = w(f) - 1 \times 2 = 0$ by $D6$. Otherwise, we have $w'(f) \geq w(f) - 2 = 0$ by $D6$.

Suppose $d_H(f) \geq 5$. Then $w(f) = 2d_H(f) - 6$. By Property 4 and Fact 1, f is incident with at most $\left\lfloor \frac{1}{2}d_H(f) \right\rfloor$ 5^--vertices. We have $w'(f) \geq w(f) - 2 \times \frac{1}{2}d_H(f) = 0$ by $D7$.

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