# THE LAGRANGIAN DENSITY OF $\{123,234,456\}$ AND THE TURÁN NUMBER OF ITS EXTENSION 

Pingae Chen<br>College of Science<br>Hunan University of Technology<br>e-mail: chenpingge@hnu.edu.cn<br>Jinhua Liang<br>College of Mathematics and Econometrics<br>Hunan University<br>e-mail: jh_liang@hnu.edu.cn

AND
Yuejian Peng ${ }^{1}$
Institute of Mathematics
Hunan University
e-mail: ypeng1@hnu.edu.cn


#### Abstract

Given a positive integer $n$ and an $r$-uniform hypergraph $F$, the Turán number ex $(n, F)$ is the maximum number of edges in an $F$-free $r$-uniform hypergraph on $n$ vertices. The Turán density of $F$ is defined as $\pi(F)=$ $\lim _{n \rightarrow \infty} \frac{e x(n, F)}{\binom{n}{r}}$. The Lagrangian density of $F$ is $\pi_{\lambda}(F)=\sup \{r!\lambda(G): G$ is $F$-free\}, where $\lambda(G)$ is the Lagrangian of $G$. Sidorenko observed that $\pi(F) \leq \pi_{\lambda}(F)$, and Pikhurko observed that $\pi(F)=\pi_{\lambda}(F)$ if every pair of vertices in $F$ is contained in an edge of $F$. Recently, Lagrangian densities of hypergraphs and Turán numbers of their extensions have been studied actively. For example, in the paper [A hypergraph Turán theorem via Lagrangians of intersecting families, J. Combin. Theory Ser. A 120 (2013) 2020-2038], Hefetz and Keevash studied the Lagrangian densitiy of the 3uniform graph spanned by $\{123,456\}$ and the Turán number of its extension. In this paper, we show that the Lagrangian density of the 3 -uniform graph


[^0]spanned by $\{123,234,456\}$ achieves only on $K_{5}^{3}$. Applying it, we get the Turán number of its extension, and show that the unique extremal hypergraph is the balanced complete 5 -partite 3 -uniform hypergraph on $n$ vertices.
Keywords: Turán number, hypergraph Lagrangian, Lagrangian density.
2010 Mathematics Subject Classification: 05C65.

## 1. Notations and Definitions

For a set $V$ and a positive integer $r$ we denote by $V^{(r)}$ the family of all $r$-subsets of $V$. An $r$-uniform graph or $r$-graph $G$ consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. We sometimes simply write $G$ as $E(G)$. Let $|G|$ denote the number of edges of $G$. An edge $e=\left\{a_{1}, a_{2}, \ldots, a_{r}\right\}$ will be simply denoted by $a_{1} a_{2} \cdots a_{r}$. An $r$-graph $H$ is a subgraph of an $r$-graph $G$, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In particular, a subgraph $H$ is spanning if $V(H)=V(G)$. A subgraph of $G$ induced by $V^{\prime} \subseteq V$, denoted as $G\left[V^{\prime}\right]$, is the $r$-graph with vertex set $V^{\prime}$ and edge set $E^{\prime}=\left\{e \in E(G): e \subseteq V^{\prime}\right\}$. Let $K_{t}^{r}$ denote the complete $r$-graph on $t$ vertices, that is, the $r$-graph on $t$ vertices containing all $r$-subsets of the vertex set as edges. Let $T_{m}^{r}(n)$ be the balanced complete $m$ partite $r$-uniform graph on $n$ vertices, i.e., $V\left(T_{m}^{r}(n)\right)=V_{1} \cup V_{2} \cup \cdots \cup V_{m}$ such that $V_{i} \cap V_{j}=\emptyset$ for every $1 \leq i<j \leq m$ and $\left|V_{1}\right| \leq\left|V_{2}\right| \leq \cdots \leq\left|V_{m}\right| \leq\left|V_{1}\right|+1$, and $E\left(T_{m}^{r}(n)\right)=\left\{e \in\binom{[n]}{r}\right.$ : for every $\left.i \in[m],\left|e \cap V_{i}\right| \leq 1\right\}$. Let $t_{m}^{r}(n)=\left|T_{m}^{r}(n)\right|$. For a positive integer $n$, let $[n]$ denote $\{1,2,3, \ldots, n\}$. Given positive integers $m$ and $r$, let $[m]_{r}=m(m-1) \cdots(m-r+1)$.

Given an $r$-graph $F$, an $r$-graph $G$ is called $F$-free if it does not contain a copy of $F$ as a subgraph. For a fixed positive integer $n$ and an $r$-graph $F$, the Turán number of $F$, denoted by ex $(n, F)$, is the maximum number of edges in an $F$-free $r$-graph with $n$ vertices. An averaging argument of Katona-NemetzSimonovits [8] shows that the sequence $\frac{e x(n, F)}{\binom{n}{r}}$ is a non-increasing sequence. Hence, $\lim _{n \rightarrow \infty} \frac{e x(n, F)}{\binom{n}{r}}$ exists. The Turán density of $F$ is defined as

$$
\pi(F)=\lim _{n \rightarrow \infty} \frac{e x(n, F)}{\binom{n}{r}}
$$

For 2-graphs, Erdős-Stone-Simonovits determined the Turán densities of all graphs except bipartite graphs. Very few results are known for hypergraphs and a recent survey on this topic can be found in Keevash's survey paper [9]. Lagrangian has been a useful tool in estimating the Turán density of a hypergraph.
Definition 1.1. Let $G$ be an $r$-graph on $[n]$ and let $\vec{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$, define the Lagrangian function of $G$ as

$$
\lambda(G, \vec{x})=\sum_{e \in E(G)} \prod_{i \in e} x_{i} .
$$

The Lagrangian of $G$, denoted by $\lambda(G)$, is defined as

$$
\lambda(G)=\max \{\lambda(G, \vec{x}): \vec{x} \in \Delta\},
$$

where

$$
\Delta=\left\{\vec{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathbb{R}^{n}: \sum_{i=1}^{n} x_{i}=1, x_{i} \geq 0 \text { for every } i \in[n]\right\} .
$$

The value $x_{i}$ is called the weight of the vertex $i$ and a vector $\vec{x} \in \Delta$ is called a feasible weighting on $G$. A feasible weighting $\vec{y} \in \Delta$ is called an optimum weighting on $G$ if $\lambda(G, \vec{y})=\lambda(G)$.

Given an $r$-graph $F$, we define the Lagrangian density $\pi_{\lambda}(F)$ of $F$ to be

$$
\pi_{\lambda}(F)=\sup \{r!\lambda(G): G \text { is } F \text {-free }\} .
$$

The Lagrangian density of an $r$-graph is closely related to its Turán density. We say that a pair of vertices $\{i, j\}$ is covered in a hypergraph $H$ if there exists $e \in H$ such that $\{i, j\} \subseteq e$. We say that a hypergraph $H$ covers pairs if every pair of vertices is covered in $H$.

Proposition $1.2[15,17] . \pi(F) \leq \pi_{\lambda}(F)$. If $F$ covers pairs, then $\pi(F)=\pi_{\lambda}(F)$.
Thus, to determine the Turán density of $K_{4}^{3}$ (a long standing conjecture of Turán) is equivalent to determine the Lagrangian density of $K_{4}^{3}$.

Let $r \geq 3, F$ be an $r$-graph and $p \geq|V(F)|$. Let $\mathcal{K}_{p}^{F}$ denote the family of $r$-graphs $H$ that contain a set $C$ of $p$ vertices, called the core, such that the subgraph of $H$ induced by $C$ contains a copy of $F$ and every pair of vertices in $C$ is covered in $H$. Let $H_{p}^{F}$ be a member of $\mathcal{K}_{p}^{F}$ obtained as follows. Label the vertices of $F$ as $v_{1}, \ldots, v_{|V(F)|}$. Add new vertices $v_{|V(F)|+1}, \ldots, v_{p}$. Let $C=\left\{v_{1}, \ldots, v_{p}\right\}$. For each pair of vertices $v_{i}, v_{j} \in C$ not covered in $F$, we add a set $B_{i j}$ of $r-2$ new vertices and the edge $\left\{v_{i}, v_{j}\right\} \cup B_{i j}$, where the $B_{i j}$ 's are pairwise disjoint over all such pairs $\{i, j\}$. We call $H_{p}^{F}$ the extension of $F$.

The Lagrangian method for hypergraph Turán problems were developed independently by Sidorenko [17] and Frankl-Füredi [3], generalizing work of Motzkin and Straus [10] and Zykov [22]. Recently, Lagrangian densities of hypergraphs and Turán numbers of their extensions have been studied by Mubayi, Pikhurko, Hefetz-Keevash, Norin-Yepremyan, Jiang, etc. In [5], Hefetz and Keevash remarked that it is interesting in its own right to determine the maximum Lagrangian of $r$-graphs with certain properties. For example, determine the Lagrangian density of an $r$-graph $F$. The Lagrangian density of the enlargement of a tree satisfying Erdős-Sos's conjecture is determined by Sidorenko [18] and Brandt-Irwin-Jiang [1]. Pikhurko [15] determined the Lagrangian density of a

4-uniform tight linear path of length 2 and applied it to confirm the conjecture of Frankl-Füredi on the Turán number of its extension, the $r$-uniform genearlized triangle for the case $r=4$. Norin and Yepremyan [13] confirmed for $r=5$ or 6 by extending the earlier result of Frankl-Füredi in [3]. Mubayi [11] and Pikhurko [16] obtained the exact Turán number of the expanded complete 2-graph, the extension of an empty graph with a core of $p$ vertices, and showed the stability. Mubayi and Pikhurko [12] also obtain the Turán numbers for the generalized fans, extension of an edge in an $r$-uniform graph with a core of $r+1$ vertices, and showed the stability. Brandt-Irwin-Jiang [1] and independently Norin and Yepremyan [14] showed that for a large family of $r$-graphs $F$ and sufficiently large $n$, ex $\left(n, H_{p}^{F}\right)=e\left(T_{r}(n, p-1)\right)$ with the unique extremal graph being $T_{r}(n, p-1)$. In [5], Hefetz and Keevash determined the Lagrangian density of a 3-uniform matching of size 2 and the Turán number of its extension. They proposed a conjecture on the Lagrangian density of an $r$-uniform matching of size 2 and the Turán number for its extension. Norin and Yepremyan confirmed this conjecture, and independently Wu-Peng-Chen [20] confirmed this conjecture for $r=4$. Jiang-Peng-Wu in [7] obtained the Lagrangian density of a 3-uniform matching of any size and the Turán number of the extension. The authors of [19] and [6] determined the Lagrangian density of a 3 -uniform linear path of length 3 or 4, the Lagrangian density of the disjoint union of a 3-uniform linear path of length 2 or 3 and a matching of any size, and the corresponding Turán numbers of their extensions. It seems to be more difficult if we replace a linear path by a tight path. The authors of [2] obtain the Lagrangian density of the disjoint union of a 3 -uniform tight path of length 2 and an edge, and the corresponding Turán number of its extension. Yan-Peng in [21] determined the Lagrangian densities of the 3 -uniform linear cycle of length $3(\{123,345,561\})$, and $F_{5}(\{123,124,345\})$.

In this paper, we obtain the Lagrangian density of a 3-uniform path of length 3 where two consecutive edges have 1 or 2 vertices in common. Precisely, let $T P_{3}$ be the 3 -graph with vertex set [6] and edge set $\{123,234,456\}$. We show that the Lagrangian density of $T P_{3}$ achieves only on $K_{5}^{3}$. Applying it, we obtain the Turán number for the extension of $T P_{3}$ and show that the unique extremal hypergraph is $T_{5}^{3}(n)$. The method in this paper uses the idea in [7] and [19] by showing that we can reduce the family of all $T P_{3}$-free 3 -graphs to the family of left compressed and dense $T P_{3}$-free 3-graphs, but much more structural analysis is needed here.

## 2. Preliminaries

In this section, we develop some useful properties of the Lagrangian function. The following fact follows immediately from the definition of the Lagrangian.
Fact 2.1. Let $G_{1}, G_{2}$ be r-graphs and $G_{1} \subseteq G_{2}$. Then $\lambda\left(G_{1}\right) \leq \lambda\left(G_{2}\right)$.
Given an $r$-graph $G$ and a set $S$ of vertices, the link of $S$ in $G$, denoted
by $L_{G}(S)$, is the hypergraph with edge set $\left\{e \in\binom{V(G) \backslash S}{r-|S|}: e \cup S \in E(G)\right\}$. In particular, $S=\{i\}$, we write $L_{G}(\{i\})$ as $L_{G}(i)$. The degree of $i$ is $d_{G}(i)=\left|L_{G}(i)\right|$, the number of edges containing $i$ in $G$. Given $i, j \in V(G)$, define

$$
L_{G}(j \backslash i)=\{e: i \notin e, e \cup\{j\} \in E(G) \text { and } e \cup\{i\} \notin E(G)\},
$$

when there is no confusion, we will drop the subscript $G$. We say $G$ on vertex set $[n]$ is left-compressed if for every $i, j, 1 \leq i<j \leq n, L_{G}(j \backslash i)=\emptyset$. Equivalently, $G$ on $[n]$ is left-compressed if $j_{1} j_{2} \cdots j_{r} \in E(G)$ implies $i_{1} i_{2} \cdots i_{r} \in E(G)$, wherever $i_{p} \leq j_{p}$ for $1 \leq p \leq r$. Let $i, j \in V(G)$, define

$$
\pi_{i j}(G)=\left(E(G) \backslash\left\{e \cup\{j\}: e \in L_{G}(j \backslash i)\right\}\right) \cup\left\{e \cup\{i\}: e \in L_{G}(j \backslash i)\right\} .
$$

By the definition of $\pi_{i j}(G)$, it's straightforward to verify the following fact.
Fact 2.2. Let $G$ be an r-graph on $[n]$. Let $\vec{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a feasible weighting on $G$. If $x_{i} \geq x_{j}$, then $\lambda\left(\pi_{i j}(G), \vec{x}\right) \geq \lambda(G, \vec{x})$.

An $r$-graph $G$ is dense if for every subgraph $G^{\prime}$ of $G$ with $\left|V\left(G^{\prime}\right)\right|<|V(G)|$ we have $\lambda\left(G^{\prime}\right)<\lambda(G)$. This is equivalent to that no coordinate in an optimum weighting is zero.
Fact 2.3 [4]. Let $G=(V, E)$ be a dense $r$-graph. Then $G$ covers pairs.
In [10], Motzkin and Straus determined the Lagrangian of any given 2-graph.
Theorem 2.4 (Motzkin and Straus [10]). If $G$ is a 2 -graph in which a maximum complete subgraph has $t$ vertices, then $\lambda(G)=\lambda\left(K_{t}^{2}\right)=\frac{1}{2}\left(1-\frac{1}{t}\right)$.

The support of a vector $\vec{x}$ is $\sigma(\vec{x})=\left\{i: x_{i} \neq 0\right.$ for $\left.i \in[n]\right\}$.
Fact 2.5 [4]. Let $G$ be an $r$-graph on [ $n$ ]. Let $\vec{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be an optimum weighting on $G$. Then

$$
\frac{\partial \lambda(G, \vec{x})}{\partial x_{i}}=r \lambda(G)
$$

for every $i \in \sigma(\vec{x})$.
Let $T_{2}^{(3)}$ be a 3-graph with two edges intersecting at two vertices.
Fact 2.6. If $G$ is a dense 3 -graph on $[n](n \geq 4)$, then $G$ contains a copy of $T_{2}^{(3)}$. Proof. Suppose that $G$ is $T_{2}^{(3)}$-free. Since $G$ is dense, then every pair of vertices is covered exactly once. Let $\vec{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be an optimum weighting on $G$. By Fact 2.5, $\frac{\partial \lambda(G, \vec{x})}{\partial x_{i}}=3 \lambda(G)$ for all $1 \leq i \leq n$. Summing over $i$ we obtain $3 n \lambda(G)=\sum_{i=1}^{n} \frac{\partial \lambda(G, \vec{x})}{\partial x_{i}}=\sum_{1 \leq i<j \leq n} x_{i} x_{j} \leq \frac{1}{2}\left(1-\frac{1}{n}\right)$. So $\lambda(G) \leq \frac{n-1}{6 n^{2}} \leq \frac{1}{32}$, it is a contradiction.

Fact 2.7. Let $G$ be an r-graph on $[n]$. Let $\vec{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a feasible weighting on $G$. Let $i, j \in[n], i \neq j$. Suppose that $L_{G}(i \backslash j)=L_{G}(j \backslash i)=\emptyset$. Let $\vec{y}=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ be defined by letting $y_{\ell}=x_{\ell}$ for every $\ell \in[n] \backslash\{i, j\}$ and letting $y_{i}=y_{j}=\frac{1}{2}\left(x_{i}+x_{j}\right)$. Then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$.
Proof. Since $L_{G}(i \backslash j)=L_{G}(j \backslash i)=\emptyset$, we have

$$
\lambda(G, \vec{y})-\lambda(G, \vec{x})=\sum_{\{i, j\} \subseteq e \in G}\left[\frac{\left(x_{i}+x_{j}\right)^{2}}{4}-x_{i} x_{j}\right] \prod_{k \in e \backslash\{i, j\}} x_{k} \geq 0
$$

Let $K_{t}^{r-}$ be an $r$-graph obtained by removing one edge from $K_{t}^{r}$.
Fact 2.8 [5]. Let $G$ be a 3-graph on [5]. If $G \neq K_{5}^{3}$, then $\lambda(G) \leq \lambda\left(K_{5}^{3-}\right) \leq$ $\lambda\left(K_{5}^{3}\right)-10^{-3}$.

As usual, if $V_{1}, \ldots, V_{s}$ are disjoint sets of vertices then $\Pi_{i=1}^{s} V_{i}=V_{1} \times V_{2} \times$ $\cdots \times V_{s}=\left\{\left(x_{1}, x_{2}, \ldots, x_{s}\right):\right.$ for every $\left.i \in[s], x_{i} \in V_{i}\right\}$. We will use $\Pi_{i=1}^{s} V_{i}$ to also denote the set of the corresponding unordered $s$-sets. If $L$ is a hypergraph on $[m$ ], then a blowup of $L$ is a hypergraph $G$ whose vertex set can be partitioned into $V_{1}, \ldots, V_{m}$ such that $E(G)=\bigcup_{e \in L} \prod_{i \in e} V_{i}$. The following proposition follows immediately from the definition and is implicit in many papers (see [9] for instance).

Proposition 2.9. Let $r \geq 2$. Let $L$ be an $r$-graph and $G$ be a blowup of $L$. Suppose $|V(G)|=n$. Then $|G| \leq \lambda(L) n^{r}$.

## 3. Lagrangian Density of $T P_{3}$ and Related Turán Number

### 3.1. Lagrangian density of $\boldsymbol{T} \boldsymbol{P}_{\mathbf{3}}$

Recall that $T P_{3}$ is the 3 -graph with vertex set [6] and edge set $\{123,234,456\}$. In this section, we will show that the maximum possible Lagrangian among all $T P_{3}$-free 3 -graphs is uniquely achieved by $K_{5}^{3}$. Our main results are as follows.
Theorem 3.1. Let $G$ be a TP $P_{3}$-free 3-graph. Then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)=\frac{2}{25}$. Furthermore, if $G$ is $K_{5}^{3}$-free, then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)-10^{-3}$.

Corollary 3.2. $\pi_{\lambda}\left(T P_{3}\right)=3!\lambda\left(K_{5}^{3}\right)$.
Proof. Since $K_{5}^{3}$ is $T P_{3}$-free, then $\pi_{\lambda}\left(T P_{3}\right) \geq 3!\lambda\left(K_{5}^{3}\right)$. On the other hand, by Theorem 3.1, $\pi_{\lambda}\left(T P_{3}\right) \leq 3!\lambda\left(K_{5}^{3}\right)$. Therefore, $\pi_{\lambda}\left(T P_{3}\right)=3!\lambda\left(K_{5}^{3}\right)$.

In order to prove Theorem 3.1, we divide the $T P_{3}$-free 3 -graphs into two categories: one is that the graphs contain at most six vertices; the other is that the graphs contain seven vertices or more than seven vertices. We will prove the following results.

Lemma 3.3. Let $G$ be a 3 -graph on $[n](n \leq 6)$. If $G$ is $T P_{3}$-free, then for every pair $i, j, 1 \leq i<j \leq n$, the following hold.
(1) $\pi_{i j}(G)$ is $T P_{3}-$ free.
(2) Furthermore, if $G$ is $K_{5}^{3}$-free and $\{i, j\}$ is covered by an edge of $G$, then $\pi_{i j}(G)$ is $K_{5}^{3}$-free.

Proof. Suppose for the contrary that $\pi_{i j}(G)$ contains a copy of $T P_{3}$, denoted by $T P$. Since $G$ is $T P_{3}$-free, there is an edge $e \in T P$ such that $i \in e \in \pi_{i j}(G), j \notin e$ and $e^{\prime}=e \backslash\{i\} \cup\{j\} \in G$. There are two cases in terms of the degree of $i$ in $T P$.

Case 1. $d_{T P}(i)=1$. If there exists no $f \in T P$ such that $j \in f$, then $T P \backslash$ $\{e\} \cup\left\{e^{\prime}\right\}$ forms a copy of $T P_{3}$ in $G$. Otherwise, there exists one edge $f$ such that $j \in f \in T P$, then $f^{\prime}=f \backslash\{j\} \cup\{i\} \in G$. So $T P \backslash\{e, f\} \cup\left\{e^{\prime}, f^{\prime}\right\}$ forms a copy of $T P_{3}$ in $G$.

Case 2. $d_{T P}(i)=2$. Let $T P=\left\{e_{1}, e_{2}, e_{3}\right\}$ and $\left|e_{1} \cap e_{2}\right|=2,\left|e_{2} \cap e_{3}\right|=1$, $\left|e_{1} \cap e_{3}\right|=0$. There are two possible cases.

Subcase 2.1. $i \in e_{1} \cap e_{2}$. Since $G$ is $T P_{3}$-free, at least one of the following cases may happen. (I) $e_{1}^{\prime}=e_{1} \backslash\{i\} \cup\{j\} \in G, e_{2}^{\prime}=e_{2} \backslash\{i\} \cup\{j\} \in G$. In this case, if $j \in e_{3} \backslash e_{2}$, then $\left\{e_{1}^{\prime}, e_{2}^{\prime}, e_{3} \backslash\{j\} \cup\{i\}\right\}$ forms a copy of $T P_{3}$. If $j \notin e_{3} \backslash e_{2}$, then $\left\{e_{1}^{\prime}, e_{2}^{\prime}, e_{3}\right\}$ forms a copy of $T P_{3}$. (II) $e_{1}^{\prime}=e_{1} \backslash\{i\} \cup\{j\} \in G$ but $e_{2}^{\prime}=e_{2} \backslash\{i\} \cup\{j\} \notin G$. In this case, if $j \in e_{3} \backslash e_{2}$, then $\left\{e_{1}^{\prime}, e_{3}, e_{3} \backslash\{j\} \cup\{i\}\right\}$ forms a copy of $T P_{3}$. If $j \notin e_{3} \backslash e_{2}$, we get $e_{1}^{\prime}, e_{2}, e_{3} \in G$ and $\left|e_{1}^{\prime} \cup e_{2} \cup e_{3}\right|=7$, contradicting the condition. (III) $e_{1}^{\prime}=e_{1} \backslash\{i\} \cup\{j\} \notin G$ but $e_{2}^{\prime}=e_{2} \backslash\{i\} \cup\{j\} \in G$. In this case, if $j \in e_{3} \backslash e_{2}$, then $\left\{e_{1}, e_{3}, e_{3} \backslash\{j\} \cup\{i\}\right\}$ forms a copy of $T P_{3}$. If $j \notin e_{3} \backslash e_{2}$, then we can get $e_{1}, e_{2}^{\prime}, e_{3} \in G$ and $\left|e_{1} \cup e_{2}^{\prime} \cup e_{3}\right|=7$, contradicting the condition.

Subcase 2.2. $i \in e_{2} \cap e_{3}$. Since $G$ is $T P_{3}$-free, at least one of the following cases may happen. (I) $e_{2}^{\prime}=e_{2} \backslash\{i\} \cup\{j\} \in G, e_{3}^{\prime}=e_{3} \backslash\{i\} \cup\{j\} \in G$. In this case, if $j \in e_{1} \backslash e_{2}$, then $\left\{e_{1} \backslash\{j\} \cup\{i\}, e_{2}^{\prime}, e_{3}^{\prime}\right\}$ forms a copy of $T P_{3}$. If $j \notin e_{1} \backslash e_{2}$, then $\left\{e_{1}, e_{2}^{\prime}, e_{3}^{\prime}\right\}$ forms a copy of $T P_{3}$. (II) $e_{2}^{\prime}=e_{2} \backslash\{i\} \cup\{j\} \notin G$ but $e_{3}^{\prime}=e_{3} \backslash\{i\} \cup\{j\} \in G$. In this case, $j \notin e_{1} \backslash e_{2}$, then we can get $e_{1}, e_{2}, e_{3}^{\prime} \in G$ and $\left|e_{1} \cup e_{2} \cup e_{3}^{\prime}\right|=7$, contradicting the condition. (III) $e_{2}^{\prime}=e_{2} \backslash\{i\} \cup\{j\} \in G$ but $e_{3}^{\prime}=e_{3} \backslash\{i\} \cup\{j\} \notin G$. In this case, $j \notin e_{1} \backslash e_{2}$, we get $e_{1}, e_{2}^{\prime}, e_{3} \in G$ and $\left|e_{1} \cup e_{2}^{\prime} \cup e_{3}\right|=7$, contradicting the condition.

Assume that $\{i, j\}$ is covered by an edge $g$ of $G$. Suppose for contradiction that $\pi_{i j}(G)$ contains a copy $K$ of $K_{5}^{3}$. Clearly, $V(K)$ must contain $i$. If $j \in V(K)$, then it is easy to see that $K$ is also in $G$, contradicting $G$ being $K_{5}^{3}$-free. By the definition of $\pi_{i j}(G)$, all the edges in $K$ not containing $i$ are also in $G$. If $j \notin V(K)$, since $n \leq 6$, we have $|g \cap V(K)|=2$. Now, we can find a copy of $T P_{3}$ in $G$, a contradiction.

Next, we need an algorithm.

Algorithm 3.4 (Dense and Left-compressed [7]).
Input: An $r$-graph $G$.
Output: A dense, left-compressed $r$-graph $G^{\prime}$.
Step 1. If $G$ is dense, then go to Step 2. Otherwise, replace $G$ by a dense subgraph $G^{\prime}$ with the same Lagrangian and go to Step 2.
Step 2. If $G$ is left-compressed, then terminate. Otherwise, let $\vec{y}$ be an optimum weighting of $G$, then there exist vertices $i, j$, where $i<j$, such that $y_{i}>y_{j}$ and $L_{G}(j \backslash i) \neq \emptyset$. Replace $G$ by $\pi_{i j}(G)$ and go to Step 1 .

Note that the algorithm terminates after finite many steps since Step 2 reduces the parameter $s(G)=\sum_{e \in G} \sum_{i \in e} i$ by at least 1 each time and Step 1 reduces the number of vertices by at least 1 each time.

Lemma 3.5. Let $G$ be a $T P_{3}$-free (and $K_{5}^{3}$-free) 3 -graph on $[n](n \leq 6)$. Then there exists a dense and left-compressed $T P_{3}$-free (and $K_{5}^{3}$-free) 3-graph $G^{\prime}$ with $\left|V\left(G^{\prime}\right)\right| \leq|V(G)|$ such that $\lambda\left(G^{\prime}\right) \geq \lambda(G)$.

Proof. We apply Algorithm 3.4 to $G$ and let $G^{\prime}$ be the final graph. Then $G^{\prime}$ is dense and left-compressed. By Fact 2.2, $\lambda\left(G^{\prime}\right) \geq \lambda(G)$. By Lemma 3.3, $G^{\prime}$ is $T P_{3}$-free (and $K_{5}^{3}$-free).

Claim 3.6. Let $G$ be a TP $P_{3}$-free 3-graph on $[n](n \leq 6)$. Then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)$. Furthermore, if $G$ is also $K_{5}^{3}$-free, then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)-10^{-3}$.

Proof. By Lemma 3.5, we may assume that $G$ is dense and left-compressed. If $n \leq 5$, by Fact 2.1 , then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)$. If $G$ is $K_{5}^{3}$-free, by Fact 2.8 , then $\lambda(G) \leq \lambda\left(K_{5}^{3-}\right) \leq \lambda\left(K_{5}^{3}\right)-10^{-3}$. Hence, we may assume that $n=6$. Let $\vec{x}=\left(x_{1}, x_{2}, \ldots, x_{6}\right)$ be an optimum weighting of $G$. It is clear that $x_{1} \geq x_{2} \geq$ $\cdots \geq x_{6}$. By Fact $2.3, G$ covers pairs. So $i 56 \in G$, for some $i<5$. Since $G$ is left-compressed, we have $156 \in G$, this implies that for every $i, j$, where $2 \leq i<$ $j \leq 6,1 i j \in G$. Suppose that $G[\{2,3,4,5,6\}]$ contains an edge. Without loss of generality, we assume that $456 \in G$. Since $123,145 \in G$, we have $\{123,145,456\}$ forms a copy of $T P_{3}$ in $G$, contradicting $G$ being $T P_{3}$-free. Hence $G=\{1 i j$ : $2 \leq i<j \leq 6\}$. Assume that $x_{1}=a$. Since $\vec{y}=\left(\frac{x_{2}}{1-a}, \ldots, \frac{x_{6}}{1-a}\right)$ is a feasible weighting on $L_{G}(1)$, By Theorem 2.4,

$$
\lambda(G)=\lambda(G, \vec{x})=a(1-a)^{2} \lambda\left(L_{G}(1), \vec{y}\right)<\frac{1}{2} a(1-a)^{2} \leq \frac{2}{27}<\lambda\left(K_{5}^{3}\right)-10^{-3}
$$

Let $Q=\left\{a_{1} b_{1} b_{2}, a_{2} b_{1} b_{2}, c_{1} c_{2} c_{3}\right\}$, we have the following result.
Claim 3.7. Let $G$ be a dense 3 -graph. If $G$ contains a subgraph $Q$, then it also contains a copy of $T P_{3}$.

Proof. Suppose that $G$ is $T P_{3}$-free. If $|V(G)|=|V(Q)|$, then we have $b_{1} c_{1} x \notin G$ for every $x \in\left\{a_{1}, a_{2}, b_{2}, c_{2}, c_{3}\right\}$. If $b_{1} c_{1} a_{1} \in G$, then $\left\{b_{1} c_{1} a_{1}, a_{1} b_{1} b_{2}, c_{1} c_{2} c_{3}\right\}$ forms a copy of $T P_{3}$, it is a contradiction. If $b_{1} c_{1} a_{2} \in G$, then $\left\{b_{1} c_{1} a_{2}, a_{2} b_{1} b_{2}, c_{1} c_{2} c_{3}\right\}$ forms a copy of $T P_{3}$. If $b_{1} c_{1} b_{2} \in G$, then $\left\{b_{1} c_{1} b_{2}, a_{1} b_{1} b_{2}, c_{1} c_{2} c_{3}\right\}$ forms a copy of $T P_{3}$. If $b_{1} c_{1} c_{2} \in G$, then $\left\{b_{1} c_{1} c_{2}, c_{1} c_{2} c_{3}, a_{1} b_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. If $b_{1} c_{1} c_{3} \in G$, then $\left\{b_{1} c_{1} c_{3}, c_{1} c_{2} c_{3}, a_{1} b_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. So the pair, $\left\{b_{1}, c_{1}\right\}$, is not covered by any edge of $G$, which is a contradiction by Fact 2.3. If $|V(G)|>|V(Q)|$, then let $v_{i} \in V(G) \backslash V(Q)$, where $1 \leq i \leq|V(G)|-|V(Q)|$. Since $G$ is $T P_{3}$-free, we have $a_{1} c_{1} x \notin G$ for every $x \in\left\{b_{1}, b_{2}, c_{2}, c_{3}, v_{i}\right\}$. If $a_{1} c_{1} b_{1} \in G$, then $\left\{a_{1} c_{1} b_{1}, a_{1} b_{1} b_{2}, c_{1} c_{2} c_{3}\right\}$ forms a copy of $T P_{3}$. If $a_{1} c_{1} b_{2} \in G$, then $\left\{a_{1} c_{1} b_{2}, a_{1} b_{1} b_{2}, c_{1} c_{2} c_{3}\right\}$ forms a copy of $T P_{3}$. If $a_{1} c_{1} c_{2} \in G$, then $\left\{a_{1} c_{1} c_{2}, c_{1} c_{2} c_{3}\right.$, $\left.a_{1} b_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} c_{1} c_{3} \in G$, then $\left\{a_{1} c_{1} c_{3}, c_{1} c_{2} c_{3}, a_{1} b_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} c_{1} v_{i} \in G$, then $\left\{a_{1} c_{1} v_{i}, a_{1} b_{1} b_{2}, a_{2} b_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. Similarly, we have $a_{1} c_{2} x \notin G$ for $x \in\left\{b_{1}, b_{2}, c_{1}, c_{3}, v_{i}\right\}$ and $a_{1} c_{3} x \notin G$ for $x \in\left\{b_{1}, b_{2}, c_{1}, c_{2}, v_{1}\right\}$. Since $G$ is dense, we have $a_{1} c_{i} a_{2} \in G(i=1,2,3)$. Next, we consider the pair $\left\{b_{1}, c_{1}\right\}$. From previous analysis, we have $b_{1} c_{1} x \notin G$ for every $x \in\left\{a_{1}, a_{2}, b_{2}, c_{2}, c_{3}\right\}$. So we just consider the edges $b_{1} c_{1} v_{i} \in G$ $(1 \leq i \leq|V(G)|-|V(Q)|)$ or not. If $b_{1} c_{1} v_{i} \in G$, then $\left\{a_{1} a_{2} c_{1}, a_{1} a_{2} c_{2}, c_{1} b_{1} v_{i}\right\}$ forms a copy of $T P_{3}$. So the pair $\left\{b_{1}, c_{1}\right\}$, is not covered by any edge of $G$, which is a contradiction by Fact 2.3 .

Claim 3.8. Let $G$ be a dense 3-graph and $|V(G)|=7$. If $G$ is $T P_{3}$-free, then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)-10^{-3}$.

Proof. Let $V(G)=\left\{a_{1}, a_{2}, b_{1}, b_{2}, v_{1}, v_{2}, v_{3}\right\}$. Since $G$ is dense, we have $G$ contains a copy of $T_{2}^{(3)}$ by Fact 2.6. Without loss of generality, we assume that $a_{1} b_{1} b_{2}, a_{2} b_{1} b_{2} \in G$. If $v_{1} v_{2} v_{3} \in G$, by Claim 3.7, then $G$ contains a copy of $T P_{3}$, it is a contradiction. If $v_{1} v_{2} v_{3} \notin G$, we consider the pairs $\left\{v_{j}, v_{k}\right\}(1 \leq j<k \leq 3)$. If $v_{j} v_{k} a_{i} \in G(i=1,2)$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, a_{i} v_{j} v_{k}\right\}$ forms a copy of $T P_{3}$ in $G$. So the pairs $\left\{v_{j}, v_{k}\right\}$ may be covered by the edge of the form $v_{j} v_{k} b_{1}$ or $v_{j} v_{k} b_{2}$. By the pigeonhole principle, there exist two pairs $\left\{v_{j}, v_{k}\right\}$ covered by $b_{1}$ or $b_{2}$. Without loss of generality, we only need to discuss the following two cases.

Case 1. $v_{1} v_{2} b_{1}, v_{2} v_{3} b_{1}, v_{1} v_{3} b_{2} \in G$. First, we consider the pairs $\left\{a_{i}, v_{j}\right\}(i=$ $1,2 ; j=1,3)$. Due to the 'symmetry' of $a_{1}, a_{2}$, and the 'symmetry' of $v_{1}, v_{3}$, we use $\left\{a_{1}, v_{1}\right\}$ as an example. If $a_{1} v_{1} a_{2} \in G$, then $\left\{v_{3} b_{1} v_{2}, b_{1} v_{2} v_{1}, v_{1} a_{1} a_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{1} b_{2} \in G$, then $\left\{v_{3} b_{1} v_{2}, b_{1} v_{2} v_{1}, v_{1} a_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{1} v_{i} \in G(i=2,3)$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, a_{1} v_{1} v_{i}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{a_{1}, v_{1}\right\}$ is $a_{1} v_{1} b_{1}$. Similarly, $a_{2} v_{1} b_{1}, a_{1} v_{3} b_{1}, a_{2} v_{3} b_{1}$ are the only edges in $G$ containing $\left\{a_{2}, v_{1}\right\},\left\{a_{1}, v_{3}\right\}$ and $\left\{a_{2}, v_{3}\right\}$, respectively.

Second, we consider $\left\{a_{1}, a_{2}\right\}$. If $a_{1} a_{2} b_{1} \in G$, then $\left\{a_{1} b_{1} b_{2}, a_{1} b_{1} a_{2}, b_{2} v_{1} v_{3}\right\}$ forms a copy of $T P_{3}$. If $a_{1} a_{2} b_{2} \in G$, then $\left\{a_{2} b_{1} b_{2}, a_{1} a_{2} b_{2}, b_{1} v_{1} v_{2}\right\}$ forms a copy of
$T P_{3}$. If $a_{1} a_{2} v_{i} \in G(i=1,3)$, then $\left\{b_{1} v_{2} v_{1}, b_{1} v_{2} v_{3}, a_{1} a_{2} v_{i}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{a_{1}, a_{2}\right\}$ is $a_{1} a_{2} v_{2}$.

Third, we consider $\left\{v_{2}, b_{2}\right\}$. If $v_{2} b_{2} a_{1} \in G$, then $\left\{v_{2} b_{2} a_{1}, a_{1} b_{1} b_{2}, b_{1} a_{2} v_{1}\right\}$ forms a copy of $T P_{3}$. If $v_{2} b_{2} a_{2} \in G$, then $\left\{v_{2} b_{2} a_{2}, a_{2} b_{1} b_{2}, b_{1} a_{1} v_{1}\right\}$ forms a copy of $T P_{3}$. If $v_{2} b_{2} v_{1} \in G$, then $\left\{v_{1} v_{2} b_{2}, v_{1} v_{2} b_{1}, b_{1} a_{1} v_{3}\right\}$ forms a copy of $T P_{3}$. If $v_{2} b_{2} v_{3} \in G$, then $\left\{v_{3} v_{2} b_{2}, v_{3} v_{2} b_{1}, b_{1} a_{1} v_{1}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{v_{2}, b_{2}\right\}$ is $v_{2} b_{2} b_{1}$.

Finally, we consider the rest of edges in $G$. If $a_{1} v_{2} b_{1} \in G$, then $\left\{a_{1} v_{2} b_{1}, a_{1} b_{1} b_{2}\right.$, $\left.v_{1} v_{3} b_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{2} v_{2} b_{1} \in G$, then $\left\{a_{2} v_{2} b_{1}, a_{2} b_{1} b_{2}, v_{1} v_{3} b_{2}\right\}$ forms a copy of $T P_{3}$. If $v_{1} v_{3} b_{1} \in G$, then $\left\{v_{1} v_{3} b_{1}, b_{1} v_{1} v_{2}, v_{2} a_{1} a_{2}\right\}$ forms a copy of $T P_{3}$. If $b_{1} b_{2} v_{i} \in G(i=1,3)$, then $\left\{b_{1} b_{2} v_{i}, b_{1} b_{2} a_{2}, a_{2} a_{1} v_{2}\right\}$ forms a copy of $T P_{3}$.

From the above, we obtain that $G=\left\{a_{1} b_{1} b_{2}, a_{2} b_{1} b_{2}, v_{1} v_{2} b_{1}, v_{2} v_{3} b_{1}, v_{1} v_{3} b_{2}\right.$, $\left.a_{1} v_{1} b_{1}, a_{1} v_{3} b_{1}, a_{2} v_{1} b_{1}, a_{2} v_{3} b_{1}, a_{1} a_{2} v_{2}, b_{1} b_{2} v_{2}\right\}$. Let $\vec{x}$ be an optimum weighting of $G$, by Fact 2.7, we may assume that $x_{a_{1}}=x_{a_{2}}=x_{v_{2}}=x, x_{b_{2}}=x_{v_{1}}=x_{v_{3}}=$ $y, x_{b_{1}}=z$ and $3 x+3 y+z=1 . \lambda(G)=\lambda(G, \vec{x})=x^{3}+y^{3}+9 x y z$. If some of $x, y, z$ is 0 , then it is easy to verify that $\lambda(G)=\frac{1}{27}<\lambda\left(K_{5}^{3}\right)-10^{-3}$. So we assume that $x, y, z>0$. Let $f(x, y, z)=x^{3}+y^{3}+9 x y z$. To get the maximum value of $f(x, y, z)$, we apply the theory of Lagrange multipliers. Let

$$
g(x, y, z, \gamma)=x^{3}+y^{3}+9 x y z-\gamma(3 x+3 y+z-1)
$$

Taking the partial derivative with respect to $x, y, z$ and $\gamma$, and let its value equal to 0 . We have

$$
\left\{\begin{array}{l}
3 x^{2}+9 y z=3 \gamma \\
3 y^{2}+9 x z=3 \gamma \\
9 x y=\gamma \\
3 x+3 y+z=1
\end{array}\right.
$$

Noting that the right hand sides of the first and the second equalities are equal, we obtain that

$$
3 x^{2}-3 y^{2}=9 z(x-y)
$$

If $x-y=0$. Solving the above system of equations, we have that $x_{0}=\frac{3}{26}, y_{0}=$ $\frac{3}{26}, z_{0}=\frac{8}{26}$. If $x-y \neq 0$, then we have that $x_{1}=\frac{15+3 \sqrt{15}}{100}, y_{1}=\frac{15-3 \sqrt{15}}{100}, z_{1}=0.1$ or $x_{2}=\frac{15-3 \sqrt{15}}{100}, y_{2}=\frac{15+3 \sqrt{15}}{100}, z_{2}=0.1$. By direct calculation, we see that the maximum occurs at $x_{0}=\frac{3}{26}, y_{0}=\frac{3}{26}, z_{0}=\frac{8}{26}$.

Hence, $\lambda(G) \leq f\left(x_{0}, y_{0}, z_{0}\right)=\frac{27}{676}<\frac{2}{25}-10^{-3}=\lambda\left(K_{5}^{3}\right)-10^{-3}$.
Case 2. $v_{1} v_{2} b_{1}, v_{2} v_{3} b_{1}, v_{1} v_{3} b_{1} \in G$. First, we consider the pairs $\left\{a_{i}, v_{j}\right\}(i=$ $1,2 ; j=1,2,3)$. Due to the 'symmetry' of $a_{1}, a_{2}$ and the 'symmetry' of $v_{1}, v_{2}$ and $v_{3}$, we use $\left\{a_{1}, v_{1}\right\}$ as an example. If $a_{1} v_{1} a_{2} \in G$, then $\left\{b_{1} v_{1} v_{2}, b_{1} v_{2} v_{3}, v_{1} a_{1} a_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{1} b_{2} \in G$, then $\left\{b_{1} v_{1} v_{2}, b_{1} v_{2} v_{3}, v_{1} a_{1} b_{2}\right\}$ forms a copy of
$T P_{3}$. If $a_{1} v_{1} v_{i} \in G(i=2,3)$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, a_{1} v_{1} v_{i}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{a_{1}, v_{1}\right\}$ is $a_{1} v_{1} b_{1}$. Similarly, $a_{2} v_{1} b_{1}, a_{1} v_{2} b_{1}, a_{2} v_{2} b_{1}$, $a_{1} v_{3} b_{1}, a_{2} v_{3} b_{1}$ are the only edges in $G$ containing $\left\{a_{2}, v_{1}\right\},\left\{a_{1}, v_{2}\right\},\left\{a_{2}, v_{2}\right\}$, $\left\{a_{1}, v_{3}\right\}$ and $\left\{a_{2}, v_{3}\right\}$, respectively.

Second, we consider $\left\{a_{1}, a_{2}\right\}$. If $a_{1} a_{2} b_{2} \in G$, then $\left\{a_{1} b_{1} b_{2}, a_{1} b_{2} a_{2}, b_{1} v_{1} v_{3}\right\}$ forms a copy of $T P_{3}$. If $a_{1} a_{2} v_{i} \in G(i=1,3)$, then $\left\{a_{1} a_{2} v_{i}, b_{1} v_{1} v_{2}, b_{1} v_{2} v_{3}\right\}$ forms a copy of $T P_{3}$. If $a_{1} a_{2} v_{2} \in G$, then $\left\{a_{1} a_{2} v_{2}, b_{1} v_{1} v_{3}, b_{1} v_{2} v_{3}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{a_{1}, a_{2}\right\}$ is $a_{1} a_{2} b_{1}$.

Third, we consider the pair $\left\{b_{2}, v_{i}\right\} \quad(i=1,2,3)$. Due to the 'symmetry' of $v_{1}, v_{2}$ and $v_{3}$, we use $\left\{b_{2}, v_{1}\right\}$ as an example. If $b_{2} v_{1} a_{1} \in G$, then $\left\{v_{1} a_{1} b_{2}, a_{1} b_{2} b_{1}\right.$, $\left.b_{1} v_{2} v_{3}\right\}$ forms a copy of $T P_{3}$. If $b_{2} v_{1} a_{2} \in G$, then $\left\{v_{1} a_{2} b_{2}, a_{2} b_{2} b_{1}, b_{1} a_{1} v_{2}\right\}$ forms a copy of $T P_{3}$. If $b_{2} v_{1} v_{2} \in G$, then $\left\{v_{1} v_{2} b_{2}, v_{1} v_{2} b_{1}, b_{1} a_{1} v_{3}\right\}$ forms a copy of $T P_{3}$. If $b_{2} v_{1} v_{3} \in G$, then $\left\{v_{3} v_{1} b_{2}, v_{3} v_{1} b_{1}, b_{1} a_{1} a_{2}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{b_{2}, v_{1}\right\}$ is $b_{2} v_{1} b_{1}$. Similarly, $b_{2} v_{2} b_{1}$ and $b_{2} v_{3} b_{1}$ are the only edges in $G$ containing $\left\{b_{2}, v_{1}\right\}$ and $\left\{b_{2}, v_{3}\right\}$, respectively.

Meanwhile, if $b_{2} v_{2} v_{3} \in G$, then $\left\{v_{2} v_{3} b_{2}, b_{1} v_{2} v_{3}, b_{1} a_{1} a_{2}\right\}$ forms a copy of $T P_{3}$.
From the above, we obtain that $G=\left\{b_{1} u v:\{u, v\} \in\left\{a_{1}, a_{2}, b_{2}, v_{1}, v_{2}, v_{3}\right\}^{(2)}\right\}$. Let $\vec{x}$ be an optimum weighting of $G$, by Fact 2.7 , we may assume that $x_{b_{1}}=$ $x, x_{a_{i}}=x_{b_{2}}=x_{v_{j}}=(1-x) / 6(i=1,2 ; j=1,2,3)$.

$$
\lambda(G)=\lambda(G, \vec{x})=15 x \cdot\left(\frac{1-x}{6}\right)^{2} \leq \frac{5}{24}\left(\frac{2 x+(1-x)+(1-x)}{3}\right)^{3}=\frac{5}{81} .
$$

Then $\lambda(G) \leq \frac{5}{81}<\frac{2}{25}-10^{-3}=\lambda\left(K_{5}^{3}\right)-10^{-3}$.
Claim 3.9. Let $G$ be a dense 3 -graph and $|V(G)| \geq 8$. If $G$ is $T P_{3}$-free, then $\lambda(G) \leq \lambda\left(K_{5}^{3}\right)-10^{-3}$.
Proof. Let $V(G)=\left\{a_{1}, a_{2}, b_{1}, b_{2}, v_{1}, v_{2}, v_{3}, v_{4}, \ldots, v_{n}\right\}(n \geq 4)$. Since G is dense, we have $G$ contains a copy of $T_{2}^{(3)}$ by Fact 2.6. Without loss of generality, we assume that $a_{1} b_{1} b_{2}, a_{2} b_{1} b_{2} \in G$. Suppose that $G\left[\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}\right]$ contains an edge $e$, then $G$ contains a copy of $T P_{3}$ by Claim 3.7, it is a contradiction. Therefore, for the pair $\left\{v_{j}, v_{k}\right\}(1 \leq j<k \leq 3)$, we have $v_{j} v_{k} v_{i} \notin G$ where $1 \leq i \leq n$, $i \neq j, k$. If $v_{j} v_{k} a_{i} \in G$ where $i=1,2$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, v_{j} v_{k} a_{i}\right\}$ forms a copy of $T P_{3}$. So the pair $\left\{v_{j}, v_{k}\right\}$ may be covered by the edge of the form $v_{j} v_{k} b_{1}$ or $v_{j} v_{k} b_{2}$. By the pigeonhole principle, there exist two pairs $\left\{v_{j}, v_{k}\right\}$ covered by $b_{1}$ or $b_{2}$. Without loss of generality, we assume that $v_{1} v_{2} b_{1}, v_{2} v_{3} b_{1} \in G$.

First, we consider $\left\{a_{1}, v_{1}\right\}$. If $a_{1} v_{1} a_{2} \in G$, then $\left\{b_{1} v_{1} v_{2}, b_{1} v_{2} v_{3}, v_{1} a_{1} a_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{1} b_{2} \in G$, then $\left\{b_{1} v_{1} v_{2}, b_{1} v_{2} v_{3}, v_{1} a_{1} b_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{1} v_{i} \in G(i=2,3, \ldots, n)$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, a_{1} v_{1} v_{i}\right\}$ forms a copy of $T P_{3}$. So we have $a_{1} v_{1} b_{1} \in G$. Similarly, $a_{2} v_{1} b_{1} \in G$ by the 'symmetry' of $a_{1}, a_{2}$.

Second, we consider $\left\{a_{1}, v_{4}\right\}$. If $a_{1} v_{4} a_{2} \in G$, then $\left\{a_{1} a_{2} v_{4}, a_{1} b_{1} v_{1}, b_{1} v_{1} v_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{4} b_{2} \in G$, then $\left\{a_{1} b_{2} v_{4}, a_{1} v_{1} b_{1}, v_{1} b_{1} v_{2}\right\}$ forms a copy of
$T P_{3}$. If $a_{1} v_{4} v_{i} \in G(1 \leq i \leq n, i \neq 4)$, then $\left\{a_{1} v_{4} v_{i}, a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}\right\}$ forms a copy of $T P_{3}$. So we have $a_{1} v_{4} b_{1} \in G$.

Third, we consider the pair $\left\{v_{1}, v_{3}\right\}$. If $v_{1} v_{3} a_{i} \in G(i=1,2)$, then $\left\{a_{1} b_{1} b_{2}\right.$, $\left.b_{1} b_{2} a_{2}, v_{1} v_{3} a_{i}\right\}$ forms a copy of $T P_{3}$. If $v_{1} v_{3} b_{2} \in G$, then $\left\{v_{4} a_{1} b_{1}, a_{1} b_{1} b_{2}, b_{2} v_{1} v_{3}\right\}$ forms a copy of $T P_{3}$. Recall that $G\left[\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}\right]$ dose not contain an edge, so $v_{1} v_{3} b_{1} \in G$. We note that the three vertices $v_{1}, v_{2}, v_{3}$ are 'symmetrical'. So the pairs $\left\{a_{i}, v_{j}\right\}(i=1,2 ; j=1,2,3)$ must be covered by the edge of the form $a_{i} v_{j} b_{1}$. We will show that $v_{1}, v_{2}, \ldots, v_{n}$ are 'symmetrical'.

Let us consider the pairs $\left\{v_{i}, v_{j}\right\}(i=1,2,3 ; j=4, \ldots, n)$. By the 'symmetry' of $v_{1}, v_{2}, v_{3}$, without loss of generality, we use $\left\{v_{1}, v_{j}\right\}$ as an example. If $v_{1} v_{j} a_{i} \in G$ $(i=1,2)$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, v_{1} v_{j} a_{i}\right\}$ forms a copy of $T P_{3}$. If $v_{1} v_{j} b_{2} \in G$, then $\left\{b_{1} v_{1} v_{2}, b_{1} v_{2} v_{3}, v_{1} v_{j} b_{2}\right\}$ forms a copy of $T P_{3}$. Recall that $G\left[\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}\right]$ dose not contain an edge. So the only edge containing $\left\{v_{i}, v_{j}\right\}(i=1,2,3 ; j=4, \ldots, n)$ is $v_{i} v_{j} b_{1}$.

Now, we consider the pairs $\left\{v_{i}, v_{j}\right\}(4 \leq i<j \leq n)$. If $v_{i} v_{j} a_{k} \in G(k=$ $1,2)$, then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, v_{i} v_{j} a_{k}\right\}$ forms a copy of $T P_{3}$. If $v_{i} v_{j} b_{2} \in G$, then $\left\{v_{i} v_{j} b_{2}, a_{1} b_{1} b_{2}, a_{1} b_{1} v_{1}\right\}$ forms a copy of $T P_{3}$. Recall that $G\left[\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}\right]$ does not contain an edge. So the only edge containing $\left\{v_{i}, v_{j}\right\}(4 \leq i<j \leq n)$ is $v_{i} v_{j} b_{1}$.

Next, we consider the pairs $\left\{a_{i}, v_{j}\right\}(i=1,2 ; j=4, \ldots, n)$. By the 'symmetry' of $a_{1}, a_{2}$, without loss of generality, we use $\left\{a_{1}, v_{j}\right\}$ as an example. If $a_{1} v_{j} a_{2} \in G$, then $\left\{b_{1} v_{1} v_{2}, b_{1} a_{1} v_{1}, a_{1} v_{j} a_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{j} b_{2} \in G$, then $\left\{a_{1} v_{j} b_{2}, a_{1} b_{1} b_{2}, b_{1} v_{1} v_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} v_{j} v_{i} \in G(i=1,2, \ldots, n$; $i \neq j$ ), then $\left\{a_{1} b_{1} b_{2}, b_{1} b_{2} a_{2}, a_{1} v_{j} v_{i}\right\}$ forms a copy of $T P_{3}$. So the only edge containing $\left\{a_{i}, v_{j}\right\}(i=1,2 ; j=4, \ldots, n)$ is $a_{i} v_{j} b_{1}$.

From the above, we obtain that $v_{1}, v_{2}, \ldots, v_{n}$ are 'symmetrical'. The discussion above implies that all pairs $\left\{v_{i}, v_{j}\right\}(1 \leq i<j \leq n)$ and $\left\{a_{k}, v_{l}\right\}(k=1,2$; $l=1,2, \ldots, n)$ must only be covered by the edges of the forms $v_{i} v_{j} b_{1}$ and $a_{k} v_{l} b_{1}$, respectively.

Next, we show that the pairs $\left\{v_{j}, b_{2}\right\}(j=1,2, \ldots, n)$ must only be covered by the edges $v_{j} b_{2} b_{1}$. If $v_{j} b_{2} a_{i} \in G(i=1,2)$, then $\left\{v_{j} b_{2} a_{i}, b_{1} b_{2} a_{i}, b_{1} v_{k} v_{l}\right\}(k, l \neq j)$ forms a copy of $T P_{3}$. If $v_{j} b_{2} v_{i} \in G(1 \leq j<i \leq n)$, then $\left\{v_{j} b_{2} v_{i}, b_{1} b_{2} a_{1}, a_{1} b_{1} v_{k}\right\}$ $(k \neq i, j)$ forms a copy of $T P_{3}$. So $v_{j} b_{2} b_{1}$ is the only edge covering the pair $\left\{v_{j}, b_{2}\right\}$.

Finally, we consider $\left\{a_{1}, a_{2}\right\}$. If $a_{1} a_{2} b_{2} \in G$ then $\left\{a_{1} a_{2} b_{2}, b_{1} a_{2} b_{2}, b_{1} v_{1} v_{2}\right\}$ forms a copy of $T P_{3}$. If $a_{1} a_{2} v_{i} \in G(i=1,2, \ldots, n)$, then $\left\{a_{1} a_{2} v_{i}, v_{i} v_{j} b_{1}, v_{j} v_{k} b_{1}\right\}$ ( $i \neq j \neq k$ ) forms a copy of $T P_{3}$. So $a_{1} a_{2} b_{1}$ is the only edge covering the pair $\left\{a_{1}, a_{2}\right\}$.

The discussion implies that $G=\left\{b_{1} u v:\{u, v\} \in\left\{a_{1}, a_{2}, b_{2}, v_{1}, v_{2}, \ldots, v_{n}\right\}^{(2)}\right\}$. Let $\vec{x}$ be an optimum weighting of $G$, by Fact 2.7 , we can assume that $x_{b_{1}}=x$,
$x_{a_{i}}=x_{b_{2}}=x_{v_{j}}=\frac{1-x}{n+3}(i=1,2, j=1,2, \ldots, n)$.

$$
\lambda(G)=\lambda(G, \vec{x})=x \cdot\binom{n+3}{2}\left(\frac{1-x}{n+3}\right)^{2}<\frac{1}{2} x \cdot(1-x)^{2} \leq 2 / 27 .
$$

Then $\lambda(G) \leq \frac{2}{27}<\frac{2}{25}-10^{-3}=\lambda\left(K_{5}^{3}\right)-10^{-3}$.
Proof of Lemma 3.1. If $G$ is not dense, we may take a dense subgraph $G^{\prime}$ of $G$, such that $\lambda\left(G^{\prime}\right)=\lambda(G)$ and $G^{\prime}$ is $T P_{3}$-free (and $K_{5}^{3}$-free). So we may assume that $G$ is dense. The conclusion follows from Claims 3.6, 3.8 and 3.9.

### 3.2. Turán number of the extension of $T P_{3}$

The main result in this section is as follows.
Theorem 3.10. For sufficiently large $n$, ex $\left(n, H_{6}^{T P_{3}}\right)=t_{5}^{3}(n)$. Moreover, if $n$ is sufficiently large and $G$ is an $H_{6}^{T P_{3}}$-free 3-graph on $[n]$ with $|G|=t_{5}^{3}(n)$, then $G=T_{5}^{3}(n)$.

To prove the theorem, we need several results from [1].
Definition 3.11 [1]. Let $m, r \geq 2$ be positive integers. Let $F$ be an $r$-graph that has at most $m+1$ vertices satisfying $\pi_{\lambda}(F) \leq \frac{[m]_{r}}{m^{r}}$. We say that $\mathcal{K}_{m+1}^{F}$ is $m$-stable if for every real $\varepsilon>0$ there are a real $\delta>0$ and an integer $n_{1}$ such that if $G$ is a $\mathcal{K}_{m+1}^{F}$-free $r$-graph with at least $n \geq n_{1}$ vertices and more than $\left(\frac{[m]_{r}}{m^{r}}-\delta\right)\binom{n}{r}$ edges, then $G$ can be made $m$-partite by deleting at most $\varepsilon n$ vertices.

Theorem 3.12 [1]. Let $m, r \geq 2$ be positive integers. Let $F$ be an $r$-graph that either has at most $m$ vertices or has $m+1$ vertices one of which has degree 1 . Suppose either $\pi_{\lambda}(F)<\frac{[m]_{r}}{m^{r}}$ or $\pi_{\lambda}(F)=\frac{[m]_{r}}{m^{r}}$ and $\mathcal{K}_{m+1}^{F}$ is $m$-stable. Then there exists a positive integer $n_{2}$ such that for all $n \geq n_{2}$ we have ex $\left(n, H_{m+1}^{F}\right)=t_{m}^{r}(n)$ and the unique extremal $r$-graph is $T_{m}^{r}(n)$.

Given an $r$-graph $G$ and a real $\alpha$ with $0<\alpha \leq 1$, we say that $G$ is $\alpha$ dense if $G$ has minimum degree at least $\alpha\binom{|V(G)|-1}{r-1}$. Let $i, j \in V(G)$, we say $i$ and $j$ are nonadjacent if $\{i, j\}$ is not covered in $G$. Given a set $U \subseteq V(G)$, we say $U$ is an equivalence class of $G$ if for every two vertices $u, v \in U, L_{G}(u)=$ $L_{G}(v)$. Given two nonadjacent nonequivalent vertices $u, v \in V(G), d_{G}(u) \geq$ $d_{G}(v)$, symmetrizing $v$ to $u$ refers to the operation of deleting all edges containing $v$ of $G$ and adding all the edges $\{u\} \cup A, A \in L_{G}(v)$ to $G$. We use the following algorithm from [1], which was originated in [15].

Algorithm 3.13 (Symmetrization and cleaning with threshold $\alpha$ ).

Input: An $r$-graph $G$.
Output: An $r$-graph $G^{*}$.
Initiation: Let $G_{0}=H_{0}=G$. Set $i=0$.
Iteration: For each vertex $u$ in $H_{i}$, let $A_{i}(u)$ denote the equivalence class that $u$ is in. If either $H_{i}$ is empty or $H_{i}$ contains no two nonadjacent nonequivalent vertices, then let $G^{*}=H_{i}$ and terminate. Otherwise let $u, v$ be two nonadjacent nonequivalent vertices in $H_{i}$, where $d_{H_{i}}(u) \geq d_{H_{i}}(v)$. We symmetrize each vertex in $A_{i}(v)$ to $u$. Let $G_{i+1}$ denote the resulting graph. If $G_{i+1}$ is $\alpha$-dense, then let $H_{i+1}=G_{i+1}$. Otherwise we let $L=G_{i+1}$ and repeat the following: let $z$ be any vertex of minimum degree in $L$. We redefine $L=L-z$ unless in forming $G_{i+1}$ from $H_{i}$ we symmetrized the equivalence class of some vertex $v$ in $H_{i}$ to some vertex in the equivalence class of $z$ in $H_{i}$. In that case, we redefine $L=L-v$ instead. We repeat the process until $L$ becomes either $\alpha$-dense or empty. Let $H_{i+1}=L$. We call the process of forming $H_{i+1}$ from $G_{i+1}$ "cleaning". Let $Z_{i+1}$ denote the set of vertices removed, so that $H_{i+1}=G_{i+1}-Z_{i+1}$. By our definition, if $H_{i+1}$ is nonempty, then it is $\alpha$-dense.

Theorem 3.14 [1]. Let $m, r \geq 2$ be positive integers. Let $F$ be an $r$-graph that has at most $m$ vertices or has $m+1$ vertices one of which has degree 1 . There exists a real $\gamma_{0}=\gamma_{0}(m, r)>0$ such that for every positive real $\gamma<\gamma_{0}$, there exist a real $\delta>0$ and an integer $n_{0}$ such that the following is true for all $n \geq n_{0}$. Let $G$ be a $\mathcal{K}_{m+1}^{F}$-free r-graph on $[n]$ with more than $\left(\frac{[n]_{r}}{m^{r}}-\delta\right)\binom{n}{r}$ edges. Let $G^{*}$ be the final $r$-graph produced by Algorithm 3.13 with threshold $\frac{[m]_{r}}{m^{r}}-\gamma$. Then $\left|V\left(G^{*}\right)\right| \geq(1-\gamma) n$ and $G^{*}$ is $\left(\frac{[m]_{r}}{m^{r}}-\gamma\right)$-dense. Furthermore, if there is a set $W \subseteq V\left(G^{*}\right)$ with $|W| \geq\left(1-\gamma_{0}\right)\left|V\left(G^{*}\right)\right|$ such that $W$ is the union of a collection of at most $m$ equivalence classes of $G^{*}$, then $G[W]$ is $m$-partite.

The following lemma is implied in [1], we give a proof for completeness.
Lemma 3.15 [1]. Let $m, r \geq 2$ be positive integers. Let $F$ be an $r$-graph that has at most $m+1$ vertices, $r-1$ vertices of an edge has degree 1 and $\pi_{\lambda}(F) \leq \frac{[m]_{r}}{m^{r}}$. Suppose there is a constant $c>0$ such that $\lambda(L) \leq \lambda\left(K_{m}^{r}\right)-c$ for every $F$-free and $K_{m}^{r}$-free $r$-graph $L$. Then $\mathcal{K}_{m+1}^{F}$ is $m$-stable.

Proof. Let $\varepsilon>0$ be given. Let $\delta, n_{0}$ be the constants guaranteed by Theorem 3.14. We can assume that $\delta$ is small enough and $n_{0}$ is large enough. Let $\gamma>0$ satisfy $\gamma<\varepsilon$ and $\delta+r \gamma<c$. Let $G$ be a $\mathcal{K}_{m+1}^{F}$-free $r$-graph on $n>n_{0}$ vertices with more than $\left(\frac{[m]_{r}}{m^{r}}-\delta\right)\binom{n}{r}$ edges. Let $G^{*}$ be the final $r$-graph produced by applying Algorithm 3.13 to $G$ with threshold $\frac{[m]_{r}}{m^{r}}-\gamma$. By Algorithm 3.13, if $S$ consists of one vertex from each equivalence class of $G^{*}$, then $G^{*}[S]$ covers pairs and $G^{*}$ is a blowup of $G^{*}[S]$.

First, suppose that $|S| \geq m+1$. If $F \subseteq G^{*}[S]$, then since $G^{*}[S]$ covers pairs, we can find a member of $\mathcal{K}_{m+1}^{F}$ in $G^{*}[S]$ by using any $(m+1)$-set that contains a copy of $F$ as the core, contradicting $G^{*}$ being $\mathcal{K}_{m+1}^{F}$-free. So $G^{*}[S]$ is $F$-free. We claim that $G^{*}[S]$ is $K_{m}^{r}$-free. Otherwise suppose $G^{*}[S]$ contains a copy of $K_{m}^{r}$. When $|V(F)|=m, K_{m}^{r}$ contains a copy of $F$ clearly. So suppose that $|V(F)|=$ $m+1$ and $F$ has $r-1$ vertices of one edge of degree 1 . Let $e=\left\{v_{1}, \ldots, v_{r}\right\} \in F$ with $d_{F}\left(v_{1}\right)=\cdots=d_{F}\left(v_{r-1}\right)=1$. Let $u_{1} \in S \backslash V\left(K_{m}^{r}\right)$ since $|S| \geq m+1$, and let $u_{2} \in V\left(K_{m}^{r}\right)$, since $G^{*}[S]$ covers pairs, there is an edge covering $\left\{u_{1}, u_{2}\right\}$ in $G^{*}[S]$, denote as $\left\{u_{1}, \ldots, u_{r}\right\}$. Assume that $V(F)=\left\{v_{1}, \ldots, v_{m+1}\right\}$. We define an injective function $f$ from $V(F)$ to $S$ with $f\left(v_{i}\right)=u_{i}$ for every $i \in[m+1]$, where $u_{r+1}, \ldots, u_{m+1}$ are arbitrary $m+1-r$ vertices in $V\left(K_{m}^{r}\right) \backslash\left\{u_{2}, \ldots, u_{r}\right\}$. It is clear that $f$ preserves edges and hence $G^{*}[S]$ contains a copy of $F$, a contradiction. Thus, by our assumption, $\lambda\left(G^{*}[S]\right) \leq \frac{1}{r!} \frac{[m]_{r}}{m^{r}}-c$. By Proposition 2.9, we have

$$
\begin{equation*}
\left|G^{*}\right| \leq \lambda\left(G^{*}[S]\right) n^{r} \leq\left(\frac{1}{r!} \frac{[m]_{r}}{m^{r}}-c\right) n^{r}<\left(\frac{[m]_{r}}{m^{r}}-c\right) \frac{n^{r}}{r!} . \tag{1}
\end{equation*}
$$

Now, during the process of obtaining $G^{*}$ from $G$, symmetrization never decreases the number of edges. Since at most $\gamma n$ vertices are deleted in the process (see Theorem 3.14),

$$
\left|G^{*}\right|>|G|-\gamma n\binom{n-1}{r-1} \geq\left(\frac{[m]_{r}}{m^{r}}-\delta-r \gamma\right)\binom{n}{r}>\left(\frac{[m]_{r}}{m^{r}}-c\right) \frac{n^{r}}{r!},
$$

contradicting (1). So $|S| \leq m$. Hence, $W=V\left(G^{*}\right)$ is the union of at most $m$ equivalence classes of $G^{*}$. By Theorem 3.14, $|W| \geq(1-\gamma) n$ and $G[W]$ is $m$ partite. Hence, $G$ can be made $m$-partite by deleting at most $\gamma n<\varepsilon n$ vertices. Thus, $\mathcal{K}_{m+1}^{F}$ is $m$-stable.

Proof of Theorem 3.10. By Theorem 3.1 and Corollary 3.2, $T P_{3}$ satisfies the conditions of Lemma 3.15. So $\mathcal{K}_{6}^{T P_{3}}$ is 5 -stable. The theorem then follows from Theorem 3.12.

## Acknowledgements

The authors would like to thank the anonymous referees for their comments on this paper. This research is supported in part by National Natural Science Foundation of China (No. 11671124).

## References

[1] A. Brandt, D. Irwin and T. Jiang, Stability and Turán numbers of a class of hypergraphs via Lagrangians, Combin. Probab. Comput. 26 (2017) 367-405.
https://doi.org/10.1017/S0963548316000444
[2] P. Chen, J. Liang and Y. Peng, Lagrangian density of $\{123,234,567\}$ and Turán numbers of its extension, submitted.
[3] P. Frankl and Z. Füredi, Extremal problems whose solutions are the blowups of the small witt-designs, J. Combin. Theory Ser. A 52 (1989) 129-147. https://doi.org/10.1016/0097-3165(89)90067-8
[4] P. Frankl and V. Rödl, Hypergraphs do not jump, Combinatorica 4 (1984) 149-159. https://doi.org/10.1007/BF02579215
[5] D. Hefetz and P. Keevash, A hypergraph Turán theorem via lagrangians of intersecting families, J. Combin. Theory Ser. A 120 (2013) 2020-2038. https://doi.org/10.1016/j.jcta.2013.07.011
[6] S. Hu, Y. Peng and B. Wu, Lagrangian densities of the disjoint union of 3-uniform linear paths and matchings and Turán numbers of their extensions, submitted.
[7] T. Jiang, Y. Peng and B. Wu, Lagrangian densities of some sparse hypergraphs and Turán numbers of their extensions, European J. Combin. 73 (2018) 20-36. https://doi.org/10.1016/j.ejc.2018.05.001
[8] G. Katona, T. Nemetz and M. Simonovits, On a problem of Turán in the theory of graphs, Mat. Lapok (N.S.) 15 (1964) 228-238.
[9] P. Keevash, Hypergrah Turán problems, in: Surveys in Combinatorics 2011, London Math. Soc. Lecture Note Ser. 392 (Cambridge Univ. Press, Cambridge, 2011) 83-140. https://doi.org/10.1017/CB09781139004114
[10] T.S. Motzkin and E.G. Straus, Maxima for graphs and a new proof of a theorem of Turán, Canad. J. Math. 17 (1965) 533-540. https://doi.org/10.4153/CJM-1965-053-6
[11] D. Mubayi, A hypergraph extension of Turán's theorem, J. Combin. Theory Ser. B 96 (2006) 122-134. https://doi.org/10.1016/j.jctb.2005.06.013
[12] D. Mubayi and O. Pikhurko, A new generalization of Mantel's theorem to $k$-graphs, J. Combin. Theory Ser. B 97 (2007) 669-678. https://doi.org/10.1016/j.jctb.2006.11.003
[13] S. Norin and L. Yepremyan, Turán number of generalized triangles, J. Combin. Theory Ser. A 146 (2017) 312-343. https://doi.org/10.1016/j.jcta.2016.09.003
[14] S. Norin and L. Yepremyan, Turán numbers of extensions, J. Combin. Theory Ser. A 155 (2018) 476-492.
https://doi.org/10.1016/j.jcta.2017.08.004
[15] O. Pikhurko, An exact Turán result for the generalized triangle, Combinatorica 28 (2008) 187-208.
https://doi.org/10.1007/s00493-008-2187-2
[16] O. Pikhurko, Exact computation of the hypergraph Turán function for expanded complete 2-graphs, J. Combin. Theory Ser. B 103 (2013) 220-225. https://doi.org/10.1016/j.jctb.2012.09.005
[17] A.F. Sidorenko, The maximal number of edges in a homogeneous hypergraph containing no prohibited subgraphs, Mat. Zametki 41 (1987) 433-455. https://doi.org/10.1027.BF01158259
[18] A.F. Sidorenko, Asymptotic solution for a new class of forbidden r-graphs, Combinatorica 9 (1989) 207-215. https://doi.org/10.1007/BF02124681
[19] B. Wu and Y. Peng, Lagrangian densities of 3-uniform linear paths and Turán numbers of their extensions, submitted.
[20] B. Wu, Y. Peng and P. Chen, On a conjecture of Hefetz and Keevash on Lagrangians of intersecting hypergraphs and Turán numbers. arXiv:1701.06126v3
[21] Z. Yan and Y. Peng, $\lambda$-perfect hypergraphs and Lagrangian densities of hypergraph cycles, Discrete Math. (2019). https://doi.org/10.1016/j.disc.2019.03.024
[22] A.A. Zykov, On some properties of linear complexes, Mat. Sb. 24 (1949) 163-188.
Received 4 August 2017
Revised 18 March 2019
Accepted 18 March 2019


[^0]:    ${ }^{1}$ Corresponding author.

