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THE LAGRANGIAN DENSITY OF $\{123, 234, 456\}$ AND THE TURÁN NUMBER OF ITS EXTENSION

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Abstract

Given a positive integer n and an r-uniform hypergraph F, the *Turán number* $ex(n, F)$ is the maximum number of edges in an F-free r-uniform hypergraph on n vertices. The *Turán density* of F is defined as $\pi(F) = \lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{r}}$. The *Lagrangian density* of F is $\pi_\lambda(F) = \sup\{r!\lambda(G) : G$ is F-free$\}$, where $\lambda(G)$ is the Lagrangian of G. Sidorenko observed that $\pi(F) \leq \pi_\lambda(F)$, and Pikhurko observed that $\pi(F) = \pi_\lambda(F)$ if every pair of vertices in F is contained in an edge of F. Recently, Lagrangian densities of hypergraphs and Turán numbers of their extensions have been studied actively. For example, in the paper [*A hypergraph Turán theorem via Lagrangians of intersecting families*, J. Combin. Theory Ser. A 120 (2013) 2020–2038], Hefetz and Keevash studied the Lagrangian densitiy of the 3-uniform graph spanned by $\{123, 456\}$ and the Turán number of its extension. In this paper, we show that the Lagrangian density of the 3-uniform graph

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spanned by $\{123, 234, 456\}$ achieves only on K_5^3 . Applying it, we get the Turán number of its extension, and show that the unique extremal hypergraph is the balanced complete 5-partite 3-uniform hypergraph on n vertices.

Keywords: Turán number, hypergraph Lagrangian, Lagrangian density.

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1. Notations and Definitions

For a set V and a positive integer r we denote by $V^{(r)}$ the family of all r-subsets of V. An *r-uniform graph* or *r-graph* G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. We sometimes simply write G as $E(G)$. Let $|G|$ denote the number of edges of G. An edge $e = \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1 a_2 \cdots a_r$. An r-graph H is a *subgraph* of an r-graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, a subgraph H is *spanning* if $V(H) = V(G)$. A subgraph of G *induced* by $V' \subseteq V$, denoted as $G[V']$, is the r-graph with vertex set V' and edge set $E' = \{e \in E(G) : e \subseteq V'\}$. Let K_t^r denote the *complete r-graph* on t vertices, that is, the r-graph on t vertices containing all r-subsets of the vertex set as edges. Let $T_m^r(n)$ be the *balanced complete m-partite r-uniform graph on n vertices*, i.e., $V(T_m^r(n)) = V_1 \cup V_2 \cup \cdots \cup V_m$ such that $V_i \cap V_j = \emptyset$ for every $1 \le i < j \le m$ and $|V_1| \le |V_2| \le \cdots \le |V_m| \le |V_1| + 1$, and $E(T_m^r(n)) = \left\{e \in \binom{[n]}{r} : \text{for every } i \in [m], |e \cap V_i| \le 1\right\}$. Let $t_m^r(n) = |T_m^r(n)|$. For a positive integer n, let $[n]$ denote $\{1, 2, 3, \ldots, n\}$. Given positive integers m and r, let $[m]_r = m(m-1)\cdots(m-r+1)$.

Given an r-graph F, an r-graph G is called *F-free* if it does not contain a copy of F as a subgraph. For a fixed positive integer n and an r-graph F, the *Turán number* of F, denoted by $ex(n, F)$, is the maximum number of edges in an F-free r-graph with n vertices. An averaging argument of Katona-Nemetz-Simonovits [8] shows that the sequence $\frac{ex(n,F)}{\binom{n}{r}}$ is a non-increasing sequence. Hence, $\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{r}}$ exists. The *Turán density* of F is defined as

$$\pi(F) = \lim_{n\to\infty} \frac{ex(n, F)}{\binom{n}{r}}.$$

For 2-graphs, Erdős-Stone-Simonovits determined the Turán densities of all graphs except bipartite graphs. Very few results are known for hypergraphs and a recent survey on this topic can be found in Keevash's survey paper [9]. Lagrangian has been a useful tool in estimating the Turán density of a hypergraph.

Definition 1.1. Let G be an r-graph on $[n]$ and let $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the Lagrangian function of G as

$$\lambda(G, \vec{x}) = \sum_{e \in E(G)} \prod_{i \in e} x_i.$$

The *Lagrangian* of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \left\{\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\, x_i \geq 0 \text{ for every } i \in [n]\right\}.$$

The value x_i is called the *weight* of the vertex i and a vector $\vec{x} \in \Delta$ is called a *feasible weighting* on G. A feasible weighting $\vec{y} \in \Delta$ is called an *optimum weighting* on G if $\lambda(G, \vec{y}) = \lambda(G)$.

Given an r-graph F, we define the *Lagrangian density* $\pi_\lambda(F)$ of F to be

$$\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ is } F\text{-free}\}.$$

The Lagrangian density of an r-graph is closely related to its Turán density. We say that a pair of vertices $\{i, j\}$ is *covered* in a hypergraph H if there exists $e \in H$ such that $\{i, j\} \subseteq e$. We say that a hypergraph H covers pairs if every pair of vertices is covered in H.

Proposition 1.2 [15, 17]**.** *$\pi(F) \leq \pi_\lambda(F)$. If F covers pairs, then $\pi(F) = \pi_\lambda(F)$.*

Thus, to determine the Turán density of K_4^3 (a long standing conjecture of Turán) is equivalent to determine the Lagrangian density of K_4^3.

Let $r \geq 3$, F be an r-graph and $p \geq |V(F)|$. Let $\mathcal{K}_p^F$ denote the family of r-graphs H that contain a set C of p vertices, called the *core*, such that the subgraph of H induced by C contains a copy of F and every pair of vertices in C is covered in H. Let H_p^F be a member of $\mathcal{K}_p^F$ obtained as follows. Label the vertices of F as $v_1, \ldots, v_{|V(F)|}$. Add new vertices $v_{|V(F)|+1}, \ldots, v_p$. Let $C = \{v_1, \ldots, v_p\}$. For each pair of vertices $v_i, v_j \in C$ not covered in F, we add a set B_{ij} of $r-2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the B_{ij} 's are pairwise disjoint over all such pairs $\{i, j\}$. We call H_p^F the *extension* of F.

The Lagrangian method for hypergraph Turán problems were developed independently by Sidorenko [17] and Frankl-Füredi [3], generalizing work of Motzkin and Straus [10] and Zykov [22]. Recently, Lagrangian densities of hypergraphs and Turán numbers of their extensions have been studied by Mubayi, Pikhurko, Hefetz-Keevash, Norin-Yepremyan, Jiang, etc. In [5], Hefetz and Keevash remarked that it is interesting in its own right to determine the maximum Lagrangian of r-graphs with certain properties. For example, determine the Lagrangian density of an r-graph F. The Lagrangian density of the enlargement of a tree satisfying Erdős-Sos's conjecture is determined by Sidorenko [18] and Brandt-Irwin-Jiang [1]. Pikhurko [15] determined the Lagrangian density of a

4-uniform tight linear path of length 2 and applied it to confirm the conjecture of Frankl-Füredi on the Turán number of its extension, the r-uniform genearlized triangle for the case $r = 4$. Norin and Yepremyan [13] confirmed for $r = 5$ or 6 by extending the earlier result of Frankl-Füredi in [3]. Mubayi [11] and Pikhurko [16] obtained the exact Turán number of the expanded complete 2-graph, the extension of an empty graph with a core of p vertices, and showed the stability. Mubayi and Pikhurko [12] also obtain the Turán numbers for the generalized fans, extension of an edge in an r-uniform graph with a core of $r + 1$ vertices, and showed the stability. Brandt-Irwin-Jiang [1] and independently Norin and Yepremyan [14] showed that for a large family of r-graphs F and sufficiently large n, $ex(n, H_p^F) = e(T_r(n, p-1))$ with the unique extremal graph being $T_r(n, p-1)$. In [5], Hefetz and Keevash determined the Lagrangian density of a 3-uniform matching of size 2 and the Turán number of its extension. They proposed a conjecture on the Lagrangian density of an r-uniform matching of size 2 and the Turán number for its extension. Norin and Yepremyan confirmed this conjecture, and independently Wu-Peng-Chen [20] confirmed this conjecture for $r = 4$. Jiang-Peng-Wu in [7] obtained the Lagrangian density of a 3-uniform matching of any size and the Turán number of the extension. The authors of [19] and [6] determined the Lagrangian density of a 3-uniform linear path of length 3 or 4, the Lagrangian density of the disjoint union of a 3-uniform linear path of length 2 or 3 and a matching of any size, and the corresponding Turán numbers of their extensions. It seems to be more difficult if we replace a linear path by a tight path. The authors of [2] obtain the Lagrangian density of the disjoint union of a 3-uniform tight path of length 2 and an edge, and the corresponding Turán number of its extension. Yan-Peng in [21] determined the Lagrangian densities of the 3-uniform linear cycle of length 3 ($\{123, 345, 561\}$), and F_5 ($\{123, 124, 345\}$).

In this paper, we obtain the Lagrangian density of a 3-uniform path of length 3 where two consecutive edges have 1 or 2 vertices in common. Precisely, let TP_3 be the 3-graph with vertex set $[6]$ and edge set $\{123, 234, 456\}$. We show that the Lagrangian density of TP_3 achieves only on K_5^3. Applying it, we obtain the Turán number for the extension of TP_3 and show that the unique extremal hypergraph is $T_5^3(n)$. The method in this paper uses the idea in [7] and [19] by showing that we can reduce the family of all TP_3-free 3-graphs to the family of left compressed and dense TP_3-free 3-graphs, but much more structural analysis is needed here.

2. Preliminaries

In this section, we develop some useful properties of the Lagrangian function. The following fact follows immediately from the definition of the Lagrangian.

Fact 2.1. *Let G_1, G_2 be r-graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

Given an r-graph G and a set S of vertices, the *link* of S in G, denoted by $L_G(S)$, is the hypergraph with edge set $\left\{e \in \binom{V(G)\setminus S}{r-|S|} : e \cup S \in E(G)\right\}$. In particular, $S = \{i\}$, we write $L_G(\{i\})$ as $L_G(i)$. The degree of i is $d_G(i) = |L_G(i)|$, the number of edges containing i in G. Given $i, j \in V(G)$, define

$$L_G(j \setminus i) = \{e : i \notin e, e \cup \{j\} \in E(G) \text{ and } e \cup \{i\} \notin E(G)\},$$

when there is no confusion, we will drop the subscript G. We say G on vertex set $[n]$ is *left-compressed* if for every i, j, $1 \leq i < j \leq n$, $L_G(j \setminus i) = \emptyset$. Equivalently, G on $[n]$ is *left-compressed* if $j_1 j_2 \cdots j_r \in E(G)$ implies $i_1 i_2 \cdots i_r \in E(G)$, wherever $i_p \leq j_p$ for $1 \leq p \leq r$. Let $i, j \in V(G)$, define

$$\pi_{ij}(G) = \big(E(G) \setminus \{e \cup \{j\} : e \in L_G(j \setminus i)\}\big) \cup \{e \cup \{i\} : e \in L_G(j \setminus i)\}.$$

By the definition of $\pi_{ij}(G)$, it's straightforward to verify the following fact.

Fact 2.2. *Let G be an r-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weighting on G. If $x_i \geq x_j$, then $\lambda(\pi_{ij}(G), \vec{x}) \geq \lambda(G, \vec{x})$.*

An r-graph G is *dense* if for every subgraph G' of G with $|V(G')| < |V(G)|$ we have $\lambda(G') < \lambda(G)$. This is equivalent to that no coordinate in an optimum weighting is zero.

Fact 2.3 [4]**.** *Let $G = (V, E)$ be a dense r-graph. Then G covers pairs.*

In [10], Motzkin and Straus determined the Lagrangian of any given 2-graph.

Theorem 2.4 (Motzkin and Straus [10])**.** *If G is a 2-graph in which a maximum complete subgraph has t vertices, then $\lambda(G) = \lambda\left(K_t^2\right) = \frac{1}{2}\left(1 - \frac{1}{t}\right)$.*

The *support* of a vector $\vec{x}$ is $\sigma(\vec{x}) = \{i : x_i \neq 0 \text{ for } i \in [n]\}$.

Fact 2.5 [4]**.** *Let G be an r-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weighting on G. Then*

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r\lambda(G)$$

for every $i \in \sigma(\vec{x})$.

Let $T_2^{(3)}$ be a 3-graph with two edges intersecting at two vertices.

Fact 2.6. *If G is a dense 3-graph on $[n]$ $(n \geq 4)$, then G contains a copy of $T_2^{(3)}$.*

Proof. Suppose that G is $T_2^{(3)}$-free. Since G is dense, then every pair of vertices is covered exactly once. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimum weighting on G. By Fact 2.5, $\frac{\partial \lambda(G,\vec{x})}{\partial x_i} = 3\lambda(G)$ for all $1 \leq i \leq n$. Summing over i we obtain $3n\lambda(G) = \sum_{i=1}^{n} \frac{\partial \lambda(G,\vec{x})}{\partial x_i} = \sum_{1 \leq i < j \leq n} x_i x_j \leq \frac{1}{2}\left(1 - \frac{1}{n}\right)$. So $\lambda(G) \leq \frac{n-1}{6n^2} \leq \frac{1}{32}$, it is a contradiction. ∎

Fact 2.7. *Let G be an r-graph on $[n]$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be a feasible weighting on G. Let $i, j \in [n]$, $i \neq j$. Suppose that $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$. Let $\vec{y} = (y_1, y_2, \ldots, y_n)$ be defined by letting $y_\ell = x_\ell$ for every $\ell \in [n] \setminus \{i, j\}$ and letting $y_i = y_j = \frac{1}{2}(x_i + x_j)$. Then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$.*

Proof. Since $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$, we have

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i,j\} \subseteq e \in G} \left[\frac{(x_i + x_j)^2}{4} - x_i x_j \right] \prod_{k \in e \setminus \{i,j\}} x_k \geq 0. \qquad \blacksquare$$

Let K_t^{r-} be an r-graph obtained by removing one edge from K_t^r.

Fact 2.8 [5]**.** *Let G be a 3-graph on* [5]*. If $G \neq K_5^3$, then $\lambda(G) \leq \lambda(K_5^{3-}) \leq \lambda(K_5^3) - 10^{-3}$.*

As usual, if $V_1, \ldots, V_s$ are disjoint sets of vertices then $\Pi_{i=1}^s V_i = V_1 \times V_2 \times \cdots \times V_s = \{(x_1, x_2, \ldots, x_s) : \text{for every } i \in [s], x_i \in V_i\}$. We will use $\Pi_{i=1}^s V_i$ to also denote the set of the corresponding unordered s-sets. If L is a hypergraph on $[m]$, then a *blowup* of L is a hypergraph G whose vertex set can be partitioned into $V_1, \ldots, V_m$ such that $E(G) = \bigcup_{e \in L} \prod_{i \in e} V_i$. The following proposition follows immediately from the definition and is implicit in many papers (see [9] for instance).

Proposition 2.9. *Let $r \geq 2$. Let L be an r-graph and G be a blowup of L. Suppose $|V(G)| = n$. Then $|G| \leq \lambda(L) n^r$.*

3. Lagrangian Density of TP_3 and Related Turán Number

3.1. Lagrangian density of TP_3

Recall that TP_3 is the 3-graph with vertex set [6] and edge set $\{123, 234, 456\}$. In this section, we will show that the maximum possible Lagrangian among all TP_3-free 3-graphs is uniquely achieved by K_5^3. Our main results are as follows.

Theorem 3.1. *Let G be a TP_3-free 3-graph. Then $\lambda(G) \leq \lambda\left(K_5^3\right) = \frac{2}{25}$. Furthermore, if G is K_5^3-free, then $\lambda(G) \leq \lambda\left(K_5^3\right) - 10^{-3}$.*

Corollary 3.2. $\pi_\lambda(TP_3) = 3!\lambda\left(K_5^3\right)$.

Proof. Since K_5^3 is TP_3-free, then $\pi_\lambda(TP_3) \geq 3!\lambda\left(K_5^3\right)$. On the other hand, by Theorem 3.1, $\pi_\lambda(TP_3) \leq 3!\lambda\left(K_5^3\right)$. Therefore, $\pi_\lambda(TP_3) = 3!\lambda\left(K_5^3\right)$. $\blacksquare$

In order to prove Theorem 3.1, we divide the TP_3-free 3-graphs into two categories: one is that the graphs contain at most six vertices; the other is that the graphs contain seven vertices or more than seven vertices. We will prove the following results.

Lemma 3.3. *Let G be a 3-graph on $[n]$ $(n \leq 6)$. If G is TP_3-free, then for every pair i, j, $1 \leq i < j \leq n$, the following hold.*

(1) *$\pi_{ij}(G)$ is TP_3-free.*

(2) *Furthermore, if G is K_5^3-free and $\{i, j\}$ is covered by an edge of G, then $\pi_{ij}(G)$ is K_5^3-free.*

Proof. Suppose for the contrary that $\pi_{ij}(G)$ contains a copy of TP_3, denoted by TP. Since G is TP_3-free, there is an edge $e \in TP$ such that $i \in e \in \pi_{ij}(G)$, $j \notin e$ and $e' = e \setminus \{i\} \cup \{j\} \in G$. There are two cases in terms of the degree of i in TP.

Case 1. $d_{TP}(i) = 1$. If there exists no $f \in TP$ such that $j \in f$, then $TP \setminus \{e\} \cup \{e'\}$ forms a copy of TP_3 in G. Otherwise, there exists one edge f such that $j \in f \in TP$, then $f' = f \setminus \{j\} \cup \{i\} \in G$. So $TP \setminus \{e, f\} \cup \{e', f'\}$ forms a copy of TP_3 in G.

Case 2. $d_{TP}(i) = 2$. Let $TP = \{e_1, e_2, e_3\}$ and $|e_1 \cap e_2| = 2$, $|e_2 \cap e_3| = 1$, $|e_1 \cap e_3| = 0$. There are two possible cases.

Subcase 2.1. $i \in e_1 \cap e_2$. Since G is TP_3-free, at least one of the following cases may happen. (I) $e_1' = e_1 \setminus \{i\} \cup \{j\} \in G$, $e_2' = e_2 \setminus \{i\} \cup \{j\} \in G$. In this case, if $j \in e_3 \setminus e_2$, then $\{e_1', e_2', e_3 \setminus \{j\} \cup \{i\}\}$ forms a copy of TP_3. If $j \notin e_3 \setminus e_2$, then $\{e_1', e_2', e_3\}$ forms a copy of TP_3. (II) $e_1' = e_1 \setminus \{i\} \cup \{j\} \in G$ but $e_2' = e_2 \setminus \{i\} \cup \{j\} \notin G$. In this case, if $j \in e_3 \setminus e_2$, then $\{e_1', e_3, e_3 \setminus \{j\} \cup \{i\}\}$ forms a copy of TP_3. If $j \notin e_3 \setminus e_2$, we get $e_1', e_2, e_3 \in G$ and $|e_1' \cup e_2 \cup e_3| = 7$, contradicting the condition. (III) $e_1' = e_1 \setminus \{i\} \cup \{j\} \notin G$ but $e_2' = e_2 \setminus \{i\} \cup \{j\} \in G$. In this case, if $j \in e_3 \setminus e_2$, then $\{e_1, e_3, e_3 \setminus \{j\} \cup \{i\}\}$ forms a copy of TP_3. If $j \notin e_3 \setminus e_2$, then we can get $e_1, e_2', e_3 \in G$ and $|e_1 \cup e_2' \cup e_3| = 7$, contradicting the condition.

Subcase 2.2. $i \in e_2 \cap e_3$. Since G is TP_3-free, at least one of the following cases may happen. (I) $e_2' = e_2 \setminus \{i\} \cup \{j\} \in G$, $e_3' = e_3 \setminus \{i\} \cup \{j\} \in G$. In this case, if $j \in e_1 \setminus e_2$, then $\{e_1 \setminus \{j\} \cup \{i\}, e_2', e_3'\}$ forms a copy of TP_3. If $j \notin e_1 \setminus e_2$, then $\{e_1, e_2', e_3'\}$ forms a copy of TP_3. (II) $e_2' = e_2 \setminus \{i\} \cup \{j\} \notin G$ but $e_3' = e_3 \setminus \{i\} \cup \{j\} \in G$. In this case, $j \notin e_1 \setminus e_2$, then we can get $e_1, e_2, e_3' \in G$ and $|e_1 \cup e_2 \cup e_3'| = 7$, contradicting the condition. (III) $e_2' = e_2 \setminus \{i\} \cup \{j\} \in G$ but $e_3' = e_3 \setminus \{i\} \cup \{j\} \notin G$. In this case, $j \notin e_1 \setminus e_2$, we get $e_1, e_2', e_3 \in G$ and $|e_1 \cup e_2' \cup e_3| = 7$, contradicting the condition.

Assume that $\{i, j\}$ is covered by an edge g of G. Suppose for contradiction that $\pi_{ij}(G)$ contains a copy K of K_5^3. Clearly, $V(K)$ must contain i. If $j \in V(K)$, then it is easy to see that K is also in G, contradicting G being K_5^3-free. By the definition of $\pi_{ij}(G)$, all the edges in K not containing i are also in G. If $j \notin V(K)$, since $n \leq 6$, we have $|g \cap V(K)| = 2$. Now, we can find a copy of TP_3 in G, a contradiction. ■

Next, we need an algorithm.

Algorithm 3.4 (Dense and Left-compressed [7])**.**
Input: An r-graph G.
Output: A dense, left-compressed r-graph G'.

Step 1. If G is dense, then go to Step 2. Otherwise, replace G by a dense subgraph G' with the same Lagrangian and go to Step 2.

Step 2. If G is left-compressed, then terminate. Otherwise, let $\vec{y}$ be an optimum weighting of G, then there exist vertices i, j, where $i < j$, such that $y_i > y_j$ and $L_G(j \setminus i) \neq \emptyset$. Replace G by $\pi_{ij}(G)$ and go to Step 1.

Note that the algorithm terminates after finite many steps since Step 2 reduces the parameter $s(G) = \sum_{e\in G}\sum_{i\in e} i$ by at least 1 each time and Step 1 reduces the number of vertices by at least 1 each time.

Lemma 3.5. *Let G be a TP_3-free* (*and K_5^3-free*) *3-graph on $[n]$* ($n \leq 6$)*. Then there exists a dense and left-compressed TP_3-free* (*and K_5^3-free*) *3-graph G' with $|V(G')| \leq |V(G)|$ such that $\lambda(G') \geq \lambda(G)$.*

Proof. We apply Algorithm 3.4 to G and let G' be the final graph. Then G' is dense and left-compressed. By Fact 2.2, $\lambda(G') \geq \lambda(G)$. By Lemma 3.3, G' is TP_3-free (and K_5^3-free). ■

Claim 3.6. *Let G be a TP_3-free 3-graph on $[n]$ ($n \leq 6$). Then $\lambda(G) \leq \lambda\left(K_5^3\right)$. Furthermore, if G is also K_5^3-free, then $\lambda(G) \leq \lambda\left(K_5^3\right) - 10^{-3}$.*

Proof. By Lemma 3.5, we may assume that G is dense and left-compressed. If $n \leq 5$, by Fact 2.1, then $\lambda(G) \leq \lambda\left(K_5^3\right)$. If G is K_5^3-free, by Fact 2.8, then $\lambda(G) \leq \lambda\left(K_5^{3-}\right) \leq \lambda\left(K_5^3\right) - 10^{-3}$. Hence, we may assume that $n = 6$. Let $\vec{x} = (x_1, x_2, \ldots, x_6)$ be an optimum weighting of G. It is clear that $x_1 \geq x_2 \geq \cdots \geq x_6$. By Fact 2.3, G covers pairs. So $i56 \in G$, for some $i < 5$. Since G is left-compressed, we have $156 \in G$, this implies that for every i, j, where $2 \leq i < j \leq 6$, $1ij \in G$. Suppose that $G[\{2,3,4,5,6\}]$ contains an edge. Without loss of generality, we assume that $456 \in G$. Since $123, 145 \in G$, we have $\{123, 145, 456\}$ forms a copy of TP_3 in G, contradicting G being TP_3-free. Hence $G = \{1ij : 2 \leq i < j \leq 6\}$. Assume that $x_1 = a$. Since $\vec{y} = \left(\frac{x_2}{1-a}, \ldots, \frac{x_6}{1-a}\right)$ is a feasible weighting on $L_G(1)$, By Theorem 2.4,

$$\lambda(G) = \lambda(G, \vec{x}) = a(1-a)^2\lambda(L_G(1), \vec{y}) < \frac{1}{2}a(1-a)^2 \leq \frac{2}{27} < \lambda\left(K_5^3\right) - 10^{-3}. \blacksquare$$

Let $Q = \{a_1b_1b_2, a_2b_1b_2, c_1c_2c_3\}$, we have the following result.

Claim 3.7. *Let G be a dense 3-graph. If G contains a subgraph Q, then it also contains a copy of TP_3.*

Proof. Suppose that G is TP_3-free. If $|V(G)| = |V(Q)|$, then we have $b_1c_1x \notin G$ for every $x \in \{a_1, a_2, b_2, c_2, c_3\}$. If $b_1c_1a_1 \in G$, then $\{b_1c_1a_1, a_1b_1b_2, c_1c_2c_3\}$ forms a copy of TP_3, it is a contradiction. If $b_1c_1a_2 \in G$, then $\{b_1c_1a_2, a_2b_1b_2, c_1c_2c_3\}$ forms a copy of TP_3. If $b_1c_1b_2 \in G$, then $\{b_1c_1b_2, a_1b_1b_2, c_1c_2c_3\}$ forms a copy of TP_3. If $b_1c_1c_2 \in G$, then $\{b_1c_1c_2, c_1c_2c_3, a_1b_1b_2\}$ forms a copy of TP_3. If $b_1c_1c_3 \in G$, then $\{b_1c_1c_3, c_1c_2c_3, a_1b_1b_2\}$ forms a copy of TP_3. So the pair, $\{b_1, c_1\}$, is not covered by any edge of G, which is a contradiction by Fact 2.3. If $|V(G)| > |V(Q)|$, then let $v_i \in V(G) \setminus V(Q)$, where $1 \leq i \leq |V(G)| - |V(Q)|$. Since G is TP_3-free, we have $a_1c_1x \notin G$ for every $x \in \{b_1, b_2, c_2, c_3, v_i\}$. If $a_1c_1b_1 \in G$, then $\{a_1c_1b_1, a_1b_1b_2, c_1c_2c_3\}$ forms a copy of TP_3. If $a_1c_1b_2 \in G$, then $\{a_1c_1b_2, a_1b_1b_2, c_1c_2c_3\}$ forms a copy of TP_3. If $a_1c_1c_2 \in G$, then $\{a_1c_1c_2, c_1c_2c_3, a_1b_1b_2\}$ forms a copy of TP_3. If $a_1c_1c_3 \in G$, then $\{a_1c_1c_3, c_1c_2c_3, a_1b_1b_2\}$ forms a copy of TP_3. If $a_1c_1v_i \in G$, then $\{a_1c_1v_i, a_1b_1b_2, a_2b_1b_2\}$ forms a copy of TP_3. Similarly, we have $a_1c_2x \notin G$ for $x \in \{b_1, b_2, c_1, c_3, v_i\}$ and $a_1c_3x \notin G$ for $x \in \{b_1, b_2, c_1, c_2, v_1\}$. Since G is dense, we have $a_1c_ia_2 \in G$ $(i = 1, 2, 3)$. Next, we consider the pair $\{b_1, c_1\}$. From previous analysis, we have $b_1c_1x \notin G$ for every $x \in \{a_1, a_2, b_2, c_2, c_3\}$. So we just consider the edges $b_1c_1v_i \in G$ $(1 \leq i \leq |V(G)| - |V(Q)|)$ or not. If $b_1c_1v_i \in G$, then $\{a_1a_2c_1, a_1a_2c_2, c_1b_1v_i\}$ forms a copy of TP_3. So the pair $\{b_1, c_1\}$, is not covered by any edge of G, which is a contradiction by Fact 2.3. ■

Claim 3.8. *Let G be a dense 3-graph and $|V(G)| = 7$. If G is TP_3-free, then $\lambda(G) \leq \lambda\left(K_5^3\right) - 10^{-3}$.*

Proof. Let $V(G) = \{a_1, a_2, b_1, b_2, v_1, v_2, v_3\}$. Since G is dense, we have G contains a copy of $T_2^{(3)}$ by Fact 2.6. Without loss of generality, we assume that $a_1b_1b_2, a_2b_1b_2 \in G$. If $v_1v_2v_3 \in G$, by Claim 3.7, then G contains a copy of TP_3, it is a contradiction. If $v_1v_2v_3 \notin G$, we consider the pairs $\{v_j, v_k\}$ $(1 \leq j < k \leq 3)$. If $v_jv_ka_i \in G$ $(i = 1, 2)$, then $\{a_1b_1b_2, b_1b_2a_2, a_iv_jv_k\}$ forms a copy of TP_3 in G. So the pairs $\{v_j, v_k\}$ may be covered by the edge of the form $v_jv_kb_1$ or $v_jv_kb_2$. By the pigeonhole principle, there exist two pairs $\{v_j, v_k\}$ covered by b_1 or b_2. Without loss of generality, we only need to discuss the following two cases.

Case 1. $v_1v_2b_1, v_2v_3b_1, v_1v_3b_2 \in G$. First, we consider the pairs $\{a_i, v_j\}$ $(i = 1, 2;\ j = 1, 3)$. Due to the 'symmetry' of a_1, a_2, and the 'symmetry' of v_1, v_3, we use $\{a_1, v_1\}$ as an example. If $a_1v_1a_2 \in G$, then $\{v_3b_1v_2, b_1v_2v_1, v_1a_1a_2\}$ forms a copy of TP_3. If $a_1v_1b_2 \in G$, then $\{v_3b_1v_2, b_1v_2v_1, v_1a_1b_2\}$ forms a copy of TP_3. If $a_1v_1v_i \in G$ $(i = 2, 3)$, then $\{a_1b_1b_2, b_1b_2a_2, a_1v_1v_i\}$ forms a copy of TP_3. So the only edge containing $\{a_1, v_1\}$ is $a_1v_1b_1$. Similarly, $a_2v_1b_1, a_1v_3b_1, a_2v_3b_1$ are the only edges in G containing $\{a_2, v_1\}$, $\{a_1, v_3\}$ and $\{a_2, v_3\}$, respectively.

Second, we consider $\{a_1, a_2\}$. If $a_1a_2b_1 \in G$, then $\{a_1b_1b_2, a_1b_1a_2, b_2v_1v_3\}$ forms a copy of TP_3. If $a_1a_2b_2 \in G$, then $\{a_2b_1b_2, a_1a_2b_2, b_1v_1v_2\}$ forms a copy of

TP_3. If $a_1a_2v_i \in G$ $(i = 1, 3)$, then $\{b_1v_2v_1, b_1v_2v_3, a_1a_2v_i\}$ forms a copy of TP_3. So the only edge containing $\{a_1, a_2\}$ is $a_1a_2v_2$.

Third, we consider $\{v_2, b_2\}$. If $v_2b_2a_1 \in G$, then $\{v_2b_2a_1, a_1b_1b_2, b_1a_2v_1\}$ forms a copy of TP_3. If $v_2b_2a_2 \in G$, then $\{v_2b_2a_2, a_2b_1b_2, b_1a_1v_1\}$ forms a copy of TP_3. If $v_2b_2v_1 \in G$, then $\{v_1v_2b_2, v_1v_2b_1, b_1a_1v_3\}$ forms a copy of TP_3. If $v_2b_2v_3 \in G$, then $\{v_3v_2b_2, v_3v_2b_1, b_1a_1v_1\}$ forms a copy of TP_3. So the only edge containing $\{v_2, b_2\}$ is $v_2b_2b_1$.

Finally, we consider the rest of edges in G. If $a_1v_2b_1 \in G$, then $\{a_1v_2b_1, a_1b_1b_2, v_1v_3b_2\}$ forms a copy of TP_3. If $a_2v_2b_1 \in G$, then $\{a_2v_2b_1, a_2b_1b_2, v_1v_3b_2\}$ forms a copy of TP_3. If $v_1v_3b_1 \in G$, then $\{v_1v_3b_1, b_1v_1v_2, v_2a_1a_2\}$ forms a copy of TP_3. If $b_1b_2v_i \in G$ $(i = 1, 3)$, then $\{b_1b_2v_i, b_1b_2a_2, a_2a_1v_2\}$ forms a copy of TP_3.

From the above, we obtain that $G = \{a_1b_1b_2, a_2b_1b_2, v_1v_2b_1, v_2v_3b_1, v_1v_3b_2, a_1v_1b_1, a_1v_3b_1, a_2v_1b_1, a_2v_3b_1, a_1a_2v_2, b_1b_2v_2\}$. Let $\vec{x}$ be an optimum weighting of G, by Fact 2.7, we may assume that $x_{a_1} = x_{a_2} = x_{v_2} = x, x_{b_2} = x_{v_1} = x_{v_3} = y, x_{b_1} = z$ and $3x + 3y + z = 1$. $\lambda(G) = \lambda(G, \vec{x}) = x^3 + y^3 + 9xyz$. If some of x, y, z is 0, then it is easy to verify that $\lambda(G) = \frac{1}{27} < \lambda(K_5^3) - 10^{-3}$. So we assume that $x, y, z > 0$. Let $f(x, y, z) = x^3 + y^3 + 9xyz$. To get the maximum value of $f(x, y, z)$, we apply the theory of Lagrange multipliers. Let

$$g(x, y, z, \gamma) = x^3 + y^3 + 9xyz - \gamma(3x + 3y + z - 1).$$

Taking the partial derivative with respect to x, y, z and γ, and let its value equal to 0. We have

$$\begin{cases} 3x^2 + 9yz = 3\gamma \\ 3y^2 + 9xz = 3\gamma \\ 9xy = \gamma \\ 3x + 3y + z = 1. \end{cases}$$

Noting that the right hand sides of the first and the second equalities are equal, we obtain that

$$3x^2 - 3y^2 = 9z(x - y).$$

If $x - y = 0$. Solving the above system of equations, we have that $x_0 = \frac{3}{26}$, $y_0 = \frac{3}{26}$, $z_0 = \frac{8}{26}$. If $x - y \neq 0$, then we have that $x_1 = \frac{15+3\sqrt{15}}{100}$, $y_1 = \frac{15-3\sqrt{15}}{100}$, $z_1 = 0.1$ or $x_2 = \frac{15-3\sqrt{15}}{100}$, $y_2 = \frac{15+3\sqrt{15}}{100}$, $z_2 = 0.1$. By direct calculation, we see that the maximum occurs at $x_0 = \frac{3}{26}$, $y_0 = \frac{3}{26}$, $z_0 = \frac{8}{26}$.

Hence, $\lambda(G) \leq f(x_0, y_0, z_0) = \frac{27}{676} < \frac{2}{25} - 10^{-3} = \lambda\left(K_5^3\right) - 10^{-3}$.

Case 2. $v_1v_2b_1, v_2v_3b_1, v_1v_3b_1 \in G$. First, we consider the pairs $\{a_i, v_j\}$ $(i = 1, 2; j = 1, 2, 3)$. Due to the 'symmetry' of a_1, a_2 and the 'symmetry' of v_1, v_2 and v_3, we use $\{a_1, v_1\}$ as an example. If $a_1v_1a_2 \in G$, then $\{b_1v_1v_2, b_1v_2v_3, v_1a_1a_2\}$ forms a copy of TP_3. If $a_1v_1b_2 \in G$, then $\{b_1v_1v_2, b_1v_2v_3, v_1a_1b_2\}$ forms a copy of

TP_3. If $a_1v_1v_i \in G$ $(i = 2, 3)$, then $\{a_1b_1b_2, b_1b_2a_2, a_1v_1v_i\}$ forms a copy of TP_3. So the only edge containing $\{a_1, v_1\}$ is $a_1v_1b_1$. Similarly, $a_2v_1b_1, a_1v_2b_1, a_2v_2b_1, a_1v_3b_1, a_2v_3b_1$ are the only edges in G containing $\{a_2, v_1\}$, $\{a_1, v_2\}$, $\{a_2, v_2\}$, $\{a_1, v_3\}$ and $\{a_2, v_3\}$, respectively.

Second, we consider $\{a_1, a_2\}$. If $a_1a_2b_2 \in G$, then $\{a_1b_1b_2, a_1b_2a_2, b_1v_1v_3\}$ forms a copy of TP_3. If $a_1a_2v_i \in G$ $(i = 1, 3)$, then $\{a_1a_2v_i, b_1v_1v_2, b_1v_2v_3\}$ forms a copy of TP_3. If $a_1a_2v_2 \in G$, then $\{a_1a_2v_2, b_1v_1v_3, b_1v_2v_3\}$ forms a copy of TP_3. So the only edge containing $\{a_1, a_2\}$ is $a_1a_2b_1$.

Third, we consider the pair $\{b_2, v_i\}$ $(i = 1, 2, 3)$. Due to the 'symmetry' of v_1, v_2 and v_3, we use $\{b_2, v_1\}$ as an example. If $b_2v_1a_1 \in G$, then $\{v_1a_1b_2, a_1b_2b_1, b_1v_2v_3\}$ forms a copy of TP_3. If $b_2v_1a_2 \in G$, then $\{v_1a_2b_2, a_2b_2b_1, b_1a_1v_2\}$ forms a copy of TP_3. If $b_2v_1v_2 \in G$, then $\{v_1v_2b_2, v_1v_2b_1, b_1a_1v_3\}$ forms a copy of TP_3. If $b_2v_1v_3 \in G$, then $\{v_3v_1b_2, v_3v_1b_1, b_1a_1a_2\}$ forms a copy of TP_3. So the only edge containing $\{b_2, v_1\}$ is $b_2v_1b_1$. Similarly, $b_2v_2b_1$ and $b_2v_3b_1$ are the only edges in G containing $\{b_2, v_1\}$ and $\{b_2, v_3\}$, respectively.

Meanwhile, if $b_2v_2v_3 \in G$, then $\{v_2v_3b_2, b_1v_2v_3, b_1a_1a_2\}$ forms a copy of TP_3.

From the above, we obtain that $G = \{b_1uv : \{u, v\} \in \{a_1, a_2, b_2, v_1, v_2, v_3\}^{(2)}\}$. Let $\vec{x}$ be an optimum weighting of G, by Fact 2.7, we may assume that $x_{b_1} = x, x_{a_i} = x_{b_2} = x_{v_j} = (1 - x)/6$ $(i = 1, 2;\ j = 1, 2, 3)$.

$$\lambda(G) = \lambda(G, \vec{x}) = 15x \cdot \left(\frac{1 - x}{6}\right)^2 \le \frac{5}{24}\left(\frac{2x + (1 - x) + (1 - x)}{3}\right)^3 = \frac{5}{81}.$$

Then $\lambda(G) \le \frac{5}{81} < \frac{2}{25} - 10^{-3} = \lambda\left(K_5^3\right) - 10^{-3}$. ■

Claim 3.9. *Let G be a dense 3-graph and $|V(G)| \ge 8$. If G is TP_3-free, then $\lambda(G) \le \lambda\left(K_5^3\right) - 10^{-3}$.*

Proof. Let $V(G) = \{a_1, a_2, b_1, b_2, v_1, v_2, v_3, v_4, \dots, v_n\}$ $(n \ge 4)$. Since G is dense, we have G contains a copy of $T_2^{(3)}$ by Fact 2.6. Without loss of generality, we assume that $a_1b_1b_2, a_2b_1b_2 \in G$. Suppose that $G[\{v_1, v_2, \dots, v_n\}]$ contains an edge e, then G contains a copy of TP_3 by Claim 3.7, it is a contradiction. Therefore, for the pair $\{v_j, v_k\}$ $(1 \le j < k \le 3)$, we have $v_jv_kv_i \notin G$ where $1 \le i \le n$, $i \ne j, k$. If $v_jv_ka_i \in G$ where $i = 1, 2$, then $\{a_1b_1b_2, b_1b_2a_2, v_jv_ka_i\}$ forms a copy of TP_3. So the pair $\{v_j, v_k\}$ may be covered by the edge of the form $v_jv_kb_1$ or $v_jv_kb_2$. By the pigeonhole principle, there exist two pairs $\{v_j, v_k\}$ covered by b_1 or b_2. Without loss of generality, we assume that $v_1v_2b_1, v_2v_3b_1 \in G$.

First, we consider $\{a_1, v_1\}$. If $a_1v_1a_2 \in G$, then $\{b_1v_1v_2, b_1v_2v_3, v_1a_1a_2\}$ forms a copy of TP_3. If $a_1v_1b_2 \in G$, then $\{b_1v_1v_2, b_1v_2v_3, v_1a_1b_2\}$ forms a copy of TP_3. If $a_1v_1v_i \in G$ $(i = 2, 3, \dots, n)$, then $\{a_1b_1b_2, b_1b_2a_2, a_1v_1v_i\}$ forms a copy of TP_3. So we have $a_1v_1b_1 \in G$. Similarly, $a_2v_1b_1 \in G$ by the 'symmetry' of a_1, a_2.

Second, we consider $\{a_1, v_4\}$. If $a_1v_4a_2 \in G$, then $\{a_1a_2v_4, a_1b_1v_1, b_1v_1v_2\}$ forms a copy of TP_3. If $a_1v_4b_2 \in G$, then $\{a_1b_2v_4, a_1v_1b_1, v_1b_1v_2\}$ forms a copy of

TP_3. If $a_1v_4v_i \in G$ $(1 \le i \le n,\ i \ne 4)$, then $\{a_1v_4v_i, a_1b_1b_2, b_1b_2a_2\}$ forms a copy of TP_3. So we have $a_1v_4b_1 \in G$.

Third, we consider the pair $\{v_1, v_3\}$. If $v_1v_3a_i \in G$ $(i = 1, 2)$, then $\{a_1b_1b_2, b_1b_2a_2, v_1v_3a_i\}$ forms a copy of TP_3. If $v_1v_3b_2 \in G$, then $\{v_4a_1b_1, a_1b_1b_2, b_2v_1v_3\}$ forms a copy of TP_3. Recall that $G[\{v_1, v_2, \ldots, v_n\}]$ dose not contain an edge, so $v_1v_3b_1 \in G$. We note that the three vertices v_1, v_2, v_3 are 'symmetrical'. So the pairs $\{a_i, v_j\}$ $(i = 1, 2;\ j = 1, 2, 3)$ must be covered by the edge of the form $a_iv_jb_1$. We will show that $v_1, v_2, \ldots, v_n$ are 'symmetrical'.

Let us consider the pairs $\{v_i, v_j\}$ $(i = 1, 2, 3;\ j = 4, \ldots, n)$. By the 'symmetry' of v_1, v_2, v_3, without loss of generality, we use $\{v_1, v_j\}$ as an example. If $v_1v_ja_i \in G$ $(i = 1, 2)$, then $\{a_1b_1b_2, b_1b_2a_2, v_1v_ja_i\}$ forms a copy of TP_3. If $v_1v_jb_2 \in G$, then $\{b_1v_1v_2, b_1v_2v_3, v_1v_jb_2\}$ forms a copy of TP_3. Recall that $G[\{v_1, v_2, \ldots, v_n\}]$ dose not contain an edge. So the only edge containing $\{v_i, v_j\}$ $(i = 1, 2, 3;\ j = 4, \ldots, n)$ is $v_iv_jb_1$.

Now, we consider the pairs $\{v_i, v_j\}$ $(4 \le i < j \le n)$. If $v_iv_ja_k \in G$ $(k = 1, 2)$, then $\{a_1b_1b_2, b_1b_2a_2, v_iv_ja_k\}$ forms a copy of TP_3. If $v_iv_jb_2 \in G$, then $\{v_iv_jb_2, a_1b_1b_2, a_1b_1v_1\}$ forms a copy of TP_3. Recall that $G[\{v_1, v_2, \ldots, v_n\}]$ does not contain an edge. So the only edge containing $\{v_i, v_j\}$ $(4 \le i < j \le n)$ is $v_iv_jb_1$.

Next, we consider the pairs $\{a_i, v_j\}$ $(i = 1, 2;\ j = 4, \ldots, n)$. By the 'symmetry' of a_1, a_2, without loss of generality, we use $\{a_1, v_j\}$ as an example. If $a_1v_ja_2 \in G$, then $\{b_1v_1v_2, b_1a_1v_1, a_1v_ja_2\}$ forms a copy of TP_3. If $a_1v_jb_2 \in G$, then $\{a_1v_jb_2, a_1b_1b_2, b_1v_1v_2\}$ forms a copy of TP_3. If $a_1v_jv_i \in G$ $(i = 1, 2, \ldots, n;\ i \ne j)$, then $\{a_1b_1b_2, b_1b_2a_2, a_1v_jv_i\}$ forms a copy of TP_3. So the only edge containing $\{a_i, v_j\}$ $(i = 1, 2;\ j = 4, \ldots, n)$ is $a_iv_jb_1$.

From the above, we obtain that $v_1, v_2, \ldots, v_n$ are 'symmetrical'. The discussion above implies that all pairs $\{v_i, v_j\}$ $(1 \le i < j \le n)$ and $\{a_k, v_l\}$ $(k = 1, 2;\ l = 1, 2, \ldots, n)$ must only be covered by the edges of the forms $v_iv_jb_1$ and $a_kv_lb_1$, respectively.

Next, we show that the pairs $\{v_j, b_2\}$ $(j = 1, 2, \ldots, n)$ must only be covered by the edges $v_jb_2b_1$. If $v_jb_2a_i \in G$ $(i = 1, 2)$, then $\{v_jb_2a_i, b_1b_2a_i, b_1v_kv_l\}$ $(k, l \ne j)$ forms a copy of TP_3. If $v_jb_2v_i \in G$ $(1 \le j < i \le n)$, then $\{v_jb_2v_i, b_1b_2a_1, a_1b_1v_k\}$ $(k \ne i, j)$ forms a copy of TP_3. So $v_jb_2b_1$ is the only edge covering the pair $\{v_j, b_2\}$.

Finally, we consider $\{a_1, a_2\}$. If $a_1a_2b_2 \in G$ then $\{a_1a_2b_2, b_1a_2b_2, b_1v_1v_2\}$ forms a copy of TP_3. If $a_1a_2v_i \in G$ $(i = 1, 2, \ldots, n)$, then $\{a_1a_2v_i, v_iv_jb_1, v_jv_kb_1\}$ $(i \ne j \ne k)$ forms a copy of TP_3. So $a_1a_2b_1$ is the only edge covering the pair $\{a_1, a_2\}$.

The discussion implies that $G = \{b_1uv : \{u, v\} \in \{a_1, a_2, b_2, v_1, v_2, \ldots, v_n\}^{(2)}\}$. Let $\vec{x}$ be an optimum weighting of G, by Fact 2.7, we can assume that $x_{b_1} = x$,

$x_{a_i} = x_{b_2} = x_{v_j} = \frac{1-x}{n+3}$ $(i = 1, 2,\ j = 1, 2, \ldots, n)$.

$$\lambda(G) = \lambda(G, \vec{x}) = x \cdot \binom{n+3}{2} \left(\frac{1-x}{n+3}\right)^2 < \frac{1}{2}x \cdot (1-x)^2 \le 2/27.$$

Then $\lambda(G) \le \frac{2}{27} < \frac{2}{25} - 10^{-3} = \lambda\left(K_5^3\right) - 10^{-3}$. ■

***Proof of Lemma* 3.1.** If G is not dense, we may take a dense subgraph G' of G, such that $\lambda(G') = \lambda(G)$ and G' is TP_3-free (and K_5^3-free). So we may assume that G is dense. The conclusion follows from Claims 3.6, 3.8 and 3.9. ■

3.2. Turán number of the extension of TP_3

The main result in this section is as follows.

Theorem 3.10. *For sufficiently large n, ex $\left(n, H_6^{TP_3}\right) = t_5^3(n)$. Moreover, if n is sufficiently large and G is an $H_6^{TP_3}$-free 3-graph on $[n]$ with $|G| = t_5^3(n)$, then $G = T_5^3(n)$.*

To prove the theorem, we need several results from [1].

Definition 3.11 [1]**.** Let $m, r \ge 2$ be positive integers. Let F be an r-graph that has at most $m+1$ vertices satisfying $\pi_\lambda(F) \le \frac{[m]_r}{m^r}$. We say that $\mathcal{K}_{m+1}^F$ is *m-stable* if for every real $\varepsilon > 0$ there are a real $\delta > 0$ and an integer n_1 such that if G is a $\mathcal{K}_{m+1}^F$-free r-graph with at least $n \ge n_1$ vertices and more than $\left(\frac{[m]_r}{m^r} - \delta\right)\binom{n}{r}$ edges, then G can be made m-partite by deleting at most εn vertices.

Theorem 3.12 [1]**.** *Let $m, r \ge 2$ be positive integers. Let F be an r-graph that either has at most m vertices or has $m+1$ vertices one of which has degree 1. Suppose either $\pi_\lambda(F) < \frac{[m]_r}{m^r}$ or $\pi_\lambda(F) = \frac{[m]_r}{m^r}$ and $\mathcal{K}_{m+1}^F$ is m-stable. Then there exists a positive integer n_2 such that for all $n \ge n_2$ we have ex $\left(n, H_{m+1}^F\right) = t_m^r(n)$ and the unique extremal r-graph is $T_m^r(n)$.*

Given an r-graph G and a real α with $0 < \alpha \le 1$, we say that G is *α-dense* if G has minimum degree at least $\alpha\binom{|V(G)|-1}{r-1}$. Let $i, j \in V(G)$, we say i and j are *nonadjacent* if $\{i, j\}$ is not covered in G. Given a set $U \subseteq V(G)$, we say U is an *equivalence class* of G if for every two vertices $u, v \in U$, $L_G(u) = L_G(v)$. Given two nonadjacent nonequivalent vertices $u, v \in V(G)$, $d_G(u) \ge d_G(v)$, *symmetrizing* v to u refers to the operation of deleting all edges containing v of G and adding all the edges $\{u\} \cup A$, $A \in L_G(v)$ to G. We use the following algorithm from [1], which was originated in [15].

Algorithm 3.13 (Symmetrization and cleaning with threshold α)**.**

Input: An r-graph G.
Output: An r-graph G^*.
Initiation: Let $G_0 = H_0 = G$. Set $i = 0$.
Iteration: For each vertex u in H_i, let $A_i(u)$ denote the equivalence class that u is in. If either H_i is empty or H_i contains no two nonadjacent nonequivalent vertices, then let $G^* = H_i$ and terminate. Otherwise let u, v be two nonadjacent nonequivalent vertices in H_i, where $d_{H_i}(u) \geq d_{H_i}(v)$. We symmetrize each vertex in $A_i(v)$ to u. Let G_{i+1} denote the resulting graph. If G_{i+1} is α-dense, then let $H_{i+1} = G_{i+1}$. Otherwise we let $L = G_{i+1}$ and repeat the following: let z be any vertex of minimum degree in L. We redefine $L = L - z$ unless in forming G_{i+1} from H_i we symmetrized the equivalence class of some vertex v in H_i to some vertex in the equivalence class of z in H_i. In that case, we redefine $L = L - v$ instead. We repeat the process until L becomes either α-dense or empty. Let $H_{i+1} = L$. We call the process of forming H_{i+1} from G_{i+1} "cleaning". Let Z_{i+1} denote the set of vertices removed, so that $H_{i+1} = G_{i+1} - Z_{i+1}$. By our definition, if H_{i+1} is nonempty, then it is α-dense.

Theorem 3.14 [1]**.** *Let $m, r \geq 2$ be positive integers. Let F be an r-graph that has at most m vertices or has $m+1$ vertices one of which has degree 1. There exists a real $\gamma_0 = \gamma_0(m, r) > 0$ such that for every positive real $\gamma < \gamma_0$, there exist a real $\delta > 0$ and an integer n_0 such that the following is true for all $n \geq n_0$. Let G be a $\mathcal{K}_{m+1}^F$-free r-graph on $[n]$ with more than $\left(\frac{[m]_r}{m^r} - \delta\right)\binom{n}{r}$ edges. Let G^* be the final r-graph produced by Algorithm 3.13 with threshold $\frac{[m]_r}{m^r} - \gamma$. Then $|V(G^*)| \geq (1-\gamma)n$ and G^* is $\left(\frac{[m]_r}{m^r} - \gamma\right)$-dense. Furthermore, if there is a set $W \subseteq V(G^*)$ with $|W| \geq (1-\gamma_0)|V(G^*)|$ such that W is the union of a collection of at most m equivalence classes of G^*, then $G[W]$ is m-partite.*

The following lemma is implied in [1], we give a proof for completeness.

Lemma 3.15 [1]**.** *Let $m, r \geq 2$ be positive integers. Let F be an r-graph that has at most $m+1$ vertices, $r-1$ vertices of an edge has degree 1 and $\pi_\lambda(F) \leq \frac{[m]_r}{m^r}$. Suppose there is a constant $c > 0$ such that $\lambda(L) \leq \lambda(K_m^r) - c$ for every F-free and K_m^r-free r-graph L. Then $\mathcal{K}_{m+1}^F$ is m-stable.*

Proof. Let $\varepsilon > 0$ be given. Let δ, n_0 be the constants guaranteed by Theorem 3.14. We can assume that δ is small enough and n_0 is large enough. Let $\gamma > 0$ satisfy $\gamma < \varepsilon$ and $\delta + r\gamma < c$. Let G be a $\mathcal{K}_{m+1}^F$-free r-graph on $n > n_0$ vertices with more than $\left(\frac{[m]_r}{m^r} - \delta\right)\binom{n}{r}$ edges. Let G^* be the final r-graph produced by applying Algorithm 3.13 to G with threshold $\frac{[m]_r}{m^r} - \gamma$. By Algorithm 3.13, if S consists of one vertex from each equivalence class of G^*, then $G^*[S]$ covers pairs and G^* is a blowup of $G^*[S]$.

First, suppose that $|S| \geq m+1$. If $F \subseteq G^*[S]$, then since $G^*[S]$ covers pairs, we can find a member of $\mathcal{K}_{m+1}^F$ in $G^*[S]$ by using any $(m+1)$-set that contains a copy of F as the core, contradicting G^* being $\mathcal{K}_{m+1}^F$-free. So $G^*[S]$ is F-free. We claim that $G^*[S]$ is K_m^r-free. Otherwise suppose $G^*[S]$ contains a copy of K_m^r. When $|V(F)| = m$, K_m^r contains a copy of F clearly. So suppose that $|V(F)| = m+1$ and F has $r-1$ vertices of one edge of degree 1. Let $e = \{v_1, \ldots, v_r\} \in F$ with $d_F(v_1) = \cdots = d_F(v_{r-1}) = 1$. Let $u_1 \in S \setminus V(K_m^r)$ since $|S| \geq m+1$, and let $u_2 \in V(K_m^r)$, since $G^*[S]$ covers pairs, there is an edge covering $\{u_1, u_2\}$ in $G^*[S]$, denote as $\{u_1, \ldots, u_r\}$. Assume that $V(F) = \{v_1, \ldots, v_{m+1}\}$. We define an injective function f from $V(F)$ to S with $f(v_i) = u_i$ for every $i \in [m+1]$, where $u_{r+1}, \ldots, u_{m+1}$ are arbitrary $m+1-r$ vertices in $V(K_m^r) \setminus \{u_2, \ldots, u_r\}$. It is clear that f preserves edges and hence $G^*[S]$ contains a copy of F, a contradiction. Thus, by our assumption, $\lambda(G^*[S]) \leq \frac{1}{r!}\frac{[m]_r}{m^r} - c$. By Proposition 2.9, we have

$$|G^*| \leq \lambda(G^*[S])n^r \leq \left(\frac{1}{r!}\frac{[m]_r}{m^r} - c\right)n^r < \left(\frac{[m]_r}{m^r} - c\right)\frac{n^r}{r!}. \tag{1}$$

Now, during the process of obtaining G^* from G, symmetrization never decreases the number of edges. Since at most γn vertices are deleted in the process (see Theorem 3.14),

$$|G^*| > |G| - \gamma n\binom{n-1}{r-1} \geq \left(\frac{[m]_r}{m^r} - \delta - r\gamma\right)\binom{n}{r} > \left(\frac{[m]_r}{m^r} - c\right)\frac{n^r}{r!},$$

contradicting (1). So $|S| \leq m$. Hence, $W = V(G^*)$ is the union of at most m equivalence classes of G^*. By Theorem 3.14, $|W| \geq (1-\gamma)n$ and $G[W]$ is m-partite. Hence, G can be made m-partite by deleting at most $\gamma n < \varepsilon n$ vertices. Thus, $\mathcal{K}_{m+1}^F$ is m-stable. ■

***Proof of Theorem* 3.10.** By Theorem 3.1 and Corollary 3.2, TP_3 satisfies the conditions of Lemma 3.15. So $\mathcal{K}_6^{TP_3}$ is 5-stable. The theorem then follows from Theorem 3.12. ■

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