# THE SPECTRUM PROBLEM FOR THE CONNECTED CUBIC GRAPHS OF ORDER 10 

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#### Abstract

We show that if $G$ is a connected cubic graph of order 10, then there exists a $G$-decomposition of $K_{v}$ if and only if $v \equiv 1$ or $10(\bmod 15)$ except when $v=10$ and $G$ is one of 5 specific graphs.


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## 1. Introduction

For a graph $G$, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of $G$, respectively. We use $K_{r \times s}$ to denote the complete simple multipartite graph with $r$ parts of size $s$, and $K_{r \times s, t}$ (or $K_{t, r \times s}$ ) to denote the complete simple multipartite graph with $r$ parts of size $s$ and one part of size $t$. If $G^{\prime}$ is a subgraph of $G$, then $G \backslash G^{\prime}$ denotes the graph obtained from $G$ by deleting the edges of $G^{\prime}$. The graph $K_{v} \backslash K_{u}$ is called a complete graph of order $v$ with a hole of size $u$ and the vertices of $K_{u}$ are called the vertices in the hole. If $a$ and $b$ are integers with $a \leq b$, let $[a, b]$ denote the set $\{a, a+1, \ldots, b\}$.

A decomposition of a graph $K$ is a set $\Delta=\left\{G_{1}, G_{2}, \ldots, G_{t}\right\}$ of subgraphs of $K$ such that each edge of $K$ appears in exactly one $G_{i}$. If each $G_{i}$ in $\Delta$ is isomorphic to a given graph $G$, the decomposition is called a $G$-decomposition of $K$ and the copies of $G$ in $\Delta$ are called $G$-blocks. A $\{G, H\}$-decomposition of $K$ is defined similarly. A $G$-decomposition of $K$ is also known as a $(K, G)$-design and a ( $K_{v}, G$ )-design is often known as a $G$-design of order $v$. If a $G$-decomposition of $K$ exists, then we may say $G$ decomposes $K$ or $K$ is decomposable by $G$.

A $\left(K_{v}, K_{k}\right)$-design is also known as a balanced incomplete block design and a $K_{k}$-decomposition of $K_{v_{1}, v_{2}, \ldots, v_{t}}$ is a group divisible design. We will make use of group divisible designs in our constructions. However, for the sake of brevity, we will refrain from adding too many details on these concepts and direct the interested reader to the Handbook of Combinatorial Designs [10] and to the summaries within on group divisible designs [11].

Given a graph $G$, a classical problem in combinatorics is to find necessary and sufficient conditions on $v$ for the existence of a $\left(K_{v}, G\right)$-design. This is known as the spectrum problem for $G$ and it has been investigated and settled for numerous classes of simple graphs $G$ (see [2] and [7] for summaries, and the website maintained by Bryant and McCourt [8] for more up to date results). When $G$ is a complete graph, the spectrum problem was settled by Kirkman [15] for $G=K_{3}$ and by Hanani [13] for $G \in\left\{K_{4}, K_{5}\right\}$. The spectrum problem for connected 2regular graphs (i.e., for cycles) was investigated by numerous authors and settled by Alspach and Gavlas in [6].

In this work, we are concerned with the spectrum problem for the connected 3 -regular (i.e., cubic) graphs of order 10 . There are 19 non-isomorphic such graphs (see Figure 1 and [18]). The spectrum problem has been settled for 6 of them (the unlabeled ones in the figure). Here we settle it for the remaining 13 graphs.

If $G$ is a cubic graph of order 10 and if there exists a $\left(K_{v}, G\right)$-design, then we must have $15 \left\lvert\,\binom{ v}{2}\right.$ (since the number of edges in $G$ is 15 and the number of edges in $K_{v}$ is $\binom{v}{2}$ ) and $3 \mid v-1$ (since $G$ is 3 -regular and $K_{v}$ is $(v-1)$-regular). Thus we must have $v \equiv 1$ or $10(\bmod 15)$.


Figure 1. The 19 connected cubic graphs of order 10. The spectrum problem for the 6 unlabeled graphs has previously been settled.

Because of the interest in the Petersen graph (Graph $G_{19}$ in Figure 1), it had been known that it does not decompose $K_{10}$ (see [14]). In 1996, Adams and Bryant [1] settled the spectrum problem for the Petersen graph by showing there exists a $G_{19}$-decomposition of $K_{v}$ if and only if $v \equiv 1$ or $10(\bmod 15)$ and $v \neq 10$. Subsequently, Adams, Bryant, and Khodkar [3] showed that 4 additional connected cubic graphs of order 10 , namely $G_{2}, G_{14}, G_{15}$, and $G_{17}$ do not decompose $K_{10}$. The aim of this manuscript is to show that these are the only cases when $v \equiv 1$ or $10(\bmod 15)$ and a $G_{i}$-decomposition of $K_{v}$ does not exist. In 2016, the spectrum problem for the 5 -Prism and 5 -Mobius $\left(G_{15}\right.$ and $G_{17}$, respectively) was settled by Meszka, Nedela, Rosa, and Skoviera in [17]. More recently, Adams et al. [4] settled the spectrum problem for Graphs $G_{1}, G_{16}$ and $G_{18}$.

It is known that if $G$ is a connected cubic graph of order 10 , then there exists a $G$-design of order $v$ for all $v \equiv 1(\bmod 30)($ see $[19]$ if $G$ is bipartite and [20]
if $G$ is tripartite). Thus to settle the spectrum problem for these 13 remaining graphs, it suffices to settle the cases $v \equiv 10,16$ and $25(\bmod 30)$.

Before proceeding, we note that several authors have considered the spectrum problem for various cubic graphs. In [13], Hanani settled the problem for $K_{4^{-}}$ designs by showing that there exists a $\left(K_{v}, K_{4}\right)$-design if and only if $v \equiv 1$ or $4(\bmod 12)$. The spectrum problem for $K_{3,3}$-designs was settled by Guy and Beineke in [12]. The spectrum problem for the 3 -prism was settled by Carter [9]. The spectrum problem for the 3 -dimensional cube was settled by Maheo in [16]. More recently, the spectrum problem for the remaining four connected cubic graphs of order 8 was settled in [5].

We will show that if $G_{i}$ is a connected cubic graph of order 10 , then there exists a $\left(K_{v}, G_{i}\right)$-design of order $v$ if and only if $v \equiv 1$ or $10(\bmod 15)$, and $v \neq 10$ when $G_{i} \in\left\{G_{2}, G_{14}, G_{15}, G_{17}, G_{19}\right\}$. Since these results are known when $G_{i} \in\left\{G_{1}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}\right\}$, we will focus on the remaining 13 graphs. Henceforth, each of the graphs $G_{i}$ with $i \in[2,14]$ in Figure 1, with vertices labeled as in the figure, will be represented by $G_{i}\left[v_{0}, v_{1}, \ldots, v_{9}\right]$. For example, $G_{2}\left[v_{0}, v_{1}, \ldots, v_{9}\right]$ is the graph with vertex set $\left\{v_{0}, v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\right.$, $\left.v_{8}, v_{9}\right\}$ and edge set $\left\{\left\{v_{0}, v_{1}\right\},\left\{v_{1}, v_{2}\right\},\left\{v_{2}, v_{3}\right\},\left\{v_{3}, v_{4}\right\},\left\{v_{4}, v_{5}\right\},\left\{v_{5}, v_{6}\right\},\left\{v_{6}, v_{7}\right\}\right.$, $\left.\left\{v_{7}, v_{8}\right\},\left\{v_{8}, v_{9}\right\},\left\{v_{9}, v_{0}\right\},\left\{v_{0}, v_{2}\right\},\left\{v_{1}, v_{9}\right\},\left\{v_{3}, v_{8}\right\},\left\{v_{4}, v_{6}\right\},\left\{v_{5}, v_{7}\right\}\right\}$. In some cases, we may write a $G$-decomposition of a graph with vertex set $V$ as a pair $(V, B)$, where $B$ is a collection of copies of $G$ that partitions the edge-set of the graph.

## 2. General Constructions

Our main constructions depend on many small examples. These examples are given in the Appendix. We first state some results on $K_{4}$-decompositions of certain complete multipartite graphs as well as a result on $\left\{K_{3}, K_{5}\right\}$-decompositions of $K_{v}$. We make use of these results in the next section. All of these results can be found in the Handbook of Combinatorial Designs [10].

Theorem 1. There exists a $K_{4}$-decomposition of $K_{n \times 2}$ if and only if $n \equiv 1$ $(\bmod 3)$ and $n \geq 7$.
Theorem 2. There exists a $K_{4}$-decomposition of $K_{n \times 3}$ if and only if $n \equiv 0$ or 1 $(\bmod 4)$ and $n \geq 4$.

Theorem 3. There exists a $K_{4}$-decomposition of $K_{n \times 2,5}$ if and only if $n \equiv 0$ $(\bmod 3)$ and $n \geq 9$.
Theorem 4. There exists a $K_{4}$-decomposition of $K_{(n-2) \times 3,6}$ if and only if $n \equiv 2$ or $3(\bmod 4)$ and $n \geq 7$.
Theorem 5. There exists a $\left\{K_{3}, K_{5}\right\}$-decomposition of $K_{v}$ if $v$ is odd.

Because of the relevance of the decompositions of $K_{10}$ results from [3], we state them in a separate lemma.

Lemma 6. Let $G$ be a connected cubic graph of order 10 (see Figure 1). There exists a $G$-decomposition of $K_{10}$ if and only if $G \notin\left\{G_{2}, G_{14}, G_{15}, G_{17}, G_{19}\right\}$.

We are ready to proceed with our results.
Lemma 7. For every integer $x \geq 2$ and each $i \in[3,13]$, there exists a $G_{i}$ decomposition of $K_{(3 x+1) \times 10}$.

Proof. By Theorem 1, there exists a $K_{4}$-decomposition of $K_{(3 x+1) \times 2}$. Replacing each vertex of $K_{(3 x+1) \times 2}$ by a set of 5 vertices and each edge of $K_{(3 x+1) \times 2}$ by a copy of $K_{5,5}$ gives a $K_{4 \times 5}$-decomposition of $K_{(3 x+1) \times 10}$. By Example 24, there exist $G_{i}$-decompositions of $K_{4 \times 5}$. Thus the result now follows.

Lemma 8. Let $i \in[3,13]$. There exists a $G_{i}$-decomposition of $K_{v}$ for all $v \equiv 10$ $(\bmod 30)$.

Proof. Let $x$ be a nonnegative integer and let $v=30 x+10$. For $v=10$ and $v=40$, the results follow from Lemma 6 and Example 20, respectively. So we may assume $x \geq 2$.

Let $H$ be the complete graph with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{3 x+1}$ with $\left|H_{j}\right|=10$ for each $j \in[1,3 x+1]$. By Lemma 7 , there is a $G_{i}$-decomposition $B_{i}^{\prime}$ of $K_{(3 x+1) \times 10}$ with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{3 x+1}$ and the obvious vertex partition. For each $i \in[3,13]$ and $j \in[1,3 x+1]$, let $B_{i, j}$ be a $G_{i}$-decomposition of $K_{10}$ with vertex set $H_{j}$. Then $\left(V, B_{i}\right)$ where $V=V(H)$ and $B_{i}=B_{i}^{\prime} \cup B_{i, 1} \cup$ $B_{i, 2} \cup \cdots \cup B_{i, 3 x+1}$ is a $G_{i}$-decomposition of $K_{30 x+10}$.

Lemma 9. For every positive integer $x$ and each $i \in[2,14]$, there exists a $G_{i}$ decomposition of $K_{(2 x+1) \times 15}$.

Proof. It is simple to see that $K_{15,15}$ decomposes $K_{(2 x+1) \times 15}$ and since $G_{14}$ decomposes $K_{15,15}$ (by Example 28), the result follows for $G_{14}$. By Theorem 5, there exists a $\left\{K_{3}, K_{5}\right\}$-decomposition of $K_{2 x+1}$. Replacing each vertex of $K_{2 x+1}$ by a set of 15 vertices and each edge of $K_{2 x+1}$ by a copy of $K_{15,15}$ gives a $\left\{K_{3 \times 15}, K_{5 \times 15}\right\}$-decomposition of $K_{(2 x+1) \times 15}$. By Examples 25 and 26, there exist $G_{i}$-decompositions of $K_{3 \times 15}$ and $K_{5 \times 15}$ for all $i \in[2,13]$. Thus the result now follows.

Lemma 10. Let $i \in[2,14]$. There exists a $G_{i}$-decomposition of $K_{v}$ for all $v \equiv 16$ $(\bmod 30)$.

Proof. Let $x$ be a nonnegative integer and let $v=30 x+16$. For $v=16$, the result follows from Example 18. So we may assume $x \geq 1$.

Let $H$ be the complete graph with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{2 x+1} \cup\{\infty\}$ with $\left|H_{j}\right|=15$ for each $j \in[1,2 x+1]$. By Lemma 9 , there is a $G_{i}$-decomposition $B_{i}^{\prime}$ of $K_{(2 x+1) \times 15}$ with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{2 x+1}$ and the obvious vertex partition. For each $i \in[2,14]$ and $j \in[1,2 x+1]$, let $B_{i, j}$ be a $G_{i}$-decomposition of $K_{16}$ with vertex set $H_{j} \cup\{\infty\}$. Then $\left(V, B_{i}\right)$ where $V=V(H)$ and $B_{i}=$ $B_{i}^{\prime} \cup B_{i, 1} \cup B_{i, 2} \cup \cdots \cup B_{i, 2 x+1}$ is a $G_{i}$-decomposition of $K_{30 x+16}$.
Lemma 11. For every integer $x \geq 3$ and each $i \in[3,13]$, there exists a $G_{i}$ decomposition of $K_{(3 x) \times 10,25}$.
Proof. By Theorem 3, there exists a $K_{4}$-decomposition of $K_{(3 x) \times 2,5}$ for $x \geq 3$. Replacing each vertex of $K_{(3 x) \times 2,5}$ by a set of 5 vertices and each edge of $K_{(3 x) \times 2,5}$ by a copy of $K_{5,5}$ gives a $K_{4 \times 5}$-decomposition of $K_{(3 x) \times 10,25}$. By Example 24, there exist $G_{i}$-decompositions of $K_{4 \times 5}$. Thus the result now follows.

Lemma 12. Let $i \in[3,13]$. There exists a $G_{i}$-decomposition of $K_{v}$ for all $v \equiv 25$ $(\bmod 30)$.
Proof. Let $x$ be a nonnegative integer and let $v=30 x+25$. There exist $G$ decompositions of $K_{25}, K_{55}$ and $K_{85}$ by Examples 19, 21 and 22, respectively. Thus it remains to consider the case $x \geq 3$.

Let $H$ be the complete graph with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{3 x+1}$ with $\left|H_{j}\right|=10$ for $j \in[1,3 x]$ and $\left|H_{3 x+1}\right|=25$. By Lemma 11, there exists a $G_{i^{-}}$ decomposition $B_{i}^{\prime}$ of $K_{(3 x) \times 10,25}$ with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{3 x+1}$ and the obvious vertex partition. For each $i \in[3,13]$ and $j \in[1,3 x]$, let $B_{i, j}$ be a $G_{i^{-}}$ decomposition of $K_{10}$ with vertex set $H_{j}$. Also, let $B_{i, 3 x+1}$ be a $G_{i}$-decomposition of $K_{25}$ with vertex set $H_{3 x+1}$. Then $\left(V, B_{i}\right)$ where $V=V(H)$ and $B_{i}=B_{i}^{\prime} \cup$ $B_{i, 1} \cup B_{i, 2} \cup \cdots \cup B_{i, 3 x+1}$ is a $G_{i}$-decomposition of $K_{30 x+25}$.

Because there is no $G_{2^{-}}$nor a $G_{14}$-decomposition of $K_{10}$, we will treat the case $v \equiv 10(\bmod 15)$ for these two graphs separately. Moreover, since $G_{14}$ is bipartite and $G_{2}$ is not, we will use different approaches for the two graphs.
Lemma 13. There exists a $G_{2}$-decomposition of $K_{x \times 15}$ when $x \equiv 0$ or $1(\bmod 4)$, $x \geq 4$, and a $G_{2}$-decomposition of $K_{(x-2) \times 15,30}$ when $x \equiv 2$ or $3(\bmod 4)$ and $x \geq 7$.
Proof. By Theorem 2, if $x \equiv 0$ or $1(\bmod 4)$, then there exists a $K_{4}$-decomposition of $K_{x \times 3}$, and by Theorem 4 , if $x \equiv 2$ or $3(\bmod 4)$ and $x \geq 7$, then there exists a $K_{4}$-decomposition of $K_{(x-2) \times 3,6}$. Replacing each vertex in these decompositions by a set of 5 vertices and each edge by a copy of a $K_{5,5}$ yields a $K_{4 \times 5}$-decomposition of $K_{x \times 15}$ when $x \equiv 0$ or $1(\bmod 4)$ and of $K_{(x-2) \times 15,30}$ when $x \equiv 2$ or $3(\bmod 4)$ and $x \geq 7$. The result follows by Example 24 .

Lemma 14. If $v \equiv 10(\bmod 15)$ and $v \neq 10$, then there exists a $G_{2}$-decomposition of $K_{v}$.

Proof. Let $x$ be a positive integer and let $v=15 x+10$. By Examples 19, 20, 21, and 23, there exist $G_{2}$-decompositions of $K_{25}, K_{40}, K_{55}$, and $K_{100}$, respectively. So we may assume that $x \notin\{1,2,3,6\}$.

First, suppose $x \equiv 0$ or $1(\bmod 4)$ and let $H$ be the complete graph with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{x+1}$ with $\left|H_{j}\right|=15$ for $j \in[1, x]$ and $\left|H_{x+1}\right|=10$. By Lemma 13, there exists a $G_{2}$-decomposition $B^{\prime}$ of $K_{x \times 15}$ with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{x}$ and the obvious vertex partition. Let $B_{1}$ be a $G_{2}$-decomposition of $K_{25}$ with vertex set $H_{1} \cup H_{x+1}$. For each $j \in[2, x]$, let $B_{j}$ be a $G_{2}$-decomposition of $K_{25} \backslash K_{10}$ with vertex set $H_{j} \cup H_{x+1}$ (with the vertices of $H_{x+1}$ being the vertices in the hole). Then ( $V, B$ ) where $V=V(H)$ and $B=B^{\prime} \cup B_{1} \cup \cdots \cup B_{x}$ is a $G_{2}$-decomposition of $K_{15 x+10}$.

Now suppose $x \equiv 2$ or $3(\bmod 4)$ and $x \notin\{2,3,6\}$. Let $H$ be the complete graph with vertex set $H_{0} \cup H_{1} \cup \cdots \cup H_{x-1}$ with $\left|H_{0}\right|=30,\left|H_{x-1}\right|=10$, and $\left|H_{j}\right|=15$ for $j \in[1, x-2]$. By Lemma 13, there exists a $G_{2}$-decomposition $B^{\prime}$ of $K_{(x-2) \times 15,30}$ with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{x-2} \cup H_{0}$ and the obvious vertex partition. Let $B_{0}$ be a $G_{2}$-decomposition of $K_{40}$ with vertex set $H_{0} \cup H_{x-1}$. For each $j \in[1, x-2]$, let $B_{j}$ be a $G_{2}$-decomposition of $K_{25} \backslash K_{10}$ with vertex set $H_{j} \cup H_{x-1}$ (with the vertices of $H_{x-1}$ being the vertices in the hole). Then ( $V, B$ ) where $V=V(H)$ and $B=B^{\prime} \cup B_{0} \cup B_{1} \cup \cdots \cup B_{x-2}$ is a $G_{2}$-decomposition of $K_{15 x+10}$.

Lemma 15. There exists a $G_{14}$-decomposition of $K_{(x-1) \times 15,24}$ for all integers $x \geq 2$.
Proof. It is simple to see that $K_{(x-1) \times 15,24}$ can be decomposed into $\binom{x-1}{2}$ copies of $K_{15,15}$ and $x-1$ copies of $K_{15,24}$. Also note that $K_{15,24}$ can be decomposed into one copy of $K_{15,15}$ and one copy of $K_{9,15}$. By Example 28, there exists a $G_{14^{-}}$ decomposition of $K_{15,15}$ and by Example 27, there exists a $G_{14}$-decomposition of $K_{9,15}$. The result now follows.

Lemma 16. If $v \equiv 10(\bmod 15)$ and $v \neq 10$, then there exists a $\left(K_{v}, G_{14}\right)$-design.
Proof. Let $x$ be a positive integer and let $v=15 x+10$. By Example 19, there exists a $G_{14}$-decomposition of $K_{25}$ and thus we may assume that $x \geq 2$.

Let $H$ be the complete graph with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{x} \cup\{\infty\}$ with $\left|H_{x}\right|=24$ and $\left|H_{j}\right|=15$ for $j \in[1, x-1]$. By Lemma 15 , there exists a $G_{14}$-decomposition $B^{\prime}$ of $K_{(x-1) \times 15,24}$ with vertex set $H_{1} \cup H_{2} \cup \cdots \cup H_{x}$ and the obvious vertex partition. Let $B_{x}$ be a $G_{14}$-decomposition of $K_{25}$ with vertex set $H_{x} \cup\{\infty\}$. For each $j \in[1, x-1]$, let $B_{j}$ be a $G_{2}$-decomposition of $K_{16}$ with vertex set $H_{j} \cup\{\infty\}$. Then $(V, B)$ where $V=V(H)$ and $B=B^{\prime} \cup B_{1} \cup B_{2} \cup \cdots \cup B_{x}$ is a $G_{14}$-decomposition of $K_{15 x+10}$.

Combining the results from Lemmas 8 to 16 and the previously known results from [1], [17], and [4], we obtain the following.

Theorem 17. Let $G$ be connected cubic graph of order 10. There exists a $G$ decomposition of $K_{v}$ if and only if $v \equiv 1$ or $10(\bmod 15)$ except when $v=10$ and $G$ is one of the 5 graphs $G_{2}, G_{14}, G_{15}, G_{17}$ or $G_{19}$ in Figure 1.

## 3. Appendix

Here we give examples of decompositions that were used in Section 2. These results were found by using computer searches.

Example 18. Let $V\left(K_{16}\right)=\mathbb{Z}_{16}$, and let
$B_{2}=\left\{G_{2}[4,14,8,11,3,12,6,5,15,0], G_{2}[14,5,9,3,1,7,2,13,8,10], G_{2}[2,10,6,4,5,8,1,15,7,3]\right.$,
$G_{2}[0,1,6,7,5,2,11,12,8,9], G_{2}[0,5,3,4,1,12,14,7,11,13], G_{2}[1,10,11,0,2,4,15,9,12,13]$,
$\left.G_{2}[6,11,9,13,4,10,12,15,3,14], G_{2}[7,10,0,8,6,13,15,14,2,9]\right\}$,
$B_{3}=\left\{G_{3}[6,5,0,11,8,4,14,7,2,15], G_{3}[3,10,9,15,12,11,1,7,5,4], G_{3}[13,6,10,2,8,1,9,0,15,3]\right.$,
$G_{3}[0,1,3,2,5,8,12,9,6,4], G_{3}[0,7,8,3,11,13,14,5,1,10], G_{3}[2,6,11,10,12,4,13,5,9,14]$,
$\left.G_{3}[10,15,8,13,7,6,12,2,1,14], G_{3}[12,14,0,13,15,4,11,9,7,3]\right\}$,
$B_{4}=\left\{G_{4}[10,2,11,1,4,7,6,14,15,0], G_{4}[12,2,15,7,5,3,9,6,8,13], G_{4}[8,7,0,11,15,5,12,9,4,10]\right.$,
$G_{4}[0,1,3,2,7,9,13,11,4,5], G_{4}[0,6,4,3,10,14,8,15,1,12], G_{4}[1,2,6,13,5,10,12,8,11,14]$,
$\left.G_{4}[7,14,12,3,6,11,9,10,1,13], G_{4}[9,14,0,13,3,15,4,8,2,5]\right\}$,
$B_{5}=\left\{G_{5}[8,0,11,2,6,4,15,12,9,13], G_{5}[10,1,9,7,3,14,6,15,11,4], G_{5}[0,1,2,4,7,10,8,12,5,3]\right.$,
$G_{5}[0,4,5,1,6,8,7,13,2,12], G_{5}[0,6,9,2,3,4,14,12,13,10], G_{5}[6,11,3,10,14,1,15,9,8,5]$,
$\left.G_{5}[11,12,1,13,14,5,15,7,2,10], G_{5}[14,15,0,7,5,9,11,13,3,8]\right\}$,
$B_{6}=\left\{G_{6}[0,2,13,14,5,6,1,11,4,9], G_{6}[9,7,6,2,11,12,4,10,14,3], G_{6}[4,0,1,3,5,7,10,9,8,2]\right.$,
$G_{6}[7,0,5,8,3,11,15,2,9,1], G_{6}[11,0,10,12,1,8,15,13,4,6], G_{6}[14,1,15,10,3,12,8,7,13,0]$,
$\left.G_{6}[12,2,5,10,13,6,9,11,14,15], G_{6}[15,6,14,8,4,3,13,5,12,7]\right\}$,
$B_{7}=\left\{G_{7}[1,13,2,11,15,9,10,8,14,5], G_{7}[12,5,9,2,0,4,14,6,1,8], G_{7}[4,7,1,3,0,5,6,10,2,12]\right.$,
$G_{7}[4,2,6,7,0,8,9,11,3,10], G_{7}[15,5,10,13,0,11,1,4,3,8], G_{7}[9,1,14,15,0,12,7,11,13,6]$,
$\left.G_{7}[11,12,9,14,3,15,4,13,7,8], G_{7}[14,13,12,6,3,5,2,7,15,10]\right\}$,
$B_{8}=\left\{G_{8}[14,8,12,6,4,15,3,5,10,9], G_{8}[13,10,7,11,14,12,5,0,15,6], G_{8}[2,1,4,8,13,5,6,7,3,0]\right.$,
$G_{8}[7,4,2,3,12,1,11,9,6,0], G_{8}[10,8,1,6,14,2,11,4,9,0], G_{8}[11,12,2,9,15,8,5,1,13,0]$,
$\left.G_{8}[8,7,5,14,13,3,10,11,15,2], G_{8}[1,14,7,12,15,10,4,13,9,3]\right\}$,
$B_{9}=\left\{G_{9}[5,2,9,8,15,10,0,11,14,3], G_{9}[4,14,10,9,15,7,6,3,13,1], G_{9}[2,1,5,7,9,4,6,8,3,0]\right.$,
$G_{9}[5,4,3,1,7,11,2,12,6,0], G_{9}[8,7,10,1,11,4,13,12,9,0], G_{9}[8,12,0,13,10,11,6,14,15,1]$,
$\left.G_{9}[10,8,13,5,15,4,12,7,14,2], G_{9}[15,11,12,5,6,2,13,14,9,3]\right\}$,
$B_{10}=\left\{G_{10}[4,14,10,13,12,1,9,5,3,2], G_{10}[2,10,6,5,15,11,8,14,3,7], G_{10}[5,1,0,2,6,4,3,8,9,7]\right.$,
$G_{10}[8,4,0,7,1,13,5,11,3,10], G_{10}[7,6,0,11,4,12,8,2,13,15], G_{10}[12,9,0,14,6,15,10,7,13,11]$,
$\left.G_{10}[2,14,1,15,0,12,6,3,13,9], G_{10}[11,9,4,15,8,1,10,12,5,14]\right\}$,
$B_{11}=\left\{G_{11}[8,7,2,3,5,11,13,1,0,15], G_{11}[1,4,15,6,3,8,12,5,9,7], G_{11}[10,5,4,0,2,1,9,3,7,6]\right.$,
$G_{11}[14,2,8,0,5,1,11,6,12,3], G_{11}[13,3,10,0,9,2,15,11,4,14], G_{11}[0,12,11,2,10,14,8,6,4,13]$
$\left.G_{11}[13,12,10,4,8,9,14,7,15,5], G_{11}[1,6,13,7,10,15,9,11,14,12]\right\}$,

$$
\begin{aligned}
B_{12}=\{ & G_{12}[6,4,2,7,9,3,8,15,1,5], G_{12}[6,7,3,11,10,8,9,0,4,13], G_{12}[0,1,4,7,12,3,13,8,5,2], \\
& G_{12}[0,5,3,14,2,6,9,10,1,7], G_{12}[0,8,2,11,15,10,6,14,1,12], G_{12}[0,13,11,4,8,14,5,10,12,15], \\
& \left.G_{12}[3,4,14,12,13,1,11,9,5,15], G_{12}[10,2,12,6,11,7,14,15,9,13]\right\}, \\
B_{13}=\{ & G_{13}[11,15,8,10,1,13,5,14,6,9], G_{13}[15,6,12,2,14,13,8,11,3,0], G_{13}[0,1,3,9,7,4,6,10,5,2], \\
& G_{13}[0,5,1,12,3,7,8,4,2,9], G_{13}[0,8,2,15,4,10,13,7,1,14], G_{13}[0,12,10,15,1,11,7,14,9,13], \\
& \left.G_{13}[2,6,13,4,11,10,3,5,12,7], G_{13}[5,6,8,9,12,15,3,4,14,11]\right\}, \\
B_{14}=\{ & G_{14}[3,6,13,0,14,5,4,12,15,7], G_{14}[5,2,7,1,11,9,6,4,3,12], G_{14}[0,6,8,14,15,11,3,9,7,10], \\
& G_{14}[0,1,4,14,13,3,10,12,9,2], G_{14}[0,8,2,13,5,7,4,10,11,12], G_{14}[0,9,8,12,6,5,1,2,10,15], \\
& \left.G_{14}[1,13,9,4,15,3,8,5,11,14], G_{14}[2,11,6,10,13,15,1,8,7,14]\right\} .
\end{aligned}
$$

For $i \in[2,14]$, a $G_{i}$-decomposition of $K_{16}$ consists of the $G_{i}$-blocks in $B_{i}$.
Example 19. Let $V\left(K_{25}\right)=\left\{r_{s}: r \in \mathbb{Z}_{5}\right.$ and $\left.s \in \mathbb{Z}_{5}\right\}$, and let
$B_{2}=\left\{G_{2}\left[0_{0}, 1_{0}, 0_{1}, 2_{0}, 4_{0}, 3_{2}, 0_{3}, 3_{0}, 0_{2}, 2_{1}\right], G_{2}\left[0_{0}, 1_{2}, 0_{3}, 2_{0}, 1_{4}, 2_{1}, 2_{2}, 0_{1}, 0_{4}, 4_{3}\right]\right.$,
$\left.G_{2}\left[0_{0}, 0_{4}, 1_{4}, 0_{1}, 1_{2}, 0_{2}, 3_{4}, 3_{2}, 4_{1}, 2_{4}\right], G_{2}\left[0_{3}, 1_{3}, 0_{1}, 4_{3}, 1_{1}, 3_{3}, 3_{4}, 2_{2}, 2_{3}, 4_{4}\right]\right\}$,
$B_{3}=\left\{G_{3}\left[0_{0}, 1_{0}, 0_{1}, 2_{0}, 4_{0}, 1_{2}, 4_{2}, 0_{2}, 1_{1}, 2_{1}\right], G_{3}\left[0_{0}, 1_{2}, 0_{3}, 1_{0}, 0_{2}, 1_{4}, 2_{1}, 0_{4}, 3_{0}, 1_{3}\right]\right.$,
$\left.G_{3}\left[0_{0}, 2_{3}, 0_{4}, 0_{1}, 2_{1}, 0_{3}, 4_{3}, 1_{2}, 4_{1}, 1_{4}\right], G_{3}\left[0_{1}, 1_{4}, 0_{3}, 3_{2}, 3_{4}, 1_{2}, 4_{4}, 2_{4}, 2_{3}, 4_{3}\right]\right\}$,
$B_{4}=\left\{G_{4}\left[0_{0}, 1_{0}, 0_{1}, 2_{0}, 4_{0}, 0_{2}, 3_{1}, 1_{2}, 1_{1}, 2_{1}\right], G_{4}\left[0_{0}, 0_{2}, 4_{2}, 0_{1}, 1_{2}, 2_{3}, 4_{3}, 3_{3}, 1_{0}, 0_{3}\right]\right.$,
$\left.G_{4}\left[0_{0}, 1_{3}, 0_{4}, 1_{0}, 3_{4}, 4_{4}, 2_{4}, 4_{2}, 1_{2}, 1_{4}\right], G_{4}\left[0_{3}, 1_{4}, 1_{1}, 2_{4}, 0_{2}, 4_{3}, 3_{1}, 3_{3}, 0_{4}, 2_{1}\right]\right\}$,
$B_{5}=\left\{G_{5}\left[0_{0}, 1_{0}, 0_{1}, 2_{0}, 4_{0}, 1_{2}, 0_{2}, 3_{2}, 1_{1}, 2_{1}\right], G_{5}\left[0_{0}, 0_{2}, 0_{3}, 1_{0}, 2_{2}, 0_{4}, 3_{3}, 1_{4}, 3_{0}, 1_{3}\right]\right.$,
$\left.G_{5}\left[0_{0}, 1_{4}, 0_{4}, 0_{1}, 2_{1}, 0_{3}, 1_{2}, 3_{3}, 1_{1}, 4_{4}\right], G_{5}\left[0_{1}, 4_{2}, 1_{4}, 4_{1}, 2_{2}, 0_{3}, 4_{3}, 3_{4}, 3_{3}, 4_{4}\right]\right\}$,
$B_{6}=\left\{G_{6}\left[2_{0}, 0_{0}, 1_{0}, 0_{1}, 1_{1}, 0_{2}, 4_{2}, 4_{1}, 1_{2}, 3_{1}\right], G_{6}\left[0_{2}, 0_{0}, 3_{1}, 4_{2}, 2_{0}, 1_{3}, 0_{4}, 3_{3}, 3_{0}, 0_{3}\right]\right.$,
$\left.G_{6}\left[0_{4}, 0_{0}, 1_{2}, 3_{3}, 0_{1}, 1_{4}, 3_{2}, 2_{4}, 1_{3}, 0_{3}\right], G_{6}\left[0_{4}, 0_{1}, 1_{3}, 4_{3}, 2_{4}, 0_{2}, 1_{4}, 3_{0}, 4_{4}, 1_{1}\right]\right\}$,
$B_{7}=\left\{G_{7}\left[1_{2}, 3_{1}, 1_{0}, 0_{1}, 0_{0}, 2_{0}, 0_{2}, 3_{0}, 1_{1}, 3_{2}\right], G_{7}\left[1_{3}, 0_{1}, 1_{2}, 0_{3}, 0_{0}, 4_{2}, 4_{1}, 2_{3}, 1_{0}, 3_{3}\right]\right.$,
$\left.G_{7}\left[2_{3}, 3_{1}, 3_{3}, 0_{4}, 0_{0}, 1_{4}, 0_{2}, 4_{4}, 1_{0}, 3_{4}\right], G_{7}\left[3_{3}, 3_{4}, 2_{1}, 4_{4}, 0_{1}, 0_{4}, 2_{2}, 2_{3}, 4_{2}, 1_{4}\right]\right\}$,
$B_{8}=\left\{G_{8}\left[0_{1}, 1_{0}, 3_{1}, 1_{2}, 0_{3}, 1_{1}, 4_{1}, 0_{2}, 2_{0}, 0_{0}\right], G_{8}\left[2_{2}, 3_{1}, 3_{2}, 4_{0}, 3_{3}, 1_{0}, 2_{3}, 1_{2}, 0_{2}, 0_{0}\right]\right.$,
$\left.G_{8}\left[0_{4}, 0_{3}, 0_{1}, 1_{3}, 4_{4}, 1_{0}, 3_{4}, 1_{1}, 1_{4}, 0_{0}\right], G_{8}\left[3_{3}, 2_{3}, 0_{3}, 1_{4}, 3_{4}, 0_{2}, 2_{4}, 3_{2}, 4_{4}, 0_{1}\right]\right\}$,
$B_{9}=\left\{G_{9}\left[0_{1}, 1_{0}, 2_{1}, 0_{2}, 3_{2}, 3_{0}, 1_{1}, 1_{2}, 2_{0}, 0_{0}\right], G_{9}\left[2_{2}, 1_{2}, 0_{1}, 1_{1}, 2_{3}, 1_{3}, 0_{4}, 2_{1}, 0_{3}, 0_{0}\right]\right.$,
$\left.G_{9}\left[3_{3}, 1_{3}, 1_{4}, 0_{1}, 0_{4}, 4_{0}, 4_{4}, 2_{1}, 2_{3}, 0_{0}\right], G_{9}\left[4_{4}, 2_{4}, 1_{3}, 3_{2}, 3_{3}, 2_{2}, 0_{4}, 4_{2}, 3_{4}, 0_{0}\right]\right\}$,
$B_{10}=\left\{G_{10}\left[2_{1}, 1_{0}, 0_{0}, 0_{1}, 3_{0}, 0_{2}, 2_{0}, 1_{2}, 3_{2}, 1_{1}\right], G_{10}\left[0_{1}, 3_{1}, 0_{0}, 0_{3}, 1_{0}, 3_{3}, 1_{2}, 2_{1}, 2_{3}, 1_{3}\right]\right.$,
$\left.G_{10}\left[0_{2}, 3_{3}, 0_{0}, 0_{4}, 1_{0}, 3_{4}, 1_{4}, 3_{1}, 4_{4}, 4_{2}\right], G_{10}\left[4_{2}, 0_{4}, 0_{1}, 4_{4}, 1_{3}, 3_{3}, 2_{4}, 2_{3}, 1_{2}, 3_{4}\right]\right\}$,
$B_{11}=\left\{G_{11}\left[1_{1}, 3_{0}, 0_{1}, 0_{0}, 1_{0}, 2_{1}, 0_{2}, 2_{0}, 2_{2}, 3_{1}\right], G_{11}\left[4_{3}, 2_{1}, 2_{2}, 0_{0}, 1_{2}, 0_{1}, 1_{3}, 0_{3}, 2_{3}, 3_{0}\right]\right.$,
$\left.G_{11}\left[4_{4}, 1_{0}, 0_{4}, 0_{0}, 2_{3}, 4_{1}, 2_{4}, 1_{4}, 3_{4}, 0_{2}\right], G_{11}\left[0_{4}, 4_{2}, 0_{3}, 0_{2}, 2_{2}, 1_{3}, 2_{4}, 2_{3}, 4_{4}, 3_{1}\right]\right\}$,
$B_{12}=\left\{G_{12}\left[0_{0}, 1_{0}, 2_{1}, 4_{0}, 0_{2}, 2_{0}, 2_{2}, 0_{3}, 3_{0}, 0_{1}\right], G_{12}\left[0_{0}, 2_{2}, 0_{1}, 0_{2}, 1_{1}, 0_{3}, 0_{4}, 1_{4}, 4_{0}, 3_{3}\right]\right.$,
$\left.G_{12}\left[0_{0}, 0_{4}, 0_{1}, 0_{3}, 3_{1}, 4_{4}, 2_{2}, 2_{3}, 1_{1}, 3_{4}\right], G_{12}\left[0_{2}, 2_{2}, 3_{2}, 2_{4}, 0_{3}, 4_{3}, 3_{4}, 4_{1}, 2_{3}, 0_{4}\right]\right\}$,
$B_{13}=\left\{G_{13}\left[0_{0}, 1_{0}, 2_{1}, 1_{1}, 2_{2}, 2_{0}, 0_{2}, 0_{3}, 3_{0}, 0_{1}\right], G_{13}\left[0_{0}, 3_{1}, 0_{1}, 1_{3}, 4_{2}, 1_{2}, 0_{3}, 4_{3}, 2_{1}, 2_{2}\right]\right.$,
$\left.G_{13}\left[0_{0}, 0_{3}, 1_{0}, 2_{4}, 3_{4}, 1_{3}, 4_{4}, 3_{3}, 0_{1}, 0_{4}\right], G_{13}\left[0_{2}, 1_{2}, 2_{3}, 4_{0}, 3_{4}, 0_{4}, 4_{2}, 1_{4}, 2_{1}, 4_{4}\right]\right\}$,
$B_{14}=\left\{G_{14}\left[0_{0}, 1_{0}, 3_{1}, 1_{2}, 2_{1}, 0_{1}, 1_{1}, 3_{0}, 0_{2}, 2_{0}\right], G_{14}\left[0_{0}, 0_{2}, 4_{1}, 3_{3}, 2_{0}, 0_{3}, 1_{0}, 0_{4}, 2_{2}, 1_{2}\right]\right.$,
$\left.G_{14}\left[0_{0}, 0_{4}, 0_{1}, 0_{3}, 1_{2}, 2_{4}, 0_{2}, 3_{3}, 2_{1}, 1_{4}\right], G_{14}\left[0_{1}, 3_{3}, 2_{2}, 1_{4}, 3_{4}, 2_{3}, 1_{3}, 4_{4}, 0_{3}, 2_{4}\right]\right\}$.

For $i \in[2,14]$, a $G_{i}$-decomposition of $K_{25}$ consists of the $G_{i}$-blocks in $B_{i}$ under the action of the map $r_{s} \mapsto(r+1(\bmod 5))_{s}$.
Example 20. Let $V\left(K_{40}\right)=\left\{r_{s}: r \in \mathbb{Z}_{13}\right.$ and $\left.s \in \mathbb{Z}_{3}\right\} \cup\{\infty\}$, and let
$B_{2}=\left\{G_{2}\left[0_{0}, 1_{0}, 3_{0}, 9_{0}, 1_{1}, 2_{1}, 4_{1}, 2_{0}, 0_{1}, 5_{0}\right], G_{2}\left[0_{0}, 1_{1}, 6_{1}, 3_{0}, 0_{1}, 1_{2}, 2_{2}, 1_{0}, 0_{2}, 7_{1}\right]\right.$, $\left.G_{2}\left[0_{0}, 9_{1}, 3_{2}, 1_{0}, 6_{2}, 0_{2}, 2_{2}, 4_{1}, 0_{1}, 4_{2}\right], G_{2}\left[0_{2}, 3_{2}, 5_{0}, 1_{2}, 2_{1}, 7_{2}, \infty, 0_{0}, 6_{2}, 3_{1}\right]\right\}$,

$$
\begin{aligned}
& B_{3}=\left\{G_{3}\left[0_{0}, 1_{0}, 3_{0}, 9_{0}, 2_{1}, 2_{0}, 4_{1}, 1_{1}, 0_{1}, 5_{0}\right], G_{3}\left[0_{0}, 1_{1}, 7_{1}, 3_{0}, 0_{1}, 1_{2}, 1_{0}, 0_{2}, 6_{0}, 9_{1}\right],\right. \\
& \left.G_{3}\left[0_{0}, 1_{2}, 2_{2}, 4_{0}, 8_{2}, 9_{1}, 5_{2}, 0_{2}, 0_{1}, 3_{2}\right], G_{3}\left[0_{2}, 4_{2}, 2_{1}, 8_{2}, 3_{0}, 9_{2}, \infty, 11_{1}, 3_{2}, 9_{1}\right]\right\}, \\
& B_{4}=\left\{G_{4}\left[0_{0}, 1_{0}, 3_{0}, 9_{0}, 1_{1}, 2_{1}, 4_{1}, 2_{0}, 0_{1}, 5_{0}\right], G_{4}\left[0_{0}, 0_{1}, 3_{1}, 2_{0}, 11_{1}, 0_{2}, 3_{2}, 2_{2}, 3_{0}, 7_{1}\right]\right. \text {, } \\
& \left.G_{4}\left[0_{0}, 10_{1}, 2_{2}, 1_{0}, 5_{2}, 11_{2}, 7_{2}, 8_{1}, 0_{1}, 3_{2}\right], G_{4}\left[0_{2}, 2_{2}, 6_{0}, 1_{2}, 9_{2}, 0_{1}, \infty, 4_{0}, 10_{2}, 2_{1}\right]\right\}, \\
& B_{5}=\left\{G_{5}\left[0_{0}, 1_{0}, 3_{0}, 9_{0}, 1_{1}, 2_{0}, 2_{1}, 11_{0}, 0_{1}, 5_{0}\right], G_{5}\left[0_{0}, 1_{1}, 7_{1}, 10_{0}, 0_{2}, 2_{1}, 1_{2}, 10_{1}, 0_{1}, 9_{1}\right]\right. \text {, } \\
& \left.G_{5}\left[0_{0}, 0_{2}, 1_{2}, 2_{0}, 8_{2}, 1_{1}, 9_{2}, 5_{2}, 0_{1}, 2_{2}\right], G_{5}\left[0_{2}, 5_{2}, 8_{0}, 4_{2}, 6_{0}, 1_{2}, 4_{1}, \infty, 8_{2}, 12_{1}\right]\right\}, \\
& B_{6}=\left\{G_{6}\left[4_{0}, 0_{0}, 1_{0}, 3_{0}, 8_{0}, 1_{1}, 3_{1}, 2_{1}, 2_{0}, 0_{1}\right], G_{6}\left[3_{1}, 0_{0}, 6_{0}, 8_{1}, 3_{0}, 0_{2}, 5_{1}, 1_{2}, 0_{1}, 10_{1}\right]\right. \text {, } \\
& \left.G_{6}\left[2_{2}, 0_{0}, 0_{2}, 1_{2}, 2_{0}, 5_{2}, 3_{1}, 8_{2}, 10_{2}, 6_{0}\right], G_{6}\left[4_{2}, 0_{1}, 4_{1}, 3_{2}, 7_{2}, 12_{0}, \infty, 12_{2}, 12_{1}, 10_{2}\right]\right\}, \\
& B_{7}=\left\{G_{7}\left[1_{1}, 0_{1}, 1_{0}, 3_{0}, 0_{0}, 4_{0}, 10_{0}, 2_{1}, 8_{0}, 5_{1}\right], G_{7}\left[0_{2}, 6_{1}, 0_{1}, 2_{1}, 0_{0}, 1_{1}, 6_{0}, 1_{2}, 4_{0}, 10_{1}\right]\right. \text {, } \\
& \left.G_{7}\left[5_{2}, 2_{2}, 3_{1}, 0_{2}, 0_{0}, 1_{2}, 3_{0}, 6_{2}, 1_{0}, 10_{2}\right], G_{7}\left[4_{0}, 8_{2}, 2_{2}, 0_{1}, 0_{2}, 2_{1}, 10_{2}, 5_{1}, 1_{2}, \infty\right]\right\} \text {, } \\
& B_{8}=\left\{G_{8}\left[3_{0}, 1_{0}, 0_{1}, 2_{0}, 3_{1}, 8_{0}, 2_{1}, 10_{0}, 4_{0}, 0_{0}\right], G_{8}\left[3_{1}, 2_{1}, 0_{1}, 5_{1}, 1_{2}, 6_{0}, 0_{2}, 1_{1}, 4_{1}, 0_{0}\right]\right. \text {, } \\
& \left.G_{8}\left[1_{2}, 0_{2}, 3_{0}, 6_{2}, 10_{2}, 5_{0}, 3_{2}, 0_{1}, 2_{2}, 0_{0}\right], G_{8}\left[4_{2}, 6_{1}, 7_{2}, 1_{2}, \infty, 0_{0}, 6_{2}, 12_{1}, 5_{2}, 0_{1}\right]\right\}, \\
& B_{9}=\left\{G_{9}\left[3_{0}, 1_{0}, 6_{0}, 0_{1}, 2_{1}, 9_{0}, 7_{1}, 1_{1}, 4_{0}, 0_{0}\right], G_{9}\left[1_{1}, 0_{1}, 1_{0}, 5_{1}, 1_{2}, 9_{0}, 12_{1}, 0_{2}, 2_{1}, 0_{0}\right]\right. \text {, } \\
& \left.G_{9}\left[1_{2}, 0_{2}, 2_{0}, 5_{2}, 10_{2}, 4_{0}, 12_{2}, 6_{2}, 2_{2}, 0_{0}\right], G_{9}\left[3_{2}, 4_{1}, 11_{2}, 5_{1}, 10_{2}, 6_{1}, \infty, 4_{0}, 0_{2}, 0_{1}\right]\right\} \text {, } \\
& B_{10}=\left\{G_{10}\left[6_{0}, 1_{0}, 0_{0}, 3_{0}, 9_{0}, 0_{1}, 4_{0}, 1_{1}, 2_{1}, 4_{1}\right], G_{10}\left[1_{0}, 0_{1}, 0_{0}, 5_{1}, 2_{0}, 0_{2}, 1_{1}, 6_{0}, 3_{2}, 1_{2}\right]\right. \text {, } \\
& \left.G_{10}\left[6_{1}, 2_{1}, 0_{0}, 2_{2}, 6_{0}, 0_{2}, 3_{2}, 0_{1}, 10_{2}, 1_{2}\right], G_{10}\left[11_{2}, 6_{1}, 0_{1}, 2_{2}, 4_{1}, 10_{2}, 4_{2}, 11_{0}, \infty, 3_{2}\right]\right\}, \\
& B_{11}=\left\{G_{11}\left[4_{1}, 9_{0}, 3_{0}, 0_{0}, 1_{0}, 6_{0}, 0_{1}, 4_{0}, 1_{1}, 11_{0}\right], G_{11}\left[1_{2}, 3_{1}, 1_{1}, 0_{0}, 0_{1}, 1_{0}, 0_{2}, 4_{1}, 7_{1}, 12_{1}\right]\right. \text {, } \\
& \left.G_{11}\left[10_{2}, 5_{0}, 2_{2}, 0_{0}, 1_{2}, 3_{0}, 7_{2}, 3_{2}, 0_{2}, 0_{1}\right], G_{11}\left[8_{1}, 11_{2}, 5_{2}, 0_{1}, 6_{1}, 1_{2}, 8_{0}, 4_{2}, 6_{2}, \infty\right]\right\}, \\
& B_{12}=\left\{G_{12}\left[7_{0}, 9_{2}, 9_{1}, 9_{0}, 0_{2}, 1_{1}, 11_{1}, 7_{1}, 5_{0}, 12_{0}\right], G_{12}\left[0_{0}, 1_{0}, 0_{1}, 5_{0}, 1_{1}, 4_{0}, 9_{1}, 1_{2}, 4_{1}, 3_{0}\right]\right. \text {, } \\
& \left.G_{12}\left[0_{0}, 3_{1}, 1_{1}, 2_{2}, 7_{2}, 4_{1}, 0_{2}, 4_{2}, 1_{0}, 1_{2}\right], G_{12}\left[0_{0}, 5_{2}, 11_{1}, 4_{1}, 6_{2}, 7_{2}, 0_{2}, \infty, 2_{0}, 8_{2}\right]\right\}, \\
& B_{13}=\left\{G_{13}\left[0_{0}, 1_{0}, 6_{0}, 4_{1}, 1_{1}, 4_{0}, 0_{1}, 8_{1}, 9_{0}, 3_{0}\right], G_{13}\left[0_{0}, 0_{1}, 5_{0}, 9_{1}, 0_{2}, 2_{1}, 8_{1}, 1_{2}, 8_{0}, 1_{1}\right]\right. \text {, } \\
& \left.G_{13}\left[0_{0}, 0_{2}, 1_{0}, 8_{2}, 5_{0}, 2_{2}, 5_{2}, 6_{1}, 3_{2}, 1_{2}\right], G_{13}\left[0_{1}, 2_{1}, 10_{2}, 7_{1}, 8_{2}, 0_{2}, 4_{2}, \infty, 0_{0}, 9_{2}\right]\right\} .
\end{aligned}
$$

For $i \in[2,13]$, a $G_{i}$-decomposition of $K_{40}$ consists of the $G_{i}$-blocks in $B_{i}$ under the action of the map $\infty \mapsto \infty$ and $r_{s} \mapsto(r+1(\bmod 13))_{s}$.

Example 21. Let $V\left(K_{55}\right)=\left\{r_{s}: r \in \mathbb{Z}_{11}\right.$ and $\left.s \in \mathbb{Z}_{5}\right\}$, and let
$B_{2}=\left\{G_{2}\left[8_{0}, 1_{0}, 10_{3}, 7_{0}, 2_{3}, 6_{3}, 6_{0}, 2_{4}, 4_{4}, 2_{1}\right], G_{2}\left[1_{0}, 10_{2}, 1_{4}, 5_{1}, 9_{1}, 6_{4}, 0_{0}, 3_{0}, 6_{2}, 8_{1}\right]\right.$,
$G_{2}\left[0_{0}, 1_{0}, 0_{1}, 3_{0}, 5_{0}, 0_{2}, 1_{2}, 1_{1}, 8_{0}, 3_{1}\right], G_{2}\left[0_{0}, 0_{2}, 2_{2}, 1_{0}, 6_{2}, 0_{3}, 3_{3}, 10_{0}, 9_{2}, 4_{2}\right]$,
$G_{2}\left[0_{0}, 5_{3}, 10_{3}, 2_{0}, 0_{4}, 1_{1}, 4_{2}, 0_{1}, 1_{4}, 2_{4}\right], G_{2}\left[0_{0}, 1_{4}, 4_{4}, 0_{1}, 3_{1}, 0_{3}, 1_{3}, 7_{1}, 2_{1}, 5_{4}\right]$,
$G_{2}\left[0_{1}, 5_{2}, 8_{2}, 1_{1}, 2_{3}, 7_{1}, 1_{4}, 3_{3}, 7_{2}, 0_{3}\right], G_{2}\left[0_{1}, 3_{3}, 6_{4}, 6_{1}, 8_{3}, 6_{3}, 8_{4}, 3_{2}, 5_{3}, 9_{4}\right]$,
$\left.G_{2}\left[0_{3}, 1_{4}, 0_{2}, 4_{4}, 5_{2}, 3_{3}, 8_{4}, 2_{2}, 10_{4}, 1_{2}\right]\right\}$,
$B_{3}=\left\{G_{3}\left[0_{0}, 1_{0}, 3_{0}, 0_{1}, 6_{0}, 6_{1}, 8_{1}, 2_{0}, 1_{1}, 5_{0}\right], G_{3}\left[0_{0}, 1_{1}, 4_{1}, 1_{0}, 4_{2}, 1_{2}, 7_{0}, 3_{2}, 2_{0}, 0_{2}\right]\right.$,
$G_{3}\left[0_{0}, 4_{2}, 6_{2}, 0_{1}, 1_{1}, 4_{3}, 5_{1}, 2_{3}, 1_{0}, 0_{3}\right], G_{3}\left[0_{0}, 2_{3}, 4_{3}, 6_{0}, 1_{3}, 1_{4}, 5_{0}, 0_{4}, 8_{0}, 5_{3}\right]$,
$G_{3}\left[0_{0}, 0_{4}, 1_{4}, 0_{1}, 0_{2}, 4_{4}, 4_{1}, 3_{4}, 5_{0}, 2_{4}\right], G_{3}\left[0_{1}, 1_{2}, 2_{2}, 4_{1}, 7_{2}, 1_{3}, 6_{3}, 2_{3}, 1_{1}, 5_{2}\right]$,
$G_{3}\left[0_{1}, 8_{2}, 5_{3}, 1_{1}, 7_{4}, 4_{4}, 1_{2}, 3_{4}, 10_{1}, 6_{3}\right], G_{3}\left[0_{2}, 5_{2}, 0_{3}, 7_{2}, 8_{3}, 9_{4}, 7_{3}, 6_{4}, 1_{2}, 1_{4}\right]$,
$\left.G_{3}\left[0_{2}, 9_{4}, 2_{3}, 5_{4}, 8_{3}, 2_{4}, 6_{4}, 9_{1}, 7_{4}, 3_{3}\right]\right\}$,
$B_{4}=\left\{G_{3}\left[5_{3}, 8_{0}, 1_{2}, 4_{2}, 10_{0}, 1_{4}, 7_{2}, 2_{2}, 9_{1}, 9_{2}\right], G_{3}\left[8_{3}, 7_{2}, 9_{2}, 10_{2}, 6_{4}, 8_{4}, 6_{1}, 3_{0}, 8_{1}, 9_{3}\right]\right.$,
$G_{3}\left[8_{4}, 5_{3}, 3_{4}, 2_{2}, 7_{3}, 6_{0}, 2_{3}, 4_{0}, 4_{4}, 5_{2}\right], G_{3}\left[0_{0}, 1_{0}, 3_{0}, 0_{1}, 2_{0}, 3_{1}, 4_{2}, 2_{1}, 1_{1}, 5_{0}\right]$,
$G_{3}\left[0_{0}, 2_{1}, 6_{1}, 2_{0}, 1_{1}, 0_{2}, 6_{3}, 0_{3}, 6_{0}, 8_{2}\right], G_{3}\left[0_{0}, 0_{2}, 4_{4}, 3_{0}, 2_{2}, 9_{2}, 0_{4}, 0_{3}, 1_{0}, 6_{4}\right]$,
$G_{3}\left[0_{0}, 2_{3}, 4_{3}, 0_{1}, 5_{1}, 0_{3}, 4_{4}, 3_{3}, 1_{1}, 6_{3}\right], G_{3}\left[0_{1}, 3_{3}, 0_{4}, 8_{4}, 7_{1}, 1_{4}, 7_{3}, 4_{2}, 10_{1}, 9_{4}\right]$,
$\left.G_{3}\left[0_{3}, 10_{4}, 4_{1}, 8_{4}, 10_{0}, 7_{4}, 10_{1}, 6_{4}, 6_{2}, 3_{3}\right]\right\}$,
$B_{5}=\left\{G_{4}\left[0_{0}, 1_{0}, 3_{0}, 0_{1}, 2_{0}, 5_{1}, 6_{0}, 6_{1}, 1_{1}, 5_{0}\right], G_{4}\left[0_{0}, 1_{1}, 2_{1}, 7_{0}, 1_{2}, 3_{2}, 2_{2}, 5_{2}, 1_{0}, 0_{2}\right]\right.$,
$G_{4}\left[0_{0}, 1_{2}, 7_{2}, 4_{0}, 2_{2}, 2_{3}, 1_{3}, 3_{3}, 1_{0}, 0_{3}\right], G_{4}\left[0_{0}, 8_{2}, 3_{3}, 5_{0}, 0_{3}, 0_{4}, 10_{3}, 2_{4}, 8_{0}, 4_{3}\right]$,

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    G4[00, 和, 14},\mp@subsup{2}{0}{},\mp@subsup{6}{4}{},\mp@subsup{2}{1}{},\mp@subsup{8}{4}{},\mp@subsup{3}{1}{},\mp@subsup{0}{1}{},\mp@subsup{2}{4}{}],\mp@subsup{G}{4}{}[\mp@subsup{0}{1}{},\mp@subsup{0}{2}{},\mp@subsup{2}{3}{},\mp@subsup{1}{1}{},\mp@subsup{2}{2}{},\mp@subsup{0}{3}{},\mp@subsup{3}{2}{},\mp@subsup{7}{3}{},\mp@subsup{3}{1}{},\mp@subsup{3}{3}{}]
```



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    G4[03, 44, 63, 54, 92, 0}\mp@subsup{0}{1}{,},\mp@subsup{9}{3}{},\mp@subsup{4}{1}{},1\mp@subsup{0}{3}{},\mp@subsup{7}{3}{}]}
    B6}={\mp@subsup{G}{6}{}[\mp@subsup{4}{0}{},\mp@subsup{0}{0}{},\mp@subsup{1}{0}{},\mp@subsup{3}{0}{},\mp@subsup{8}{0}{},\mp@subsup{1}{1}{},\mp@subsup{3}{1}{},\mp@subsup{2}{1}{},\mp@subsup{2}{0}{},\mp@subsup{0}{1}{}],\mp@subsup{G}{6}{}[\mp@subsup{3}{1}{},\mp@subsup{0}{0}{},2\mp@subsup{2}{1}{},\mp@subsup{6}{1}{},\mp@subsup{0}{1}{},\mp@subsup{2}{2}{},\mp@subsup{3}{2}{},\mp@subsup{1}{2}{},\mp@subsup{1}{0}{},\mp@subsup{0}{2}{}]
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    G6[84, 03, 64, 103, 94, 62, 104 , 42, 54, 101]},
B7 ={G7[[60, 0}\mp@subsup{\mp@code{1}}{1}{},\mp@subsup{1}{0}{},\mp@subsup{3}{0}{},\mp@subsup{0}{0}{},\mp@subsup{4}{0}{},\mp@subsup{1}{1}{},\mp@subsup{6}{1}{},\mp@subsup{8}{0}{},\mp@subsup{8}{1}{}],\mp@subsup{G}{7}{}[\mp@subsup{2}{2}{},\mp@subsup{0}{2}{},\mp@subsup{1}{1}{},\mp@subsup{4}{1}{},\mp@subsup{0}{0}{},\mp@subsup{3}{1}{},\mp@subsup{2}{1}{},\mp@subsup{3}{2}{},\mp@subsup{0}{1}{},\mp@subsup{6}{2}{}]
    G7[72, 3}\mp@subsup{3}{0}{},\mp@subsup{0}{2}{},\mp@subsup{1}{2}{},\mp@subsup{0}{0}{},\mp@subsup{2}{2}{},\mp@subsup{4}{0}{},1\mp@subsup{0}{2}{},\mp@subsup{5}{0}{},\mp@subsup{0}{3}{}],\mp@subsup{G}{7}{}[7\mp@subsup{7}{3}{},\mp@subsup{3}{0}{},\mp@subsup{0}{3}{},\mp@subsup{1}{3}{},\mp@subsup{0}{0}{},\mp@subsup{2}{3}{},\mp@subsup{4}{0}{},\mp@subsup{0}{4}{},\mp@subsup{5}{0}{},1\mp@subsup{0}{3}{}]
```



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    G7[14, 31, 0}\mp@subsup{0}{3}{},\mp@subsup{0}{4}{},\mp@subsup{0}{1}{},\mp@subsup{1}{3}{},\mp@subsup{9}{1}{},\mp@subsup{3}{3}{},\mp@subsup{1}{1}{},\mp@subsup{5}{4}{}],\mp@subsup{G}{7}{}[\mp@subsup{2}{3}{},\mp@subsup{0}{2}{},\mp@subsup{6}{4}{},\mp@subsup{8}{4}{},\mp@subsup{0}{1}{},\mp@subsup{1}{4}{},\mp@subsup{2}{2}{},\mp@subsup{7}{3}{},\mp@subsup{1}{2}{},\mp@subsup{5}{4}{}]
    G7}[\mp@subsup{6}{4}{},\mp@subsup{4}{2}{},\mp@subsup{3}{3}{},\mp@subsup{0}{4}{},\mp@subsup{0}{2}{},\mp@subsup{9}{4}{},\mp@subsup{2}{3}{},\mp@subsup{1}{4}{},\mp@subsup{6}{3}{},\mp@subsup{9}{2}{}]}
B8}={\mp@subsup{G}{8}{}[\mp@subsup{3}{0}{},\mp@subsup{1}{0}{},\mp@subsup{0}{1}{},\mp@subsup{2}{0}{},\mp@subsup{3}{1}{},\mp@subsup{8}{0}{},\mp@subsup{2}{1}{},\mp@subsup{1}{1}{},\mp@subsup{4}{0}{},\mp@subsup{0}{0}{}],\mp@subsup{G}{8}{}[\mp@subsup{0}{2}{},\mp@subsup{2}{1}{},\mp@subsup{6}{1}{},\mp@subsup{0}{1}{},\mp@subsup{3}{2}{},\mp@subsup{1}{0}{},\mp@subsup{2}{2}{},1\mp@subsup{0}{0}{},\mp@subsup{3}{1}{},\mp@subsup{0}{0}{}]
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    G}\mp@subsup{G}{8}{[3},\mp@subsup{3}{3}{},\mp@subsup{1}{3}{},\mp@subsup{3}{2}{},\mp@subsup{1}{2}{},\mp@subsup{6}{4}{},\mp@subsup{5}{1}{},\mp@subsup{2}{4}{},\mp@subsup{0}{2}{},\mp@subsup{2}{3}{},\mp@subsup{0}{1}{}],\mp@subsup{G}{8}{}[\mp@subsup{3}{4}{},\mp@subsup{4}{3}{},\mp@subsup{1}{2}{},\mp@subsup{9}{2}{},\mp@subsup{6}{4}{},\mp@subsup{0}{3}{},\mp@subsup{5}{2}{},\mp@subsup{2}{3}{},\mp@subsup{4}{4}{},\mp@subsup{0}{1}{}]
    G8[73,43, 04, 63, 44, 3, 3, 34, 64, 74, 0 2]},
B9}={\mp@subsup{G}{9}{}[\mp@subsup{3}{0}{},\mp@subsup{1}{0}{},\mp@subsup{6}{0}{},\mp@subsup{2}{1}{},\mp@subsup{7}{0}{},\mp@subsup{0}{1}{},\mp@subsup{1}{1}{},\mp@subsup{5}{1}{},\mp@subsup{4}{0}{},\mp@subsup{0}{0}{}],G\mp@subsup{G}{9}{}[2\mp@subsup{2}{1}{},\mp@subsup{0}{1}{},\mp@subsup{5}{1}{},\mp@subsup{0}{2}{},\mp@subsup{2}{2}{},1\mp@subsup{0}{1}{},\mp@subsup{9}{2}{},\mp@subsup{1}{2}{},\mp@subsup{3}{1}{},\mp@subsup{0}{0}{}]
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    G9[44, 53, 74 , 30, 14, ,42, 94, 3, , 64, 01]},
B10}={\mp@subsup{G}{10}{}[\mp@subsup{6}{0}{},\mp@subsup{1}{0}{},\mp@subsup{0}{0}{},\mp@subsup{3}{0}{},\mp@subsup{0}{1}{},\mp@subsup{2}{1}{},\mp@subsup{4}{0}{},\mp@subsup{3}{1}{},\mp@subsup{0}{2}{},\mp@subsup{6}{1}{}],\mp@subsup{G}{10}{[}[\mp@subsup{6}{0}{},\mp@subsup{1}{1}{},\mp@subsup{0}{0}{},2,2,\mp@subsup{8}{0}{},\mp@subsup{1}{2}{},\mp@subsup{3}{1}{},1\mp@subsup{0}{0}{},\mp@subsup{6}{2}{},\mp@subsup{0}{2}{}]
    G}\mp@subsup{G}{10}{[}[\mp@subsup{1}{0}{},\mp@subsup{0}{2}{},\mp@subsup{0}{0}{},\mp@subsup{2}{2}{},\mp@subsup{5}{0}{},\mp@subsup{0}{3}{},\mp@subsup{3}{2}{},\mp@subsup{0}{1}{},\mp@subsup{2}{3}{},\mp@subsup{1}{3}{}],\mp@subsup{G}{10}{}[\mp@subsup{3}{0}{},\mp@subsup{1}{3}{},\mp@subsup{0}{0}{},\mp@subsup{3}{3}{},\mp@subsup{7}{0}{},\mp@subsup{0}{4}{},\mp@subsup{2}{3}{},\mp@subsup{8}{0}{},\mp@subsup{3}{4}{},\mp@subsup{2}{4}{}]
    G}\mp@subsup{G}{10}{[[0}\mp@subsup{0}{1}{},\mp@subsup{4}{3}{},\mp@subsup{0}{0}{},\mp@subsup{0}{4}{},\mp@subsup{2}{0}{},\mp@subsup{4}{4}{},\mp@subsup{1}{4}{},\mp@subsup{2}{1}{},\mp@subsup{5}{4}{},\mp@subsup{7}{1}{}],\mp@subsup{G}{10}{}[2, 1, 12,\mp@subsup{0}{1}{},\mp@subsup{2}{2}{},\mp@subsup{6}{1}{},\mp@subsup{0}{3}{},\mp@subsup{4}{2}{},\mp@subsup{9}{1}{},\mp@subsup{6}{3}{},\mp@subsup{1}{3}{}]
    G10}[02,\mp@subsup{6}{3}{},\mp@subsup{0}{1}{},\mp@subsup{0}{4}{},\mp@subsup{3}{1}{},\mp@subsup{4}{4}{},\mp@subsup{7}{3}{},\mp@subsup{2}{2}{},1\mp@subsup{0}{4}{},\mp@subsup{0}{3}{}],\mp@subsup{G}{10}{}[\mp@subsup{0}{2}{},\mp@subsup{2}{4}{},\mp@subsup{0}{1}{},\mp@subsup{6}{4}{},\mp@subsup{1}{2}{},\mp@subsup{4}{2}{},\mp@subsup{5}{4}{},\mp@subsup{2}{2}{},\mp@subsup{4}{3}{},\mp@subsup{1}{3}{}]
    G10[0}\mp@subsup{0}{3}{},\mp@subsup{4}{3}{},\mp@subsup{0}{2}{},1\mp@subsup{0}{4}{},1\mp@subsup{0}{3}{},\mp@subsup{1}{4}{},\mp@subsup{6}{4}{},\mp@subsup{6}{2}{},\mp@subsup{2}{4}{},\mp@subsup{4}{4}{}]}
B
    G11 [0}\mp@subsup{0}{3}{},\mp@subsup{6}{0}{},\mp@subsup{3}{2}{},\mp@subsup{0}{0}{},\mp@subsup{2}{2}{},\mp@subsup{4}{0}{},\mp@subsup{0}{2}{},\mp@subsup{4}{2}{},\mp@subsup{1}{2}{},\mp@subsup{8}{1}{}],\mp@subsup{G}{11}{}[43,1,\mp@subsup{1}{0}{},\mp@subsup{0}{3}{},\mp@subsup{0}{0}{},\mp@subsup{5}{2}{},\mp@subsup{3}{1}{},\mp@subsup{0}{2}{},\mp@subsup{1}{3}{},\mp@subsup{3}{3}{},1\mp@subsup{0}{1}{}]
```



```
    G}\mp@subsup{G}{11}{}[\mp@subsup{1}{4}{},\mp@subsup{9}{1}{},\mp@subsup{2}{4}{},\mp@subsup{0}{1}{},\mp@subsup{2}{3}{},\mp@subsup{4}{1}{},\mp@subsup{9}{4}{},\mp@subsup{6}{3}{},\mp@subsup{5}{3}{},\mp@subsup{0}{2}{}],\mp@subsup{G}{11}{}[\mp@subsup{2}{4}{},\mp@subsup{3}{2}{},\mp@subsup{0}{3}{},\mp@subsup{0}{2}{},\mp@subsup{2}{2}{},\mp@subsup{1}{3}{},\mp@subsup{9}{3}{},\mp@subsup{5}{2}{},\mp@subsup{0}{4}{},\mp@subsup{6}{2}{}]
    G11[74},\mp@subsup{4}{2}{},\mp@subsup{4}{4}{},\mp@subsup{0}{2}{},\mp@subsup{2}{3}{},\mp@subsup{9}{4}{},\mp@subsup{2}{4}{},\mp@subsup{7}{3}{},\mp@subsup{6}{4}{},\mp@subsup{9}{3}{}]}
```






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    G12[0}\mp@subsup{0}{2}{},\mp@subsup{3}{2}{},\mp@subsup{4}{4}{},\mp@subsup{9}{1}{},\mp@subsup{8}{4}{},\mp@subsup{2}{2}{},\mp@subsup{9}{4}{},\mp@subsup{0}{4}{},\mp@subsup{8}{1}{},\mp@subsup{5}{4}{}]}
B13}={\mp@subsup{G}{13}{}[\mp@subsup{0}{0}{},\mp@subsup{1}{0}{},\mp@subsup{6}{0}{},\mp@subsup{1}{1}{},\mp@subsup{3}{1}{},\mp@subsup{4}{0}{},\mp@subsup{2}{1}{},\mp@subsup{8}{0}{},\mp@subsup{0}{1}{},\mp@subsup{3}{0}{}],\mp@subsup{G}{13}{}[\mp@subsup{0}{0}{},\mp@subsup{0}{1}{},\mp@subsup{4}{1}{},\mp@subsup{4}{2}{},\mp@subsup{1}{2}{},\mp@subsup{2}{1}{},\mp@subsup{0}{2}{},\mp@subsup{9}{2}{},\mp@subsup{6}{1}{},\mp@subsup{1}{1}{}]
    G}\mp@subsup{G}{13}{}[\mp@subsup{0}{0}{},\mp@subsup{0}{2}{},\mp@subsup{1}{0}{},\mp@subsup{4}{2}{},\mp@subsup{0}{3}{},\mp@subsup{2}{2}{},\mp@subsup{6}{2}{},\mp@subsup{1}{3}{},\mp@subsup{3}{0}{},\mp@subsup{1}{2}{2}],\mp@subsup{G}{13}{}[\mp@subsup{0}{0}{},\mp@subsup{4}{2}{},\mp@subsup{7}{0}{},\mp@subsup{3}{2}{},\mp@subsup{6}{3}{},\mp@subsup{6}{2}{},\mp@subsup{0}{3}{},\mp@subsup{7}{3}{},\mp@subsup{1}{0}{},\mp@subsup{3}{3}{}]
```



```
    G}\mp@subsup{G}{13}{}[\mp@subsup{0}{0}{},\mp@subsup{5}{4}{},\mp@subsup{0}{1}{},1\mp@subsup{0}{3}{},\mp@subsup{9}{2}{},\mp@subsup{6}{4}{},\mp@subsup{4}{2}{},\mp@subsup{2}{4}{},\mp@subsup{1}{1}{},\mp@subsup{7}{4}{}],\mp@subsup{G}{13}{}[0\mp@subsup{0}{1}{},\mp@subsup{2}{3}{},\mp@subsup{4}{1}{},\mp@subsup{9}{3}{},\mp@subsup{0}{4}{},\mp@subsup{3}{3}{},\mp@subsup{0}{3}{},1\mp@subsup{0}{4}{},\mp@subsup{7}{1}{},\mp@subsup{4}{3}{}]
    G13[0}\mp@subsup{0}{1}{},\mp@subsup{0}{4}{},\mp@subsup{6}{4}{},\mp@subsup{7}{1}{},\mp@subsup{4}{4}{},\mp@subsup{6}{3}{},\mp@subsup{0}{3}{},\mp@subsup{5}{4}{},\mp@subsup{1}{2}{},\mp@subsup{7}{4}{}]}
```

For $i \in[2,13]$, a $G_{i}$-decomposition of $K_{55}$ consists of the $G_{i}$-blocks in $B_{i}$ under
the action of the map $r_{s} \mapsto(r+1(\bmod 11))_{s}$.
Example 22. Let $V\left(K_{85}\right)=\left\{r_{s}: r \in \mathbb{Z}_{17}\right.$ and $\left.s \in \mathbb{Z}_{5}\right\}$, and let
$B_{2}=\left\{G_{2}\left[8_{2}, 2_{3}, 10_{1}, 1_{3}, 16_{0}, 3_{0}, 15_{3}, 9_{3}, 3_{2}, 7_{3}\right], G_{2}\left[5_{0}, 2_{1}, 6_{2}, 12_{0}, 7_{0}, 6_{1}, 10_{4}, 1_{3}, 9_{0}, 13_{1}\right]\right.$,
$G_{2}\left[10_{0}, 0_{4}, 12_{1}, 14_{1}, 7_{4}, 7_{1}, 4_{2}, 16_{1}, 13_{4}, 10_{1}\right], G_{2}\left[4_{0}, 6_{4}, 10_{1}, 10_{2}, 8_{2}, 12_{0}, 11_{0}, 7_{4}, 11_{3}, 11_{1}\right]$,
$G_{2}\left[0_{0}, 2_{0}, 8_{0}, 1_{0}, 10_{1}, 5_{1}, 0_{2}, 10_{0}, 6_{1}, 3_{1}\right], G_{2}\left[0_{0}, 10_{1}, 11_{1}, 13_{0}, 0_{2}, 4_{2}, 7_{2}, 1_{0}, 1_{2}, 2_{2}\right]$,
$G_{2}\left[0_{0}, 8_{2}, 16_{2}, 1_{0}, 11_{2}, 2_{3}, 6_{3}, 15_{0}, 10_{2}, 0_{3}\right], G_{2}\left[0_{0}, 1_{3}, 3_{3}, 5_{0}, 2_{3}, 9_{3}, 6_{4}, 2_{0}, 3_{4}, 10_{3}\right]$,
$G_{2}\left[0_{0}, 5_{3}, 0_{4}, 1_{0}, 12_{3}, 7_{1}, 10_{2}, 0_{1}, 14_{3}, 5_{4}\right], G_{2}\left[0_{0}, 6_{4}, 8_{4}, 11_{0}, 5_{4}, 1_{2}, 13_{2}, 2_{1}, 4_{4}, 9_{4}\right]$,
$G_{2}\left[0_{1}, 4_{1}, 6_{2}, 5_{1}, 1_{2}, 6_{3}, 0_{4}, 2_{1}, 1_{3}, 2_{3}\right], G_{2}\left[0_{1}, 8_{2}, 1_{3}, 8_{1}, 15_{3}, 0_{4}, 1_{4}, 0_{2}, 14_{3}, 11_{3}\right]$,
$\left.G_{2}\left[0_{1}, 3_{3}, 1_{4}, 7_{1}, 13_{4}, 7_{2}, 3_{4}, 1_{2}, 16_{4}, 8_{4}\right], G_{2}\left[0_{3}, 1_{4}, 4_{2}, 14_{4}, 9_{2}, 3_{4}, 16_{4}, 8_{2}, 8_{3}, 7_{4}\right]\right\}$,
$B_{3}=\left\{G_{3}\left[0_{0}, 1_{0}, 3_{0}, 9_{0}, 1_{1}, 4_{0}, 2_{1}, 0_{1}, 12_{0}, 5_{0}\right], G_{3}\left[0_{0}, 0_{1}, 3_{1}, 1_{0}, 7_{1}, 2_{1}, 0_{2}, 13_{1}, 3_{0}, 4_{1}\right]\right.$,
$G_{3}\left[0_{0}, 7_{1}, 13_{1}, 2_{0}, 4_{2}, 5_{2}, 8_{0}, 0_{2}, 1_{0}, 1_{2}\right], G_{3}\left[0_{0}, 3_{2}, 5_{2}, 10_{0}, 0_{2}, 1_{3}, 7_{0}, 0_{3}, 12_{0}, 6_{2}\right]$,
$G_{3}\left[0_{0}, 8_{2}, 0_{3}, 1_{0}, 4_{3}, 11_{3}, 0_{1}, 5_{3}, 3_{0}, 1_{3}\right], G_{3}\left[0_{0}, 9_{3}, 12_{3}, 15_{0}, 1_{4}, 7_{1}, 15_{1}, 0_{4}, 0_{1}, 13_{3}\right]$,
$G_{3}\left[0_{0}, 0_{4}, 1_{4}, 2_{0}, 8_{4}, 2_{4}, 1_{1}, 11_{4}, 6_{0}, 4_{4}\right], G_{3}\left[0_{0}, 7_{4}, 10_{4}, 2_{0}, 16_{4}, 0_{2}, 8_{1}, 13_{4}, 0_{1}, 12_{4}\right]$,
$G_{3}\left[0_{1}, 1_{2}, 5_{2}, 2_{1}, 9_{2}, 0_{2}, 3_{3}, 16_{2}, 4_{1}, 6_{2}\right], G_{3}\left[0_{1}, 8_{2}, 3_{3}, 1_{1}, 0_{2}, 2_{3}, 11_{2}, 0_{3}, 5_{1}, 6_{3}\right]$,
$G_{3}\left[0_{1}, 0_{3}, 8_{3}, 1_{1}, 11_{3}, 3_{4}, 11_{2}, 5_{4}, 6_{1}, 15_{3}\right], G_{3}\left[0_{2}, 7_{2}, 14_{3}, 1_{2}, 0_{3}, 11_{4}, 1_{3}, 8_{4}, 2_{1}, 0_{4}\right]$,
$\left.G_{3}\left[0_{2}, 5_{3}, 1_{4}, 3_{2}, 7_{4}, 0_{4}, 2_{3}, 8_{4}, 5_{2}, 2_{4}\right], G_{3}\left[0_{3}, 5_{3}, 0_{4}, 9_{2}, 5_{4}, 3_{3}, 14_{3}, 15_{4}, 6_{4}, 4_{4}\right]\right\}$,
$B_{4}=\left\{G_{4}\left[0_{2}, 9_{1}, 13_{3}, 1_{3}, 6_{4}, 10_{4}, 3_{1}, 7_{2}, 2_{2}, 10_{1}\right], G_{4}\left[1_{4}, 0_{1}, 13_{1}, 2_{3}, 2_{1}, 12_{2}, 0_{3}, 10_{4}, 7_{0}, 13_{4}\right]\right.$,
$G_{4}\left[9_{3}, 7_{3}, 5_{2}, 16_{4}, 9_{0}, 11_{0}, 6_{3}, 15_{0}, 10_{1}, 0_{0}\right], G_{4}\left[14_{1}, 11_{1}, 12_{0}, 1_{1}, 16_{4}, 1_{2}, 11_{0}, 1_{0}, 14_{3}, 14_{0}\right]$,
$G_{4}\left[2_{0}, 9_{1}, 4_{4}, 0_{3}, 1_{2}, 15_{1}, 14_{3}, 0_{0}, 11_{4}, 1_{0}\right], G_{4}\left[3_{3}, 13_{3}, 3_{4}, 6_{2}, 3_{2}, 8_{0}, 12_{3}, 7_{0}, 8_{4}, 2_{3}\right]$,
$G_{4}\left[0_{0}, 3_{0}, 8_{0}, 4_{0}, 7_{1}, 9_{1}, 10_{2}, 14_{1}, 5_{0}, 1_{1}\right], G_{4}\left[0_{0}, 4_{1}, 2_{2}, 2_{0}, 13_{1}, 1_{2}, 0_{2}, 7_{2}, 3_{0}, 6_{2}\right]$,
$G_{4}\left[0_{0}, 8_{2}, 10_{2}, 1_{0}, 12_{2}, 3_{3}, 0_{3}, 13_{3}, 2_{0}, 1_{3}\right], G_{4}\left[0_{0}, 6_{3}, 0_{4}, 1_{0}, 9_{4}, 15_{4}, 8_{4}, 7_{4}, 12_{0}, 4_{4}\right]$,
$G_{4}\left[0_{1}, 5_{1}, 3_{3}, 2_{1}, 8_{1}, 14_{2}, 0_{4}, 15_{3}, 1_{1}, 10_{3}\right], G_{4}\left[0_{1}, 8_{1}, 0_{4}, 3_{1}, 0_{2}, 7_{4}, 15_{4}, 3_{3}, 9_{1}, 2_{4}\right]$,
$\left.G_{4}\left[0_{2}, 4_{2}, 15_{3}, 7_{1}, 2_{3}, 10_{4}, 6_{2}, 1_{4}, 12_{2}, 0_{4}\right], G_{4}\left[0_{2}, 6_{3}, 14_{3}, 5_{2}, 5_{3}, 6_{4}, 3_{4}, 10_{2}, 1_{2}, 9_{4}\right]\right\}$,
$B_{5}=\left\{G_{5}\left[3_{2}, 5_{0}, 6_{3}, 10_{3}, 2_{0}, 16_{3}, 1_{3}, 16_{4}, 4_{0}, 8_{0}\right], G_{5}\left[7_{3}, 6_{4}, 1_{1}, 0_{0}, 7_{2}, 0_{1}, 10_{4}, 2_{3}, 10_{2}, 5_{2}\right]\right.$,
$G_{5}\left[2_{1}, 7_{1}, 4_{2}, 5_{3}, 0_{1}, 1_{1}, 14_{4}, 1_{2}, 7_{0}, 9_{3}\right], G_{5}\left[4_{3}, 1_{3}, 14_{0}, 2_{0}, 16_{2}, 12_{0}, 2_{4}, 4_{1}, 14_{1}, 0_{1}\right]$,
$G_{5}\left[0_{0}, 1_{0}, 7_{0}, 5_{0}, 13_{0}, 5_{1}, 1_{1}, 10_{1}, 2_{0}, 0_{1}\right], G_{5}\left[0_{0}, 4_{1}, 6_{1}, 1_{0}, 13_{1}, 4_{2}, 0_{2}, 9_{2}, 3_{0}, 10_{1}\right]$,
$G_{5}\left[0_{0}, 0_{2}, 2_{2}, 6_{0}, 11_{2}, 2_{3}, 14_{2}, 4_{3}, 10_{0}, 3_{2}\right], G_{5}\left[0_{0}, 9_{2}, 3_{3}, 3_{0}, 1_{3}, 2_{3}, 9_{3}, 2_{4}, 6_{0}, 5_{3}\right]$,
$G_{5}\left[0_{0}, 10_{3}, 4_{4}, 1_{0}, 0_{4}, 9_{4}, 2_{4}, 14_{4}, 3_{0}, 5_{4}\right], G_{5}\left[0_{2}, 6_{2}, 4_{4}, 9_{2}, 11_{4}, 3_{2}, 14_{4}, 5_{2}, 9_{3}, 16_{4}\right]$,
$G_{5}\left[0_{1}, 4_{2}, 5_{2}, 6_{1}, 12_{2}, 11_{3}, 0_{3}, 2_{4}, 2_{1}, 11_{2}\right], G_{5}\left[0_{1}, 15_{2}, 13_{3}, 1_{1}, 4_{3}, 0_{4}, 11_{3}, 8_{4}, 6_{1}, 1_{4}\right]$,
$\left.G_{5}\left[0_{1}, 3_{4}, 0_{3}, 12_{3}, 4_{1}, 8_{4}, 12_{4}, 10_{4}, 1_{1}, 15_{3}\right], G_{5}\left[0_{0}, 9_{4}, 8_{4}, 0_{1}, 0_{2}, 0_{3}, 3_{2}, 0_{4}, 1_{1}, 15_{4}\right]\right\}$,
$B_{6}=\left\{G_{6}\left[4_{0}, 0_{0}, 1_{0}, 3_{0}, 10_{0}, 0_{1}, 2_{1}, 1_{1}, 15_{0}, 9_{0}\right], G_{6}\left[1_{1}, 0_{0}, 0_{1}, 5_{1}, 3_{0}, 9_{1}, 0_{2}, 14_{1}, 11_{1}, 2_{0}\right]\right.$,
$G_{6}\left[12_{1}, 0_{0}, 10_{1}, 14_{1}, 3_{1}, 4_{2}, 2_{2}, 1_{2}, 1_{0}, 0_{2}\right], G_{6}\left[3_{2}, 0_{0}, 2_{2}, 5_{2}, 1_{0}, 7_{2}, 13_{1}, 8_{2}, 0_{2}, 5_{0}\right]$,
$G_{6}\left[9_{2}, 0_{0}, 8_{2}, 13_{2}, 0_{1}, 7_{1}, 6_{2}, 0_{2}, 0_{3}, 15_{0}\right], G_{6}\left[3_{3}, 0_{0}, 0_{3}, 1_{3}, 3_{0}, 7_{3}, 14_{0}, 11_{3}, 9_{3}, 4_{0}\right]$,
$G_{6}\left[7_{3}, 0_{0}, 6_{3}, 9_{3}, 15_{0}, 0_{4}, 13_{1}, 1_{4}, 10_{2}, 0_{1}\right], G_{6}\left[4_{4}, 0_{0}, 0_{4}, 1_{4}, 3_{0}, 8_{4}, 0_{1}, 9_{4}, 2_{4}, 5_{0}\right]$,
$G_{6}\left[8_{4}, 0_{0}, 7_{4}, 9_{4}, 2_{1}, 0_{3}, 0_{1}, 1_{3}, 10_{4}, 14_{0}\right], G_{6}\left[33_{3}, 0_{1}, 2_{3}, 8_{3}, 3_{1}, 12_{3}, 2_{2}, 7_{3}, 0_{3}, 6_{1}\right]$,
$G_{6}\left[12_{3}, 0_{1}, 10_{3}, 0_{4}, 1_{1}, 7_{4}, 1_{3}, 2_{4}, 10_{2}, 3_{2}\right], G_{6}\left[12_{4}, 0_{1}, 3_{4}, 11_{4}, 13_{1}, 9_{4}, 3_{2}, 10_{4}, 6_{3}, 0_{2}\right]$,
$\left.G_{6}\left[2_{3}, 0_{2}, 1_{3}, 13_{3}, 1_{2}, 2_{4}, 16_{3}, 1_{4}, 2_{2}, 0_{4}\right], G_{6}\left[12_{4}, 0_{4}, 6_{4}, 12_{3}, 8_{4}, 15_{2}, 2_{4}, 6_{2}, 5_{3}, 1_{2}\right]\right\}$,
$B_{7}=\left\{G_{7}\left[0_{1}, 9_{0}, 1_{0}, 3_{0}, 0_{0}, 4_{0}, 15_{0}, 2_{1}, 10_{0}, 3_{1}\right], G_{7}\left[8_{1}, 2_{0}, 0_{1}, 5_{1}, 0_{0}, 1_{1}, 5_{0}, 0_{2}, 8_{0}, 2_{1}\right]\right.$,
$G_{7}\left[2_{2}, 3_{1}, 7_{1}, 0_{2}, 0_{0}, 12_{1}, 2_{1}, 3_{2}, 1_{0}, 4_{2}\right], G_{7}\left[0_{3}, 7_{0}, 1_{2}, 4_{2}, 0_{0}, 5_{2}, 12_{0}, 2_{3}, 8_{0}, 14_{2}\right]$,
$G_{7}\left[4_{2}, 0_{1}, 7_{2}, 14_{2}, 0_{0}, 8_{2}, 2_{1}, 13_{2}, 1_{1}, 16_{1}\right], G_{7}\left[7_{3}, 3_{0}, 0_{3}, 1_{3}, 0_{0}, 2_{3}, 4_{0}, 13_{3}, 5_{0}, 11_{3}\right]$,
$G_{7}\left[5_{4}, 0_{1}, 12_{3}, 0_{4}, 0_{0}, 1_{4}, 3_{0}, 6_{4}, 1_{0}, 7_{4}\right], G_{7}\left[5_{4}, 11_{0}, 4_{4}, 7_{4}, 0_{0}, 8_{4}, 13_{0}, 9_{4}, 0_{1}, 3_{2}\right]$,
$G_{7}\left[6_{3}, 0_{2}, 9_{2}, 0_{3}, 0_{1}, 1_{3}, 3_{1}, 5_{3}, 1_{1}, 11_{3}\right], G_{7}\left[3_{4}, 9_{1}, 5_{3}, 8_{3}, 0_{1}, 6_{3}, 12_{1}, 1_{4}, 1_{1}, 10_{3}\right]$,
$G_{7}\left[16_{4}, 4_{1}, 2_{4}, 10_{4}, 0_{1}, 3_{4}, 0_{2}, 1_{4}, 14_{1}, 11_{4}\right], G_{7}\left[15_{4}, 12_{3}, 4_{4}, 10_{4}, 0_{2}, 12_{4}, 11_{3}, 13_{4}, 5_{2}, 0_{3}\right]$,
$\left.G_{7}\left[4_{2}, 1_{4}, 4_{3}, 11_{3}, 0_{2}, 7_{3}, 2_{4}, 8_{2}, 0_{4}, 4_{4}\right], G_{7}\left[1_{4}, 2_{3}, 4_{2}, 0_{3}, 0_{2}, 5_{2}, 11_{2}, 3_{3}, 1_{2}, 11_{3}\right]\right\}$,
$B_{8}=\left\{G_{8}\left[3_{0}, 1_{0}, 9_{0}, 0_{1}, 3_{1}, 10_{0}, 2_{1}, 15_{0}, 4_{0}, 0_{0}\right], G_{8}\left[1_{1}, 0_{1}, 3_{0}, 6_{1}, 16_{1}, 5_{0}, 11_{1}, 4_{0}, 2_{1}, 0_{0}\right]\right.$,
$G_{8}\left[1_{2}, 0_{2}, 3_{0}, 6_{2}, 11_{2}, 5_{0}, 9_{2}, 4_{0}, 2_{2}, 0_{0}\right], G_{8}\left[10_{2}, 8_{2}, 14_{0}, 0_{3}, 2_{2}, 0_{1}, 0_{2}, 1_{1}, 9_{2}, 0_{0}\right]$,
$G_{8}\left[1_{3}, 0_{3}, 3_{0}, 7_{3}, 12_{3}, 5_{0}, 10_{3}, 4_{0}, 2_{3}, 0_{0}\right], G_{8}\left[11_{3}, 9_{3}, 0_{1}, 4_{1}, 1_{4}, 16_{0}, 0_{4}, 2_{1}, 10_{3}, 0_{0}\right]$,
$G_{8}\left[3_{4}, 0_{4}, 5_{0}, 10_{4}, 15_{4}, 6_{0}, 14_{4}, 8_{0}, 4_{4}, 0_{0}\right], G_{8}\left[11_{4}, 10_{4}, 3_{1}, 9_{1}, 14_{2}, 0_{1}, 3_{2}, 5_{1}, 15_{4}, 0_{0}\right]$,
$G_{8}\left[4_{2}, 8_{1}, 15_{2}, 2_{2}, 4_{3}, 9_{1}, 3_{3}, 0_{2}, 6_{2}, 0_{1}\right], G_{8}\left[4_{3}, 0_{3}, 2_{1}, 7_{3}, 5_{4}, 1_{1}, 4_{4}, 4_{1}, 1_{3}, 0_{1}\right]$,
$G_{8}\left[13_{3}, 2_{3}, 7_{2}, 14_{2}, 12_{4}, 6_{1}, 8_{4}, 0_{2}, 6_{3}, 0_{1}\right], G_{8}\left[13_{3}, 5_{3}, 13_{2}, 4_{3}, 3_{4}, 3_{2}, 0_{4}, 10_{2}, 7_{3}, 0_{2}\right]$,
$\left.G_{8}\left[3_{4}, 1_{4}, 10_{1}, 15_{4}, 9_{4}, 4_{2}, 8_{4}, 0_{3}, 2_{4}, 0_{2}\right], G_{8}\left[1_{4}, 7_{3}, 13_{4}, 0_{2}, 9_{4}, 6_{3}, 6_{4}, 16_{3}, 4_{4}, 0_{3}\right]\right\}$,
$B_{9}=\left\{G_{9}\left[7_{3}, 9_{0}, 4_{1}, 12_{1}, 4_{3}, 15_{4}, 14_{0}, 10_{0}, 10_{2}, 7_{0}\right], G_{9}\left[11_{0}, 1_{0}, 9_{3}, 1_{4}, 12_{3}, 1_{2}, 4_{1}, 16_{3}, 4_{0}, 3_{2}\right]\right.$,
$G_{9}\left[9_{4}, 11_{1}, 13_{4}, 15_{3}, 16_{3}, 11_{0}, 12_{0}, 5_{1}, 8_{2}, 10_{4}\right], G_{9}\left[4_{3}, 3_{4}, 9_{1}, 14_{2}, 13_{1}, 12_{3}, 10_{0}, 9_{3}, 3_{0}, 11_{0}\right]$,
$G_{9}\left[0_{1}, 3_{0}, 9_{0}, 1_{1}, 2_{1}, 1_{0}, 5_{1}, 7_{1}, 5_{0}, 0_{0}\right], G_{9}\left[1_{2}, 5_{1}, 14_{0}, 5_{2}, 10_{1}, 3_{0}, 0_{2}, 10_{2}, 6_{1}, 0_{0}\right]$,
$G_{9}\left[5_{2}, 4_{2}, 9_{0}, 1_{3}, 13_{3}, 12_{0}, 9_{3}, 3_{3}, 6_{2}, 0_{0}\right], G_{9}\left[0_{4}, 3_{3}, 0_{1}, 6_{1}, 3_{4}, 1_{0}, 7_{4}, 7_{1}, 13_{3}, 0_{0}\right]$,
$G_{9}\left[5_{4}, 3_{4}, 7_{0}, 1_{4}, 5_{1}, 0_{1}, 11_{2}, 14_{4}, 4_{4}, 0_{0}\right], G_{9}\left[1_{3}, 4_{1}, 6_{2}, 2_{2}, 13_{2}, 3_{1}, 5_{3}, 9_{2}, 0_{2}, 0_{1}\right]$,
$G_{9}\left[10_{3}, 8_{2}, 3_{2}, 2_{3}, 16_{3}, 5_{1}, 0_{4}, 7_{3}, 9_{2}, 0_{1}\right], G_{9}\left[4_{4}, 4_{3}, 1_{2}, 0_{4}, 5_{3}, 0_{2}, 7_{4}, 1_{4}, 13_{3}, 0_{1}\right]$,
$\left.G_{9}\left[13_{4}, 0_{3}, 7_{4}, 16_{0}, 9_{4}, 3_{2}, 1_{4}, 6_{2}, 11_{4}, 0_{2}\right], G_{9}\left[6_{3}, 8_{3}, 11_{4}, 1_{1}, 2_{4}, 5_{2}, 14_{4}, 5_{1}, 8_{4}, 0_{2}\right]\right\}$,
$B_{10}=\left\{G_{10}\left[10_{3}, 5_{0}, 3_{2}, 6_{3}, 4_{0}, 2_{0}, 8_{0}, 1_{3}, 4_{2}, 3_{3}\right], G_{10}\left[2_{1}, 1_{0}, 5_{1}, 6_{2}, 4_{2}, 16_{4}, 0_{2}, 7_{3}, 1_{1}, 6_{4}\right]\right.$,
$G_{10}\left[10_{4}, 10_{2}, 5_{2}, 2_{2}, 0_{1}, 2_{1}, 11_{0}, 7_{1}, 9_{3}, 4_{2}\right], G_{10}\left[14_{4}, 7_{0}, 5_{3}, 0_{1}, 1_{1}, 7_{2}, 11_{0}, 1_{2}, 2_{0}, 1_{3}\right]$,
$G_{10}\left[8_{0}, 1_{0}, 0_{0}, 4_{0}, 12_{0}, 0_{1}, 5_{0}, 2_{1}, 12_{1}, 6_{1}\right], G_{10}\left[88_{0}, 2_{1}, 0_{0}, 7_{1}, 1_{0}, 10_{2}, 3_{1}, 0_{1}, 4_{2}, 5_{2}\right]$,
$G_{10}\left[8_{0}, 1_{2}, 0_{0}, 8_{2}, 2_{0}, 5_{3}, 4_{2}, 4_{1}, 8_{3}, 0_{3}\right], G_{10}\left[4_{0}, 0_{3}, 0_{0}, 4_{3}, 9_{0}, 0_{4}, 7_{3}, 0_{1}, 2_{4}, 1_{4}\right]$,
$G_{10}\left[1_{0}, 0_{4}, 0_{0}, 2_{4}, 4_{0}, 7_{4}, 1_{4}, 1_{1}, 8_{4}, 12_{4}\right], G_{10}\left[2_{1}, 5_{4}, 0_{0}, 12_{4}, 0_{1}, 11_{2}, 9_{4}, 3_{1}, 3_{3}, 1_{2}\right]$,
$G_{10}\left[1_{2}, 8_{1}, 0_{1}, 1_{3}, 2_{1}, 14_{3}, 8_{2}, 12_{1}, 10_{3}, 9_{3}\right], G_{10}\left[0_{2}, 11_{3}, 0_{1}, 9_{4}, 1_{2}, 10_{3}, 10_{4}, 5_{2}, 15_{4}, 1_{4}\right]$,
$\left.G_{10}\left[0_{4}, 6_{2}, 0_{2}, 4_{3}, 10_{4}, 16_{3}, 4_{2}, 7_{4}, 3_{3}, 15_{3}\right], G_{10}\left[7_{1}, 3_{4}, 0_{3}, 8_{4}, 4_{2}, 6_{4}, 11_{3}, 8_{3}, 13_{4}, 5_{4}\right]\right\}$,
$B_{11}=\left\{G_{11}\left[1_{1}, 9_{0}, 3_{0}, 0_{0}, 1_{0}, 6_{0}, 13_{0}, 4_{0}, 0_{1}, 2_{1}\right], G_{11}\left[12_{1}, 5_{0}, 3_{1}, 0_{0}, 0_{1}, 1_{0}, 5_{1}, 1_{1}, 7_{1}, 0_{2}\right]\right.$,
$G_{11}\left[2_{2}, 1_{1}, 10_{1}, 0_{0}, 5_{1}, 12_{1}, 1_{2}, 14_{1}, 0_{2}, 1_{0}\right], G_{11}\left[11_{2}, 6_{0}, 3_{2}, 0_{0}, 2_{2}, 4_{0}, 0_{2}, 4_{2}, 1_{2}, 3_{1}\right]$,
$G_{11}\left[2_{3}, 15_{0}, 8_{2}, 0_{0}, 6_{2}, 12_{0}, 0_{3}, 9_{2}, 1_{3}, 1_{0}\right], G_{11}\left[0_{4}, 9_{0}, 6_{3}, 0_{0}, 2_{3}, 4_{0}, 0_{3}, 8_{3}, 1_{3}, 6_{0}\right]$,
$G_{11}\left[4_{4}, 1_{0}, 0_{4}, 0_{0}, 10_{3}, 0_{1}, 0_{2}, 1_{4}, 3_{4}, 4_{1}\right], G_{11}\left[1_{4}, 8_{0}, 5_{4}, 0_{0}, 4_{4}, 6_{0}, 13_{4}, 6_{4}, 0_{4}, 3_{1}\right]$,
$G_{11}\left[2_{3}, 0_{2}, 9_{2}, 0_{1}, 2_{2}, 7_{2}, 0_{3}, 11_{2}, 1_{3}, 1_{1}\right], G_{11}\left[0_{3}, 6_{1}, 3_{3}, 0_{1}, 2_{3}, 4_{1}, 9_{3}, 4_{3}, 1_{3}, 4_{2}\right]$,
$G_{11}\left[2_{4}, 4_{2}, 8_{3}, 0_{1}, 6_{3}, 14_{1}, 0_{4}, 7_{3}, 1_{4}, 6_{1}\right], G_{11}\left[0_{4}, 0_{2}, 6_{4}, 0_{1}, 1_{4}, 9_{1}, 16_{4}, 2_{4}, 6_{2}, 0_{3}\right]$,
$\left.G_{11}\left[8_{4}, 3_{2}, 2_{4}, 0_{2}, 0_{3}, 1_{2}, 4_{4}, 3_{3}, 7_{4}, 13_{2}\right], G_{11}\left[2_{4}, 13_{3}, 10_{4}, 0_{2}, 15_{3}, 4_{3}, 0_{4}, 9_{4}, 5_{4}, 14_{3}\right]\right\}$,
$B_{12}=\left\{G_{12}\left[1_{3}, 3_{1}, 13_{2}, 7_{3}, 8_{1}, 10_{2}, 13_{0}, 9_{0}, 8_{0}, 8_{3}\right], G_{12}\left[6_{4}, 16_{2}, 1_{2}, 11_{1}, 0_{2}, 7_{3}, 12_{2}, 8_{1}, 15_{1}, 15_{0}\right]\right.$,
$G_{12}\left[10_{4}, 8_{4}, 16_{0}, 3_{2}, 1_{4}, 2_{1}, 5_{4}, 4_{1}, 11_{0}, 16_{1}\right], G_{12}\left[0_{0}, 2_{0}, 9_{0}, 6_{1}, 4_{1}, 3_{0}, 1_{1}, 10_{0}, 0_{1}, 8_{0}\right]$,
$G_{12}\left[0_{0}, 2_{1}, 3_{0}, 5_{2}, 14_{1}, 3_{1}, 1_{2}, 8_{2}, 2_{0}, 6_{1}\right], G_{12}\left[0_{0}, 0_{2}, 2_{0}, 13_{2}, 9_{2}, 3_{2}, 0_{3}, 4_{2}, 9_{0}, 5_{2}\right]$,
$G_{12}\left[0_{0}, 8_{2}, 15_{0}, 2_{3}, 7_{3}, 9_{2}, 8_{3}, 4_{3}, 2_{0}, 1_{3}\right], G_{12}\left[0_{0}, 3_{3}, 6_{0}, 1_{3}, 2_{3}, 5_{3}, 0_{4}, 1_{4}, 1_{0}, 11_{3}\right]$,
$G_{12}\left[0_{3}, 1_{4}, 6_{2}, 11_{3}, 7_{4}, 14_{2}, 11_{4}, 1_{1}, 14_{3}, 5_{1}\right], G_{12}\left[0_{0}, 4_{4}, 0_{1}, 3_{2}, 9_{1}, 6_{4}, 4_{2}, 4_{3}, 3_{1}, 10_{4}\right]$,
$G_{12}\left[0_{0}, 7_{4}, 1_{1}, 12_{3}, 10_{2}, 13_{4}, 13_{2}, 0_{4}, 0_{1}, 12_{4}\right], G_{12}\left[0_{1}, 0_{3}, 3_{1}, 7_{3}, 5_{4}, 2_{3}, 6_{4}, 13_{4}, 0_{2}, 6_{3}\right]$,
$\left.G_{12}\left[0_{1}, 3_{3}, 11_{2}, 3_{4}, 16_{4}, 8_{3}, 0_{4}, 10_{3}, 6_{2}, 5_{4}\right], G_{12}\left[0_{0}, 7_{3}, 0_{1}, 0_{2}, 5_{1}, 3_{4}, 0_{4}, 8_{4}, 3_{0}, 1_{4}\right]\right\}$,
$B_{13}=\left\{G_{13}\left[3_{2}, 5_{0}, 10_{3}, 3_{3}, 1_{3}, 8_{0}, 2_{0}, 4_{2}, 4_{0}, 6_{3}\right], G_{13}\left[5_{1}, 1_{0}, 2_{1}, 1_{1}, 6_{4}, 0_{2}, 16_{4}, 5_{2}, 4_{2}, 6_{2}\right]\right.$,
$G_{13}\left[0_{0}, 10_{2}, 0_{1}, 9_{2}, 7_{1}, 10_{4}, 2_{1}, 5_{3}, 2_{3}, 7_{2}\right], G_{13}\left[7_{0}, 0_{1}, 4_{4}, 14_{1}, 2_{0}, 15_{1}, 1_{2}, 4_{1}, 1_{1}, 11_{0}\right]$,
$G_{13}\left[0_{0}, 1_{0}, 8_{0}, 8_{1}, 3_{1}, 5_{0}, 2_{1}, 2_{2}, 11_{0}, 3_{0}\right], G_{13}\left[0_{0}, 2_{1}, 14_{0}, 0_{2}, 11_{2}, 3_{1}, 8_{2}, 0_{3}, 2_{0}, 1_{2}\right]$,
$G_{13}\left[0_{0}, 4_{2}, 7_{0}, 1_{3}, 8_{1}, 6_{2}, 3_{3}, 4_{1}, 0_{1}, 13_{2}\right], G_{13}\left[0_{0}, 0_{3}, 1_{0}, 7_{3}, 0_{4}, 3_{3}, 8_{3}, 3_{4}, 9_{0}, 4_{3}\right]$,
$G_{13}\left[0_{0}, 9_{3}, 0_{1}, 6_{2}, 3_{4}, 1_{4}, 6_{1}, 4_{4}, 1_{0}, 0_{4}\right], G_{13}\left[0_{0}, 4_{4}, 6_{0}, 2_{4}, 10_{4}, 6_{4}, 13_{4}, 8_{4}, 13_{0}, 5_{4}\right]$,
$G_{13}\left[0_{1}, 7_{2}, 0_{2}, 13_{2}, 7_{3}, 1_{3}, 12_{2}, 3_{4}, 2_{1}, 0_{3}\right], G_{13}\left[0_{1}, 2_{3}, 6_{1}, 13_{3}, 3_{4}, 4_{3}, 2_{4}, 5_{4}, 3_{1}, 11_{3}\right]$,
$\left.G_{13}\left[0_{1}, 6_{3}, 1_{2}, 10_{4}, 7_{3}, 9_{4}, 5_{3}, 4_{3}, 6_{2}, 6_{4}\right], G_{13}\left[0_{3}, 5_{4}, 12_{1}, 6_{4}, 12_{3}, 10_{2}, 11_{4}, 1_{2}, 13_{4}, 9_{2}\right]\right\}$.

For $i \in[2,13]$, a $G_{i}$-decomposition of $K_{85}$ consists of the $G_{i}$-blocks in $B_{i}$ under the action of the map $r_{s} \mapsto(r+1(\bmod 17))_{s}$.

Example 23. Let $V\left(K_{100}\right)=\left\{r_{s}: r \in \mathbb{Z}_{33}\right.$ and $\left.s \in \mathbb{Z}_{3}\right\} \cup\{\infty\}$, and let

$$
\begin{aligned}
B_{2}=\{ & G_{2}\left[13_{0}, 12_{1}, 23_{1}, 22_{1}, 13_{2}, 5_{2}, 30_{1}, 3_{0}, 9_{0}, 4_{1}\right], G_{2}\left[1_{1}, 18_{0}, 9_{0}, 32_{0}, 4_{2}, 13_{2}, 9_{2}, 29_{1}, 15_{2}, 12_{2}\right], \\
& G_{2}\left[5_{1}, 19_{1}, 17_{2}, 18_{0}, 14_{2}, 28_{2}, 13_{1}, 28_{1}, 15_{2}, 8_{2}\right], G_{2}\left[3_{1}, 17_{2}, 16_{0}, 5_{2}, 20_{0}, 22_{1}, 27_{1}, 10_{0}, 27_{2}, 17_{0}\right], \\
& G_{2}\left[13_{1}, 9_{1}, 8_{0}, 6_{1}, 3_{0}, 20_{0}, 22_{0}, 19_{0}, 30_{0}, 15_{1}\right], G_{2}\left[0_{0}, 4_{0}, 12_{0}, 5_{0}, 18_{0}, 6_{1}, 22_{1}, 25_{0}, 10_{0}, 0_{1}\right], \\
& G_{2}\left[0_{0}, 6_{1}, 15_{1}, 4_{0}, 12_{1}, 0_{2}, 16_{2}, 7_{0}, 2_{2}, 26_{1}\right], G_{2}\left[0_{0}, 22_{1}, 7_{2}, 1_{0}, 4_{2}, 10_{2}, 25_{2}, 6_{0}, 14_{2}, 12_{2}\right], \\
& \left.G_{2}\left[0_{0}, 11_{2}, 21_{2}, 11_{1}, 6_{2}, 7_{1}, 10_{1}, 4_{2}, 32_{1}, 24_{2}\right], G_{2}\left[0_{2}, 1_{2}, 19_{0}, 6_{2}, 14_{0}, 4_{2}, \infty, 11_{1}, 4_{1}, 27_{1}\right]\right\} .
\end{aligned}
$$

Then a $G_{2}$-decomposition of $K_{100}$ consists of the $G_{2}$-blocks in $B_{2}$ under the action of the map $\infty \mapsto \infty$ and $r_{s} \mapsto(r+1(\bmod 33))_{s}$.
Example 24. Let $V\left(K_{4 \times 5}\right)=\left\{r_{s}: r \in \mathbb{Z}_{5}\right.$ and $\left.s \in \mathbb{Z}_{4}\right\}$ with the obvious vertex partition and let

$$
\begin{aligned}
B_{2} & =\left\{G_{2}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 3_{2}, 1_{1}, 0_{3}, 4_{0}, 2_{1}, 1_{2}\right], G_{2}\left[0_{0}, 4_{1}, 0_{3}, 3_{1}, 1_{3}, 4_{0}, 2_{2}, 2_{3}, 1_{2}, 4_{3}\right]\right\}, \\
B_{3} & =\left\{G_{3}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 3_{2}, 1_{1}, 0_{3}, 2_{1}, 3_{0}, 1_{2}\right], G_{3}\left[0_{1}, 4_{2}, 0_{3}, 0_{2}, 2_{1}, 3_{3}, 0_{0}, 4_{3}, 2_{0}, 2_{3}\right]\right\}, \\
B_{4} & =\left\{G_{4}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 3_{2}, 0_{3}, 1_{1}, 1_{3}, 3_{0}, 1_{2}\right], G_{4}\left[0_{1}, 3_{3}, 2_{2}, 3_{1}, 4_{0}, 0_{3}, 0_{0}, 1_{1}, 2_{3}, 3_{2}\right]\right\}, \\
B_{5} & =\left\{G_{5}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 2_{1}, 0_{3}, 3_{1}, 2_{3}, 3_{0}, 1_{2}\right], G_{5}\left[0_{0}, 4_{1}, 2_{2}, 1_{3}, 1_{2}, 2_{1}, 4_{2}, 2_{3}, 4_{0}, 0_{3}\right]\right\}, \\
B_{6} & =\left\{G_{6}\left[1_{1}, 0_{0}, 0_{1}, 0_{2}, 1_{0}, 2_{2}, 0_{3}, 3_{2}, 4_{1}, 2_{0}\right], G_{6}\left[3_{3}, 0_{0}, 2_{3}, 3_{2}, 4_{3}, 4_{0}, 0_{3}, 2_{1}, 3_{1}\right]\right\}, \\
B_{7} & =\left\{G_{7}\left[0_{3}, 2_{0}, 0_{1}, 0_{2}, 0_{0}, 1_{1}, 4_{0}, 1_{3}, 2_{2}\right], G_{7}\left[3_{3}, 0_{1}, 2_{2}, 4_{3}, 0_{0}, 3_{2}, 4_{1}, 0_{3}, 4_{2}, 3_{1}\right]\right\}, \\
B_{8} & =\left\{G_{8}\left[0_{2}, 0_{1}, 2_{0}, 3_{2}, 1_{3}, 1_{0}, 0_{3}, 4_{0}, 1_{1}, 0_{0}\right], G_{8}\left[3_{3}, 2_{2}, 1_{1}, 0_{2}, 4_{3}, 2_{1}, 0_{3}, 0_{1}, 3_{2}, 0_{0}\right]\right\}, \\
B_{9} & =\left\{G_{9}\left[0_{2}, 0_{1}, 1_{0}, 2_{2}, 4_{1}, 2_{0}, 0_{3}, 3_{2}, 1_{1}, 0_{0}\right], G_{9}\left[0_{3}, 3_{1}, 3_{3}, 2_{0}, 1_{2}, 2_{1}, 1_{3}, 0_{2}, 4_{3}, 0_{0}\right]\right\}, \\
B_{10} & =\left\{G_{10}\left[1_{0}, 0_{1}, 0_{0}, 0_{2}, 2_{0}, 3_{2}, 1_{1}, 2_{2}, 0_{3}, 3_{1}\right], G_{10}\left[2_{3}, 3_{1}, 0_{0}, 1_{3}, 2_{0}, 1_{2}, 0_{3}, 4_{1}, 4_{3}, 2_{2}\right]\right\}, \\
B_{11} & =\left\{G_{11}\left[4_{1}, 2_{0}, 0_{2}, 0_{0}, 0_{1}, 1_{0}, 2_{2}, 1_{1}, 4_{2}, 0_{3}\right], G_{11}\left[4_{1}, 1_{2}, 1_{3}, 0_{0}, 4_{2}, 3_{3}, 1_{0}, 0_{3}, 2_{1}, 4_{3}\right]\right\}, \\
B_{12} & =\left\{G_{12}\left[0_{0}, 0_{1}, 1_{0}, 3_{1}, 2_{2}, 1_{1}, 4_{2}, 0_{3}, 2_{0}, 0_{2}\right], G_{12}\left[0_{0}, 3_{1}, 4_{3}, 4_{1}, 1_{2}, 0_{3}, 3_{2}, 3_{3}, 4_{0}, 1_{3}\right]\right\}, \\
B_{13} & =\left\{G_{13}\left[0_{0}, 0_{1}, 1_{0}, 3_{1}, 0_{3}, 1_{1}, 2_{2}, 1_{3}, 2_{0}, 0_{2}\right], G_{13}\left[0_{0}, 3_{1}, 3_{3}, 2_{1}, 4_{2}, 1_{3}, 1_{0}, 0_{2}, 0_{3}, 2_{2}\right]\right\} .
\end{aligned}
$$

For $i \in[2,13]$, a $G_{i}$-decomposition of $K_{4 \times 5}$ consists of the $G_{i}$-blocks in $B_{i}$ under the action of the map $r_{s} \mapsto(r+1(\bmod 5))_{s}$.

Example 25. Let $V\left(K_{3 \times 15}\right)=\left\{r_{s}: r \in \mathbb{Z}_{15}\right.$ and $\left.s \in \mathbb{Z}_{3}\right\}$ with the obvious vertex partition and let

$$
\begin{aligned}
B_{2}=\{ & G_{2}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 3_{2}, 6_{0}, 8_{1}, 4_{2}, 2_{1}, 1_{2}\right], G_{2}\left[0_{0}, 3_{2}, 5_{1}, 8_{0}, 3_{1}, 4_{0}, 12_{2}, 7_{1}, 13_{2}, 6_{1}\right], \\
& \left.G_{2}\left[0_{1}, 3_{2}, 8_{0}, 1_{1}, 7_{0}, 5_{1}, 13_{2}, 9_{0}, 5_{2}, 11_{0}\right]\right\}, \\
B_{3}=\{ & \left\{G_{3}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 3_{1}, 5_{2}, 8_{0}, 2_{1}, 3_{0}, 1_{2}\right], G_{3}\left[0_{0}, 3_{1}, 6_{2}, 1_{0}, 13_{1}, 12_{2}, 2_{0}, 7_{1}, 3_{0}, 7_{2}\right],\right. \\
& \left.G_{3}\left[0_{0}, 13_{1}, 8_{2}, 0_{1}, 12_{2}, 9_{0}, 1_{1}, 7_{2}, 2_{1}, 11_{2}\right]\right\}, \\
B_{4}=\{ & G_{4}\left[0_{2}, 0_{1}, 11_{0}, 14_{2}, 13_{0}, 6_{1}, 14_{0}, 1_{1}, 7_{2}, 9_{0}\right], G_{4}\left[0_{0}, 0_{1}, 2_{2}, 2_{0}, 1_{1}, 12_{2}, 3_{1}, 0_{2}, 6_{0}, 5_{2}\right], \\
& \left.G_{4}\left[0_{0}, 11_{2}, 1_{1}, 3_{0}, 12_{1}, 10_{2}, 6_{1}, 1_{0}, 13_{2}, 10_{1}\right]\right\}, \\
B_{5}= & \left\{G_{5}\left[0_{0}, 0_{1}, 0_{2}, 1_{0}, 2_{1}, 5_{2}, 3_{1}, 7_{2}, 3_{0}, 1_{2}\right], G_{5}\left[0_{0}, 3_{1}, 9_{2}, 1_{0}, 12_{1}, 6_{2}, 13_{1}, 8_{2}, 3_{0}, 10_{2}\right],\right. \\
& \left.G_{5}\left[0_{1}, 13_{2}, 1_{0}, 7_{2}, 8_{1}, 0_{0}, 1_{1}, 4_{0}, 9_{1}, 2_{0}\right]\right\}, \\
B_{6}=\{ & \left\{G_{6}\left[1_{1}, 0_{0}, 0_{1}, 0_{2}, 1_{0}, 2_{2}, 7_{0}, 3_{2}, 4_{1}, 2_{0}\right], G_{6}\left[4_{1}, 0_{0}, 3_{1}, 5_{2}, 1_{0}, 7_{2}, 10_{0}, 4_{2}, 0_{1}, 8_{0}\right],\right. \\
& \left.G_{6}\left[11_{2}, 0_{1}, 13_{2}, 6_{0}, 14_{2}, 2_{1}, 12_{2}, 7_{1}, 14_{0}, 5_{1}\right]\right\}, \\
B_{7}=\{ & G_{7}\left[6_{1}, 2_{0}, 0_{1}, 0_{2}, 0_{0}, 1_{1}, 4_{0}, 7_{1}, 1_{0}, 2_{2}\right], G_{7}\left[3_{0}, 6_{2}, 5_{1}, 2_{2}, 0_{0}, 7_{1}, 0_{2}, 12_{1}, 4_{0}, 10_{2}\right], \\
& \left.G_{7}\left[6_{0}, 1_{2}, 9_{1}, 11_{2}, 0_{0}, 10_{1}, 1_{2}, 10_{0}, 6_{1}, 0_{2}\right]\right\}, \\
B_{8}=\{ & G_{8}\left[0_{2}, 0_{1}, 2_{0}, 4_{1}, 6_{2}, 1_{0}, 2_{2}, 4_{0}, 1_{1}, 0_{0}\right], G_{8}\left[4_{2}, 3_{1}, 8_{0}, 0_{1}, 8_{2}, 1_{0}, 7_{2}, 11_{0}, 4_{1}, 0_{0}\right], \\
& \left.G_{8}\left[9_{0}, 4_{2}, 1_{1}, 7_{2}, 12_{1}, 2_{2}, 8_{1}, 3_{0}, 12_{2}, 0_{1}\right]\right\},
\end{aligned}
$$

```
\(B_{9}=\left\{G_{9}\left[0_{2}, 0_{1}, 1_{2}, 3_{0}, 5_{1}, 1_{0}, 7_{2}, 4_{0}, 1_{1}, 0_{0}\right], G_{9}\left[1_{2}, 3_{1}, 0_{2}, 10_{0}, 14_{2}, 7_{0}, 6_{1}, 13_{0}, 5_{1}, 0_{0}\right]\right.\),
        \(\left.G_{9}\left[10_{2}, 6_{1}, 13_{2}, 2_{1}, 6_{0}, 0_{1}, 14_{2}, 8_{1}, 11_{2}, 0_{0}\right]\right\}\),
\(B_{10}=\left\{G_{10}\left[1_{0}, 0_{1}, 0_{0}, 0_{2}, 2_{0}, 3_{2}, 1_{1}, 3_{0}, 6_{2}, 5_{1}\right], G_{10}\left[6_{0}, 3_{1}, 0_{0}, 6_{2}, 0_{1}, 13_{2}, 5_{1}, 10_{0}, 9_{2}, 2_{1}\right]\right.\),
        \(\left.G_{10}\left[5_{2}, 8_{1}, 0_{0}, 12_{2}, 4_{0}, 10_{1}, 9_{2}, 4_{1}, 0_{2}, 10_{0}\right]\right\}\),
\(B_{11}=\left\{G_{11}\left[4_{1}, 2_{0}, 0_{2}, 0_{0}, 0_{1}, 1_{0}, 2_{2}, 1_{1}, 4_{2}, 7_{0}\right], G_{11}\left[10_{2}, 4_{0}, 3_{2}, 0_{0}, 4_{1}, 6_{0}, 0_{2}, 5_{1}, 9_{2}, 2_{1}\right]\right.\),
        \(\left.G_{11}\left[2_{2}, 11_{1}, 8_{2}, 0_{0}, 6_{1}, 10_{0}, 6_{2}, 10_{1}, 3_{0}, 12_{1}\right]\right\}\),
\(B_{12}=\left\{G_{12}\left[0_{0}, 0_{1}, 1_{0}, 3_{1}, 2_{2}, 1_{1}, 4_{0}, 6_{2}, 2_{1}, 0_{2}\right], G_{12}\left[0_{0}, 3_{1}, 0_{2}, 4_{1}, 8_{0}, 3_{2}, 9_{1}, 14_{2}, 1_{0}, 5_{2}\right]\right.\),
        \(\left.G_{12}\left[0_{1}, 8_{2}, 14_{0}, 6_{1}, 13_{2}, 5_{0}, 11_{2}, 0_{0}, 5_{1}, 11_{0}\right]\right\}\),
\(B_{13}=\left\{G_{13}\left[0_{0}, 0_{1}, 1_{0}, 6_{1}, 4_{0}, 1_{1}, 2_{2}, 10_{0}, 2_{1}, 0_{2}\right], G_{13}\left[0_{0}, 3_{1}, 0_{2}, 2_{0}, 6_{2}, 4_{1}, 9_{0}, 14_{2}, 3_{0}, 2_{2}\right]\right.\),
        \(\left.G_{13}\left[0_{1}, 3_{2}, 9_{1}, 0_{2}, 7_{1}, 11_{2}, 1_{0}, 10_{1}, 2_{2}, 9_{0}\right]\right\}\).
```

For $i \in[2,13]$, a $G_{i}$-decomposition of $K_{3 \times 15}$ consists of the $G_{i}$-blocks in $B_{i}$ under the action of the map $r_{s} \mapsto(r+1 \bmod 15)_{s}$.
Example 26. Let $V\left(K_{5 \times 15}\right)=\left\{r_{s}: r \in \mathbb{Z}_{15}\right.$ and $\left.s \in \mathbb{Z}_{5}\right\}$ with the obvious vertex partition. Let

```
B2}={\mp@subsup{G}{2}{[}[\mp@subsup{1}{4}{},\mp@subsup{4}{1}{},\mp@subsup{4}{3}{},1\mp@subsup{4}{0}{},\mp@subsup{9}{1}{},1\mp@subsup{4}{2}{},1\mp@subsup{2}{0}{},\mp@subsup{9}{3}{},\mp@subsup{8}{2}{},\mp@subsup{3}{0}{}],\mp@subsup{G}{2}{}[\mp@subsup{6}{4}{},\mp@subsup{6}{2}{},1\mp@subsup{0}{3}{},\mp@subsup{6}{1}{},1\mp@subsup{0}{0}{},\mp@subsup{1}{4}{},\mp@subsup{9}{1}{},\mp@subsup{5}{0}{},\mp@subsup{7}{4}{},\mp@subsup{7}{0}{}]
```



```
    G2[10},\mp@subsup{4}{3}{},\mp@subsup{6}{4}{},1\mp@subsup{4}{3}{},1\mp@subsup{3}{4}{},1\mp@subsup{2}{2}{},\mp@subsup{3}{1}{},\mp@subsup{0}{3}{},\mp@subsup{0}{4}{},1\mp@subsup{1}{2}{}],\mp@subsup{G}{2}{}[00,\mp@subsup{0}{1}{},\mp@subsup{0}{2}{},\mp@subsup{2}{0}{},\mp@subsup{4}{1}{},\mp@subsup{2}{2}{},\mp@subsup{0}{3}{},\mp@subsup{7}{0}{},\mp@subsup{5}{1}{},\mp@subsup{4}{2}{}]
```



```
    G2[0}\mp@subsup{\mp@code{1}}{1}{},\mp@subsup{8}{2}{},\mp@subsup{3}{4}{},1\mp@subsup{4}{0}{},\mp@subsup{8}{4}{},\mp@subsup{2}{1}{},1\mp@subsup{4}{2}{},\mp@subsup{2}{4}{},\mp@subsup{6}{2}{},1\mp@subsup{3}{4}{}],\mp@subsup{G}{2}{}[0\mp@subsup{0}{3}{},54,1\mp@subsup{3}{0}{},\mp@subsup{5}{3}{},1\mp@subsup{2}{1}{},1\mp@subsup{3}{3}{},\mp@subsup{6}{4}{},\mp@subsup{9}{0}{},1\mp@subsup{4}{4}{},\mp@subsup{6}{2}{}]}
```






```
        G3[0
B4}={\mp@subsup{G}{4}{}[84,\mp@subsup{3}{3}{},\mp@subsup{9}{2}{},\mp@subsup{7}{1}{},\mp@subsup{8}{0}{},1\mp@subsup{1}{3}{},1\mp@subsup{4}{1}{},1\mp@subsup{3}{2}{},\mp@subsup{5}{3}{},\mp@subsup{6}{0}{}],\mp@subsup{G}{4}{}[1\mp@subsup{2}{0}{},1\mp@subsup{3}{1}{},\mp@subsup{9}{4}{},\mp@subsup{6}{0}{},1\mp@subsup{4}{3}{},\mp@subsup{0}{1}{},\mp@subsup{1}{4}{},\mp@subsup{6}{2}{},1\mp@subsup{4}{1}{},\mp@subsup{4}{2}{}]
```






```
B5}={\mp@subsup{G}{5}{}[\mp@subsup{0}{1}{},1\mp@subsup{3}{0}{},\mp@subsup{5}{3}{},1\mp@subsup{4}{4}{},1\mp@subsup{3}{2}{},\mp@subsup{3}{1}{},\mp@subsup{2}{3}{},\mp@subsup{9}{1}{},\mp@subsup{8}{0}{},1\mp@subsup{0}{3}{}],\mp@subsup{G}{5}{}[30,91,1\mp@subsup{3}{4}{},1\mp@subsup{0}{2}{},\mp@subsup{1}{4}{},\mp@subsup{4}{1}{},\mp@subsup{4}{4}{},1\mp@subsup{1}{2}{},\mp@subsup{5}{3}{},\mp@subsup{9}{2}{}]
```






```
B6}={\mp@subsup{G}{6}{}[1\mp@subsup{0}{0}{},\mp@subsup{5}{3}{},\mp@subsup{7}{1}{},\mp@subsup{3}{0}{},1\mp@subsup{0}{1}{},\mp@subsup{5}{2}{},\mp@subsup{8}{4}{},1\mp@subsup{0}{3}{},\mp@subsup{6}{2}{},1\mp@subsup{4}{3}{}],\mp@subsup{G}{6}{}[32,1\mp@subsup{4}{0}{},1\mp@subsup{2}{2}{},1\mp@subsup{4}{1}{},1\mp@subsup{3}{0}{},5\mp@subsup{5}{3}{},1\mp@subsup{1}{4}{},\mp@subsup{3}{3}{},1\mp@subsup{3}{1}{},1\mp@subsup{0}{3}{}]
```




```
        G6[12, 0},\mp@subsup{0}{2}{},\mp@subsup{0}{3}{},\mp@subsup{1}{0}{},\mp@subsup{2}{3}{},\mp@subsup{7}{4}{},\mp@subsup{7}{3}{},\mp@subsup{4}{0}{},\mp@subsup{6}{4}{}],\mp@subsup{G}{6}{}[1\mp@subsup{2}{2}{},\mp@subsup{0}{0}{},\mp@subsup{6}{2}{},\mp@subsup{4}{4}{},\mp@subsup{3}{0}{},\mp@subsup{8}{4}{},\mp@subsup{6}{1}{},1\mp@subsup{2}{4}{},\mp@subsup{1}{3}{},1\mp@subsup{0}{1}{}]
        G6[74, 01, 4 4, 14, 60, 0}\mp@subsup{\mp@code{3}}{3}{},\mp@subsup{5}{2}{},\mp@subsup{5}{4}{},\mp@subsup{5}{1}{},1\mp@subsup{4}{2}{}],\mp@subsup{G}{6}{}[\mp@subsup{9}{4}{},\mp@subsup{0}{3}{},\mp@subsup{2}{4}{},\mp@subsup{7}{1}{},\mp@subsup{9}{3}{},1\mp@subsup{2}{0}{},\mp@subsup{5}{4}{},\mp@subsup{0}{1}{},\mp@subsup{1}{3}{},1\mp@subsup{3}{1}{}]}
B7 ={G7[1\mp@subsup{1}{0}{},1\mp@subsup{4}{2}{},\mp@subsup{2}{1}{},\mp@subsup{9}{2}{},\mp@subsup{6}{4}{},\mp@subsup{9}{1}{},\mp@subsup{9}{3}{},\mp@subsup{7}{2}{},\mp@subsup{5}{1}{},\mp@subsup{1}{3}{}],\mp@subsup{G}{7}{}[\mp@subsup{7}{2}{},\mp@subsup{7}{3}{},1\mp@subsup{2}{4}{},\mp@subsup{1}{2}{},\mp@subsup{6}{0}{},\mp@subsup{6}{2}{},1\mp@subsup{2}{1}{},1\mp@subsup{1}{2}{},\mp@subsup{6}{3}{},\mp@subsup{6}{4}{}],
```






$$
\begin{aligned}
& B_{8}=\left\{G_{8}\left[11_{0}, 1_{3}, 12_{2}, 1_{1}, 13_{4}, 14_{2}, 9_{3}, 10_{1}, 4_{0}, 2_{4}\right], G_{8}\left[10_{3}, 7_{0}, 4_{4}, 10_{0}, 12_{4}, 8_{1}, 0_{4}, 11_{0}, 4_{1}, 3_{4}\right],\right. \\
& G_{8}\left[1_{3}, 1_{1}, 7_{2}, 6_{0}, 4_{3}, 10_{4}, 4_{1}, 3_{2}, 14_{4}, 8_{2}\right], G_{8}\left[1_{2}, 0_{1}, 1_{3}, 10_{1}, 8_{0}, 13_{1}, 6_{0}, 0_{3}, 2_{2}, 4_{3}\right] \text {, } \\
& G_{8}\left[11_{0}, 12_{1}, 6_{2}, 2_{1}, 0_{4}, 11_{2}, 11_{3}, 5_{2}, 0_{0}, 8_{3}\right], G_{8}\left[2_{2}, 0_{1}, 4_{0}, 13_{1}, 5_{3}, 3_{0}, 10_{2}, 6_{0}, 3_{1}, 0_{0}\right] \text {, } \\
& G_{8}\left[10_{2}, 10_{1}, 5_{3}, 4_{0}, 12_{4}, 12_{0}, 8_{3}, 13_{0}, 9_{2}, 0_{0}\right], G_{8}\left[6_{3}, 12_{2}, 2_{1}, 10_{2}, 12_{4}, 1_{1}, 11_{4}, 3_{1}, 0_{3}, 0_{0}\right] \text {, } \\
& \left.G_{8}\left[2_{4}, 5_{2}, 3_{4}, 6_{3}, 10_{4}, 2_{2}, 5_{4}, 10_{3}, 13_{2}, 0_{1}\right], G_{8}\left[9_{4}, 7_{3}, 10_{4}, 0_{0}, 7_{4}, 11_{3}, 3_{4}, 2_{1}, 5_{3}, 0_{2}\right]\right\} \text {, } \\
& B_{9}=\left\{G_{9}\left[3_{1}, 5_{0}, 0_{1}, 6_{4}, 6_{1}, 2_{4}, 1_{3}, 8_{4}, 9_{2}, 10_{3}\right], G_{9}\left[3_{3}, 11_{0}, 14_{1}, 4_{2}, 1_{1}, 8_{2}, 7_{3}, 3_{4}, 1_{3}, 11_{2}\right]\right. \text {, } \\
& G_{9}\left[13_{0}, 2_{2}, 13_{1}, 3_{3}, 0_{0}, 9_{3}, 8_{1}, 1_{3}, 2_{0}, 13_{4}\right], G_{9}\left[3_{0}, 13_{2}, 3_{1}, 5_{3}, 5_{0}, 14_{2}, 12_{1}, 11_{2}, 5_{1}, 2_{4}\right] \text {, } \\
& G_{9}\left[4_{2}, 6_{1}, 10_{3}, 13_{4}, 11_{0}, 6_{3}, 12_{4}, 2_{2}, 5_{0}, 11_{4}\right], G_{9}\left[1_{2}, 0_{1}, 1_{0}, 13_{3}, 4_{1}, 2_{0}, 7_{4}, 14_{3}, 1_{1}, 0_{0}\right] \text {, } \\
& G_{9}\left[0_{2}, 2_{4}, 10_{1}, 9_{3}, 14_{4}, 10_{0}, 14_{3}, 11_{4}, 11_{2}, 0_{1}\right], G_{9}\left[13_{4}, 3_{3}, 1_{4}, 6_{0}, 9_{4}, 4_{2}, 10_{3}, 10_{2}, 3_{4}, 0_{2}\right] \text {, } \\
& \left.G_{9}\left[13_{2}, 4_{1}, 8_{0}, 1_{3}, 7_{2}, 5_{0}, 11_{3}, 0_{4}, 5_{1}, 0_{0}\right], G_{9}\left[3_{2}, 6_{1}, 9_{0}, 3_{4}, 2_{2}, 11_{0}, 13_{3}, 7_{4}, 9_{1}, 0_{0}\right]\right\} \text {, } \\
& B_{10}=\left\{G_{10}\left[11_{2}, 4_{1}, 6_{0}, 2_{4}, 12_{3}, 13_{0}, 9_{1}, 2_{2}, 4_{4}, 0_{0}\right], G_{10}\left[8_{2}, 11_{1}, 4_{3}, 7_{2}, 7_{1}, 1_{4}, 11_{2}, 2_{3}, 10_{2}, 2_{4}\right]\right. \text {, } \\
& G_{10}\left[10_{1}, 0_{2}, 12_{4}, 14_{0}, 6_{2}, 2_{0}, 10_{3}, 5_{1}, 7_{3}, 11_{4}\right], G_{10}\left[2_{0}, 8_{4}, 2_{2}, 5_{3}, 9_{2}, 8_{1}, 6_{3}, 3_{1}, 6_{0}, 5_{4}\right] \text {, } \\
& G_{10}\left[9_{1}, 9_{3}, 0_{0}, 10_{4}, 9_{2}, 13_{4}, 7_{3}, 4_{0}, 12_{4}, 1_{3}\right], G_{10}\left[5_{1}, 1_{3}, 0_{1}, 10_{4}, 3_{0}, 4_{3}, 3_{4}, 13_{1}, 7_{3}, 5_{4}\right] \text {, } \\
& G_{10}\left[0_{2}, 2_{1}, 0_{0}, 12_{2}, 2_{0}, 2_{3}, 7_{1}, 1_{2}, 11_{4}, 14_{3}\right], G_{10}\left[14_{3}, 5_{2}, 0_{0}, 5_{4}, 5_{0}, 7_{4}, 8_{2}, 6_{4}, 11_{3}, 9_{0}\right] \text {, } \\
& \left.G_{10}\left[11_{1}, 8_{4}, 7_{0}, 12_{1}, 11_{4}, 0_{2}, 5_{3}, 5_{2}, 3_{1}, 0_{3}\right], G_{10}\left[1_{0}, 0_{1}, 0_{0}, 6_{2}, 6_{0}, 0_{2}, 1_{1}, 7_{0}, 2_{3}, 11_{1}\right]\right\} \text {, } \\
& B_{11}=\left\{G_{11}\left[5_{4}, 4_{0}, 4_{4}, 9_{0}, 6_{3}, 2_{2}, 12_{3}, 5_{2}, 12_{4}, 14_{1}\right], G_{11}\left[5_{3}, 0_{4}, 4_{2}, 14_{4}, 14_{1}, 6_{4}, 3_{0}, 1_{2}, 7_{3}, 6_{1}\right]\right. \text {, } \\
& G_{11}\left[5{ }_{3}, 14_{4}, 13_{1}, 5_{0}, 12_{4}, 6_{0}, 10_{4}, 3_{3}, 2_{0}, 11_{2}\right], G_{11}\left[8,0_{2}, 9_{0}, 2_{2}, 8_{4}, 8_{2}, 12_{4}, 10_{3}, 1_{0}, 4_{2}\right] \text {, } \\
& G_{11}\left[10_{4}, 9_{2}, 9_{0}, 9_{1}, 11_{4}, 6_{3}, 14_{1}, 8_{2}, 6_{1}, 11_{3}\right], G_{11}\left[6_{1}, 1_{0}, 2_{2}, 0_{0}, 1_{1}, 2_{0}, 6_{2}, 2_{1}, 8_{2}, 11_{0}\right] \text {, } \\
& G_{11}\left[10_{3}, 4_{0}, 0_{3}, 0_{0}, 6_{1}, 8_{0}, 3_{3}, 7_{1}, 7_{3}, 5_{2}\right], G_{11}\left[10_{4}, 14_{0}, 4_{3}, 0_{0}, 9_{1}, 12_{0}, 6_{4}, 11_{1}, 4_{4}, 1_{1}\right] \text {, } \\
& \left.G_{11}\left[3_{2}, 5_{4}, 8_{3}, 0_{0}, 5_{2}, 8_{1}, 6_{2}, 7_{3}, 1_{1}, 3_{3}\right], G_{11}\left[1_{4}, 5_{3}, 11_{4}, 0_{1}, 8_{3}, 4_{1}, 7_{3}, 8_{2}, 5_{4}, 11_{2}\right]\right\} \text {, } \\
& B_{12}=\left\{G_{12}\left[14_{4}, 8_{1}, 3_{2}, 2_{4}, 14_{0}, 1_{2}, 6_{0}, 9_{1}, 2_{3}, 5_{2}\right], G_{12}\left[9_{0}, 4_{1}, 7_{4}, 13_{1}, 2_{0}, 2_{3}, 11_{0}, 9_{4}, 2_{2}, 5_{4}\right]\right. \text {, } \\
& G_{12}\left[4_{4}, 10_{3}, 7_{0}, 6_{1}, 10_{2}, 5_{3}, 5_{4}, 8_{2}, 4_{3}, 5_{0}\right], G_{12}\left[0_{3}, 4_{1}, 4_{2}, 13_{3}, 12_{0}, 2_{2}, 7_{4}, 12_{2}, 14_{3}, 11_{2}\right] \text {, } \\
& G_{12}\left[5_{1}, 2_{4}, 13_{2}, 3_{3}, 8_{1}, 3_{0}, 12_{4}, 13_{1}, 11_{2}, 13_{0}\right], G_{12}\left[0_{0}, 0_{1}, 2_{0}, 14_{1}, 10_{2}, 1_{1}, 0_{2}, 8_{3}, 1_{0}, 1_{2}\right] \text {, } \\
& G_{12}\left[0_{3}, 2_{4}, 7_{1}, 9_{4}, 14_{3}, 11_{1}, 10_{3}, 12_{0}, 13_{4}, 13_{1}\right], G_{12}\left[0_{3}, 7_{4}, 7_{2}, 7_{3}, 13_{4}, 3_{0}, 11_{4}, 0_{2}, 6_{3}, 0_{1}\right] \text {, } \\
& \left.G_{12}\left[0_{0}, 4_{1}, 10_{0}, 3_{1}, 4_{3}, 6_{1}, 3_{3}, 1_{4}, 1_{0}, 12_{2}\right], G_{12}\left[0_{0}, 9_{2}, 3_{1}, 6_{2}, 8_{4}, 4_{3}, 1_{4}, 5_{3}, 10_{0}, 2_{4}\right]\right\}, \\
& B_{13}=\left\{G_{13}\left[4_{0}, 13_{3}, 7_{2}, 10_{3}, 6_{1}, 0_{3}, 8_{0}, 8_{1}, 5_{0}, 9_{2}\right], G_{13}\left[2_{4}, 13_{3}, 9_{4}, 7_{2}, 10_{1}, 12_{0}, 12_{3}, 6_{0}, 10_{4}, 14_{1}\right]\right. \text {, } \\
& G_{13}\left[14_{4}, 2_{2}, 11_{1}, 9_{4}, 14_{2}, 13_{1}, 1_{2}, 1_{1}, 8_{3}, 7_{0}\right], G_{13}\left[12_{3}, 13_{2}, 4_{1}, 4_{3}, 13_{4}, 9_{1}, 14_{3}, 6_{0}, 6_{2}, 2_{4}\right] \text {, } \\
& G_{13}\left[13_{4}, 5_{2}, 9_{0}, 11_{4}, 10_{3}, 4_{1}, 6_{2}, 8_{3}, 0_{2}, 2_{0}\right], G_{13}\left[0_{0}, 2_{1}, 3_{0}, 0_{1}, 1_{3}, 4_{1}, 11_{2}, 8_{3}, 4_{0}, 6_{2}\right] \text {, } \\
& G_{13}\left[0_{0}, 14_{3}, 12_{4}, 6_{0}, 5_{4}, 10_{3}, 14_{2}, 14_{4}, 0_{2}, 1_{4}\right], G_{13}\left[0_{1}, 11_{2}, 13_{1}, 6_{2}, 4_{3}, 14_{2}, 5_{4}, 13_{3}, 4_{4}, 11_{3}\right] \text {, } \\
& \left.G_{13}\left[0_{0}, 5_{1}, 9_{0}, 2_{1}, 1_{4}, 6_{1}, 1_{2}, 4_{4}, 1_{0}, 3_{3}\right], G_{13}\left[0_{0}, 7_{1}, 12_{0}, 6_{2}, 0_{4}, 9_{1}, 6_{4}, 7_{3}, 2_{0}, 12_{4}\right]\right\} .
\end{aligned}
$$

For $i \in[2,13]$, a $G_{i}$-decomposition of $K_{5 \times 15}$ consists of the $G_{i}$-blocks in $B_{i}$ under the action of the map $r_{s} \mapsto(r+1(\bmod 15))_{s}$.

Example 27. Let $V\left(K_{9,15}\right)=[0,8] \cup[9,23]$ with the obvious vertex partition. Let

$$
\begin{aligned}
B_{14}=\{ & G_{14}[8,18,2,20,5,14,4,17,7,23], G_{14}[20,3,11,4,12,6,9,0,17,8], \\
& G_{14}[9,5,13,1,19,2,17,6,22,7], G_{14}[0,10,1,15,4,13,3,16,2,11], \\
& G_{14}[0,14,2,12,7,16,5,11,6,15], G_{14}[0,18,3,9,8,22,1,16,6,23] \\
& G_{14}[1,18,4,21,7,20,0,19,5,23], G_{14}[2,13,6,10,4,22,3,19,8,21], \\
& \left.G_{14}[5,10,7,14,3,21,1,12,8,15]\right\} .
\end{aligned}
$$

Then a $G_{14}$-decomposition of $K_{9,15}$ consists of the $G_{14}$-blocks in $B_{14}$.

Example 28. Let $V\left(K_{15,15}\right)=\left\{r_{s}: r \in \mathbb{Z}_{15}, s \in \mathbb{Z}_{2}\right\}$ with the obvious vertex partition. Let $B_{14}=\left\{G_{14}\left[2_{0}, 14_{1}, 4_{0}, 8_{1}, 3_{0}, 10_{1}, 7_{0}, 9_{1}, 0_{0}, 0_{1}\right]\right\}$. Then a $G_{14^{-}}$ decomposition of $K_{15,15}$ consists of the $G_{14}$-blocks in $B_{14}$ under the action of the map $r_{s} \mapsto(r+1(\bmod 15))_{s}$.

Example 29. Let $V\left(K_{25} \backslash K_{10}\right)=\mathbb{Z}_{25}$ with [0, 9] being the vertices in the hole. Let

$$
\begin{aligned}
B_{2}= & \left\{G_{2}[12,16,13,11,14,4,23,24,15,21], G_{2}[16,17,1,20,14,22,24,2,23,7],\right. \\
& G_{2}[14,18,7,21,22,5,13,10,23,3], G_{2}[2,14,10,4,12,20,24,5,16,15], \\
& G_{2}[16,18,20,9,23,13,19,0,21,2], G_{2}[15,20,17,6,22,3,10,24,13,4], \\
& G_{2}[0,10,11,1,13,15,18,5,14,12], G_{2}[0,14,16,3,19,18,21,4,11,17], \\
& G_{2}[0,15,22,1,24,11,19,2,12,23], G_{2}[1,10,15,3,17,21,24,6,12,18], \\
& G_{2}[2,13,17,4,19,10,16,9,22,20], G_{2}[3,21,13,9,17,12,22,7,11,20], \\
& G_{2}[5,12,19,1,23,8,17,10,21,11], G_{2}[8,14,21,5,23,6,11,18,17,19], \\
& G_{2}[10,20,7,13,8,12,15,9,14,6], G_{2}[15,19,6,16,23,18,22,8,24,7], \\
& \left.G_{2}[18,24,9,19,22,11,16,8,20,0]\right\} .
\end{aligned}
$$

Then a $G_{2}$-decomposition of $K_{25} \backslash K_{10}$ consists of the $G_{2}$-blocks in $B_{2}$.

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