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THE SPECTRUM PROBLEM FOR THE CONNECTED CUBIC GRAPHS OF ORDER 10

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Abstract

We show that if G is a connected cubic graph of order 10, then there exists a G-decomposition of K_v if and only if $v \equiv 1$ or 10 (mod 15) except when $v = 10$ and G is one of 5 specific graphs.

Keywords: spectrum problem, graph decomposition, cubic graphs.

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1. Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $K_{r\times s}$ to denote the complete simple multipartite graph with r parts of size s, and $K_{r\times s,t}$ (or $K_{t,r\times s}$) to denote the complete simple multipartite graph with r parts of size s and one part of size t. If G' is a subgraph of G, then $G \setminus G'$ denotes the graph obtained from G by deleting the edges of G'. The graph $K_v \setminus K_u$ is called a complete graph of order v with a *hole* of size u and the vertices of K_u are called the *vertices in the hole*. If a and b are integers with $a \le b$, let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$.

A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K such that each edge of K appears in exactly one G_i. If each G_i in Δ is isomorphic to a given graph G, the decomposition is called a *G-decomposition* of K and the copies of G in Δ are called *G-blocks*. A $\{G, H\}$-decomposition of K is defined similarly. A G-decomposition of K is also known as a *(K, G)-design* and a (K_v, G)-design is often known as a *G-design of order v*. If a G-decomposition of K exists, then we may say G *decomposes* K or K is *decomposable* by G.

A (K_v, K_k)-design is also known as a *balanced incomplete block design* and a K_k-decomposition of $K_{v_1,v_2,\ldots,v_t}$ is a *group divisible design*. We will make use of group divisible designs in our constructions. However, for the sake of brevity, we will refrain from adding too many details on these concepts and direct the interested reader to the *Handbook of Combinatorial Designs* [10] and to the summaries within on group divisible designs [11].

Given a graph G, a classical problem in combinatorics is to find necessary and sufficient conditions on v for the existence of a (K_v, G)-design. This is known as the *spectrum problem* for G and it has been investigated and settled for numerous classes of simple graphs G (see [2] and [7] for summaries, and the website maintained by Bryant and McCourt [8] for more up to date results). When G is a complete graph, the spectrum problem was settled by Kirkman [15] for $G = K_3$ and by Hanani [13] for $G \in \{K_4, K_5\}$. The spectrum problem for connected 2-regular graphs (i.e., for cycles) was investigated by numerous authors and settled by Alspach and Gavlas in [6].

In this work, we are concerned with the spectrum problem for the connected 3-regular (i.e., *cubic*) graphs of order 10. There are 19 non-isomorphic such graphs (see Figure 1 and [18]). The spectrum problem has been settled for 6 of them (the unlabeled ones in the figure). Here we settle it for the remaining 13 graphs.

If G is a cubic graph of order 10 and if there exists a (K_v, G)-design, then we must have $15 | \binom{v}{2}$ (since the number of edges in G is 15 and the number of edges in K_v is $\binom{v}{2}$) and $3 | v-1$ (since G is 3-regular and K_v is $(v-1)$-regular). Thus we must have $v \equiv 1$ or $10 \pmod{15}$.

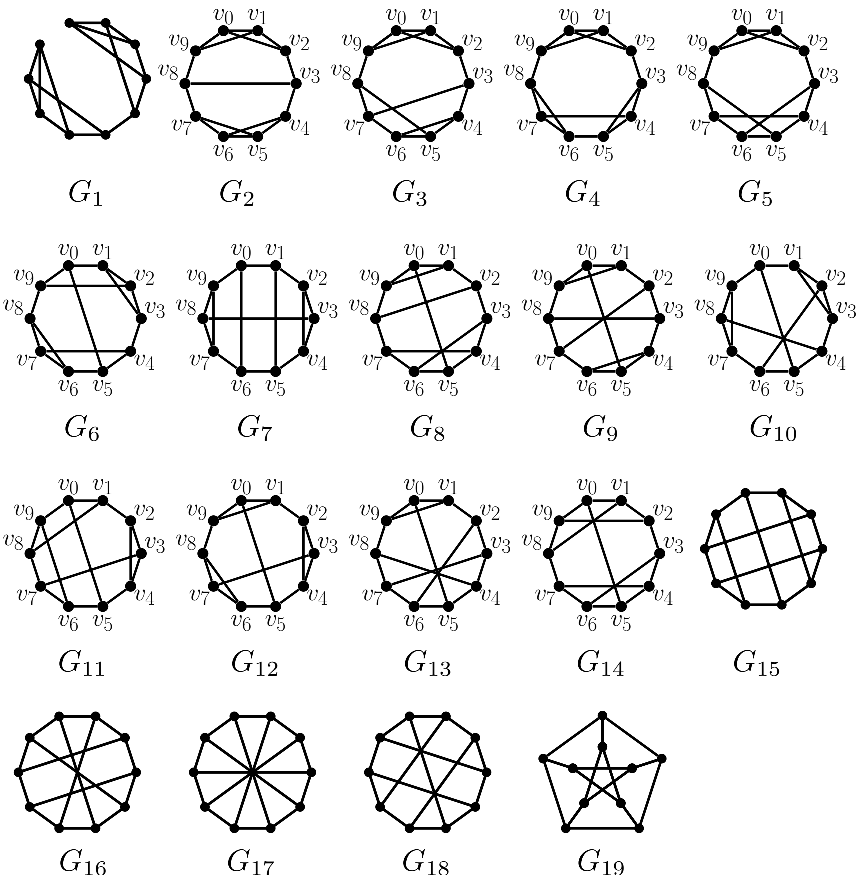

Figure 1. The 19 connected cubic graphs of order 10. The spectrum problem for the 6 unlabeled graphs has previously been settled.

Because of the interest in the Petersen graph (Graph G_{19} in Figure 1), it had been known that it does not decompose K_{10} (see [14]). In 1996, Adams and Bryant [1] settled the spectrum problem for the Petersen graph by showing there exists a G_{19}-decomposition of K_v if and only if $v \equiv 1$ or 10 (mod 15) and $v \neq 10$. Subsequently, Adams, Bryant, and Khodkar [3] showed that 4 additional connected cubic graphs of order 10, namely G_2, G_{14}, G_{15}, and G_{17} do not decompose K_{10}. The aim of this manuscript is to show that these are the only cases when $v \equiv 1$ or 10 (mod 15) and a G_i-decomposition of K_v does not exist. In 2016, the spectrum problem for the 5-Prism and 5-Mobius (G_{15} and G_{17}, respectively) was settled by Meszka, Nedela, Rosa, and Skoviera in [17]. More recently, Adams *et al.* [4] settled the spectrum problem for Graphs G_1, G_{16} and G_{18}.

It is known that if G is a connected cubic graph of order 10, then there exists a G-design of order v for all $v \equiv 1$ (mod 30) (see [19] if G is bipartite and [20] if G is tripartite). Thus to settle the spectrum problem for these 13 remaining graphs, it suffices to settle the cases $v \equiv 10, 16$ and $25 \pmod{30}$.

Before proceeding, we note that several authors have considered the spectrum problem for various cubic graphs. In [13], Hanani settled the problem for K_4-designs by showing that there exists a (K_v, K_4)-design if and only if $v \equiv 1$ or $4 \pmod{12}$. The spectrum problem for $K_{3,3}$-designs was settled by Guy and Beineke in [12]. The spectrum problem for the 3-prism was settled by Carter [9]. The spectrum problem for the 3-dimensional cube was settled by Maheo in [16]. More recently, the spectrum problem for the remaining four connected cubic graphs of order 8 was settled in [5].

We will show that if G_i is a connected cubic graph of order 10, then there exists a (K_v, G_i)-design of order v if and only if $v \equiv 1$ or $10 \pmod{15}$, and $v \neq 10$ when $G_i \in \{G_2, G_{14}, G_{15}, G_{17}, G_{19}\}$. Since these results are known when $G_i \in \{G_1, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}\}$, we will focus on the remaining 13 graphs. Henceforth, each of the graphs G_i with $i \in [2, 14]$ in Figure 1, with vertices labeled as in the figure, will be represented by $G_i[v_0, v_1, \ldots, v_9]$. For example, $G_2[v_0, v_1, \ldots, v_9]$ is the graph with vertex set $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set $\big\{\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_8\}, \{v_8, v_9\}, \{v_9, v_0\}, \{v_0, v_2\}, \{v_1, v_9\}, \{v_3, v_8\}, \{v_4, v_6\}, \{v_5, v_7\}\big\}$. In some cases, we may write a G-decomposition of a graph with vertex set V as a pair (V, B), where B is a collection of copies of G that partitions the edge-set of the graph.

2. General Constructions

Our main constructions depend on many small examples. These examples are given in the Appendix. We first state some results on K_4-decompositions of certain complete multipartite graphs as well as a result on $\{K_3, K_5\}$-decompositions of K_v. We make use of these results in the next section. All of these results can be found in the *Handbook of Combinatorial Designs* [10].

Theorem 1. *There exists a K_4-decomposition of $K_{n\times 2}$ if and only if $n \equiv 1 \pmod 3$ and $n \geq 7$.*

Theorem 2. *There exists a K_4-decomposition of $K_{n\times 3}$ if and only if $n \equiv 0$ or $1 \pmod 4$ and $n \geq 4$.*

Theorem 3. *There exists a K_4-decomposition of $K_{n\times 2,5}$ if and only if $n \equiv 0 \pmod 3$ and $n \geq 9$.*

Theorem 4. *There exists a K_4-decomposition of $K_{(n-2)\times 3,6}$ if and only if $n \equiv 2$ or $3 \pmod 4$ and $n \geq 7$.*

Theorem 5. *There exists a $\{K_3, K_5\}$-decomposition of K_v if v is odd.*

Because of the relevance of the decompositions of K_{10} results from [3], we state them in a separate lemma.

Lemma 6. *Let G be a connected cubic graph of order* 10 (*see Figure* 1). *There exists a G-decomposition of K_{10} if and only if $G \notin \{G_2, G_{14}, G_{15}, G_{17}, G_{19}\}$.*

We are ready to proceed with our results.

Lemma 7. *For every integer $x \geq 2$ and each $i \in [3, 13]$, there exists a G_i-decomposition of $K_{(3x+1)\times 10}$.*

Proof. By Theorem 1, there exists a K_4-decomposition of $K_{(3x+1)\times 2}$. Replacing each vertex of $K_{(3x+1)\times 2}$ by a set of 5 vertices and each edge of $K_{(3x+1)\times 2}$ by a copy of $K_{5,5}$ gives a $K_{4\times 5}$-decomposition of $K_{(3x+1)\times 10}$. By Example 24, there exist G_i-decompositions of $K_{4\times 5}$. Thus the result now follows. ■

Lemma 8. *Let $i \in [3, 13]$. There exists a G_i-decomposition of K_v for all $v \equiv 10$ (mod 30).*

Proof. Let x be a nonnegative integer and let $v = 30x + 10$. For $v = 10$ and $v = 40$, the results follow from Lemma 6 and Example 20, respectively. So we may assume $x \geq 2$.

Let H be the complete graph with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{3x+1}$ with $|H_j| = 10$ for each $j \in [1, 3x + 1]$. By Lemma 7, there is a G_i-decomposition B'_i of $K_{(3x+1)\times 10}$ with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{3x+1}$ and the obvious vertex partition. For each $i \in [3, 13]$ and $j \in [1, 3x + 1]$, let $B_{i,j}$ be a G_i-decomposition of K_{10} with vertex set H_j. Then (V, B_i) where $V = V(H)$ and $B_i = B'_i \cup B_{i,1} \cup B_{i,2} \cup \cdots \cup B_{i,3x+1}$ is a G_i-decomposition of K_{30x+10}. ■

Lemma 9. *For every positive integer x and each $i \in [2, 14]$, there exists a G_i-decomposition of $K_{(2x+1)\times 15}$.*

Proof. It is simple to see that $K_{15,15}$ decomposes $K_{(2x+1)\times 15}$ and since G_{14} decomposes $K_{15,15}$ (by Example 28), the result follows for G_{14}. By Theorem 5, there exists a $\{K_3, K_5\}$-decomposition of K_{2x+1}. Replacing each vertex of K_{2x+1} by a set of 15 vertices and each edge of K_{2x+1} by a copy of $K_{15,15}$ gives a $\{K_{3\times 15}, K_{5\times 15}\}$-decomposition of $K_{(2x+1)\times 15}$. By Examples 25 and 26, there exist G_i-decompositions of $K_{3\times 15}$ and $K_{5\times 15}$ for all $i \in [2, 13]$. Thus the result now follows. ■

Lemma 10. *Let $i \in [2, 14]$. There exists a G_i-decomposition of K_v for all $v \equiv 16$ (mod 30).*

Proof. Let x be a nonnegative integer and let $v = 30x + 16$. For $v = 16$, the result follows from Example 18. So we may assume $x \geq 1$.

Let H be the complete graph with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{2x+1} \cup \{\infty\}$ with $|H_j| = 15$ for each $j \in [1, 2x+1]$. By Lemma 9, there is a G_i-decomposition B_i' of $K_{(2x+1)\times 15}$ with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{2x+1}$ and the obvious vertex partition. For each $i \in [2, 14]$ and $j \in [1, 2x + 1]$, let $B_{i,j}$ be a G_i-decomposition of K_{16} with vertex set $H_j \cup \{\infty\}$. Then (V, B_i) where $V = V(H)$ and $B_i = B_i' \cup B_{i,1} \cup B_{i,2} \cup \cdots \cup B_{i,2x+1}$ is a G_i-decomposition of K_{30x+16}. ∎

Lemma 11. *For every integer $x \geq 3$ and each $i \in [3, 13]$, there exists a G_i-decomposition of $K_{(3x)\times 10,25}$.*

Proof. By Theorem 3, there exists a K_4-decomposition of $K_{(3x)\times 2,5}$ for $x \geq 3$. Replacing each vertex of $K_{(3x)\times 2,5}$ by a set of 5 vertices and each edge of $K_{(3x)\times 2,5}$ by a copy of $K_{5,5}$ gives a $K_{4\times 5}$-decomposition of $K_{(3x)\times 10,25}$. By Example 24, there exist G_i-decompositions of $K_{4\times 5}$. Thus the result now follows. ∎

Lemma 12. *Let $i \in [3, 13]$. There exists a G_i-decomposition of K_v for all $v \equiv 25 \pmod{30}$.*

Proof. Let x be a nonnegative integer and let $v = 30x + 25$. There exist G-decompositions of K_{25}, K_{55} and K_{85} by Examples 19, 21 and 22, respectively. Thus it remains to consider the case $x \geq 3$.

Let H be the complete graph with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{3x+1}$ with $|H_j| = 10$ for $j \in [1, 3x]$ and $|H_{3x+1}| = 25$. By Lemma 11, there exists a G_i-decomposition B_i' of $K_{(3x)\times 10,25}$ with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{3x+1}$ and the obvious vertex partition. For each $i \in [3, 13]$ and $j \in [1, 3x]$, let $B_{i,j}$ be a G_i-decomposition of K_{10} with vertex set H_j. Also, let $B_{i,3x+1}$ be a G_i-decomposition of K_{25} with vertex set H_{3x+1}. Then (V, B_i) where $V = V(H)$ and $B_i = B_i' \cup B_{i,1} \cup B_{i,2} \cup \cdots \cup B_{i,3x+1}$ is a G_i-decomposition of K_{30x+25}. ∎

Because there is no G_2- nor a G_{14}-decomposition of K_{10}, we will treat the case $v \equiv 10 \pmod{15}$ for these two graphs separately. Moreover, since G_{14} is bipartite and G_2 is not, we will use different approaches for the two graphs.

Lemma 13. *There exists a G_2-decomposition of $K_{x\times 15}$ when $x \equiv 0$ or $1 \pmod 4$, $x \geq 4$, and a G_2-decomposition of $K_{(x-2)\times 15,30}$ when $x \equiv 2$ or $3 \pmod 4$ and $x \geq 7$.*

Proof. By Theorem 2, if $x \equiv 0$ or $1 \pmod 4$, then there exists a K_4-decomposition of $K_{x\times 3}$, and by Theorem 4, if $x \equiv 2$ or $3 \pmod 4$ and $x \geq 7$, then there exists a K_4-decomposition of $K_{(x-2)\times 3,6}$. Replacing each vertex in these decompositions by a set of 5 vertices and each edge by a copy of a $K_{5,5}$ yields a $K_{4\times 5}$-decomposition of $K_{x\times 15}$ when $x \equiv 0$ or $1 \pmod 4$ and of $K_{(x-2)\times 15,30}$ when $x \equiv 2$ or $3 \pmod 4$ and $x \geq 7$. The result follows by Example 24. ∎

Lemma 14. *If $v \equiv 10 \pmod{15}$ and $v \neq 10$, then there exists a G_2-decomposition of K_v.*

Proof. Let x be a positive integer and let $v = 15x+10$. By Examples 19, 20, 21, and 23, there exist G_2-decompositions of K_{25}, K_{40}, K_{55}, and K_{100}, respectively. So we may assume that $x \notin \{1, 2, 3, 6\}$.

First, suppose $x \equiv 0$ or $1 \pmod 4$ and let H be the complete graph with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{x+1}$ with $|H_j| = 15$ for $j \in [1, x]$ and $|H_{x+1}| = 10$. By Lemma 13, there exists a G_2-decomposition B' of $K_{x\times 15}$ with vertex set $H_1 \cup H_2 \cup \cdots \cup H_x$ and the obvious vertex partition. Let B_1 be a G_2-decomposition of K_{25} with vertex set $H_1 \cup H_{x+1}$. For each $j \in [2, x]$, let B_j be a G_2-decomposition of $K_{25} \setminus K_{10}$ with vertex set $H_j \cup H_{x+1}$ (with the vertices of H_{x+1} being the vertices in the hole). Then (V, B) where $V = V(H)$ and $B = B' \cup B_1 \cup \cdots \cup B_x$ is a G_2-decomposition of K_{15x+10}.

Now suppose $x \equiv 2$ or $3 \pmod 4$ and $x \notin \{2, 3, 6\}$. Let H be the complete graph with vertex set $H_0 \cup H_1 \cup \cdots \cup H_{x-1}$ with $|H_0| = 30$, $|H_{x-1}| = 10$, and $|H_j| = 15$ for $j \in [1, x-2]$. By Lemma 13, there exists a G_2-decomposition B' of $K_{(x-2)\times 15,30}$ with vertex set $H_1 \cup H_2 \cup \cdots \cup H_{x-2} \cup H_0$ and the obvious vertex partition. Let B_0 be a G_2-decomposition of K_{40} with vertex set $H_0 \cup H_{x-1}$. For each $j \in [1, x-2]$, let B_j be a G_2-decomposition of $K_{25} \setminus K_{10}$ with vertex set $H_j \cup H_{x-1}$ (with the vertices of H_{x-1} being the vertices in the hole). Then (V, B) where $V = V(H)$ and $B = B' \cup B_0 \cup B_1 \cup \cdots \cup B_{x-2}$ is a G_2-decomposition of K_{15x+10}. ■

Lemma 15. *There exists a G_{14}-decomposition of $K_{(x-1)\times 15,24}$ for all integers $x \geq 2$.*

Proof. It is simple to see that $K_{(x-1)\times 15,24}$ can be decomposed into $\binom{x-1}{2}$ copies of $K_{15,15}$ and $x-1$ copies of $K_{15,24}$. Also note that $K_{15,24}$ can be decomposed into one copy of $K_{15,15}$ and one copy of $K_{9,15}$. By Example 28, there exists a G_{14}-decomposition of $K_{15,15}$ and by Example 27, there exists a G_{14}-decomposition of $K_{9,15}$. The result now follows. ■

Lemma 16. *If $v \equiv 10 \pmod{15}$ and $v \neq 10$, then there exists a (K_v, G_{14})-design.*

Proof. Let x be a positive integer and let $v = 15x+10$. By Example 19, there exists a G_{14}-decomposition of K_{25} and thus we may assume that $x \geq 2$.

Let H be the complete graph with vertex set $H_1 \cup H_2 \cup \cdots \cup H_x \cup \{\infty\}$ with $|H_x| = 24$ and $|H_j| = 15$ for $j \in [1, x-1]$. By Lemma 15, there exists a G_{14}-decomposition B' of $K_{(x-1)\times 15,24}$ with vertex set $H_1 \cup H_2 \cup \cdots \cup H_x$ and the obvious vertex partition. Let B_x be a G_{14}-decomposition of K_{25} with vertex set $H_x \cup \{\infty\}$. For each $j \in [1, x-1]$, let B_j be a G_2-decomposition of K_{16} with vertex set $H_j \cup \{\infty\}$. Then (V, B) where $V = V(H)$ and $B = B' \cup B_1 \cup B_2 \cup \cdots \cup B_x$ is a G_{14}-decomposition of K_{15x+10}. ■

Combining the results from Lemmas 8 to 16 and the previously known results from [1], [17], and [4], we obtain the following.

Theorem 17. *Let G be connected cubic graph of order* 10*. There exists a G-decomposition of K_v if and only if $v \equiv 1$ or* 10 (mod 15) *except when $v = 10$ and G is one of the 5 graphs G_2, G_{14}, G_{15}, G_{17} or G_{19} in Figure* 1.

3. Appendix

Here we give examples of decompositions that were used in Section 2. These results were found by using computer searches.

Example 18. Let $V(K_{16}) = \mathbb{Z}_{16}$, and let

$$\begin{aligned}
B_2 = \{&G_2[4,14,8,11,3,12,6,5,15,0], G_2[14,5,9,3,1,7,2,13,8,10], G_2[2,10,6,4,5,8,1,15,7,3],\\
&G_2[0,1,6,7,5,2,11,12,8,9], G_2[0,5,3,4,1,12,14,7,11,13], G_2[1,10,11,0,2,4,15,9,12,13],\\
&G_2[6,11,9,13,4,10,12,15,3,14], G_2[7,10,0,8,6,13,15,14,2,9]\},\\
B_3 = \{&G_3[6,5,0,11,8,4,14,7,2,15], G_3[3,10,9,15,12,11,1,7,5,4], G_3[13,6,10,2,8,1,9,0,15,3],\\
&G_3[0,1,3,2,5,8,12,9,6,4], G_3[0,7,8,3,11,13,14,5,1,10], G_3[2,6,11,10,12,4,13,5,9,14],\\
&G_3[10,15,8,13,7,6,12,2,1,14], G_3[12,14,0,13,15,4,11,9,7,3]\},\\
B_4 = \{&G_4[10,2,11,1,4,7,6,14,15,0], G_4[12,2,15,7,5,3,9,6,8,13], G_4[8,7,0,11,15,5,12,9,4,10],\\
&G_4[0,1,3,2,7,9,13,11,4,5], G_4[0,6,4,3,10,14,8,15,1,12], G_4[1,2,6,13,5,10,12,8,11,14],\\
&G_4[7,14,12,3,6,11,9,10,1,13], G_4[9,14,0,13,3,15,4,8,2,5]\},\\
B_5 = \{&G_5[8,0,11,2,6,4,15,12,9,13], G_5[10,1,9,7,3,14,6,15,11,4], G_5[0,1,2,4,7,10,8,12,5,3],\\
&G_5[0,4,5,1,6,8,7,13,2,12], G_5[0,6,9,2,3,4,14,12,13,10], G_5[6,11,3,10,14,1,15,9,8,5],\\
&G_5[11,12,1,13,14,5,15,7,2,10], G_5[14,15,0,7,5,9,11,13,3,8]\},\\
B_6 = \{&G_6[0,2,13,14,5,6,1,11,4,9], G_6[9,7,6,2,11,12,4,10,14,3], G_6[4,0,1,3,5,7,10,9,8,2],\\
&G_6[7,0,5,8,3,11,15,2,9,1], G_6[11,0,10,12,1,8,15,13,4,6], G_6[14,1,15,10,3,12,8,7,13,0],\\
&G_6[12,2,5,10,13,6,9,11,14,15], G_6[15,6,14,8,4,3,13,5,12,7]\},\\
B_7 = \{&G_7[1,13,2,11,15,9,10,8,14,5], G_7[12,5,9,2,0,4,14,6,1,8], G_7[4,7,1,3,0,5,6,10,2,12],\\
&G_7[4,2,6,7,0,8,9,11,3,10], G_7[15,5,10,13,0,11,1,4,3,8], G_7[9,1,14,15,0,12,7,11,13,6],\\
&G_7[11,12,9,14,3,15,4,13,7,8], G_7[14,13,12,6,3,5,2,7,15,10]\},\\
B_8 = \{&G_8[14,8,12,6,4,15,3,5,10,9], G_8[13,10,7,11,14,12,5,0,15,6], G_8[2,1,4,8,13,5,6,7,3,0],\\
&G_8[7,4,2,3,12,1,11,9,6,0], G_8[10,8,1,6,14,2,11,4,9,0], G_8[11,12,2,9,15,8,5,1,13,0],\\
&G_8[8,7,5,14,13,3,10,11,15,2], G_8[1,14,7,12,15,10,4,13,9,3]\},\\
B_9 = \{&G_9[5,2,9,8,15,10,0,11,14,3], G_9[4,14,10,9,15,7,6,3,13,1], G_9[2,1,5,7,9,4,6,8,3,0],\\
&G_9[5,4,3,1,7,11,2,12,6,0], G_9[8,7,10,1,11,4,13,12,9,0], G_9[8,12,0,13,10,11,6,14,15,1],\\
&G_9[10,8,13,5,15,4,12,7,14,2], G_9[15,11,12,5,6,2,13,14,9,3]\},\\
B_{10} = \{&G_{10}[4,14,10,13,12,1,9,5,3,2], G_{10}[2,10,6,5,15,11,8,14,3,7], G_{10}[5,1,0,2,6,4,3,8,9,7],\\
&G_{10}[8,4,0,7,1,13,5,11,3,10], G_{10}[7,6,0,11,4,12,8,2,13,15], G_{10}[12,9,0,14,6,15,10,7,13,11],\\
&G_{10}[2,14,1,15,0,12,6,3,13,9], G_{10}[11,9,4,15,8,1,10,12,5,14]\},\\
B_{11} = \{&G_{11}[8,7,2,3,5,11,13,1,0,15], G_{11}[1,4,15,6,3,8,12,5,9,7], G_{11}[10,5,4,0,2,1,9,3,7,6],\\
&G_{11}[14,2,8,0,5,1,11,6,12,3], G_{11}[13,3,10,0,9,2,15,11,4,14], G_{11}[0,12,11,2,10,14,8,6,4,13]\\
&G_{11}[13,12,10,4,8,9,14,7,15,5], G_{11}[1,6,13,7,10,15,9,11,14,12]\},
\end{aligned}$$

$$\begin{aligned}
B_{12} = \{&G_{12}[6,4,2,7,9,3,8,15,1,5], G_{12}[6,7,3,11,10,8,9,0,4,13], G_{12}[0,1,4,7,12,3,13,8,5,2],\\
&G_{12}[0,5,3,14,2,6,9,10,1,7], G_{12}[0,8,2,11,15,10,6,14,1,12], G_{12}[0,13,11,4,8,14,5,10,12,15],\\
&G_{12}[3,4,14,12,13,1,11,9,5,15], G_{12}[10,2,12,6,11,7,14,15,9,13]\},\\
B_{13} = \{&G_{13}[11,15,8,10,1,13,5,14,6,9], G_{13}[15,6,12,2,14,13,8,11,3,0], G_{13}[0,1,3,9,7,4,6,10,5,2],\\
&G_{13}[0,5,1,12,3,7,8,4,2,9], G_{13}[0,8,2,15,4,10,13,7,1,14], G_{13}[0,12,10,15,1,11,7,14,9,13],\\
&G_{13}[2,6,13,4,11,10,3,5,12,7], G_{13}[5,6,8,9,12,15,3,4,14,11]\},\\
B_{14} = \{&G_{14}[3,6,13,0,14,5,4,12,15,7], G_{14}[5,2,7,1,11,9,6,4,3,12], G_{14}[0,6,8,14,15,11,3,9,7,10],\\
&G_{14}[0,1,4,14,13,3,10,12,9,2], G_{14}[0,8,2,13,5,7,4,10,11,12], G_{14}[0,9,8,12,6,5,1,2,10,15],\\
&G_{14}[1,13,9,4,15,3,8,5,11,14], G_{14}[2,11,6,10,13,15,1,8,7,14]\}.
\end{aligned}$$

For $i \in [2,14]$, a G_i-decomposition of K_{16} consists of the G_i-blocks in B_i.

Example 19. Let $V(K_{25}) = \{r_s : r \in \mathbb{Z}_5 \text{ and } s \in \mathbb{Z}_5\}$, and let

$$\begin{aligned}
B_2 = \{&G_2[0_0,1_0,0_1,2_0,4_0,3_2,0_3,3_0,0_2,2_1], G_2[0_0,1_2,0_3,2_0,1_4,2_1,2_2,0_1,0_4,4_3],\\
&G_2[0_0,0_4,1_4,0_1,1_2,0_2,3_4,3_2,4_1,2_4], G_2[0_3,1_3,0_1,4_3,1_1,3_3,3_4,2_2,2_3,4_4]\},\\
B_3 = \{&G_3[0_0,1_0,0_1,2_0,4_0,1_2,4_2,0_2,1_1,2_1], G_3[0_0,1_2,0_3,1_0,0_2,1_4,2_1,0_4,3_0,1_3],\\
&G_3[0_0,2_3,0_4,0_1,2_1,0_3,4_3,1_2,4_1,1_4], G_3[0_1,1_4,0_3,3_2,3_4,1_2,4_4,2_4,2_3,4_3]\},\\
B_4 = \{&G_4[0_0,1_0,0_1,2_0,4_0,0_2,3_1,1_2,1_1,2_1], G_4[0_0,0_2,4_2,0_1,1_2,2_3,4_3,3_3,1_0,0_3],\\
&G_4[0_0,1_3,0_4,1_0,3_4,4_4,2_4,4_2,1_2,1_4], G_4[0_3,1_4,1_1,2_4,0_2,4_3,3_1,3_3,0_4,2_1]\},\\
B_5 = \{&G_5[0_0,1_0,0_1,2_0,4_0,1_2,0_2,3_2,1_1,2_1], G_5[0_0,0_2,0_3,1_0,2_2,0_4,3_3,1_4,3_0,1_3],\\
&G_5[0_0,1_4,0_4,0_1,2_1,0_3,1_2,3_3,1_1,4_4], G_5[0_1,4_2,1_4,4_1,2_2,0_3,4_3,3_4,3_3,4_4]\},\\
B_6 = \{&G_6[2_0,0_0,1_0,0_1,1_1,0_2,4_2,4_1,1_2,3_1], G_6[0_2,0_0,3_1,4_2,2_0,1_3,0_4,3_3,3_0,0_3],\\
&G_6[0_4,0_0,1_2,3_3,0_1,1_4,3_2,2_4,1_3,0_3], G_6[0_4,0_1,1_3,4_3,2_4,0_2,1_4,3_0,4_4,1_1]\},\\
B_7 = \{&G_7[1_2,3_1,1_0,0_1,0_0,2_0,0_2,3_0,1_1,3_2], G_7[1_3,0_1,1_2,0_3,0_0,4_2,4_1,2_3,1_0,3_3],\\
&G_7[2_3,3_1,3_3,0_4,0_0,1_4,0_2,4_4,1_0,3_4], G_7[3_3,3_4,2_1,4_4,0_1,0_4,2_2,2_3,4_2,1_4]\},\\
B_8 = \{&G_8[0_1,1_0,3_1,1_2,0_3,1_1,4_1,0_2,2_0,0_0], G_8[2_2,3_1,3_2,4_0,3_3,1_0,2_3,1_2,0_2,0_0],\\
&G_8[0_4,0_3,0_1,1_3,4_4,1_0,3_4,1_1,1_4,0_0], G_8[3_3,2_3,0_3,1_4,3_4,0_2,2_4,3_2,4_4,0_1]\},\\
B_9 = \{&G_9[0_1,1_0,2_1,0_2,3_2,3_0,1_1,1_2,2_0,0_0], G_9[2_2,1_2,0_1,1_1,2_3,1_3,0_4,2_1,0_3,0_0],\\
&G_9[3_3,1_3,1_4,0_1,0_4,4_0,4_4,2_1,2_3,0_0], G_9[4_4,2_4,1_3,3_2,3_3,2_2,0_4,4_2,3_4,0_0]\},\\
B_{10} = \{&G_{10}[2_1,1_0,0_0,0_1,3_0,0_2,2_0,1_2,3_2,1_1], G_{10}[0_1,3_1,0_0,0_3,1_0,3_3,1_2,2_1,2_3,1_3],\\
&G_{10}[0_2,3_3,0_0,0_4,1_0,3_4,1_4,3_1,4_4,4_2], G_{10}[4_2,0_4,0_1,4_4,1_3,3_3,2_4,2_3,1_2,3_4]\},\\
B_{11} = \{&G_{11}[1_1,3_0,0_1,0_0,1_0,2_1,0_2,2_0,2_2,3_1], G_{11}[4_3,2_1,2_2,0_0,1_2,0_1,1_3,0_3,2_3,3_0],\\
&G_{11}[4_4,1_0,0_4,0_0,2_3,4_1,2_4,1_4,3_4,0_2], G_{11}[0_4,4_2,0_3,0_2,2_2,1_3,2_4,2_3,4_4,3_1]\},\\
B_{12} = \{&G_{12}[0_0,1_0,2_1,4_0,0_2,2_0,2_2,0_3,3_0,0_1], G_{12}[0_0,2_2,0_1,0_2,1_1,0_3,0_4,1_4,4_0,3_3],\\
&G_{12}[0_0,0_4,0_1,0_3,3_1,4_4,2_2,2_3,1_1,3_4], G_{12}[0_2,2_2,3_2,2_4,0_3,4_3,3_4,4_1,2_3,0_4]\},\\
B_{13} = \{&G_{13}[0_0,1_0,2_1,1_1,2_2,2_0,0_2,0_3,3_0,0_1], G_{13}[0_0,3_1,0_1,1_3,4_2,1_2,0_3,4_3,2_1,2_2],\\
&G_{13}[0_0,0_3,1_0,2_4,3_4,1_3,4_4,3_3,0_1,0_4], G_{13}[0_2,1_2,2_3,4_0,3_4,0_4,4_2,1_4,2_1,4_4]\},\\
B_{14} = \{&G_{14}[0_0,1_0,3_1,1_2,2_1,0_1,1_1,3_0,0_2,2_0], G_{14}[0_0,0_2,4_1,3_3,2_0,0_3,1_0,0_4,2_2,1_2],\\
&G_{14}[0_0,0_4,0_1,0_3,1_2,2_4,0_2,3_3,2_1,1_4], G_{14}[0_1,3_3,2_2,1_4,3_4,2_3,1_3,4_4,0_3,2_4]\}.
\end{aligned}$$

For $i \in [2,14]$, a G_i-decomposition of K_{25} consists of the G_i-blocks in B_i under the action of the map $r_s \mapsto (r+1 \pmod 5)_s$.

Example 20. Let $V(K_{40}) = \{r_s : r \in \mathbb{Z}_{13} \text{ and } s \in \mathbb{Z}_3\} \cup \{\infty\}$, and let

$$\begin{aligned}
B_2 = \{&G_2[0_0,1_0,3_0,9_0,1_1,2_1,4_1,2_0,0_1,5_0], G_2[0_0,1_1,6_1,3_0,0_1,1_2,2_2,1_0,0_2,7_1],\\
&G_2[0_0,9_1,3_2,1_0,6_2,0_2,2_2,4_1,0_1,4_2], G_2[0_2,3_2,5_0,1_2,2_1,7_2,\infty,0_0,6_2,3_1]\},
\end{aligned}$$

$$\begin{aligned}
B_3 = \{&G_3[0_0,1_0,3_0,9_0,2_1,2_0,4_1,1_1,0_1,5_0], G_3[0_0,1_1,7_1,3_0,0_1,1_2,1_0,0_2,6_0,9_1],\\
&G_3[0_0,1_2,2_2,4_0,8_2,9_1,5_2,0_2,0_1,3_2], G_3[0_2,4_2,2_1,8_2,3_0,9_2,\infty,11_1,3_2,9_1]\},\\
B_4 = \{&G_4[0_0,1_0,3_0,9_0,1_1,2_1,4_1,2_0,0_1,5_0], G_4[0_0,0_1,3_1,2_0,11_1,0_2,3_2,2_2,3_0,7_1],\\
&G_4[0_0,10_1,2_2,1_0,5_2,11_2,7_2,8_1,0_1,3_2], G_4[0_2,2_2,6_0,1_2,9_2,0_1,\infty,4_0,10_2,2_1]\},\\
B_5 = \{&G_5[0_0,1_0,3_0,9_0,1_1,2_0,2_1,11_0,0_1,5_0], G_5[0_0,1_1,7_1,10_0,0_2,2_1,1_2,10_1,0_1,9_1],\\
&G_5[0_0,0_2,1_2,2_0,8_2,1_1,9_2,5_2,0_1,2_2], G_5[0_2,5_2,8_0,4_2,6_0,1_2,4_1,\infty,8_2,12_1]\},\\
B_6 = \{&G_6[4_0,0_0,1_0,3_0,8_0,1_1,3_1,2_1,2_0,0_1], G_6[3_1,0_0,6_0,8_1,3_0,0_2,5_1,1_2,0_1,10_1],\\
&G_6[2_2,0_0,0_2,1_2,2_0,5_2,3_1,8_2,10_2,6_0], G_6[4_2,0_1,4_1,3_2,7_2,12_0,\infty,12_2,12_1,10_2]\},\\
B_7 = \{&G_7[1_1,0_1,1_0,3_0,0_0,4_0,10_0,2_1,8_0,5_1], G_7[0_2,6_1,0_1,2_1,0_0,1_1,6_0,1_2,4_0,10_1],\\
&G_7[5_2,2_2,3_1,0_2,0_0,1_2,3_0,6_2,1_0,10_2], G_7[4_0,8_2,2_2,0_1,0_2,2_1,10_2,5_1,1_2,\infty]\},\\
B_8 = \{&G_8[3_0,1_0,0_1,2_0,3_1,8_0,2_1,10_0,4_0,0_0], G_8[3_1,2_1,0_1,5_1,1_2,6_0,0_2,1_1,4_1,0_0],\\
&G_8[1_2,0_2,3_0,6_2,10_2,5_0,3_2,0_1,2_2,0_0], G_8[4_2,6_1,7_2,1_2,\infty,0_0,6_2,12_1,5_2,0_1]\},\\
B_9 = \{&G_9[3_0,1_0,6_0,0_1,2_1,9_0,7_1,1_1,4_0,0_0], G_9[1_1,0_1,1_0,5_1,1_2,9_0,12_1,0_2,2_1,0_0],\\
&G_9[1_2,0_2,2_0,5_2,10_2,4_0,12_2,6_2,2_2,0_0], G_9[3_2,4_1,11_2,5_1,10_2,6_1,\infty,4_0,0_2,0_1]\},\\
B_{10} = \{&G_{10}[6_0,1_0,0_0,3_0,9_0,0_1,4_0,1_1,2_1,4_1], G_{10}[1_0,0_1,0_0,5_1,2_0,0_2,1_1,6_0,3_2,1_2],\\
&G_{10}[6_1,2_1,0_0,2_2,6_0,0_2,3_2,0_1,10_2,1_2], G_{10}[11_2,6_1,0_1,2_2,4_1,10_2,4_2,11_0,\infty,3_2]\},\\
B_{11} = \{&G_{11}[4_1,9_0,3_0,0_0,1_0,6_0,0_1,4_0,1_1,11_0], G_{11}[1_2,3_1,1_1,0_0,0_1,1_0,0_2,4_1,7_1,12_1],\\
&G_{11}[10_2,5_0,2_2,0_0,1_2,3_0,7_2,3_2,0_2,0_1], G_{11}[8_1,11_2,5_2,0_1,6_1,1_2,8_0,4_2,6_2,\infty]\},\\
B_{12} = \{&G_{12}[7_0,9_2,9_1,9_0,0_2,1_1,11_1,7_1,5_0,12_0], G_{12}[0_0,1_0,0_1,5_0,1_1,4_0,9_1,1_2,4_1,3_0],\\
&G_{12}[0_0,3_1,1_1,2_2,7_2,4_1,0_2,4_2,1_0,1_2], G_{12}[0_0,5_2,11_1,4_1,6_2,7_2,0_2,\infty,2_0,8_2]\},\\
B_{13} = \{&G_{13}[0_0,1_0,6_0,4_1,1_1,4_0,0_1,8_1,9_0,3_0], G_{13}[0_0,0_1,5_0,9_1,0_2,2_1,8_1,1_2,8_0,1_1],\\
&G_{13}[0_0,0_2,1_0,8_2,5_0,2_2,5_2,6_1,3_2,1_2], G_{13}[0_1,2_1,10_2,7_1,8_2,0_2,4_2,\infty,0_0,9_2]\}.
\end{aligned}$$

For $i \in [2,13]$, a G_i-decomposition of K_{40} consists of the G_i-blocks in B_i under the action of the map $\infty \mapsto \infty$ and $r_s \mapsto (r+1 \pmod{13})_s$.

Example 21. Let $V(K_{55}) = \{r_s : r \in \mathbb{Z}_{11} \text{ and } s \in \mathbb{Z}_5\}$, and let

$$\begin{aligned}
B_2 = \{&G_2[8_0,1_0,10_3,7_0,2_3,6_3,6_0,2_4,4_4,2_1], G_2[1_0,10_2,1_4,5_1,9_1,6_4,0_0,3_0,6_2,8_1],\\
&G_2[0_0,1_0,0_1,3_0,5_0,0_2,1_2,1_1,8_0,3_1], G_2[0_0,0_2,2_2,1_0,6_2,0_3,3_3,10_0,9_2,4_2],\\
&G_2[0_0,5_3,10_3,2_0,0_4,1_1,4_2,0_1,1_4,2_4], G_2[0_0,1_4,4_4,0_1,3_1,0_3,1_3,7_1,2_1,5_4],\\
&G_2[0_1,5_2,8_2,1_1,2_3,7_1,1_4,3_3,7_2,0_3], G_2[0_1,3_3,6_4,6_1,8_3,6_3,8_4,3_2,5_3,9_4],\\
&G_2[0_3,1_4,0_2,4_4,5_2,3_3,8_4,2_2,10_4,1_2]\},\\
B_3 = \{&G_3[0_0,1_0,3_0,0_1,6_0,6_1,8_1,2_0,1_1,5_0], G_3[0_0,1_1,4_1,1_0,4_2,1_2,7_0,3_2,2_0,0_2],\\
&G_3[0_0,4_2,6_2,0_1,1_1,4_3,5_1,2_3,1_0,0_3], G_3[0_0,2_3,4_3,6_0,1_3,1_4,5_0,0_4,8_0,5_3],\\
&G_3[0_0,0_4,1_4,0_1,0_2,4_4,4_1,3_4,5_0,2_4], G_3[0_1,1_2,2_2,4_1,7_2,1_3,6_3,2_3,1_1,5_2],\\
&G_3[0_1,8_2,5_3,1_1,7_4,4_4,1_2,3_4,10_1,6_3], G_3[0_2,5_2,0_3,7_2,8_3,9_4,7_3,6_4,1_2,1_4],\\
&G_3[0_2,9_4,2_3,5_4,8_3,2_4,6_4,9_1,7_4,3_3]\},\\
B_4 = \{&G_3[5_3,8_0,1_2,4_2,10_0,1_4,7_2,2_2,9_1,9_2], G_3[8_3,7_2,9_2,10_2,6_4,8_4,6_1,3_0,8_1,9_3],\\
&G_3[8_4,5_3,3_4,2_2,7_3,6_0,2_3,4_0,4_4,5_2], G_3[0_0,1_0,3_0,0_1,2_0,3_1,4_2,2_1,1_1,5_0],\\
&G_3[0_0,2_1,6_1,2_0,1_1,0_2,6_3,0_3,6_0,8_2], G_3[0_0,0_2,4_4,3_0,2_2,9_2,0_4,0_3,1_0,6_4],\\
&G_3[0_0,2_3,4_3,0_1,5_1,0_3,4_4,3_3,1_1,6_3], G_3[0_1,3_3,0_4,8_4,7_1,1_4,7_3,4_2,10_1,9_4],\\
&G_3[0_3,10_4,4_1,8_4,10_0,7_4,10_1,6_4,6_2,3_3]\},\\
B_5 = \{&G_4[0_0,1_0,3_0,0_1,2_0,5_1,6_0,6_1,1_1,5_0], G_4[0_0,1_1,2_1,7_0,1_2,3_2,2_2,5_2,1_0,0_2],\\
&G_4[0_0,1_2,7_2,4_0,2_2,2_3,1_3,3_3,1_0,0_3], G_4[0_0,8_2,3_3,5_0,0_3,0_4,10_3,2_4,8_0,4_3],
\end{aligned}$$

$$
\begin{aligned}
& G_4[0_0,0_4,1_4,2_0,6_4,2_1,8_4,3_1,0_1,2_4], G_4[0_1,0_2,2_3,1_1,2_2,0_3,3_2,7_3,3_1,3_3],\\
& G_4[0_1,3_2,0_4,2_0,9_4,6_2,10_4,8_2,2_1,1_4], G_4[0_1,8_2,7_4,9_1,5_2,0_4,5_3,10_4,3_4,8_4],\\
& G_4[0_3,4_4,6_3,5_4,9_2,0_1,9_3,4_1,10_3,7_3]\},\\
B_6 = \{& G_6[4_0,0_0,1_0,3_0,8_0,1_1,3_1,2_1,2_0,0_1], G_6[3_1,0_0,2_1,6_1,0_1,2_2,3_2,1_2,1_0,0_2],\\
& G_6[3_2,0_0,1_2,4_2,7_0,0_3,0_1,1_3,0_2,5_0], G_6[1_3,0_0,5_2,0_3,1_0,3_3,0_2,7_3,5_1,2_1],\\
& G_6[8_3,0_0,3_3,7_3,9_0,0_4,8_1,1_4,4_2,0_1], G_6[4_4,0_0,0_4,1_4,3_0,8_4,0_1,10_4,2_4,5_0],\\
& G_6[5_3,0_1,5_2,4_3,6_1,2_3,2_4,0_4,4_2,0_2], G_6[1_4,0_1,6_3,0_4,0_2,0_3,2_4,9_3,4_3,2_2],\\
& G_6[8_4,0_3,6_4,10_3,9_4,6_2,10_4,4_2,5_4,10_1]\},\\
B_7 = \{& G_7[6_0,0_1,1_0,3_0,0_0,4_0,1_1,6_1,8_0,8_1], G_7[2_2,0_2,1_1,4_1,0_0,3_1,2_1,3_2,0_1,6_2],\\
& G_7[7_2,3_0,0_2,1_2,0_0,2_2,4_0,10_2,5_0,0_3], G_7[7_3,3_0,0_3,1_3,0_0,2_3,4_0,0_4,5_0,10_3],\\
& G_7[7_4,3_0,0_4,1_4,0_0,2_4,0_1,5_4,7_0,10_4], G_7[2_3,0_3,2_2,7_2,0_1,4_2,6_1,1_3,2_1,6_3],\\
& G_7[1_4,3_1,0_3,0_4,0_1,1_3,9_1,3_3,1_1,5_4], G_7[2_3,0_2,6_4,8_4,0_1,1_4,2_2,7_3,1_2,5_4],\\
& G_7[6_4,4_2,3_3,0_4,0_2,9_4,2_3,1_4,6_3,9_2]\},\\
B_8 = \{& G_8[3_0,1_0,0_1,2_0,3_1,8_0,2_1,1_1,4_0,0_0], G_8[0_2,2_1,6_1,0_1,3_2,1_0,2_2,10_0,3_1,0_0],\\
& G_8[6_2,5_2,0_1,0_2,1_3,8_0,0_3,10_0,7_2,0_0], G_8[5_3,0_3,0_1,1_2,1_4,6_0,0_4,8_0,6_3,0_0],\\
& G_8[8_3,7_3,0_1,4_2,8_4,3_1,2_4,1_0,0_4,0_0], G_8[8_4,2_4,0_1,6_2,0_3,1_1,0_2,3_1,9_4,0_0],\\
& G_8[3_3,1_3,3_2,1_2,6_4,5_1,2_4,0_2,2_3,0_1], G_8[3_4,4_3,1_2,9_2,6_4,0_3,5_2,2_3,4_4,0_1],\\
& G_8[7_3,4_3,0_4,6_3,4_4,3_3,3_4,6_4,7_4,0_2]\},\\
B_9 = \{& G_9[3_0,1_0,6_0,2_1,7_0,0_1,1_1,5_1,4_0,0_0], G_9[2_1,0_1,5_1,0_2,2_2,10_1,9_2,1_2,3_1,0_0],\\
& G_9[1_2,0_2,1_0,7_2,1_3,3_0,8_2,0_3,2_2,0_0], G_9[0_3,3_2,6_0,10_3,10_2,3_0,6_3,0_4,4_2,0_0],\\
& G_9[2_3,1_3,5_0,0_4,4_4,7_0,6_4,1_4,5_3,0_0], G_9[1_4,0_4,0_1,0_2,1_3,2_1,3_3,1_2,2_4,0_0],\\
& G_9[1_4,2_2,8_1,3_3,0_3,3_1,5_3,2_4,4_2,0_1], G_9[4_3,0_3,4_1,6_4,9_1,5_4,7_3,7_4,3_3,0_1],\\
& G_9[4_4,5_3,7_4,3_0,1_4,4_2,9_4,3_2,6_4,0_1]\},\\
B_{10} = \{& G_{10}[6_0,1_0,0_0,3_0,0_1,2_1,4_0,3_1,0_2,6_1], G_{10}[6_0,1_1,0_0,2_1,8_0,1_2,3_1,10_0,6_2,0_2],\\
& G_{10}[1_0,0_2,0_0,2_2,5_0,0_3,3_2,0_1,2_3,1_3], G_{10}[3_0,1_3,0_0,3_3,7_0,0_4,2_3,8_0,3_4,2_4],\\
& G_{10}[0_1,4_3,0_0,0_4,2_0,4_4,1_4,2_1,5_4,7_1], G_{10}[2_1,1_2,0_1,2_2,6_1,0_3,4_2,9_1,6_3,1_3],\\
& G_{10}[0_2,6_3,0_1,0_4,3_1,4_4,7_3,2_2,10_4,0_3], G_{10}[0_2,2_4,0_1,6_4,1_2,4_2,5_4,2_2,4_3,1_3],\\
& G_{10}[0_3,4_3,0_2,10_4,10_3,1_4,6_4,6_2,2_4,4_4]\},\\
B_{11} = \{& G_{11}[1_1,0_1,3_0,0_0,1_0,6_0,2_1,4_0,3_1,9_0], G_{11}[2_2,7_1,2_1,0_0,0_1,4_1,0_2,1_1,1_2,1_0],\\
& G_{11}[0_3,6_0,3_2,0_0,2_2,4_0,0_2,4_2,1_2,8_1], G_{11}[4_3,1_0,0_3,0_0,5_2,3_1,0_2,1_3,3_3,10_1],\\
& G_{11}[1_4,0_1,8_3,0_0,4_3,6_0,0_4,6_3,0_3,1_1], G_{11}[8_4,4_0,1_4,0_0,0_4,1_0,10_4,2_4,7_4,1_1],\\
& G_{11}[1_4,9_1,2_4,0_1,2_3,4_1,9_4,6_3,5_3,0_2], G_{11}[2_4,3_2,0_3,0_2,2_2,1_3,9_3,5_2,0_4,6_2],\\
& G_{11}[7_4,4_2,4_4,0_2,2_3,9_4,2_4,7_3,6_4,9_3]\},\\
B_{12} = \{& G_{12}[0_3,10_0,2_4,7_0,6_0,7_2,1_0,5_1,10_3,4_4], G_{12}[2_3,8_4,4_1,6_3,5_1,10_4,3_3,7_1,7_2,10_1],\\
& G_{12}[1_4,0_0,0_2,3_4,0_3,1_2,10_0,6_2,1_1,5_1], G_{12}[0_0,2_0,1_1,0_2,4_1,4_0,5_1,8_0,0_1,5_0],\\
& G_{12}[0_0,7_1,5_1,6_2,0_2,4_2,3_2,0_3,6_0,5_2], G_{12}[0_0,9_2,5_1,5_3,7_2,2_3,4_3,8_3,1_0,0_3],\\
& G_{12}[0_0,4_3,7_0,4_4,5_4,6_3,6_4,1_4,2_0,2_4], G_{12}[0_1,3_2,1_4,8_2,0_3,6_3,5_3,7_4,7_1,4_3],\\
& G_{12}[0_2,3_2,4_4,9_1,8_4,2_2,9_4,0_4,8_1,5_4]\},\\
B_{13} = \{& G_{13}[0_0,1_0,6_0,1_1,3_1,4_0,2_1,8_0,0_1,3_0], G_{13}[0_0,0_1,4_1,4_2,1_2,2_1,0_2,9_2,6_1,1_1],\\
& G_{13}[0_0,0_2,1_0,4_2,0_3,2_2,6_2,1_3,3_0,1_2], G_{13}[0_0,4_2,7_0,3_2,6_3,6_2,0_3,7_3,1_0,3_3],\\
& G_{13}[0_0,0_3,1_0,1_4,5_2,7_3,0_4,1_2,0_1,1_3], G_{13}[0_0,1_4,3_0,7_4,2_2,3_4,0_4,0_3,0_1,2_4],\\
& G_{13}[0_0,5_4,0_1,10_3,9_2,6_4,4_2,2_4,1_1,7_4], G_{13}[0_1,2_3,4_1,9_3,0_4,3_3,0_3,10_4,7_1,4_3],\\
& G_{13}[0_1,0_4,6_4,7_1,4_4,6_3,0_3,5_4,1_2,7_4]\}.
\end{aligned}
$$

For $i \in [2, 13]$, a G_i-decomposition of K_{55} consists of the G_i-blocks in B_i under the action of the map $r_s \mapsto (r+1 \pmod{11})_s$.

Example 22. Let $V(K_{85}) = \{r_s : r \in \mathbb{Z}_{17} \text{ and } s \in \mathbb{Z}_5\}$, and let

$$
\begin{aligned}
B_2 = \{&G_2[8_2, 2_3, 10_1, 1_3, 16_0, 3_0, 15_3, 9_3, 3_2, 7_3], G_2[5_0, 2_1, 6_2, 12_0, 7_0, 6_1, 10_4, 1_3, 9_0, 13_1],\\
&G_2[10_0, 0_4, 12_1, 14_1, 7_4, 7_1, 4_2, 16_1, 13_4, 10_1], G_2[4_0, 6_4, 10_1, 10_2, 8_2, 12_0, 11_0, 7_4, 11_3, 11_1],\\
&G_2[0_0, 2_0, 8_0, 1_0, 10_1, 5_1, 0_2, 10_0, 6_1, 3_1], G_2[0_0, 10_1, 11_1, 13_0, 0_2, 4_2, 7_2, 1_0, 1_2, 2_2],\\
&G_2[0_0, 8_2, 16_2, 1_0, 11_2, 2_3, 6_3, 15_0, 10_2, 0_3], G_2[0_0, 1_3, 3_3, 5_0, 2_3, 9_3, 6_4, 2_0, 3_4, 10_3],\\
&G_2[0_0, 5_3, 0_4, 1_0, 12_3, 7_1, 10_2, 0_1, 14_3, 5_4], G_2[0_0, 6_4, 8_4, 11_0, 5_4, 1_2, 13_2, 2_1, 4_4, 9_4],\\
&G_2[0_1, 4_1, 6_2, 5_1, 1_2, 6_3, 0_4, 2_1, 1_3, 2_3], G_2[0_1, 8_2, 1_3, 8_1, 15_3, 0_4, 1_4, 0_2, 14_3, 11_3],\\
&G_2[0_1, 3_3, 1_4, 7_1, 13_4, 7_2, 3_4, 1_2, 16_4, 8_4], G_2[0_3, 1_4, 4_2, 14_4, 9_2, 3_4, 16_4, 8_2, 8_3, 7_4]\},\\
B_3 = \{&G_3[0_0, 1_0, 3_0, 9_0, 1_1, 4_0, 2_1, 0_1, 12_0, 5_0], G_3[0_0, 0_1, 3_1, 1_0, 7_1, 2_1, 0_2, 13_1, 3_0, 4_1],\\
&G_3[0_0, 7_1, 13_1, 2_0, 4_2, 5_2, 8_0, 0_2, 1_0, 1_2], G_3[0_0, 3_2, 5_2, 10_0, 0_2, 1_3, 7_0, 0_3, 12_0, 6_2],\\
&G_3[0_0, 8_2, 0_3, 1_0, 4_3, 11_3, 0_1, 5_3, 3_0, 1_3], G_3[0_0, 9_3, 12_3, 15_0, 1_4, 7_1, 15_1, 0_4, 0_1, 13_3],\\
&G_3[0_0, 0_4, 1_4, 2_0, 8_4, 2_4, 1_1, 11_4, 6_0, 4_4], G_3[0_0, 7_4, 10_4, 2_0, 16_4, 0_2, 8_1, 13_4, 0_1, 12_4],\\
&G_3[0_1, 1_2, 5_2, 2_1, 9_2, 0_2, 3_3, 16_2, 4_1, 6_2], G_3[0_1, 8_2, 3_3, 1_1, 0_2, 2_3, 11_2, 0_3, 5_1, 6_3],\\
&G_3[0_1, 0_3, 8_3, 1_1, 11_3, 3_4, 11_2, 5_4, 6_1, 15_3], G_3[0_2, 7_2, 14_3, 1_2, 0_3, 11_4, 1_3, 8_4, 2_1, 0_4],\\
&G_3[0_2, 5_3, 1_4, 3_2, 7_4, 0_4, 2_3, 8_4, 5_2, 2_4], G_3[0_3, 5_3, 0_4, 9_2, 5_4, 3_3, 14_3, 15_4, 6_4, 4_4]\},\\
B_4 = \{&G_4[0_2, 9_1, 13_3, 1_3, 6_4, 10_4, 3_1, 7_2, 2_2, 10_1], G_4[1_4, 0_1, 13_1, 2_3, 2_1, 12_2, 0_3, 10_4, 7_0, 13_4],\\
&G_4[9_3, 7_3, 5_2, 16_4, 9_0, 11_0, 6_3, 15_0, 10_1, 0_0], G_4[14_1, 11_1, 12_0, 1_1, 16_4, 1_2, 11_0, 1_0, 14_3, 14_0],\\
&G_4[2_0, 9_1, 4_4, 0_3, 1_2, 15_1, 14_3, 0_0, 11_4, 1_0], G_4[3_3, 13_3, 3_4, 6_2, 3_2, 8_0, 12_3, 7_0, 8_4, 2_3],\\
&G_4[0_0, 3_0, 8_0, 4_0, 7_1, 9_1, 10_2, 14_1, 5_0, 1_1], G_4[0_0, 4_1, 2_2, 2_0, 13_1, 1_2, 0_2, 7_2, 3_0, 6_2],\\
&G_4[0_0, 8_2, 10_2, 1_0, 12_2, 3_3, 0_3, 13_3, 2_0, 1_3], G_4[0_0, 6_3, 0_4, 1_0, 9_4, 15_4, 8_4, 7_4, 12_0, 4_4],\\
&G_4[0_1, 5_1, 3_3, 2_1, 8_1, 14_2, 0_4, 15_3, 1_1, 10_3], G_4[0_1, 8_1, 0_4, 3_1, 0_2, 7_4, 15_4, 3_3, 9_1, 2_4],\\
&G_4[0_2, 4_2, 15_3, 7_1, 2_3, 10_4, 6_2, 1_4, 12_2, 0_4], G_4[0_2, 6_3, 14_3, 5_2, 5_3, 6_4, 3_4, 10_2, 1_2, 9_4]\},\\
B_5 = \{&G_5[3_2, 5_0, 6_3, 10_3, 2_0, 16_3, 1_3, 16_4, 4_0, 8_0], G_5[7_3, 6_4, 1_1, 0_0, 7_2, 0_1, 10_4, 2_3, 10_2, 5_2],\\
&G_5[2_1, 7_1, 4_2, 5_3, 0_1, 1_1, 14_4, 1_2, 7_0, 9_3], G_5[4_3, 1_3, 14_0, 2_0, 16_2, 12_0, 2_4, 4_1, 14_1, 0_1],\\
&G_5[0_0, 1_0, 7_0, 5_0, 13_0, 5_1, 1_1, 10_1, 2_0, 0_1], G_5[0_0, 4_1, 6_1, 1_0, 13_1, 4_2, 0_2, 9_2, 3_0, 10_1],\\
&G_5[0_0, 0_2, 2_2, 6_0, 11_2, 2_3, 14_2, 4_3, 10_0, 3_2], G_5[0_0, 9_2, 3_3, 3_0, 1_3, 2_3, 9_3, 2_4, 6_0, 5_3],\\
&G_5[0_0, 10_3, 4_4, 1_0, 0_4, 9_4, 2_4, 14_4, 3_0, 5_4], G_5[0_2, 6_2, 4_4, 9_2, 11_4, 3_2, 14_4, 5_2, 9_3, 16_4],\\
&G_5[0_1, 4_2, 5_2, 6_1, 12_2, 11_3, 0_3, 2_4, 2_1, 11_2], G_5[0_1, 15_2, 13_3, 1_1, 4_3, 0_4, 11_3, 8_4, 6_1, 1_4],\\
&G_5[0_1, 3_4, 0_3, 12_3, 4_1, 8_4, 12_4, 10_4, 1_1, 15_3], G_5[0_0, 9_4, 8_4, 0_1, 0_2, 0_3, 3_2, 0_4, 1_1, 15_4]\},\\
B_6 = \{&G_6[4_0, 0_0, 1_0, 3_0, 10_0, 0_1, 2_1, 1_1, 15_0, 9_0], G_6[1_1, 0_0, 0_1, 5_1, 3_0, 9_1, 0_2, 14_1, 11_1, 2_0],\\
&G_6[12_1, 0_0, 10_1, 14_1, 3_1, 4_2, 2_2, 1_2, 1_0, 0_2], G_6[3_2, 0_0, 2_2, 5_2, 1_0, 7_2, 13_1, 8_2, 0_2, 5_0],\\
&G_6[9_2, 0_0, 8_2, 13_2, 0_1, 7_1, 6_2, 0_2, 0_3, 15_0], G_6[3_3, 0_0, 0_3, 1_3, 3_0, 7_3, 14_0, 11_3, 9_3, 4_0],\\
&G_6[7_3, 0_0, 6_3, 9_3, 15_0, 0_4, 13_1, 1_4, 10_2, 0_1], G_6[4_4, 0_0, 0_4, 1_4, 3_0, 8_4, 0_1, 9_4, 2_4, 5_0],\\
&G_6[8_4, 0_0, 7_4, 9_4, 2_1, 0_3, 0_1, 1_3, 10_4, 14_0], G_6[3_3, 0_1, 2_3, 8_3, 3_1, 12_3, 2_2, 7_3, 0_3, 6_1],\\
&G_6[12_3, 0_1, 10_3, 0_4, 1_1, 7_4, 1_3, 2_4, 10_2, 3_2], G_6[12_4, 0_1, 3_4, 11_4, 13_1, 9_4, 3_2, 10_4, 6_3, 0_2],\\
&G_6[2_3, 0_2, 1_3, 13_3, 1_2, 2_4, 16_3, 1_4, 2_2, 0_4], G_6[12_4, 0_4, 6_4, 12_3, 8_4, 15_2, 2_4, 6_2, 5_3, 1_2]\},\\
B_7 = \{&G_7[0_1, 9_0, 1_0, 3_0, 0_0, 4_0, 15_0, 2_1, 10_0, 3_1], G_7[8_1, 2_0, 0_1, 5_1, 0_0, 1_1, 5_0, 0_2, 8_0, 2_1],\\
&G_7[2_2, 3_1, 7_1, 0_2, 0_0, 12_1, 2_1, 3_2, 1_0, 4_2], G_7[0_3, 7_0, 1_2, 4_2, 0_0, 5_2, 12_0, 2_3, 8_0, 14_2],\\
&G_7[4_2, 0_1, 7_2, 14_2, 0_0, 8_2, 2_1, 13_2, 1_1, 16_1], G_7[7_3, 3_0, 0_3, 1_3, 0_0, 2_3, 4_0, 13_3, 5_0, 11_3],\\
&G_7[5_4, 0_1, 12_3, 0_4, 0_0, 1_4, 3_0, 6_4, 1_0, 7_4], G_7[5_4, 11_0, 4_4, 7_4, 0_0, 8_4, 13_0, 9_4, 0_1, 3_2],\\
&G_7[6_3, 0_2, 9_2, 0_3, 0_1, 1_3, 3_1, 5_3, 1_1, 11_3], G_7[3_4, 9_1, 5_3, 8_3, 0_1, 6_3, 12_1, 1_4, 1_1, 10_3],\\
&G_7[16_4, 4_1, 2_4, 10_4, 0_1, 3_4, 0_2, 1_4, 14_1, 11_4], G_7[15_4, 12_3, 4_4, 10_4, 0_2, 12_4, 11_3, 13_4, 5_2, 0_3],\\
&G_7[4_2, 1_4, 4_3, 11_3, 0_2, 7_3, 2_4, 8_2, 0_4, 4_4], G_7[1_4, 2_3, 4_2, 0_3, 0_2, 5_2, 11_2, 3_3, 1_2, 11_3]\},
\end{aligned}
$$

$$\begin{aligned}
B_8 = \{ & G_8[3_0,1_0,9_0,0_1,3_1,10_0,2_1,15_0,4_0,0_0], G_8[1_1,0_1,3_0,6_1,16_1,5_0,11_1,4_0,2_1,0_0],\\
& G_8[1_2,0_2,3_0,6_2,11_2,5_0,9_2,4_0,2_2,0_0], G_8[10_2,8_2,14_0,0_3,2_2,0_1,0_2,1_1,9_2,0_0],\\
& G_8[1_3,0_3,3_0,7_3,12_3,5_0,10_3,4_0,2_3,0_0], G_8[11_3,9_3,0_1,4_1,1_4,16_0,0_4,2_1,10_3,0_0],\\
& G_8[3_4,0_4,5_0,10_4,15_4,6_0,14_4,8_0,4_4,0_0], G_8[11_4,10_4,3_1,9_1,14_2,0_1,3_2,5_1,15_4,0_0],\\
& G_8[4_2,8_1,15_2,2_2,4_3,9_1,3_3,0_2,6_2,0_1], G_8[4_3,0_3,2_1,7_3,5_4,1_1,4_4,4_1,1_3,0_1],\\
& G_8[13_3,2_3,7_2,14_2,12_4,6_1,8_4,0_2,6_3,0_1], G_8[13_3,5_3,13_2,4_3,3_4,3_2,0_4,10_2,7_3,0_2],\\
& G_8[3_4,1_4,10_1,15_4,9_4,4_2,8_4,0_3,2_4,0_2], G_8[1_4,7_3,13_4,0_2,9_4,6_3,6_4,16_3,4_4,0_3]\},\\
B_9 = \{ & G_9[7_3,9_0,4_1,12_1,4_3,15_4,14_0,10_0,10_2,7_0], G_9[11_0,1_0,9_3,1_4,12_3,1_2,4_1,16_3,4_0,3_2],\\
& G_9[9_4,11_1,13_4,15_3,16_3,11_0,12_0,5_1,8_2,10_4], G_9[4_3,3_4,9_1,14_2,13_1,12_3,10_0,9_3,3_0,11_0],\\
& G_9[0_1,3_0,9_0,1_1,2_1,1_0,5_1,7_1,5_0,0_0], G_9[1_2,5_1,14_0,5_2,10_1,3_0,0_2,10_2,6_1,0_0],\\
& G_9[5_2,4_2,9_0,1_3,13_3,12_0,9_3,3_3,6_2,0_0], G_9[0_4,3_3,0_1,6_1,3_4,1_0,7_4,7_1,13_3,0_0],\\
& G_9[5_4,3_4,7_0,1_4,5_1,0_1,11_2,14_4,4_4,0_0], G_9[1_3,4_1,6_2,2_2,13_2,3_1,5_3,9_2,0_2,0_1],\\
& G_9[10_3,8_2,3_2,2_3,16_3,5_1,0_4,7_3,9_2,0_1], G_9[4_4,4_3,1_2,0_4,5_3,0_2,7_4,1_4,13_3,0_1],\\
& G_9[13_4,0_3,7_4,16_0,9_4,3_2,1_4,6_2,11_4,0_2], G_9[6_3,8_3,11_4,1_1,2_4,5_2,14_4,5_1,8_4,0_2]\},\\
B_{10} = \{ & G_{10}[10_3,5_0,3_2,6_3,4_0,2_0,8_0,1_3,4_2,3_3], G_{10}[2_1,1_0,5_1,6_2,4_2,16_4,0_2,7_3,1_1,6_4],\\
& G_{10}[10_4,10_2,5_2,2_2,0_1,2_1,11_0,7_1,9_3,4_2], G_{10}[14_4,7_0,5_3,0_1,1_1,7_2,11_0,1_2,2_0,1_3],\\
& G_{10}[8_0,1_0,0_0,4_0,12_0,0_1,5_0,2_1,12_1,6_1], G_{10}[8_0,2_1,0_0,7_1,1_0,10_2,3_1,0_1,4_2,5_2],\\
& G_{10}[8_0,1_2,0_0,8_2,2_0,5_3,4_2,4_1,8_3,0_3], G_{10}[4_0,0_3,0_0,4_3,9_0,0_4,7_3,0_1,2_4,1_4],\\
& G_{10}[1_0,0_4,0_0,2_4,4_0,7_4,1_4,1_1,8_4,12_4], G_{10}[2_1,5_4,0_0,12_4,0_1,11_2,9_4,3_1,3_3,1_2],\\
& G_{10}[1_2,8_1,0_1,1_3,2_1,14_3,8_2,12_1,10_3,9_3], G_{10}[0_2,11_3,0_1,9_4,1_2,10_3,10_4,5_2,15_4,1_4],\\
& G_{10}[0_4,6_2,0_2,4_3,10_4,16_3,4_2,7_4,3_3,15_3], G_{10}[7_1,3_4,0_3,8_4,4_2,6_4,11_3,8_3,13_4,5_4]\},\\
B_{11} = \{ & G_{11}[1_1,9_0,3_0,0_0,1_0,6_0,13_0,4_0,0_1,2_1], G_{11}[12_1,5_0,3_1,0_0,0_1,1_0,5_1,1_1,7_1,0_2],\\
& G_{11}[2_2,1_1,10_1,0_0,5_1,12_1,1_2,14_1,0_2,1_0], G_{11}[11_2,6_0,3_2,0_0,2_2,4_0,0_2,4_2,1_2,3_1],\\
& G_{11}[2_3,15_0,8_2,0_0,6_2,12_0,0_3,9_2,1_3,1_0], G_{11}[0_4,9_0,6_3,0_0,2_3,4_0,0_3,8_3,1_3,6_0],\\
& G_{11}[4_4,1_0,0_4,0_0,10_3,0_1,0_2,1_4,3_4,4_1], G_{11}[1_4,8_0,5_4,0_0,4_4,6_0,13_4,6_4,0_4,3_1],\\
& G_{11}[2_3,0_2,9_2,0_1,2_2,7_2,0_3,11_2,1_3,1_1], G_{11}[0_3,6_1,3_3,0_1,2_3,4_1,9_3,4_3,1_3,4_2],\\
& G_{11}[2_4,4_2,8_3,0_1,6_3,14_1,0_4,7_3,1_4,6_1], G_{11}[0_4,0_2,6_4,0_1,1_4,9_1,16_4,2_4,6_2,0_3],\\
& G_{11}[8_4,3_2,2_4,0_2,0_3,1_2,4_4,3_3,7_4,13_2], G_{11}[2_4,13_3,10_4,0_2,15_3,4_3,0_4,9_4,5_4,14_3]\},\\
B_{12} = \{ & G_{12}[1_3,3_1,13_2,7_3,8_1,10_2,13_0,9_0,8_0,8_3], G_{12}[6_4,16_2,1_2,11_1,0_2,7_3,12_2,8_1,15_1,15_0],\\
& G_{12}[10_4,8_4,16_0,3_2,1_4,2_1,5_4,4_1,11_0,16_1], G_{12}[0_0,2_0,9_0,6_1,4_1,3_0,1_1,10_0,0_1,8_0],\\
& G_{12}[0_0,2_1,3_0,5_2,14_1,3_1,1_2,8_2,2_0,6_1], G_{12}[0_0,0_2,2_0,13_2,9_2,3_2,0_3,4_2,9_0,5_2],\\
& G_{12}[0_0,8_2,15_0,2_3,7_3,9_2,8_3,4_3,2_0,1_3], G_{12}[0_0,3_3,6_0,1_3,2_3,5_3,0_4,1_4,1_0,11_3],\\
& G_{12}[0_3,1_4,6_2,11_3,7_4,14_2,11_4,1_1,14_3,5_1], G_{12}[0_0,4_4,0_1,3_2,9_1,6_4,4_2,4_3,3_1,10_4],\\
& G_{12}[0_0,7_4,1_1,12_3,10_2,13_4,13_2,0_4,0_1,12_4], G_{12}[0_1,0_3,3_1,7_3,5_4,2_3,6_4,13_4,0_2,6_3],\\
& G_{12}[0_1,3_3,11_2,3_4,16_4,8_3,0_4,10_3,6_2,5_4], G_{12}[0_0,7_3,0_1,0_2,5_1,3_4,0_4,8_4,3_0,1_4]\},\\
B_{13} = \{ & G_{13}[3_2,5_0,10_3,3_3,1_3,8_0,2_0,4_2,4_0,6_3], G_{13}[5_1,1_0,2_1,1_1,6_4,0_2,16_4,5_2,4_2,6_2],\\
& G_{13}[0_0,10_2,0_1,9_2,7_1,10_4,2_1,5_3,2_3,7_2], G_{13}[7_0,0_1,4_4,14_1,2_0,15_1,1_2,4_1,1_1,11_0],\\
& G_{13}[0_0,1_0,8_0,8_1,3_1,5_0,2_1,2_2,11_0,3_0], G_{13}[0_0,2_1,14_0,0_2,11_2,3_1,8_2,0_3,2_0,1_2],\\
& G_{13}[0_0,4_2,7_0,1_3,8_1,6_2,3_3,4_1,0_1,13_2], G_{13}[0_0,0_3,1_0,7_3,0_4,3_3,8_3,3_4,9_0,4_3],\\
& G_{13}[0_0,9_3,0_1,6_2,3_4,1_4,6_1,4_4,1_0,0_4], G_{13}[0_0,4_4,6_0,2_4,10_4,6_4,13_4,8_4,13_0,5_4],\\
& G_{13}[0_1,7_2,0_2,13_2,7_3,1_3,12_2,3_4,2_1,0_3], G_{13}[0_1,2_3,6_1,13_3,3_4,4_3,2_4,5_4,3_1,11_3],\\
& G_{13}[0_1,6_3,1_2,10_4,7_3,9_4,5_3,4_3,6_2,6_4], G_{13}[0_3,5_4,12_1,6_4,12_3,10_2,11_4,1_2,13_4,9_2]\}.
\end{aligned}$$

For $i \in [2,13]$, a G_i-decomposition of K_{85} consists of the G_i-blocks in B_i under the action of the map $r_s \mapsto (r+1 \pmod{17})_s$.

Example 23. Let $V(K_{100}) = \{r_s : r \in \mathbb{Z}_{33} \text{ and } s \in \mathbb{Z}_3\} \cup \{\infty\}$, and let

$$\begin{aligned} B_2 = \{&G_2[13_0, 12_1, 23_1, 22_1, 13_2, 5_2, 30_1, 3_0, 9_0, 4_1], G_2[1_1, 18_0, 9_0, 32_0, 4_2, 13_2, 9_2, 29_1, 15_2, 12_2],\\ &G_2[5_1, 19_1, 17_2, 18_0, 14_2, 28_2, 13_1, 28_1, 15_2, 8_2], G_2[3_1, 17_2, 16_0, 5_2, 20_0, 22_1, 27_1, 10_0, 27_2, 17_0],\\ &G_2[13_1, 9_1, 8_0, 6_1, 3_0, 20_0, 22_0, 19_0, 30_0, 15_1], G_2[0_0, 4_0, 12_0, 5_0, 18_0, 6_1, 22_1, 25_0, 10_0, 0_1],\\ &G_2[0_0, 6_1, 15_1, 4_0, 12_1, 0_2, 16_2, 7_0, 2_2, 26_1], G_2[0_0, 22_1, 7_2, 1_0, 4_2, 10_2, 25_2, 6_0, 14_2, 12_2],\\ &G_2[0_0, 11_2, 21_2, 11_1, 6_2, 7_1, 10_1, 4_2, 32_1, 24_2], G_2[0_2, 1_2, 19_0, 6_2, 14_0, 4_2, \infty, 11_1, 4_1, 27_1]\}. \end{aligned}$$

Then a G_2-decomposition of K_{100} consists of the G_2-blocks in B_2 under the action of the map $\infty \mapsto \infty$ and $r_s \mapsto (r + 1 \pmod{33})_s$.

Example 24. Let $V(K_{4\times5}) = \{r_s : r \in \mathbb{Z}_5 \text{ and } s \in \mathbb{Z}_4\}$ with the obvious vertex partition and let

$$\begin{aligned} B_2 =&\{G_2[0_0, 0_1, 0_2, 1_0, 3_2, 1_1, 0_3, 4_0, 2_1, 1_2], G_2[0_0, 4_1, 0_3, 3_1, 1_3, 4_0, 2_2, 2_3, 1_2, 4_3]\},\\ B_3 =&\{G_3[0_0, 0_1, 0_2, 1_0, 3_2, 1_1, 0_3, 2_1, 3_0, 1_2], G_3[0_1, 4_2, 0_3, 0_2, 2_1, 3_3, 0_0, 4_3, 2_0, 2_3]\},\\ B_4 =&\{G_4[0_0, 0_1, 0_2, 1_0, 3_2, 0_3, 1_1, 1_3, 3_0, 1_2], G_4[0_1, 3_3, 2_2, 3_1, 4_0, 0_3, 0_0, 1_1, 2_3, 3_2]\},\\ B_5 =&\{G_5[0_0, 0_1, 0_2, 1_0, 2_1, 0_3, 3_1, 2_3, 3_0, 1_2], G_5[0_0, 4_1, 2_2, 1_3, 1_2, 2_1, 4_2, 2_3, 4_0, 0_3]\},\\ B_6 =&\{G_6[1_1, 0_0, 0_1, 0_2, 1_0, 2_2, 0_3, 3_2, 4_1, 2_0], G_6[3_3, 0_0, 2_3, 3_2, 4_3, 4_0, 0_3, 2_1, 0_2, 3_1]\},\\ B_7 =&\{G_7[0_3, 2_0, 0_1, 0_2, 0_0, 1_1, 4_0, 1_3, 1_0, 2_2], G_7[3_3, 0_1, 2_2, 4_3, 0_0, 3_2, 4_1, 0_3, 4_2, 3_1]\},\\ B_8 =&\{G_8[0_2, 0_1, 2_0, 3_2, 1_3, 1_0, 0_3, 4_0, 1_1, 0_0], G_8[3_3, 2_2, 1_1, 0_2, 4_3, 2_1, 0_3, 0_1, 3_2, 0_0]\},\\ B_9 =&\{G_9[0_2, 0_1, 1_0, 2_2, 4_1, 2_0, 0_3, 3_2, 1_1, 0_0], G_9[0_3, 3_1, 3_3, 2_0, 1_2, 2_1, 1_3, 0_2, 4_3, 0_0]\},\\ B_{10} =&\{G_{10}[1_0, 0_1, 0_0, 0_2, 2_0, 3_2, 1_1, 2_2, 0_3, 3_1], G_{10}[2_3, 3_1, 0_0, 1_3, 2_0, 1_2, 0_3, 4_1, 4_3, 2_2]\},\\ B_{11} =&\{G_{11}[4_1, 2_0, 0_2, 0_0, 0_1, 1_0, 2_2, 1_1, 4_2, 0_3], G_{11}[4_1, 1_2, 1_3, 0_0, 4_2, 3_3, 1_0, 0_3, 2_1, 4_3]\},\\ B_{12} =&\{G_{12}[0_0, 0_1, 1_0, 3_1, 2_2, 1_1, 4_2, 0_3, 2_0, 0_2], G_{12}[0_0, 3_1, 4_3, 4_1, 1_2, 0_3, 3_2, 3_3, 4_0, 1_3]\},\\ B_{13} =&\{G_{13}[0_0, 0_1, 1_0, 3_1, 0_3, 1_1, 2_2, 1_3, 2_0, 0_2], G_{13}[0_0, 3_1, 3_3, 2_1, 4_2, 1_3, 1_0, 0_2, 0_3, 2_2]\}. \end{aligned}$$

For $i \in [2, 13]$, a G_i-decomposition of $K_{4\times5}$ consists of the G_i-blocks in B_i under the action of the map $r_s \mapsto (r + 1 \pmod{5})_s$.

Example 25. Let $V(K_{3\times15}) = \{r_s : r \in \mathbb{Z}_{15} \text{ and } s \in \mathbb{Z}_3\}$ with the obvious vertex partition and let

$$\begin{aligned} B_2 = \{&G_2[0_0, 0_1, 0_2, 1_0, 3_2, 6_0, 8_1, 4_2, 2_1, 1_2], G_2[0_0, 3_2, 5_1, 8_0, 3_1, 4_0, 12_2, 7_1, 13_2, 6_1],\\ &G_2[0_1, 3_2, 8_0, 1_1, 7_0, 5_1, 13_2, 9_0, 5_2, 11_0]\},\\ B_3 = \{&G_3[0_0, 0_1, 0_2, 1_0, 3_1, 5_2, 8_0, 2_1, 3_0, 1_2], G_3[0_0, 3_1, 6_2, 1_0, 13_1, 12_2, 2_0, 7_1, 3_0, 7_2],\\ &G_3[0_0, 13_1, 8_2, 0_1, 12_2, 9_0, 1_1, 7_2, 2_1, 11_2]\},\\ B_4 = \{&G_4[0_2, 0_1, 11_0, 14_2, 13_0, 6_1, 14_0, 1_1, 7_2, 9_0], G_4[0_0, 0_1, 2_2, 2_0, 1_1, 12_2, 3_1, 0_2, 6_0, 5_2],\\ &G_4[0_0, 11_2, 1_1, 3_0, 12_1, 10_2, 6_1, 1_0, 13_2, 10_1]\},\\ B_5 = \{&G_5[0_0, 0_1, 0_2, 1_0, 2_1, 5_2, 3_1, 7_2, 3_0, 1_2], G_5[0_0, 3_1, 9_2, 1_0, 12_1, 6_2, 13_1, 8_2, 3_0, 10_2],\\ &G_5[0_1, 13_2, 1_0, 7_2, 8_1, 0_0, 10_1, 4_0, 9_1, 2_0]\},\\ B_6 = \{&G_6[1_1, 0_0, 0_1, 0_2, 1_0, 2_2, 7_0, 3_2, 4_1, 2_0], G_6[4_1, 0_0, 3_1, 5_2, 1_0, 7_2, 10_0, 4_2, 0_1, 8_0],\\ &G_6[11_2, 0_1, 13_2, 6_0, 14_2, 2_1, 12_2, 7_1, 14_0, 5_1]\},\\ B_7 = \{&G_7[6_1, 2_0, 0_1, 0_2, 0_0, 1_1, 4_0, 7_1, 1_0, 2_2], G_7[3_0, 6_2, 5_1, 2_2, 0_0, 7_1, 0_2, 12_1, 4_0, 10_2],\\ &G_7[6_0, 1_2, 9_1, 11_2, 0_0, 10_1, 14_2, 10_0, 6_1, 0_2]\},\\ B_8 = \{&G_8[0_2, 0_1, 2_0, 4_1, 6_2, 1_0, 2_2, 4_0, 1_1, 0_0], G_8[4_2, 3_1, 8_0, 0_1, 8_2, 1_0, 7_2, 11_0, 4_1, 0_0],\\ &G_8[9_0, 4_2, 1_1, 7_2, 12_1, 2_2, 8_1, 3_0, 12_2, 0_1]\}, \end{aligned}$$

$$B_9 = \{G_9[0_2, 0_1, 1_2, 3_0, 5_1, 1_0, 7_2, 4_0, 1_1, 0_0], G_9[1_2, 3_1, 0_2, 10_0, 14_2, 7_0, 6_1, 13_0, 5_1, 0_0],$$
$$G_9[10_2, 6_1, 13_2, 2_1, 6_0, 0_1, 14_2, 8_1, 11_2, 0_0]\},$$
$$B_{10} = \{G_{10}[1_0, 0_1, 0_0, 0_2, 2_0, 3_2, 1_1, 3_0, 6_2, 5_1], G_{10}[6_0, 3_1, 0_0, 6_2, 0_1, 13_2, 5_1, 10_0, 9_2, 2_1],$$
$$G_{10}[5_2, 8_1, 0_0, 12_2, 4_0, 10_1, 9_2, 4_1, 0_2, 10_0]\},$$
$$B_{11} = \{G_{11}[4_1, 2_0, 0_2, 0_0, 0_1, 1_0, 2_2, 1_1, 4_2, 7_0], G_{11}[10_2, 4_0, 3_2, 0_0, 4_1, 6_0, 0_2, 5_1, 9_2, 2_1],$$
$$G_{11}[2_2, 11_1, 8_2, 0_0, 6_1, 10_0, 6_2, 10_1, 3_0, 12_1]\},$$
$$B_{12} = \{G_{12}[0_0, 0_1, 1_0, 3_1, 2_2, 1_1, 4_0, 6_2, 2_1, 0_2], G_{12}[0_0, 3_1, 0_2, 4_1, 8_0, 3_2, 9_1, 14_2, 1_0, 5_2],$$
$$G_{12}[0_1, 8_2, 14_0, 6_1, 13_2, 5_0, 11_2, 0_0, 5_1, 11_0]\},$$
$$B_{13} = \{G_{13}[0_0, 0_1, 1_0, 6_1, 4_0, 1_1, 2_2, 10_0, 2_1, 0_2], G_{13}[0_0, 3_1, 0_2, 2_0, 6_2, 4_1, 9_0, 14_2, 3_0, 2_2],$$
$$G_{13}[0_1, 3_2, 9_1, 0_2, 7_1, 11_2, 1_0, 10_1, 2_2, 9_0]\}.$$

For $i \in [2, 13]$, a G_i-decomposition of $K_{3\times 15}$ consists of the G_i-blocks in B_i under the action of the map $r_s \mapsto (r + 1 \bmod 15)_s$.

Example 26. Let $V(K_{5\times 15}) = \{r_s : r \in \mathbb{Z}_{15}$ and $s \in \mathbb{Z}_5\}$ with the obvious vertex partition. Let

$$B_2 = \{G_2[1_4, 4_1, 4_3, 14_0, 9_1, 14_2, 12_0, 9_3, 8_2, 3_0], G_2[6_4, 6_2, 10_3, 6_1, 10_0, 1_4, 9_1, 5_0, 7_4, 7_0],$$
$$G_2[7_3, 1_1, 7_2, 6_3, 4_1, 10_0, 9_3, 13_4, 5_0, 11_2], G_2[11_3, 13_0, 14_4, 9_1, 8_4, 0_1, 2_2, 14_3, 3_0, 6_2],$$
$$G_2[10_1, 4_3, 6_4, 14_3, 13_4, 12_2, 3_1, 0_3, 0_4, 11_2], G_2[0_0, 0_1, 0_2, 2_0, 4_1, 2_2, 0_3, 7_0, 5_1, 4_2],$$
$$G_2[0_0, 5_1, 1_2, 5_0, 12_1, 4_3, 14_4, 4_0, 14_3, 12_2], G_2[0_0, 8_1, 3_3, 0_1, 3_2, 9_3, 0_4, 2_2, 4_4, 6_3],$$
$$G_2[0_1, 8_2, 3_4, 14_0, 8_4, 2_1, 14_2, 2_4, 6_2, 13_4], G_2[0_3, 5_4, 13_0, 5_3, 12_1, 13_3, 6_4, 9_0, 14_4, 6_2]\},$$
$$B_3 = \{G_3[11_4, 1_3, 10_0, 10_2, 4_1, 12_2, 13_4, 7_1, 3_4, 7_0], G_3[6_1, 9_4, 12_0, 10_4, 13_1, 3_3, 11_0, 8_2, 6_0, 4_2],$$
$$G_3[9_0, 7_3, 9_1, 8_2, 2_3, 0_0, 6_4, 3_0, 6_2, 5_1], G_3[8_0, 0_1, 7_4, 7_0, 6_1, 8_2, 5_3, 6_2, 12_0, 2_4],$$
$$G_3[13_0, 0_4, 8_2, 6_4, 2_1, 9_3, 2_4, 6_3, 4_2, 10_1], G_3[0_0, 1_1, 1_2, 9_0, 0_1, 0_3, 12_0, 2_1, 4_0, 8_2],$$
$$G_3[0_0, 4_1, 0_3, 1_0, 11_1, 4_3, 0_2, 6_3, 0_1, 1_3], G_3[0_0, 4_3, 7_4, 14_0, 7_3, 11_2, 6_4, 9_3, 0_1, 10_4],$$
$$G_3[0_1, 5_2, 3_3, 2_2, 9_3, 11_4, 14_2, 2_4, 3_1, 1_4], G_3[0_2, 0_3, 5_1, 2_2, 10_3, 0_4, 7_2, 6_4, 8_3, 9_4]\},$$
$$B_4 = \{G_4[8_4, 3_3, 9_2, 7_1, 8_0, 11_3, 14_1, 13_2, 5_3, 6_0], G_4[12_0, 13_1, 9_4, 6_0, 14_3, 0_1, 1_4, 6_2, 14_1, 4_2],$$
$$G_4[1_1, 11_2, 9_4, 4_1, 14_4, 6_0, 4_4, 3_3, 4_2, 0_4], G_4[6_4, 8_3, 12_0, 7_4, 4_3, 0_2, 14_1, 14_0, 14_3, 3_1],$$
$$G_4[14_4, 5_2, 7_0, 13_2, 5_0, 1_4, 11_0, 6_3, 8_1, 10_3], G_4[0_0, 2_1, 0_2, 1_0, 4_1, 12_2, 3_3, 8_2, 5_0, 2_2],$$
$$G_4[0_0, 4_1, 1_2, 6_0, 1_1, 10_2, 1_4, 10_3, 1_0, 7_3], G_4[0_0, 6_1, 9_2, 13_1, 6_0, 8_3, 2_4, 11_1, 3_0, 4_4],$$
$$G_4[0_0, 4_3, 11_1, 0_4, 10_2, 8_3, 3_4, 10_3, 9_1, 6_4], G_4[0_3, 0_4, 8_1, 8_4, 0_2, 11_3, 9_2, 12_3, 11_4, 14_2]\},$$
$$B_5 = \{G_5[0_1, 13_0, 5_3, 14_4, 13_2, 3_1, 2_3, 9_1, 8_0, 10_3], G_5[3_0, 9_1, 13_4, 10_2, 1_4, 4_1, 4_4, 11_2, 5_3, 9_2],$$
$$G_5[3_1, 5_4, 5_2, 0_1, 2_0, 0_3, 10_0, 1_3, 2_2, 7_0], G_5[11_2, 2_3, 13_0, 1_2, 4_3, 4_1, 5_4, 12_1, 3_4, 6_0],$$
$$G_5[11_0, 11_3, 12_4, 0_2, 2_1, 8_2, 12_1, 7_4, 0_1, 7_2], G_5[0_0, 0_1, 1_2, 1_0, 4_1, 10_3, 8_1, 2_4, 2_0, 9_2],$$
$$G_5[0_2, 10_3, 2_4, 7_2, 7_3, 11_4, 12_3, 12_4, 9_0, 13_4], G_5[0_2, 14_4, 8_3, 6_2, 9_1, 12_0, 2_4, 7_3, 5_4, 12_3],$$
$$G_5[0_0, 4_1, 1_3, 6_0, 14_1, 10_3, 3_2, 10_4, 1_0, 7_4], G_5[0_0, 9_1, 8_2, 4_0, 3_2, 1_0, 0_4, 4_3, 0_1, 3_3]\},$$
$$B_6 = \{G_6[10_0, 5_3, 7_1, 3_0, 10_1, 5_2, 8_4, 10_3, 6_2, 14_3], G_6[3_2, 14_0, 12_2, 14_1, 13_0, 5_3, 11_4, 3_3, 13_1, 10_3],$$
$$G_6[4_4, 12_0, 5_1, 4_2, 12_1, 5_2, 14_4, 3_0, 2_2, 6_0], G_6[4_2, 5_4, 1_2, 12_1, 6_4, 3_1, 8_2, 0_0, 11_1, 10_3],$$
$$G_6[13_0, 4_1, 1_4, 1_0, 12_3, 0_2, 14_3, 13_4, 3_2, 6_3], G_6[5_1, 0_0, 2_1, 5_2, 2_0, 11_2, 3_4, 0_3, 4_1, 7_0],$$
$$G_6[1_2, 0_0, 0_2, 0_3, 1_0, 2_3, 7_4, 7_3, 4_0, 6_4], G_6[12_2, 0_0, 6_2, 4_4, 3_0, 8_4, 6_1, 12_4, 1_3, 10_1],$$
$$G_6[7_4, 0_1, 4_3, 1_4, 6_0, 0_3, 5_2, 5_4, 5_1, 14_2], G_6[9_4, 0_3, 2_4, 7_1, 9_3, 12_0, 5_4, 0_1, 1_3, 13_1]\},$$
$$B_7 = \{G_7[1_0, 14_2, 2_1, 9_2, 6_4, 9_1, 9_3, 7_2, 5_1, 1_3], G_7[7_2, 7_3, 12_4, 1_2, 6_0, 6_2, 12_1, 11_2, 6_3, 6_4],$$
$$G_7[2_1, 14_3, 5_1, 9_3, 12_4, 6_2, 8_4, 10_0, 3_4, 3_2], G_7[2_4, 13_2, 12_3, 14_0, 13_4, 11_0, 7_3, 6_0, 4_1, 10_3],$$
$$G_7[1_0, 3_3, 2_1, 7_4, 1_3, 4_0, 1_1, 6_3, 8_1, 2_4], G_7[9_1, 6_0, 1_1, 1_2, 0_0, 2_1, 3_0, 10_1, 2_0, 12_3],$$
$$G_7[6_0, 0_2, 4_1, 7_2, 0_0, 9_1, 2_2, 6_3, 1_0, 10_4], G_7[12_4, 0_3, 3_2, 6_3, 0_0, 4_2, 7_0, 3_4, 3_0, 10_3],$$
$$G_7[3_4, 2_2, 9_3, 7_4, 0_0, 10_4, 8_1, 1_4, 4_0, 9_2], G_7[1_4, 3_1, 3_4, 2_1, 0_2, 13_3, 5_1, 8_4, 9_3, 3_2]\},$$

$$\begin{aligned}
B_8 = \{&G_8[11_0,1_3,12_2,1_1,13_4,14_2,9_3,10_1,4_0,2_4], G_8[10_3,7_0,4_4,10_0,12_4,8_1,0_4,11_0,4_1,3_4],\\
&G_8[1_3,1_1,7_2,6_0,4_3,10_4,4_1,3_2,14_4,8_2], G_8[1_2,0_1,1_3,10_1,8_0,13_1,6_0,0_3,2_2,4_3],\\
&G_8[11_0,12_1,6_2,2_1,0_4,11_2,11_3,5_2,0_0,8_3], G_8[2_2,0_1,4_0,13_1,5_3,3_0,10_2,6_0,3_1,0_0],\\
&G_8[10_2,10_1,5_3,4_0,12_4,12_0,8_3,13_0,9_2,0_0], G_8[6_3,12_2,2_1,10_2,12_4,1_1,11_4,3_1,0_3,0_0],\\
&G_8[2_4,5_2,3_4,6_3,10_4,2_2,5_4,10_3,13_2,0_1], G_8[9_4,7_3,10_4,0_0,7_4,11_3,3_4,2_1,5_3,0_2]\},\\
B_9 = \{&G_9[3_1,5_0,0_1,6_4,6_1,2_4,1_3,8_4,9_2,10_3], G_9[3_3,11_0,14_1,4_2,1_1,8_2,7_3,3_4,1_3,11_2],\\
&G_9[13_0,2_2,13_1,3_3,0_0,9_3,8_1,1_3,2_0,13_4], G_9[3_0,13_2,3_1,5_3,5_0,14_2,12_1,11_2,5_1,2_4],\\
&G_9[4_2,6_1,10_3,13_4,11_0,6_3,12_4,2_2,5_0,11_4], G_9[1_2,0_1,1_0,13_3,4_1,2_0,7_4,14_3,1_1,0_0],\\
&G_9[0_2,2_4,10_1,9_3,14_4,10_0,14_3,11_4,11_2,0_1], G_9[13_4,3_3,1_4,6_0,9_4,4_2,10_3,10_2,3_4,0_2],\\
&G_9[13_2,4_1,8_0,1_3,7_2,5_0,11_3,0_4,5_1,0_0], G_9[3_2,6_1,9_0,3_4,2_2,11_0,13_3,7_4,9_1,0_0]\},\\
B_{10} = \{&G_{10}[11_2,4_1,6_0,2_4,12_3,13_0,9_1,2_2,4_4,0_0], G_{10}[8_2,11_1,4_3,7_2,7_1,1_4,11_2,2_3,10_2,2_4],\\
&G_{10}[10_1,0_2,12_4,14_0,6_2,2_0,10_3,5_1,7_3,11_4], G_{10}[2_0,8_4,2_2,5_3,9_2,8_1,6_3,3_1,6_0,5_4],\\
&G_{10}[9_1,9_3,0_0,10_4,9_2,13_4,7_3,4_0,12_4,1_3], G_{10}[5_1,1_3,0_1,10_4,3_0,4_3,3_4,13_1,7_3,5_4],\\
&G_{10}[0_2,2_1,0_0,12_2,2_0,2_3,7_1,1_2,11_4,14_3], G_{10}[14_3,5_2,0_0,5_4,5_0,7_4,8_2,6_4,11_3,9_0],\\
&G_{10}[11_1,8_4,7_0,12_1,11_4,0_2,5_3,5_2,3_1,0_3], G_{10}[1_0,0_1,0_0,6_2,6_0,0_2,1_1,7_0,2_3,11_1]\},\\
B_{11} = \{&G_{11}[5_4,4_0,4_4,9_0,6_3,2_2,12_3,5_2,12_4,14_1], G_{11}[5_3,0_4,4_2,14_4,14_1,6_4,3_0,1_2,7_3,6_1],\\
&G_{11}[5_3,14_4,13_1,5_0,12_4,6_0,10_4,3_3,2_0,11_2], G_{11}[8_1,0_2,9_0,2_2,8_4,8_2,12_4,10_3,1_0,4_2],\\
&G_{11}[10_4,9_2,9_0,9_1,11_4,6_3,14_1,8_2,6_1,11_3], G_{11}[6_1,1_0,2_2,0_0,1_1,2_0,6_2,2_1,8_2,11_0],\\
&G_{11}[10_3,4_0,0_3,0_0,6_1,8_0,3_3,7_1,7_3,5_2], G_{11}[10_4,14_0,4_3,0_0,9_1,12_0,6_4,11_1,4_4,1_1],\\
&G_{11}[3_2,5_4,8_3,0_0,5_2,8_1,6_2,7_3,1_1,3_3], G_{11}[1_4,5_3,11_4,0_1,8_3,4_1,7_3,8_2,5_4,11_2]\},\\
B_{12} = \{&G_{12}[14_4,8_1,3_2,2_4,14_0,1_2,6_0,9_1,2_3,5_2], G_{12}[9_0,4_1,7_4,13_1,2_0,2_3,11_0,9_4,2_2,5_4],\\
&G_{12}[4_4,10_3,7_0,6_1,10_2,5_3,5_4,8_2,4_3,5_0], G_{12}[0_3,4_1,4_2,13_3,12_0,2_2,7_4,12_2,14_3,11_2],\\
&G_{12}[5_1,2_4,13_2,3_3,8_1,3_0,12_4,13_1,11_2,13_0], G_{12}[0_0,0_1,2_0,14_1,10_2,1_1,0_2,8_3,1_0,1_2],\\
&G_{12}[0_3,2_4,7_1,9_4,14_3,11_1,10_3,12_0,13_4,13_1], G_{12}[0_3,7_4,7_2,7_3,13_4,3_0,11_4,0_2,6_3,0_1],\\
&G_{12}[0_0,4_1,10_0,3_1,4_3,6_1,3_3,1_4,1_0,12_2], G_{12}[0_0,9_2,3_1,6_2,8_4,4_3,1_4,5_3,10_0,2_4]\},\\
B_{13} = \{&G_{13}[4_0,13_3,7_2,10_3,6_1,0_3,8_0,8_1,5_0,9_2], G_{13}[2_4,13_3,9_4,7_2,10_1,12_0,12_3,6_0,10_4,14_1],\\
&G_{13}[14_4,2_2,11_1,9_4,14_2,13_1,1_2,1_1,8_3,7_0], G_{13}[12_3,13_2,4_1,4_3,13_4,9_1,14_3,6_0,6_2,2_4],\\
&G_{13}[13_4,5_2,9_0,11_4,10_3,4_1,6_2,8_3,0_2,2_0], G_{13}[0_0,2_1,3_0,0_1,1_3,4_1,11_2,8_3,4_0,6_2],\\
&G_{13}[0_0,14_3,12_4,6_0,5_4,10_3,14_2,14_4,0_2,1_4], G_{13}[0_1,11_2,13_1,6_2,4_3,14_2,5_4,13_3,4_4,11_3],\\
&G_{13}[0_0,5_1,9_0,2_1,1_4,6_1,1_2,4_4,1_0,3_3], G_{13}[0_0,7_1,12_0,6_2,0_4,9_1,6_4,7_3,2_0,12_4]\}.
\end{aligned}$$

For $i \in [2,13]$, a G_i-decomposition of $K_{5\times 15}$ consists of the G_i-blocks in B_i under the action of the map $r_s \mapsto (r+1 \pmod{15})_s$.

Example 27. Let $V(K_{9,15}) = [0,8] \cup [9,23]$ with the obvious vertex partition. Let

$$\begin{aligned}
B_{14} = \{&G_{14}[8,18,2,20,5,14,4,17,7,23], G_{14}[20,3,11,4,12,6,9,0,17,8],\\
&G_{14}[9,5,13,1,19,2,17,6,22,7], G_{14}[0,10,1,15,4,13,3,16,2,11],\\
&G_{14}[0,14,2,12,7,16,5,11,6,15], G_{14}[0,18,3,9,8,22,1,16,6,23],\\
&G_{14}[1,18,4,21,7,20,0,19,5,23], G_{14}[2,13,6,10,4,22,3,19,8,21],\\
&G_{14}[5,10,7,14,3,21,1,12,8,15]\}.
\end{aligned}$$

Then a G_{14}-decomposition of $K_{9,15}$ consists of the G_{14}-blocks in B_{14}.

Example 28. Let $V(K_{15,15}) = \{r_s : r \in \mathbb{Z}_{15}, s \in \mathbb{Z}_2\}$ with the obvious vertex partition. Let $B_{14} = \{G_{14}[2_0, 14_1, 4_0, 8_1, 3_0, 10_1, 7_0, 9_1, 0_0, 0_1]\}$. Then a G_{14}-decomposition of $K_{15,15}$ consists of the G_{14}-blocks in B_{14} under the action of the map $r_s \mapsto (r + 1 \pmod{15})_s$.

Example 29. Let $V(K_{25}\backslash K_{10}) = \mathbb{Z}_{25}$ with $[0, 9]$ being the vertices in the hole. Let

$$\begin{aligned} B_2 = \{&G_2[12, 16, 13, 11, 14, 4, 23, 24, 15, 21], G_2[16, 17, 1, 20, 14, 22, 24, 2, 23, 7],\\ &G_2[14, 18, 7, 21, 22, 5, 13, 10, 23, 3], G_2[2, 14, 10, 4, 12, 20, 24, 5, 16, 15],\\ &G_2[16, 18, 20, 9, 23, 13, 19, 0, 21, 2], G_2[15, 20, 17, 6, 22, 3, 10, 24, 13, 4],\\ &G_2[0, 10, 11, 1, 13, 15, 18, 5, 14, 12], G_2[0, 14, 16, 3, 19, 18, 21, 4, 11, 17],\\ &G_2[0, 15, 22, 1, 24, 11, 19, 2, 12, 23], G_2[1, 10, 15, 3, 17, 21, 24, 6, 12, 18],\\ &G_2[2, 13, 17, 4, 19, 10, 16, 9, 22, 20], G_2[3, 21, 13, 9, 17, 12, 22, 7, 11, 20],\\ &G_2[5, 12, 19, 1, 23, 8, 17, 10, 21, 11], G_2[8, 14, 21, 5, 23, 6, 11, 18, 17, 19],\\ &G_2[10, 20, 7, 13, 8, 12, 15, 9, 14, 6], G_2[15, 19, 6, 16, 23, 18, 22, 8, 24, 7],\\ &G_2[18, 24, 9, 19, 22, 11, 16, 8, 20, 0]\}. \end{aligned}$$

Then a G_2-decomposition of $K_{25}\backslash K_{10}$ consists of the G_2-blocks in B_2.

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