# SUFFICIENT CONDITIONS FOR A DIGRAPH TO ADMIT A $(1, \leq \ell)$-IDENTIFYING CODE 

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#### Abstract

A $(1, \leq \ell)$-identifying code in a digraph $D$ is a subset $C$ of vertices of $D$ such that all distinct subsets of vertices of cardinality at most $\ell$ have distinct closed in-neighbourhoods within $C$. In this paper, we give some sufficient conditions for a digraph of minimum in-degree $\delta^{-} \geq 1$ to admit a $(1, \leq \ell)$ identifying code for $\ell \in\left\{\delta^{-}, \delta^{-}+1\right\}$. As a corollary, we obtain the result by Laihonen that states that a graph of minimum degree $\delta \geq 2$ and girth at least 7 admits a $(1, \leq \delta)$-identifying code. Moreover, we prove that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if the girth of the digraph is at least 5 . We also characterize all the 2-in-regular digraphs admitting a $(1, \leq \ell)$-identifying code for $\ell \in\{2,3\}$.


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## 1. Introduction

The aim of this paper is to study identifying codes in digraphs. We consider simple digraphs (or directed graphs) without loops or multiple edges. Unless otherwise stated, we follow the book by Bang-Jensen and Gutin [4] for terminology and definitions.

Let $D=(V, A)$ be a digraph with vertex set $V(D)=V$ and $\operatorname{arc}$ set $A(D)=A$. A vertex $u$ is adjacent to a vertex $v$ if $(u, v) \in A$. If both $\operatorname{arcs}(u, v),(v, u) \in A$, then we say that they form a digon. A digraph is symmetric if $(u, v) \in A$ implies $(v, u) \in A$. A digon is often said a symmetric arc of $D$. A digraph $D$ is said to be oriented graph if $D$ has no digon. The girth $g$ of a digraph is the length of a shortest directed cycle. Hence, an oriented graph has girth $g \geq 3$. Moreover, observe that every graph $G$ with vertex set $V$ and edge set $E$ can be seen as a symmetric digraph, replacing each edge $u v \in E$ by the digon $(u, v)$ and $(v, u)$. The out-neighborhood of a vertex $u$ is $N^{+}(u)=\{v \in V:(u, v) \in A\}$ and the inneighborhood of $u$ is $N^{-}(u)=\{v \in V:(v, u) \in A\}$. The closed in-neighbourhood of $u$ is $N^{-}[u]=\{u\} \cup N^{-}(u)$. The out-degree of $u$ is $d^{+}(u)=\left|N^{+}(u)\right|$ and its in-degree $d^{-}(u)=\left|N^{-}(u)\right|$. We denote by $\delta^{+}(D)$ the minimum out-degree of the vertices in $D$, and by $\delta^{-}(D)$ the minimum in-degree. The minimum degree is $\delta(D)=\min \left\{\delta^{+}(D), \delta^{-}(D)\right\}$. A digraph $D$ is said to be $d$-in-regular if $d^{-}(v)=d$ for all $v \in V$, and $d$-regular if $d^{+}(v)=d^{-}(v)=d$ for all $v \in V$.

Given a vertex subset $U \subseteq V$, let $N^{-}[U]=\bigcup_{u \in U} N^{-}[u]$. For a given integer $\ell \geq 1$, a vertex subset $C \subseteq V$ is a $(1, \leq \ell)$-identifying code in $D$ when for all distinct subsets $X, Y \subseteq V$, with $1 \leq|X|,|Y| \leq \ell$, we have

$$
\begin{equation*}
N^{-}[X] \cap C \neq N^{-}[Y] \cap C \tag{1}
\end{equation*}
$$

The definition of a $(1, \leq \ell)$-identifying code for graphs was introduced by Karpovsky, Chakrabarty and Levitin [16], and it is obtained by omitting the superscript sign minus in the neighborhoods in (1). Thus, the definition for digraphs is a natural extension of the concept of $(1, \leq \ell)$-identifying codes in graphs. A $(1, \leq 1)$-identifying code is known as an identifying code. Thus, an identifying code of a graph is a set of vertices such that any two vertices of the graph have distinct closed neighborhoods within this set. Identifying codes model faultdiagnosis in multiprocessor systems, and these are used in other applications such as the design of emergency sensor networks. Identifying codes in graphs have received much more attention by researchers. Honkala and Laihonen [15] studied identifying codes in the king grid that are robust against edge deletions. Identifying codes have been considered for vertex-transitive graphs and strongly regular graphs by Gravier et al. [14], and for graphs of girth at least five by Balbuena et al. [3]. More recently, techniques of spectral graph theory have been applied to the identifying codes [2]. Other results on identifying codes in specific families
of graphs, as well as on the smallest cardinality of an identifying code $C$, can be seen in Bertrand et al. [5], Charon et al. [6], Exoo et al. [9, 10], and the online bibliography of Lobstein [19].

A graph $G$ is said to admit a $(1, \leq \ell)$-identifying code if there is a subset of vertices $C \subseteq V(G)$ such that $C$ is a ( $1, \leq \ell$ )-identifying code in $G$. Not all graphs admit $(1, \leq \ell)$-identifying codes. For instance, Laihonen [17] pointed out that a graph containing an isolated edge cannot admit a $(1, \leq 1)$-identifying code, because clearly, if $u v \in E(G)$ is isolated, then $N[u]=\{u, v\}=N[v]$. In fact, a graph containing an isolated complete bipartite graph $K_{r, d}$, with $r \leq d$, cannot admit a $(1, \leq d)$-identifying code. It is not difficult to see that if $G$ admits a $(1, \leq \ell)$-identifying code, then $C=V$ is also a $(1, \leq \ell)$-identifying code. Hence, a graph admits a $(1, \leq \ell)$-identifying code if and only if the sets $N[X]$ are mutually different for all $X \subseteq V$, with $|X| \leq \ell$. Laihonen and Ranto [18] proved that if $G$ is a connected graph with at least three vertices admitting a $(1, \leq \ell)$-identifying code, then the minimum degree is $\delta(G) \geq \ell$. Gravier and Moncel [13] showed the existence of a graph with minimum degree exactly $\ell$ admitting a $(1, \leq \ell)$ identifying code. Laihonen [17] proved the following result.

Theorem 1 [17]. Let $k \geq 2$ be an integer.

1. If a $k$-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.
2. If a $k$-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.

Araujo-Pardo et al. [1] characterized the bipartite $k$-regular graphs of girth at least 6 having a $(1, \leq k)$-identifying code.

Identifying codes for digraphs were considered by Charon et al. [7, 8], and Frieze et al. [12] studied identifying codes in random networks. Foucaud et al. [11] studied identifying codes in digraphs under the name of separating sets, and they called identifying codes the separating sets that also are dominating sets. These authors characterized the finite digraphs that only admit their whole vertex set as an identifying code in this meaning.

In this paper, we give some sufficient conditions for a digraph of minimum in-degree $\delta^{-} \geq 1$ to admit a $(1, \leq \ell)$-identifying code for $\ell \in\left\{\delta^{-}, \delta^{-}+1\right\}$. As a corollary, we obtain Theorem 1. Moreover, we prove that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if the girth of the digraph is at least 5 . We also characterize all the 2 -in-regular digraphs admitting a $(1, \leq \ell)$ identifying code for $\ell \in\{2,3\}$.

## 2. Identifying Codes

In this paper we study the concept of a $(1, \leq \ell)$-identifying code for digraphs
given in (1). We begin by noting that if $C$ is a $(1, \leq \ell)$-identifying code in a digraph $D$, then the whole set of vertices $V$ also is. Thus, we have the following straightforward observation.

Lemma 2. A digraph $D=(V, A)$ admits some $(1, \leq \ell)$-identifying code if and only if for all distinct subsets $X, Y \subseteq V$ with $|X|,|Y| \leq \ell$, we have

$$
\begin{equation*}
N^{-}[X] \neq N^{-}[Y] \tag{2}
\end{equation*}
$$

As already mentioned in the introduction, Laihonen and Ranto [18] proved that if $G$ is a connected graph with at least three vertices admitting a $(1, \leq \ell)$ identifying code, then the minimum degree is $\delta(G) \geq \ell$. We present the following similar result for digraphs.

Proposition 3. Let $D$ be a digraph admitting a $(1, \leq \ell)$-identifying code. Let $u$ be a vertex such that $d^{+}(u) \geq 1$. Then, $\ell \leq d^{-}(u)+1$. Furthermore, if $u$ belongs to a digon, then $\ell \leq d^{-}(u)$.

Proof. Let $u \in V(D)$ be such that $d^{+}(u) \geq 1$ and $v \in N^{+}(u)$. Then, both sets $X=N^{-}(u) \cup\{u, v\}$ and $Y=N^{-}(u) \cup\{v\}$ have the same closed in-neighbourhood. Consequently, $\ell \leq d^{-}(u)+1$. Furthermore, if $v \in N^{-}(u)$, then $X^{\prime}=N^{-}(u) \cup\{u\}$ and $Y^{\prime}=N^{-}(u)$ have the same closed in-neighbourhood implying that $\ell \leq$ $d^{-}(u)$.

Corollary 4. Let $D$ be a digraph of minimum in-degree $\delta^{-}$admitting a $(1, \leq$ $\left.\delta^{-}+1\right)$-identifying code. Then, any vertex $u$ with $d^{-}(u)=\delta^{-}$does not lay on a digon.

Corollary 5. Let $D$ be a digraph admitting a $(1, \leq \ell)$-identifying code. Then, $\ell \leq \min \left\{d^{-}(u)+1 \mid u \in V(D)\right.$ and $\left.d^{+}(u) \geq 1\right\}$.

We recall that a transitive tournament of three vertices is denoted by $T T_{3}$, see $F_{1}$ of Figure 1.

Remark 6. Let $D$ be a $T T_{3}$-free digraph. Then, for every arc $(x, y)$ of $D$, we have $N^{-}(x) \cap N^{-}(y)=\emptyset$ and $N^{+}(x) \cap N^{+}(y)=\emptyset$.

Remark 7. Two distinct vertices $u$ and $v$ of $D$ are called twins if $N^{-}[u]=N^{-}[v]$. Hence, a digraph $D$ admits a $(1, \leq 1)$-identifying code if and only if $D$ is twin-free.

Theorem 8. Let $D$ be a twin-free digraph with minimum in-degree $\delta^{-} \geq 1$.
(i) Suppose that $\delta^{-} \geq 2$ and $D$ does not contain any subdigraph as $F_{1}$ nor $F_{2}$ of Figure 1, then $D$ admits a $\left(1, \leq \delta^{-}-1\right)$-identifying code.
(ii) Suppose that $D$ is an oriented graph and does not contain any subdigraph as $F_{1}$ nor $F_{2}$ of Figure 1 , then $D$ admits a $\left(1, \leq \delta^{-}\right)$-identifying code.
(iii) If $D$ does not contain any subdigraph from $F_{1}$ to $F_{9}$ of Figure 1, then $D$ admits $a\left(1, \leq \delta^{-}\right)$-identifying code.
(iv) Suppose that $\delta^{-} \geq 2$ and the vertices of in-degree $\delta^{-}$do not lay on a digon. If $D$ does not contain any subdigraph as those of Figure 1 , then $D$ admits a $\left(1, \leq \delta^{-}+1\right)$-identifying code.
(v) Suppose that $\delta^{-}=1$ and the vertices of in-degree 1 do not lay on directed cycles of length less than five. If $D$ does not contain any subdigraph as $F_{1}$, $F_{3}, F_{4}, F_{5}, F_{6}$ nor $F_{11}$ of Figure 1 , then $D$ admits a $(1, \leq 2)$-identifying code.


Figure 1. All the forbidden subdigraphs of Theorem 8.

Proof. By Remark 7, $D$ admits a $(1, \leq 1)$-identifying code because $D$ is twinfree. In what follows, for brevity, we made reference to the different cases $F_{1}-F_{11}$ of Figure 1 without mentioning the figure.

We reason assuming that $D$ does not admit a $(1, \leq \ell)$-identifying code for $\ell \in\left\{\delta^{-}-1, \delta^{-}, \delta^{-}+1\right\}$. Then there are two different subsets $X$ and $Y$ with $|Y|,|X| \leq \ell$ such that $N^{-}[X]=N^{-}[Y]$. Since $X \neq Y$ we can choose $x \in X \backslash Y$ with $N^{-}(x)=\left\{v_{1}, \ldots, v_{\tau}\right\}$ for $\tau \geq \delta^{-}$. As $N^{-}(x) \subseteq N^{-}[X]=N^{-}[Y]$, for all $v_{i}$, $i=1, \ldots, \tau$, there exists a vertex $y_{i} \in Y$ such that $y_{i} \in N^{+}\left(v_{i}\right)$ or $y_{i}=v_{i} \in Y$. Moreover, all vertices $y_{i}$ are mutually different, since otherwise some subdigraph $F_{1}$ or $F_{2}$ would be contained in $D$. Hence, $|Y| \geq \delta^{-}$, which contradicts the hypothesis of (i), and the proof of (i) is completed.

Observe that both (ii) and (iii) are proved if $\delta^{-}=1$, so we may assume that $\delta^{-} \geq 2$ in these two cases.

We continue the proof assuming that $\ell \in\left\{\delta^{-}, \delta^{-}+1\right\}$. Since $x \in N^{-}[X]=$ $N^{-}[Y]$, there is $y \in Y$ such that $y \in N^{+}(x)$. Observe that $y \notin N^{-}(x)$ because by hypothesis of (ii) the digraph is an oriented graph. Moreover, $y$ is different from each $y_{i}$ because $D$ is free of $F_{1}$, implying that $|Y| \geq \delta^{-}+1$, which contradicts the hypothesis of (ii), and the proof of (ii) is completed.

Next, to see (iii) let us show that $|X| \geq \delta^{-}+1$. To do that, let us see that for each $v_{i} \in N^{-}(x)$ one can associate to it a vertex $z_{i} \in X \backslash\{x\}$ in such a way that $z_{i} \neq z_{j}$ for all $i \neq j$. Let us consider the following partition of $N^{-}(x): N^{-}(x) \cap(Y \backslash X), N^{-}(x) \cap X$ and $N^{-}(x) \cap(V \backslash(X \cup Y))$. We have the following cases (see Figure 8).

Case 1. $v_{i} \in N^{-}(x) \cap(Y \backslash X)$. Since $\delta^{-} \geq 2$, there is $w_{i} \in N^{-}\left(v_{i}\right) \backslash\{x\} \subseteq$ $N^{-}[Y] \backslash\{x\}=N^{-}[X] \backslash\{x\}$. Hence, if $w_{i} \in X$, then $z_{i}=w_{i}$ and $z_{i} \neq x$; and if $w_{i} \notin X$, since $w_{i} \in N^{-}[Y]=N^{-}[X]$, there exists $z_{i} \in X$ such that $z_{i} \in N^{+}\left(w_{i}\right)$. In this case we may assume that $z_{i} \neq x$, because $D$ is free of $F_{1}$.

Case 2. $v_{i} \in N^{-}(x) \cap X$. Then $z_{i}=v_{i}$ and $z_{i} \neq x$.
Case 3. $v_{i} \in N^{-}(x) \cap(V \backslash(X \cup Y)) \subseteq N^{-}[X] \backslash(X \cup Y)=N^{-}[Y] \backslash(X \cup Y)$. Then we consider the vertices $y_{i} \in Y$ such that $y_{i} \in N^{+}\left(v_{i}\right)$ and $y_{i} \neq y_{j}$ for $i \neq j$. If $y_{i} \in X$, then $z_{i}=y_{i}$, and $y_{i} \neq x$ because $x \in X \backslash Y$. If $y_{i} \in Y \backslash X$, then there exists $z_{i} \in X$ such that $z_{i} \in N^{+}\left(y_{i}\right)$. Observe that $z_{i}$ is different from $x$, because $D$ is free of $F_{1}$.

Case 1


Case 2
$N^{-}(x) \cap X$

Case 3


Figure 2. All the cases in the proof of (iii) of Theorem 8.
Now let us see that all $z_{i}$ are different. For this, let $i, j \in\{1, \ldots, \tau\}$ such that $i \neq j$. If $v_{i}, v_{j} \in N^{-}(x) \cap(Y \backslash X)$ and $z_{i}=z_{j}$, then (see Figure 2, Case 1) it could be $w_{j}=z_{j}=z_{i}=w_{i} \in X$, and $D$ would contain the subdigraph $F_{3}$, contradicting the hypothesis of (iii). It could be $z_{j}=z_{i}=w_{i} \in X$ and $w_{j} \notin X$, then $D$ would contain the subdigraph $F_{5}$, a contradiction. Finally, it could be $w_{i}, w_{j} \notin X$, $z_{i}=z_{j}$ and $z_{i} \in N^{+}\left(w_{i}\right) \cap N^{+}\left(w_{j}\right)$, then $D$ would contain the subdigraph $F_{8}$, a
contradiction. Therefore, all the $z_{i}$ are different in Case 1. If $v_{i}, v_{j} \in N^{-}(x) \cap X$, then it is clear that $z_{i} \neq z_{j}$ in Case 2. If $v_{i}, v_{j} \in N^{-}(x) \cap(V \backslash(X \cup Y))$ and $z_{i}=z_{j}$, then (see Figure 2, Case 3) it could be $z_{j}=y_{i} \in X$, and $D$ contains the subdigraph $F_{6}$. Hence, $y_{i}, y_{j} \in Y \backslash X$ and $D$ contains the subdigraph $F_{8}$. Therefore, all the $z_{i}$ are different in Case 3. It remains to prove that for all $i, j \in\{1, \ldots, \tau\}$, with $i \neq j, z_{i} \neq z_{j}$ when $v_{i}$ and $v_{j}$ are in a different partite subsets of the considered partition of $N^{-}(x)$. Thus, if $z_{i}=z_{j}$ for some $i \neq j$, with $v_{i} \in N^{-}(x) \cap(Y \backslash X)$ and $v_{j} \in N^{-}(x) \cap X$, then $D$ contains one of the subdigraphs $F_{1}$ or $F_{3}$ (see Figure 2, Cases 1 and 2); if $v_{i} \in N^{-}(x) \cap(Y \backslash X)$ and $v_{j} \in N^{-}(x) \cap(V \backslash(X \cup Y))$, then $D$ contains one of the subdigraphs $F_{4}$, $F_{6}, F_{7}$ or $F_{9}$ (see Figure 2, Cases 1 and 3); and finally, if $v_{i} \in N^{-}(x) \cap X$ and $v_{j} \in N^{-}(x) \cap(V \backslash(X \cup Y))$, then $D$ contains one of the subdigraphs $F_{1}$ or $F_{4}$ (see Figure 2, Cases 2 and 3). In any case, we can conclude that $X$ has at least $\delta^{-}+1$ vertices, which is a contradiction because $|Y|,|X| \leq \delta^{-}$in case (iii), and the proof of this case is completed.

To prove (iv), we assume that $|X|=\delta^{-}+1$ and $|Y| \leq \delta^{-}+1$. Since by hypothesis $\delta^{-} \geq 2$, reasoning as in (iii) it follows that $X=\left\{z_{1}, z_{2}, \ldots, z_{\delta^{-}}, x\right\}$ and $d^{-}(x)=\delta^{-}$. Hence, by hypothesis $x$ does not lay on a digon. Let $y \in N^{+}(x)$ with $y \in Y \backslash N^{-}(x)$. First, let us show that $y \in Y \cap X$. Suppose that $y \in Y \backslash X$. Observe that for all $u \in Y \backslash X$, it can be proved analogously that $d^{-}(u)=\delta^{-}$. Since $\delta^{-} \geq 2$, there is $z \in N^{-}(y) \backslash\{x\}$. Let us show that $z \notin X$. Otherwise, suppose $z \in X$, then $z=z_{j}$ for some $j=1, \ldots, \delta^{-}$. If $v_{j} \in N^{-}(x) \cap(Y \backslash X)$, then $D$ contains $F_{4}$ or $F_{5}$ (see Figure 2, Case 1), if $v_{j} \in N^{-}(x) \cap X$, then $D$ contains $F_{1}$ (see Figure 2, Case 2); and if $v_{j} \in N^{-}(x) \backslash(X \cup Y)$, then $D$ contains $F_{3}$ or $F_{5}$ (see Figure 2, Case 3). Therefore, $z \notin X$. Hence, $z \in N^{-}\left(z_{i}\right)$ for some $i \in\left\{1, \ldots, \delta^{-}\right\}$. If $v_{i} \in N^{-}(x) \cap(Y \backslash X)$, then $D$ contains $F_{7}$ or $F_{10}$; if $v_{i} \in N^{-}(x) \cap X$, then $D$ contains $F_{4}$; and if $v_{i} \in N^{-}(x) \backslash(X \cup Y)$, then $D$ contains $F_{6}$ or $F_{11}$, a contradiction. This implies that $y \in X \cap Y$ as we claimed. So $y=z_{i}$ for some $i=1, \ldots, \delta^{-}$. Notice that if $v_{i} \notin N^{-}(x) \cap(Y \backslash X)$, then $x$ would be contained in a digon, or $D$ contains $F_{1}$ or $F_{3}$, a contradiction. If $v_{i} \in Y \backslash X$, then reasoning for $v_{i}$ as for $x$, we obtain that every $t \in N^{+}\left(v_{i}\right) \cap X$ satisfies that $t \in X \cap Y$. However, $x \in N^{+}\left(v_{i}\right) \cap X$, but $x \notin Y$, which is a contradiction and the proof of (iv) is done.

To prove (v) we assume that $\delta^{-}=1$ and $|X|=2$. Clearly, the following claims holds if $\delta^{-} \geq 2$; moreover, since there are no vertices of in-degree 1 laying on a digon and by Remark 6, the claim follows.

Claim 9. Let $(u, v) \in A(D)$. Then, there is $w \in N^{-}(u) \backslash N^{-}[v]$.
First observe that if $|Y|=1$, say $Y=\{y\}$, then $x \in N^{-}(y)$ and by Claim 9 , there is $w \in N^{-}(x) \backslash N^{-}[y]$, implying that $N^{-}[X] \neq N^{-}[Y]$, a contradiction. Then $|Y|=|X|=2$.

Let $X=\left\{x, x^{\prime}\right\}, x \in X \backslash Y$, and $Y=\left\{y, y^{\prime}\right\}$ with $y \in N^{+}(x)$. Let us prove that the $\operatorname{arc}(x, y)$ is not on a digon. Otherwise, suppose that $(x, y),(y, x) \in$ $A(D)$. By Claim 9, there exist $w, z \in V(D)$ such that $z \in N^{-}(x) \backslash N^{-}[y]$, and $w \in N^{-}(y) \backslash N^{-}[x]$. Hence, $z \in N^{-}\left[y^{\prime}\right]$ and $w \in N^{-}\left[x^{\prime}\right]$. If $z \notin Y$, then $z \neq y^{\prime}$ and $z \in N^{-}\left(y^{\prime}\right)$. Moreover, since $D$ is free of $F_{1}, y^{\prime} \in N^{-}\left[x^{\prime}\right] \subset N^{-}[X]$. If $x^{\prime}=y^{\prime}$, then $w \neq x^{\prime}$ because $D$ is free of $F_{3} ; w \in N^{-}\left(x^{\prime}\right)$, implying that $D$ contains $F_{6}$, therefore $x^{\prime} \neq y^{\prime}$ (and so $y^{\prime} \in Y \backslash X$ ). Moreover, we can assume that $w \notin\left\{y^{\prime}, x^{\prime}\right\}$, otherwise $D$ contains $F_{4}$ or $F_{5}$. Thus, $w \in N^{-}\left(x^{\prime}\right)$ implying that $D$ contains $F_{11}$, concluding that $z \in Y$. Hence, let us assume that $Y=\{y, z\}$, and analogously $X=\{x, w\}$. By Claim 9, there is $u \in\left(N^{-}(z) \backslash N^{-}[x]\right) \cap N^{-}[w]$, because $N^{-}[Y]=N^{-}[X]$, then $D$ contains $F_{3}$ if $u=w$ or $F_{5}$ if $u \in N^{-}(w)$. Therefore, the arc $(x, y)$ is not on a digon.

Suppose that $X \cap Y \neq \emptyset$. First assume that $X=\left\{x, x^{\prime}\right\}$ and $Y=\left\{y, x^{\prime}\right\}$. Taking into account that $N^{-}[Y]=N^{-}[X]$ we have $x \in N^{-}(y) \cup N^{-}\left(x^{\prime}\right)$ and $y \in N^{-}\left(x^{\prime}\right)$ because $(x, y)$ is not on a digon. By Claim 9 there is $w \in N^{-}(x) \backslash$ $N^{-}[y]$ and $w \in N^{-}\left[x^{\prime}\right]$ (because $N^{-}[X]=N^{-}[Y]$ ). If $w=x^{\prime}$, then $\left(x y x^{\prime} x\right)$ is a 3 -cycle in $D$, and by hypothesis there is $u \in N^{-}(x) \backslash\left\{x^{\prime}\right\}$. By Remark 6, $u \notin N^{-}(y) \cup N^{-}\left(x^{\prime}\right)$, a contradiction. Then, $w \neq x^{\prime}$, implying that $D$ contains $F_{4}$. Secondly, assume that $X=\{x, y\}$ and $Y=\left\{y, y^{\prime}\right\}$. By Claim 9 there is $w \in N^{-}(x) \backslash N^{-}[y]$ and $w \in N^{-}\left[y^{\prime}\right]$. If $w=y^{\prime}$, there is $w^{\prime} \in N^{-}\left(y^{\prime}\right) \backslash N^{-}[x]$ by Claim 9, and $D$ would contain a $F_{4}$. Thus $w \neq y^{\prime}$ and $w \in N^{-}\left(y^{\prime}\right)$, and since $y^{\prime} \in N^{-}(x) \cup N^{-}(y), D$ would contain a $F_{1}$ or $F_{3}$, a contradiction.

Suppose that $X \cap Y=\emptyset$. Let $X=\left\{x, x^{\prime}\right\}$ and $Y=\left\{y, y^{\prime}\right\}$. Then, $y \in N^{-}\left(x^{\prime}\right)$, and since $y \in Y \backslash X$, reasoning for $y$ as for $x$, the arc $\left(y, x^{\prime}\right)$ is like the arc $(x, y)$ and so it is not lying on a digon. Then $x^{\prime} \in N^{-}\left(y^{\prime}\right)$ and similarly, $y^{\prime} \in N^{-}(x)$. By hypothesis there are no vertices of in-degree 1 lying on a 4 -cycle, it follows that there is $z \in N^{-}(x) \backslash\left\{y^{\prime}\right\}$, but by Remark $6, N^{-}(x) \cap\left(N^{-}(y) \cup N^{-}\left(y^{\prime}\right)\right)=\emptyset$ implying that $N^{-}[X] \neq N^{-}[Y]$, a contradiction.

If for each graph $G$, we consider its corresponding symmetric digraph obtained by replacing each edge $u v$ of $G$ by the $\operatorname{arcs}(u, v)$ and $(v, u)$, then we obtain the following corollary from Theorem 8.

Corollary 10. Let $G$ be a graph of girth $g$ and minimum degree $\delta \geq 2$. Then

1. If $g \geq 7$, then $G$ admits $a(1, \leq \delta)$-identifying code.
2. If $g \geq 5$, then $G$ admits $a(1, \leq \delta-1)$-identifying code.

Observe that Theorem 1 by Laihonen is a consequence of Corollary 10 .

## 3. 1-In-Regular and 2-In-Regular Digraphs

In this section, we characterize the $d$-in-regular digraphs admitting a $(1, \leq d)$ identifying code and a $(1, \leq d+1)$-identifying code for $d \in\{1,2\}$. Recall that by Proposition 3, if $D$ is a $d$-in-regular digraph admiting a $(1, \leq \ell)$-identifying code, then $\ell \leq d+1$. We start by giving a characterization of 1 -in-regular digraphs admitting a $(1, \leq 2)$-identifying code. Observe that every 1 -in-regular digraph $D$ admits a ( $1, \leq 1$ )-identifying code if and only if $D$ does not contain digons.

Theorem 11. Every 1-in-regular digraph $D$ admits a $(1, \leq 2)$-identifying code if and only if the girth of $D$ is at least 5 .

Proof. Let $\left(u_{1} u_{2} u_{3} u_{1}\right)$ be a directed triangle or $\left(v_{1} v_{2} v_{3} v_{4} v_{1}\right)$ a 4-cycle in $D$. Then the sets $X_{1}=\left\{u_{1}, u_{3}\right\}, Y_{1}=\left\{u_{2}, u_{3}\right\}, X_{2}=\left\{v_{1}, v_{3}\right\}$ and $Y_{2}=\left\{v_{2}, v_{4}\right\}$ are such that $N^{-}\left[X_{i}\right]=N^{-}\left[Y_{i}\right]$, for $i=1,2$. Therefore, if $D$ contains a $k$-cycle, for some $k=2$, 3 or 4 , then $D$ does not admit a ( $1, \leq 2$ )-identifying code. Conversely, suppose that the girth of $D$ is at least 5 . Since $D$ is 1-in-regular, it follows that $D$ does not contain any subdigraph isomorphic to $F_{1}, F_{3}, F_{4}, F_{5}, F_{6}$ nor $F_{11}$ of Figure 1 , then by Theorem $8, D$ admits a ( $1, \leq 2$ )-identifying code. This completes the proof.

The following result gives a complete characterization of all 2-in-regular digraphs admitting a $(1, \leq 1)$-identifying code and a characterization of all 2 -inregular digraphs admitting a $(1, \leq 2)$-identifying code.

Theorem 12. Let $D$ be a 2-in-regular digraph.
(i) $D$ admits a $(1, \leq 1)$-identifying code if and only if it does not contain any subdigraph isomorphic to $H_{1}$ of Figure 3.
(ii) $D$ admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to one of the digraphs of Figure 3.

Proof. In what follows, for brevity, we made reference to the different cases $H_{1-}$ $H_{13}$ of Figure 3 without mentioning the figure. First note that any digraph with twins and minimum in-degree at least 2 , necessarily contains $H_{1}$. Hence, the proof of (i) follows by Remark 7, because the vertices $x, y$ of $H_{1}$ are twins. To prove (ii), first let $X=\left\{x, x^{\prime}\right\}$ (or $X=\{x\}$ ) and $Y=\left\{y, y^{\prime}\right\}$. It is straightforward to check that $N^{-}[X]=N^{-}[Y]$ in each one of the digraphs shown in Figure 3. For the converse, we assume that $D$ does not contain any subdigraph isomorphic to the digraphs depicted in Figure 3, and $N^{-}[X]=N^{-}[Y]$ for $X \neq Y$ such that $1 \leq|Y| \leq|X| \leq 2$. According to (i), we can assume that $|X|=2$, consequently $3 \leq\left|N^{-}[X]\right| \leq 6$. Notice that if $|Y|=1$, then $\left|N^{-}[Y]\right|=3$, and so $\left|N^{-}[X]\right|=3$ yielding that $D$ contains $H_{1}$. Therefore, we assume that $|Y|=|X|=2$. Let $X=\left\{x, x^{\prime}\right\}$ and $Y=\left\{y, y^{\prime}\right\}$ with $x \in X \backslash Y$. Let $N^{-}(x)=\left\{v_{1}, v_{2}\right\}$ and $y \in Y$
such that $y \in N^{+}(x)$. As we did in the proof of Theorem 8, we consider the different cases according to the partition of $N^{-}(x): N^{-}(x) \cap(Y \backslash X), N^{-}(x) \cap X$ and $N^{-}(x) \cap(V \backslash(X \cup Y))$.

$H_{1}$

$H_{2}$

$H_{3}$

$H_{4}$

$H_{5}$


$H_{8}$

$H_{9}$

$H_{10}$


Figure 3. The forbidden subdigraphs in a 2 -in-regular digraph admitting a $(1, \leq 2)$ identifying code.

Case 1. $v_{1}, v_{2} \in Y \backslash X$. Let $y=v_{1}$ and $y^{\prime}=v_{2}$ and observe that in this case $x^{\prime} \notin Y$. As $D$ is $H_{1}$-free and $H_{3}$-free, $\left(N^{-}(y) \backslash\{x\}\right) \cap N^{-}\left[y^{\prime}\right]=\emptyset$ and there is no arc between $y^{\prime}$ and $N^{-}(y) \backslash\{x\}$. Let $w \in N^{-}(y) \backslash\{x\}$ and $w^{\prime} \in N^{-}\left(y^{\prime}\right) \backslash\{x\}$. Then $w, w^{\prime} \in N^{-}\left[x^{\prime}\right]$.

Subcase 1.1. Suppose that $\left\{w, w^{\prime}\right\} \cap\left\{x^{\prime}\right\}=\emptyset$. Hence, $N^{-}\left(x^{\prime}\right)=\left\{w, w^{\prime}\right\}$. Since $x^{\prime} \in N^{-}[Y]$, it follows that $x^{\prime} \in N^{-}\left(y^{\prime}\right)$ implying that $D$ contains $H_{13}$, a contradiction.

Subcase 1.2. Suppose that $x^{\prime}=w$. Hence, $w^{\prime} \in N^{-}\left(x^{\prime}\right)$. If there is $z \in$ $N^{-}\left(x^{\prime}\right) \backslash\left(X \cup Y \cup\left\{w^{\prime}\right\}\right)$, then $z \in N^{-}\left(y^{\prime}\right)$, implying that $D$ contains $H_{10}$, a contradiction. Therefore, $N^{-}[X]=X \cup Y \cup\left\{w^{\prime}\right\}$, implying that $N^{-}\left(x^{\prime}\right)=\left\{w^{\prime}, x\right\}$
or $N^{-}\left(x^{\prime}\right)=\left\{w^{\prime}, y\right\}$. First suppose that $N^{-}\left(x^{\prime}\right)=\left\{w^{\prime}, x\right\}$. If $x \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{5}$ and if $y \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{4}$, a contradiction. Therefore, $N^{-}\left(x^{\prime}\right)=\left\{w^{\prime}, y\right\}$. If $x \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{6}$ and if $y \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{5}$, a contradiction.

Subcase 1.3. Suppose that $x^{\prime}=w^{\prime}$. Hence, $w \in N^{-}\left(x^{\prime}\right)$. If there is $z \in$ $N^{-}\left(x^{\prime}\right) \backslash\left(X \cup Y \cup\left\{w^{\prime}\right\}\right)$, then $z \in N^{-}\left(y^{\prime}\right)$, implying that $D$ contains $H_{13}$, a contradiction. Therefore, $N^{-}[X]=X \cup Y \cup\left\{w^{\prime}\right\}$. Hence, $N^{-}\left(x^{\prime}\right)=\{w, x\}$ or $N^{-}\left(x^{\prime}\right)=\{w, y\}$. First suppose that $N^{-}\left(x^{\prime}\right)=\{w, x\}$. If $x \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{4}$ and if $y \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{9}$, a contradiction. Therefore, $N^{-}\left(x^{\prime}\right)=\{w, y\}$. Hence, if $x \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{4}$ and if $y \in N^{-}\left(y^{\prime}\right)$, then $D$ contains $H_{7}$, a contradiction.

Case 2. $v_{1}, v_{2} \in X$. Since $|X|=2$ this case is not possible.
Case 3. $v_{1}, v_{2} \notin(X \cup Y)$. Since $x \in N^{-}(y)$, we have $\left|N^{-}(y) \cap\left\{v_{1}, v_{2}\right\}\right| \leq 1$ implying that $\left\{v_{1}, v_{2}\right\} \cap N^{-}\left(y^{\prime}\right) \neq \emptyset$. Without loss of generality, suppose that $v_{1} \in N^{-}\left(y^{\prime}\right)$.

Subcase 3.1. If $y \in Y \backslash X$, then $y \in N^{-}\left(x^{\prime}\right)$. If $y^{\prime} \in X \cap Y$, i.e., $y^{\prime}=x^{\prime}$, then $v_{2} \in N^{-}(y)$, implying that $D$ contains $H_{4}$. If $y^{\prime} \in Y \backslash X$, then $N^{-}\left(x^{\prime}\right)=$ $\left\{y, y^{\prime}\right\}$ and $x^{\prime} \in N^{-}(y) \cup N^{-}\left(y^{\prime}\right)$. If $x^{\prime} \in N^{-}(y)$, then $v_{2} \in N^{-}\left(y^{\prime}\right)$, implying that $D$ contains $H_{10}$. And, if $x^{\prime} \in N^{-}\left(y^{\prime}\right)$, then $v_{2} \in N^{-}(y)$, implying that $D$ contains $H_{13}$.

Subcase 3.2. If $y \in X \cap Y$ i.e. $x^{\prime}=y$, then $y^{\prime} \in N^{-}(y)$ and $v_{1}, v_{2} \in N^{-}\left(y^{\prime}\right)$, hence $D$ contains $H_{9}$, a contradiction.

Case 4. $v_{1} \in Y \backslash X$ and $v_{2} \in X$, that is, $v_{2}=x^{\prime}$. Observe that if $v_{1} \in N^{+}(x)$, since $D$ is $H_{1}$-free, there is $w \in V(D) \backslash X$ such that $w \in N^{-}\left(v_{1}\right) \subset N^{-}[Y]$. Thus, $w \in N^{-}\left(x^{\prime}\right)$, implying that $D$ contains $H_{3}$, a contradiction. Then $v_{1} \notin N^{+}(x)$ and so $v_{1}=y^{\prime}$, and moreover $y \in N^{-}\left(x^{\prime}\right)$. If $x^{\prime} \in N^{+}(x)$, then $N^{-}[X]=$ $\left\{x, x^{\prime}, y, y^{\prime}\right\}$, yielding that $y \in N^{-}\left(y^{\prime}\right)$, contradicting that $D$ is $H_{3}$-free. Therefore, $N^{+}(x) \cap\left\{y^{\prime}, x^{\prime}\right\}=\emptyset$ and recall that $y \in N^{-}\left(x^{\prime}\right)$. Moreover, reasoning for $y$ as for $x$ in Case 1, we get that $x^{\prime} \notin N^{-}(y)$. Moreover, if $y^{\prime} \in N^{-}(y)$, then $D$ contains $H_{2}$, a contradiction. Therefore, there is $w \in N^{-}(y) \backslash(X \cup Y)$. Hence, $w \in N^{-}\left(x^{\prime}\right)$, implying that $D$ contains $H_{2}$, a contradiction.

Case 5. $v_{1} \in Y \backslash X$ and $v_{2} \notin(X \cup Y)$.
Subcase 5.1. If $v_{1} \in N^{+}(x)$, then we can assume that $v_{1}=y$. Since $D$ is $H_{1}$-free, $v_{2} \in N^{-}\left(y^{\prime}\right)$ and there is $w \in V(D) \backslash\left\{x, v_{2}\right\}$ such that $N^{-}(y)=\{x, w\}$. Observe that since $D$ is $H_{3}$-free, $v_{2} \notin N^{-}(w)$, we have $w \neq y^{\prime}$. Moreover, since $D$ is $H_{6}$-free, $w \notin N^{-}\left(y^{\prime}\right)$. Hence, $w \in N^{-}\left[x^{\prime}\right]$, implying that $x^{\prime} \neq y^{\prime}$. Observe that reasoning for $y$ as for $x$ in Case 1, we get that $w \neq x^{\prime}$. Then, $w \in N^{-}\left(x^{\prime}\right)$
and, since $x^{\prime}, y^{\prime} \in N^{-}[X]=N^{-}[Y]$, it follows that $x^{\prime} \in N^{-}\left(y^{\prime}\right)$ and $y^{\prime} \in N^{-}\left(x^{\prime}\right)$, therefore $D$ contains $H_{11}$, a contradiction.

Subcase 5.2. If $v_{1} \notin N^{+}(x)$, then $v_{1}=y^{\prime}$ and $y \in N^{-}\left[x^{\prime}\right]$. First suppose that $y=x^{\prime}$. If $N^{-}\left(y^{\prime}\right) \subseteq X \cup\left\{v_{2}\right\}$, then $N^{-}\left(y^{\prime}\right)=\left\{x^{\prime}, v_{2}\right\}$ implying that $D$ contains $H_{2}$. Hence, there is $w \in N^{-}\left(y^{\prime}\right) \backslash\left(X \cup\left\{v_{2}\right\}\right)$. Then $w \in N^{-}\left(x^{\prime}\right)$ and $v_{2} \in N^{-}\left(y^{\prime}\right)$, implying that $D$ contains $H_{4}$, a contradiction. Therefore, $y \neq x^{\prime}$, implying that $y \in N^{-}\left(x^{\prime}\right)$. Reasoning for $y$ as for $x$ in Case 1 and Case 4, it follows that $N^{-}(y) \cap\left\{x^{\prime}, y^{\prime}\right\}=\emptyset$. Then $x^{\prime} \in N^{-}\left(y^{\prime}\right)$. Moreover, since $v_{2} \in N^{-}(x)$, $v_{2} \in N^{-}(y) \cup N^{-}\left(y^{\prime}\right)$. Also, reasoning for $x^{\prime}$ as for $x$ in Case 1 and Case 4 it follows that $N^{-}\left(x^{\prime}\right) \cap\left\{x, y^{\prime}\right\}=\emptyset$. Hence, if $v_{2} \in N^{-}(y) \cap N^{-}\left(y^{\prime}\right)$, then $N^{-}[Y]=$ $X \cup Y \cup\left\{v_{2}\right\}$, implying that $v_{2} \in N^{-}\left(x^{\prime}\right)$. Then $D$ contains $H_{8}$, a contradiction. If $v_{2} \in N^{-}\left(y^{\prime}\right) \backslash N^{-}(y)$, then there is $z \in N^{-}(y) \backslash\left(X \cup Y \cup\left\{v_{2}\right\}\right)$, implying that $N^{-}\left(x^{\prime}\right)=\{y, z\}$ and $D$ contains $H_{12}$. Analogously if $v_{2} \in N^{-}(y) \backslash N^{-}\left(y^{\prime}\right)$. And the proof of this case is completed.

Case 6. $v_{1} \in X$ and $v_{2} \notin(X \cup Y)$. That is, $v_{1}=x^{\prime}$. If $x^{\prime} \in X \backslash Y$, then $y \in N^{-}\left(x^{\prime}\right)$. Since $y \in Y \backslash X$, reasoning for $x^{\prime}$ as for $x$ in Cases 1,4 and 5 , we reach a contradiction. Hence, $x^{\prime} \in X \cap Y$. If $x^{\prime}=y$, then $y^{\prime} \in N^{-}\left(x^{\prime}\right)$ and $v_{2} \in N^{-}\left(y^{\prime}\right)$, implying that $D$ contains $H_{3}$. Therefore, $x^{\prime} \neq y$ and hence, $y \in Y \backslash X$. Since $x \in N^{-}(y)$, reasoning for $y$ as for $x$ in Cases 1, 4 and 5, we reach a contradiction.

Corollary 13. Every $T T_{3}$-free 2 -in-regular oriented graph admits a ( $1, \leq 2$ )identifying code if and only if it does not contain any subdigraph isomorphic to $H_{9}$ of Figure 3.

Observe that Corollary 13 is an improvement of Theorem 8(ii) for 2-in-regular oriented digraphs. Now, a $T T_{3}$-free and 2 -in-regular oriented graph can have two distinct vertices $u, v$ with $\left|N^{-}(u) \cap N^{-}(v)\right|=2$, that is, a subdigraph $F_{2}$ of Figure 1 , but in this case there is no vertex $w \in V$ such that $u, v \in N^{-}(w)$.

In the following theorem we characterize the 2 -in-regular digraphs admitting a ( $1, \leq 3$ )-identifying code.

Theorem 14. Let $D$ be a 2 -in-regular digraph. Then $D$ has a $(1, \leq 3)$-identifying code if and only if it is a $T T_{3}$-free oriented graph, and does not contain any subdigraph isomorphic to one of the digraphs of Figure 4.

Proof. By Proposition 3, if $D$ contains a digon, then $D$ does not admit a $(1, \leq 3)$ identifying code. Suppose that $D$ contains a $T T_{3}$, let say $w \in N^{-}(u) \cap N^{-}(v)$ and $(u, v) \in A(D)$, and let $z \in V(D)$ be such that $N^{-}(u)=\{w, z\}$. Hence, the sets $X=\{z, u, v\}$ and $Y=\{z, v\}$ have the same closed in-neiborhood. Furthermore, for every digraph shown in Figure 4 let $X=\left\{x_{1}, x_{2}, x_{3}\right\}$ (or $X=\left\{x_{1}, x_{2}\right\}$ )


$J_{6}$

$J_{11}$

$J_{7}$

$J_{8}$






Figure 4. All the forbidden subdigraphs of Theorem 14.
and $Y=\left\{y_{1}, y_{2}, y_{3}\right\}$. It is straightforward to check that $N^{-}[X]=N^{-}[Y]$ in each case. To the converse, we reason by a contradiction. Let $D$ be a $T T_{3}$-free oriented graph without the subdigraphs of Figure 4. Let $X, Y \subseteq V(D), X \neq Y$, with $N^{-}[X]=N^{-}[Y]$ and such that $1 \leq|X| \leq|Y| \leq 3$. Since $D$ does not contain a subdigraph isomorphic to $J_{1}$ of Figure 4, then it does not contain a subdigraph $H_{9}$ of Figure 3. By Corollary $13, D$ admits a ( $1, \leq 2$ )-identifying code. Hence, $|Y|=3,\left|N^{-}[Y]\right| \geq 6$ and $|X| \geq 2$. In what follows, for brevity, we always make reference to the different cases $J_{1}-J_{15}$ of Figure 4 without mentioning the figure. Let us prove the following claim.

Claim 15. Let $a, b \in V(D), a \neq b$, be such that $N^{-}(a) \subseteq N^{-}[b]$. Then, $N^{-}(a)=$ $N^{-}(b)$ and $N^{+}(a)=N^{+}(b)=\emptyset$.

Proof. If $b \in N^{-}(a)$, then $D$ contains a $T T_{3}$, which is a contradiction. Hence, $N^{-}(a)=N^{-}(b)$ and $N^{+}(a)=N^{+}(b)=\emptyset$, because otherwise $D$ contains $J_{1}$.

Suppose $X=\left\{x_{1}, x_{2}\right\}$. Then $\left|N^{-}[X]\right|=6$ (because $N^{-}[X]=N^{-}[Y]$ ) and $N^{-}\left[x_{1}\right] \cap N^{-}\left[x_{2}\right]=\emptyset$. Let $N^{-}\left(x_{1}\right)=\{u, v\}$ and $N^{-}\left(x_{2}\right)=\{z, t\}$, so that $N^{-}[X]=\left\{x_{1}, x_{2}, u, v, z, t\right\}=N^{-}[Y]$. Without loss of generality, we may assume that $u \in Y$. Since $D$ has neither digon nor $T T_{3}, N^{-}(u) \subseteq N^{-}\left[x_{2}\right]$, which implies by Claim 15 that $N^{-}(u)=N^{-}\left(x_{2}\right)$ and $N^{+}(u)=\emptyset$, a contradiction. Therefore, $|X|=|Y|=3$. Let us denote $X=\left\{x_{1}, x_{2}, x_{3}\right\}$. We prove the following claims.

Claim 16. Let $a, b, c \in V(D)$. If $N^{-}[a] \subseteq N^{-}[b] \cup N^{-}[c]$, then $a \in\{b, c\}$.
Proof. If $a \notin\{b, c\}$, then without loss of generality, let us assume that $a \in$ $N^{-}(b)$. Hence, by Remark $6, N^{-}(a) \subseteq N^{-}[c]$, which contradicts Claim 15 because $N^{+}(a) \neq \emptyset$.

Claim 17. $N^{-}\left(x_{i}\right) \neq N^{-}\left(x_{j}\right)$ for all $1 \leq i<j \leq 3$.
Proof. Suppose that $N^{-}\left(x_{1}\right)=N^{-}\left(x_{2}\right)$. Then $N^{+}\left(x_{1}\right)=N^{+}\left(x_{2}\right)=\emptyset$, because $D$ is $J_{1}$-free, which implies $x_{1}, x_{2} \in Y$. Since $\left|N^{-}[X]\right| \geq 6$, there is $z \in N^{-}\left(x_{3}\right) \backslash$ $\left(N^{-}\left[x_{1}\right] \cup N^{-}\left[x_{2}\right]\right)$. Because $\left\{x_{3}, z\right\} \subseteq N^{-}[Y], D$ must contain a digon if $z=$ $y_{3} \in Y$, or a $T T_{3}$ if $\left\{x_{3}, z\right\}=N^{-}\left(y_{3}\right)$, which is a contradiction. Therefore, $N^{-}\left(x_{1}\right) \neq N^{-}\left(x_{2}\right)$.

Claim 18. If $7 \leq\left|N^{-}[X]\right| \leq 8, N^{-}\left(x_{i}\right) \cap N^{-}\left(x_{j}\right)=\{v\}, i \neq j$, and there are exactly two or no arc between the elements of $X$, then $\left|Y \cap\left\{x_{i}, x_{j}\right\}\right| \leq 1$.

Proof. We proceed by contradiction. Assume $Y=\left\{x_{1}, x_{2}, y\right\}$. First suppose that there is no arc between the elements of $X$. If $v \in N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{3}\right)$, then according to Claim $17,\left|N^{-}[X]\right|=7$ and $N^{-}\left[x_{3}\right] \subseteq N^{-}\left[x_{1}\right] \cup N^{-}[y]$, which contradicts Claim 16. Hence, $N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{3}\right)=\emptyset$. If $\left|N^{-}[X]\right|=7$, let $N^{-}\left(x_{1}\right)=\{u, v\}, N^{-}\left(x_{2}\right)=\{v, z\}$ and $N^{-}\left(x_{3}\right)=\{z, w\}$. Since $N^{-}(v) \cap$ $N^{-}[X] \subseteq\left\{x_{3}, w\right\}$, by Remark $6, v \notin Y$, and analogously $z \notin Y$. Consequently, $N^{-}\left[x_{3}\right] \subseteq N^{-}\left[x_{2}\right] \cup N^{-}[y]$, which contradicts Claim 16. If $\left|N^{-}[X]\right|=8$, then $N^{-}\left(x_{3}\right) \subseteq N^{-}[y]$, a contradiction to Claim 15 because $y \notin X$ and so $N^{+}(y) \neq \emptyset$. Finally assume that there are two arcs between the elements of $X$. Notice that by Remark 6, both arcs between the elements of $X$ are incident in $x_{3}$. Furthermore, since $7 \leq\left|N^{-}[X]\right| \leq 8$ and $N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right)=\{v\}, v=x_{3}$ and $\left|N^{-}[X]\right|=7$, we have $N^{-}\left(x_{3}\right) \subseteq N^{-}[y]$, a contradiction to Claim 15 .

Let $N^{-}\left(x_{1}\right)=\{u, v\}$. We distinguish the following cases according to the number of arcs between the vertices of $X$.

Case 1. There are no arcs between the elements of $X$.
Subcase 1.1. Suppose $\left|N^{-}[X]\right|=6$. Then, $N^{-}[X]=\left\{x_{1}, x_{2}, x_{3}, u, v, z\right\}$, so Claim 17 implies that $\left|N^{-}\left(x_{i}\right) \cap N^{-}\left(x_{j}\right)\right|=1$ for all $i \neq j$. Let $N^{-}\left(x_{2}\right)=\{v, z\}$. Observe that $v \notin N^{-}\left(x_{3}\right)$, otherwise $N^{-}\left(x_{3}\right)=N^{-}\left(x_{i}\right)$ for some $i \in\{1,2\}$,
contradicting Claim 17. Therefore $N^{-}\left(x_{3}\right)=\{u, z\}$. Let $y \in Y \backslash X$. Then $y \in\{u, v, z\}$. We can check that $\left|N^{-}(y) \cap N^{-}[X]\right| \leq 1$ for all $y \in\{u, v, z\}$, because $D$ is a $T T_{3}$-free oriented graph, which is a contradiction.

Subcase 1.2. Suppose $\left|N^{-}[X]\right|=7$. Then $N^{-}[X]=\left\{x_{1}, x_{2}, x_{3}, u, v, z, w\right\}$. By Claim 17, there are two cases to be considered, namely, $\mid N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right) \cap$ $N^{-}\left(x_{3}\right) \mid=1$ and $\left|N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{3}\right)\right|=0$.

Subsubcase 1.2.1. If $\left|N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{3}\right)\right|=1$, without loss of generality, $N^{-}\left(x_{2}\right)=\{v, z\}$ and $N^{-}\left(x_{3}\right)=\{v, w\}$. Since $D$ is an oriented graph and does not contain $T T_{3}, N^{-}(v) \cap N^{-}[X]=\emptyset$, which means that $v \notin Y$ and $v \in N^{-}(Y)$. Since $N^{+}(v) \cap\{u, z, w\}=\emptyset$, it follows that $Y \cap X \neq \emptyset$. By Claim 18, $|X \cap Y|=1$. Without loss of generality, suppose that $X \cap Y=\left\{x_{1}\right\}$. If $Y=\left\{x_{1}, z, w\right\}$, then $x_{2} \in N^{-}(w)$ and $x_{3} \in N^{-}(z)$, implying that $D$ contains $J_{4}$. If $Y=\left\{x_{1}, u, z\right\}$, then $x_{2} \in N^{-}(u)$ and $N^{-}(u) \subseteq\left\{x_{2}, x_{3}, w\right\}$. If $N^{-}(u)=\left\{x_{2}, x_{3}\right\}$, then $w \in N^{-}(z)$ and hence $D$ contains $J_{6}$. If $N^{-}(u)=\left\{x_{2}, w\right\}$, then $x_{3} \in N^{-}(z)$ and $D$ contains $J_{5}$.

Subsubcase 1.2.2. If $\left|N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{3}\right)\right|=0$, without loss of generality, $N^{-}\left(x_{2}\right)=\{v, z\}$ and $N^{-}\left(x_{3}\right)=\{z, w\}$. By Claim 18, $\left|Y \cap\left\{x_{1}, x_{2}\right\}\right| \leq 1$ and $\left|Y \cap\left\{x_{2}, x_{3}\right\}\right| \leq 1$. Moreover, if $\left\{x_{1}, x_{3}\right\} \subseteq Y$, then since $x_{2} \in N^{-}[Y]$, we have $\{u, w\} \cap Y \neq \emptyset$; without loss of generality, let us assume that $Y=$ $\left\{x_{1}, x_{3}, u\right\}$. Then $x_{2} \in N^{-}(u)$ and $N^{-}(u) \subseteq\left\{x_{2}, x_{3}, w\right\}$. If $N^{-}(u)=\left\{x_{2}, x_{3}\right\}$, then $D$ contains $J_{8}$, and if $N^{-}(u)=\left\{x_{2}, w\right\}$, then $D$ contains $J_{10}$. Therefore, $|Y \cap X| \leq 1$. First, assume that $|Y \cap X|=1$. Suppose that $X \cap Y=\left\{x_{1}\right\}$ and let $Y=\left\{x_{1}, y, y^{\prime}\right\}$. Then $N^{-}\left[x_{3}\right] \subseteq N^{-}[y] \cup N^{-}\left[y^{\prime}\right]$, which contradicts Claim 16 . Hence, $X \cap Y \neq\left\{x_{1}\right\}$, and similarly $X \cap Y \neq\left\{x_{3}\right\}$. Then $X \cap Y=\left\{x_{2}\right\}$. If $v \in Y$, then there is $y \in Y \backslash\left\{x_{2}, v\right\}$ such that $N^{-}(y)=\left\{x_{1}, u\right\}$ contradicting Remark 6. Hence, $v \notin Y$, and analogously $z \notin Y$. Therefore $Y=\left\{x_{2}, u, w\right\}$, and then $x_{1} \in N^{-}(w)$, implying that $N^{-}(u)=\left\{x_{3}, x_{2}\right\}$. Consequently, $D$ contains $J_{8}$. Finally if $|Y \cap X|=0$, by symmetry, we only have to consider the following two cases. If $Y=\{u, v, z\}$, then $x_{2} \in N^{-}(u)$ and $x_{1} \in N^{-}(z)$, implying that $D$ contains $J_{4}$. If $Y=\{u, z, w\}$, then $x_{3} \in N^{-}(u)$ implying that $N^{-}(u)=\left\{x_{2}, x_{3}\right\}$, and $D$ contains $J_{8}$.

Subcase 1.3. Suppose $\left|N^{-}[X]\right|=8$. Without loss of generality, $N^{-}\left(x_{2}\right)=$ $\{v, z\}$, and $N^{-}\left(x_{3}\right)=\{t, w\}$. Observe that $v \notin Y$, otherwise $N^{-}(v) \subseteq N^{-}\left[x_{3}\right]$ in a contradiction to Claim 15. If $Y \cap X=\emptyset$, then we can assume that $t \in Y$ and $v \in N^{-}(t)$. Consequently, $\{u, z\} \cap N^{-}(t)=\emptyset$, otherwise $D$ contains $J_{1}$, therefore $\left\{x_{3}, w\right\} \cap N^{-}(t) \neq \emptyset$, a contradiction. Therefore $Y \cap X \neq \emptyset$. If $|Y \cap X|=2$, then by Claim 18, $\left\{x_{1}, x_{3}\right\} \subseteq Y$ or $\left\{x_{2}, x_{3}\right\} \subseteq Y$. If $Y=\left\{y, x_{2}, x_{3}\right\}$, then $N^{-}\left[x_{1}\right] \subseteq N^{-}\left[x_{2}\right] \cup N^{-}[y]$, contradicting Claim 16. Then $Y \neq\left\{y, x_{2}, x_{3}\right\}$, and similarly $Y \neq\left\{y, x_{1}, x_{3}\right\}$. Thus $|Y \cap X|=1$. If $Y=\left\{x_{1}, y, y^{\prime}\right\}$ or $Y=\left\{x_{3}, y, y^{\prime}\right\}$,
then $N^{-}\left[x_{3}\right] \subseteq N^{-}[y] \cup N^{-}\left[y^{\prime}\right]$ or $N^{-}\left[x_{1}\right] \subseteq N^{-}[y] \cup N^{-}\left[y^{\prime}\right]$, respectively, which contradicts Claim 16.

Subcase 1.4. Suppose $\left|N^{-}[X]\right|=9$. Hence, the in-neighborhoods of the elements of $X$ must be disjoint, the same is true for $Y$. Let $N^{-}\left(x_{i}\right)=\left\{u_{i}, v_{i}\right\}$, for $i=1,2,3$. Observe that if $1 \leq|X \cap Y| \leq 2$, then $N^{-}\left[x_{i}\right] \subseteq N^{-}[y] \cup N^{-}\left[y^{\prime}\right]$ for some $i \in\{1,2,3\}$ and $y, y^{\prime} \in Y \backslash\left\{x_{i}\right\}$, in a contradiction to Claim 16. Therefore, $X \cap Y=\emptyset$. Without loss of generality, there are two cases to be considered.

Subsubcase 1.4.1. If $Y=\left\{u_{1}, v_{1}, u_{2}\right\}$, then $x_{1} \in N^{-}\left(u_{2}\right)$. If $x_{3} \in N^{-}\left(u_{1}\right)$, then without loss of generality, $u_{3} \in N^{-}\left(v_{1}\right)$ and $v_{3} \in N^{-}\left(u_{2}\right)$; moreover, $x_{2} \in$ $N^{-}\left(v_{1}\right)$ and $v_{2} \in N^{-}\left(u_{1}\right)$ or $x_{2} \in N^{-}\left(u_{1}\right)$ and $v_{2} \in N^{-}\left(v_{1}\right)$, implying that $D$ contains $J_{14}$ or $J_{15}$, respectively. If $x_{3} \in N^{-}\left(u_{2}\right)$, then we may assume that $u_{3} \in N^{-}\left(u_{1}\right)$ and $v_{3} \in N^{-}\left(v_{1}\right)$, and so $x_{2} \in N^{-}\left(u_{1}\right)$ and $v_{2} \in N^{-}\left(v_{1}\right)$, implying that $D$ contains $J_{15}$.

Subsubcase 1.4.2. Let $Y=\left\{u_{1}, u_{2}, u_{3}\right\}$. Without loss of generality, suppose $x_{2} \in N^{-}\left(u_{1}\right)$. Then by Remark $6, N^{-}\left(u_{1}\right) \backslash\left\{x_{2}\right\} \subseteq N^{-}\left[x_{3}\right]$. Since there is no arc between the elements of $Y$, there are two possibilities to be considered.

- If $N^{-}\left(u_{1}\right)=\left\{x_{2}, x_{3}\right\}$, then $v_{3} \in N^{-}\left(u_{2}\right)$ and $v_{2} \in N^{-}\left(u_{3}\right)$. Hence, $x_{1} \in$ $N^{-}\left(u_{2}\right)$ and $v_{1} \in N^{-}\left(u_{3}\right)$, or $v_{1} \in N^{-}\left(u_{2}\right)$ and $x_{1} \in N^{-}\left(u_{3}\right)$; in any case $D$ contains $J_{14}$.
- If $N^{-}\left(u_{1}\right)=\left\{x_{2}, v_{3}\right\}$, then $x_{3} \in N^{-}\left(u_{2}\right)$, and $v_{2} \in N^{-}\left(u_{3}\right)$. If $x_{1} \in N^{-}\left(u_{2}\right)$, then $v_{1} \in N^{-}\left(u_{3}\right)$, implying that $D$ contains $J_{14}$. Finally, if $x_{1} \in N^{-}\left(u_{3}\right)$, then $v_{1} \in N^{-}\left(u_{2}\right)$, implying that $D$ contains $J_{13}$.

Case 2. There is just one arc between the elements of $X$, say $\left(x_{1}, x_{2}\right) \in$ $A(D)$. Then $\left|N^{-}(X)\right| \in\{6,7,8\}$, and $N^{-}\left(x_{1}\right) \cap N^{-}\left(x_{2}\right)=\emptyset$ by Remark 6 . Let $N^{-}\left(x_{2}\right)=\left\{x_{1}, z\right\}$ and let us distinguish the following cases.

Subcase 2.1. $\left|N^{-}[X]\right|=6$. Hence, $N^{-}[X]=\left\{x_{1}, x_{2}, x_{3}, u, v, z\right\}$, and by Claim 17 let us assume without loss of generality, that $N^{-}\left(x_{3}\right)=\{v, z\}$. Moreover, since $D$ is an oriented graph and does not contain $J_{1}, N^{-}(z) \cap N^{-}[X] \subseteq\{u\}$ and $N^{-}(v) \cap N^{-}[X] \subseteq\left\{x_{2}\right\}$, therefore $z, v \notin Y$; hence $u \in Y$. Since $D$ is a $T T_{3}$-free oriented graph, $N^{-}(u) \subseteq\left\{x_{2}, x_{3}, z\right\}$. Moreover, by Remark $6, z \notin N^{-}(u)$. Hence, $N^{-}(u)=\left\{x_{2}, x_{3}\right\}$, implying that $D$ contains $J_{2}$.

Subcase 2.2. $\left|N^{-}[X]\right|=7$. Then there is $w \in N^{-}\left(x_{3}\right) \backslash(X \cup\{u, v, z\})$. By symmetry, $N^{-}\left(x_{3}\right)=\{z, w\}$ or $N^{-}\left(x_{3}\right)=\{v, w\}$. First suppose that $N^{-}\left(x_{3}\right)=$ $\{z, w\}$. Since $D$ is a $T T_{3}$-free oriented graph, if $z \in Y$, then $N^{-}(z)=N^{-}\left(x_{1}\right)$, which is a contradiction by Claim 15. Hence $z \notin Y$. Analogously, if $w \in Y$ and $x_{2} \in N^{-}(w)$, then $N^{-}(w) \subseteq\left\{x_{2}, u, v\right\}$, implying that $D$ contains $J_{7}$; and if $x_{2} \notin N^{-}(w)$, then $N^{-}(w) \subseteq N^{-}\left[x_{1}\right]$, contradicting Claim 15. Thus, $w \notin Y$. If $v \in Y$, then $N^{-}(v) \subseteq\left(N^{-}\left[x_{3}\right] \cup\left\{x_{2}\right\}\right)$. Hence, by Claim 15, $x_{2} \in N^{-}(v)$,
implying that $N^{-}(v) \subseteq\left\{x_{2}, x_{3}, w\right\}$, but if $N^{-}(v)=\left\{x_{2}, x_{3}\right\}$ or $N^{-}(v)=\left\{x_{2}, w\right\}$, then $D$ contains $J_{2}$ or $J_{9}$, respectively. Therefore, $v \notin Y$, and by symmetry we can conclude also that $u \notin Y$, a contradiction.

Assume now that $N^{-}\left(x_{3}\right)=\{v, w\}$. Observe that $N^{-}(v) \cap N^{-}[X] \subseteq\left\{x_{2}, z\right\}$. Then $v \notin Y$. If $u \in Y$, then $N^{-}(u) \subseteq\left\{x_{2}, x_{3}, z, w\right\}$, but it could not be neither $\left\{x_{2}, z\right\}$ nor $\left\{x_{3}, w\right\}$ (by Remark 6). If $N^{-}(u)=\left\{x_{2}, x_{3}\right\}$, then $D$ contains $J_{3}$; if $N^{-}(u)=\left\{x_{2}, w\right\}$, then $D$ contains $J_{9}$; if $N^{-}(u)=\left\{x_{3}, z\right\}$, then $D$ contains $J_{7}$; and if $N^{-}(u)=\{z, w\}$, then $D$ contains $J_{10}$. Therefore, $u \notin Y$. If $w \in Y$, then $N^{-}(w) \subseteq\left\{x_{1}, x_{2}, u, z\right\}$. Hence, by Remark 6 and Claim 15, $N^{-}(w)=\{u, z\}$ or $N^{-}(w)=\left\{u, x_{2}\right\}$, implying that $D$ contains $J_{12}$ or $J_{5}$, respectively. Therefore, $w \notin Y$. If $z \in Y$, then $N^{-}(z) \subseteq\left(N^{-}\left[x_{3}\right] \cup N^{-}\left(x_{1}\right)\right)$. Hence, by Claim 15 and Remark $6, N^{-}(z)=\{u, w\}$ or $N^{-}(z)=\left\{u, x_{3}\right\}$, yielding that $D$ contains $J_{11}$ or $J_{6}$, respectively. Hence, $z \notin Y$, a contradiction.

Subcase 2.3. $\left|N^{-}[X]\right|=8$. In this case, $N^{-}\left(x_{3}\right)=\{t, w\}$ for $t, w \notin N^{-}\left[x_{1}\right] \cup$ $N^{-}\left[x_{2}\right]$. First, observe that if $Y \cap\left\{x_{1}, x_{2}\right\}=\emptyset$, then without loss of generality, $t \in Y, x_{1} \in N^{-}(t)$, yielding that $N^{-}(t)=N^{-}\left(x_{2}\right)$, a contradiction to Claim 15. Therefore $Y \cap\left\{x_{1}, x_{2}\right\} \neq \emptyset$. Hence, since $N^{-}\left[x_{3}\right] \cap\left(N^{-}\left[x_{1}\right] \cup N^{-}\left[x_{2}\right]\right)=\emptyset$, it follows that $N^{-}\left[x_{3}\right] \subseteq N^{-}[y] \cup N^{-}\left[y^{\prime}\right]$, with $y, y^{\prime} \in Y$, yielding by Claim 16 that $x_{3} \in Y$. If $x_{2} \notin Y$, then $Y=\left\{x_{1}, x_{3}, y\right\}$ and $\left\{x_{2}, z\right\}=N^{-}(y)$, which is a contradiction to Remark 6. Therefore, $Y=\left\{x_{2}, x_{3}, y\right\}$, yielding that $N^{-}\left(x_{1}\right) \subseteq$ $N^{-}[y]$, contradicting Claim 16.

Case 3. There are exactly two arcs between the elements of $X$. Then $\left|N^{-}[X]\right| \in\{6,7\}$. Let us distinguish the following cases.

Subcase 3.1. Assume that $\left(x_{1} x_{2} x_{3}\right)$ is a path of $D$. Then $N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{3}\right)=$ $N^{-}\left(x_{2}\right) \cap N^{-}\left(x_{1}\right)=\emptyset$ by Remark 6. Hence, $N^{-}\left(x_{2}\right)=\left\{z, x_{1}\right\}$.

Subsubcase 3.1.1. $\left|N^{-}[X]\right|=6$. Without loss of generality, we may assume that $N^{-}\left(x_{3}\right)=\left\{x_{2}, u\right\}$. Observe that if $u \in Y$, then $N^{-}(u)=\left\{x_{2}, z\right\}$, a contradiction to Remark 6 , and then $u \notin Y$. If $v \in Y$, then $x_{2} \notin N^{-}(v)$ again by Remark 6. Hence, if $v \in Y$, then $N^{-}(v)=\left\{x_{3}, z\right\}$, yielding that $D$ contains $J_{4}$. Therefore, $z \in Y$ and $|Y \cap X|=2$. By Remark 6 and Claim 15, $N^{-}(z)=\left\{x_{3}, v\right\}$, implying that $D$ contains $J_{3}$.

Subsubcase 3.1.2. $\left|N^{-}[X]\right|=7$. Then $N^{-}\left(x_{3}\right)=\left\{x_{2}, w\right\}$ for some $w \notin$ $N^{-}\left[x_{1}\right] \cup N^{-}\left[x_{2}\right]$. If $w \in Y$, then $N^{-}(w) \subseteq\left(N^{-}\left[x_{1}\right] \cup\{z\}\right)$ and, by Claim 15 and Remark $6, z \in N^{-}(w)$ and $N^{-}(w) \subseteq\{u, v, z\}$. This implies that $D$ contains $J_{6}$. Therefore, $w \notin Y$. If $z \in Y$, then $N^{-}(z) \subseteq N^{-}\left(x_{1}\right) \cup\left\{x_{3}, w\right\}$. Hence, by Claim 15 and Remark 6, without loss of generality, $N^{-}(z)=\{v, w\}$ or $N^{-}(z)=\left\{v, x_{3}\right\}$, implying that $D$ contains $J_{8}$ or $J_{2}$, respectively. Therefore, $z \notin Y$. If $u \in Y$, then $N^{-}(u) \subseteq N^{-}\left[x_{3}\right] \cup\{z\}$. By Claim 15 and Remark 6, $N^{-}(u)=\left\{z, x_{3}\right\}$ or $N^{-}(u)=\{z, w\}$, yielding that $D$ contains $J_{4}$ or $J_{5}$, respectively. Therefore, $u \notin Y$ and, by symmetry $v \notin Y$, hence, $Y \backslash X=\emptyset$, a contradiction.

Subcase 3.2. Assume that $N^{-}\left(x_{2}\right)=\left\{x_{1}, x_{3}\right\}$. If $\left|N^{-}[X]\right|=6$, then, without loss of generality, suppose $N^{-}\left(x_{3}\right)=\{v, z\}$. Observe that $v \notin Y$, otherwise $N^{-}(v)=\left\{x_{2}\right\}$. If $z \in Y$, then $N^{-}(z) \subseteq\left\{x_{1}, x_{2}, u\right\}$ and by Remark $6, N^{-}(z)=$ $\left\{x_{2}, u\right\}$, yielding that $D$ contains $J_{3}$. Hence, $z \notin Y$, and reasoning similarly, $u \notin Y$, a contradiction. If $\left|N^{-}[X]\right|=7$, then $N^{-}\left(x_{3}\right)=\{z, w\}$ for some $w \notin$ $\left\{x_{1}, x_{2}, x_{3}, u, v, z\right\}$. If $u \in Y$, then $N^{-}(u) \subseteq N^{-}\left[x_{3}\right] \cup\left\{x_{2}\right\}$, and by Claim 15 and Remark $6, x_{2} \in N^{-}(u)$ and $N^{-}(u) \subseteq\left\{x_{2}, z, w\right\}$, implying that $D$ contains $J_{3}$. Therefore, $u \notin Y$. Analogously, $v, z, w \notin Y$, yielding that $Y \backslash X=\emptyset$, a contradiction.

Subcase 3.3. Without loss of generality, let us assume that $\left(x_{1}, x_{2}\right),\left(x_{1}, x_{3}\right) \in$ A. If $\left|N^{-}[X]\right|=6$, then $N^{-}\left(x_{2}\right)=\left\{x_{1}, z\right\}=N^{-}\left(x_{3}\right)$, which contradicts Claim 17. Hence, $\left|N^{-}[X]\right|=7$. Let $N^{-}\left(x_{2}\right)=\left\{x_{1}, z\right\}$ and $N^{-}\left(x_{3}\right)=\left\{x_{1}, w\right\}$. Observe that we also may assume that there are exactly two arcs between the elements of $Y$ and there is some $y \in Y$ satisfying the same as $x_{1}$, that is, $N^{+}[y] \cap Y=Y-y$. Therefore, if $x_{1} \in Y$, we can assume that $Y=\left\{x_{1}, u, w\right\}$ and $N^{+}(u) \cap Y=$ $\left\{x_{1}, w\right\}$. Then $N^{-}(u) \subseteq\left\{x_{3}, x_{2}, z\right\}$ and by Remark $6, x_{3} \in N^{-}(u)$, implying that $N^{-}(u)=\left\{x_{3}, x_{2}\right\}$ or $N^{-}(u)=\left\{x_{3}, z\right\}$ yielding that $D$ contains $J_{2}$ or $J_{9}$, respectively. Moreover, since $N^{+}\left(x_{1}\right) \cap N^{-}[X]=\left\{x_{2}, x_{3}\right\}$, it follows that $Y \cap$ $\left\{x_{2}, x_{3}\right\} \neq \emptyset$. Furthermore, by Claim 18, $\left|Y \cap\left\{x_{2}, x_{3}\right\}\right|=1$. Without loss of generality, suppose $Y \cap X=\left\{x_{2}\right\}$. If $Y=\left\{x_{2}, z, u\right\}$, then $u \in N^{+}(z)$, yielding that $N^{-}(z) \subseteq\left\{v, x_{3}, w\right\}$. By Remark $6, N^{-}(z)=\{v, w\}$ or $N^{-}(z)=\left\{v, x_{3}\right\}$, implying that $D$ contains $J_{5}$ or $J_{4}$, respectively. If $Y=\left\{x_{2}, z, w\right\}$, then $z \in$ $N^{-}(w)$ and $x_{3} \in N^{-}(z)$. Then, without loss of generality, $u \in N^{-}(z)$ yielding that $D$ contains $J_{3}$. Therefore, $z \notin Y$. If $Y=\left\{x_{2}, u, v\right\}$, then, without loss of generality, $x_{3} \in N^{-}(u)$ and $w \in N^{-}(v)$, implying that $D$ contains $J_{3}$. Finally, if $Y=\left\{x_{2}, u, w\right\}$, then $x_{3} \in N^{-}(u)$ and $v \in N^{-}(w)$, yielding that $D$ contains $J_{2}$.

Case 4. There are three arcs between the elements of $X$. Hence, $\left|N^{-}[X]\right|=6$ and since $D$ is $T T_{3}$-free, we may assume that $\left(x_{1} x_{2} x_{3} x_{1}\right)$ is a directed triangle. Then $N^{-}\left(x_{i}\right) \cap N^{-}\left(x_{j}\right)=\emptyset$, for all $i \neq j$. Let $N^{-}\left(x_{1}\right)=\left\{x_{2}, u\right\}, N^{-}\left(x_{2}\right)=$ $\left\{x_{3}, v\right\}$ and $N^{-}\left(x_{3}\right)=\left\{x_{1}, z\right\}$. Notice that if $z \in N^{-}(u)$ or $v \in N^{-}(u)$, then $D$ contains $J_{2}$ or $J_{3}$, respectively. Therefore, since $D$ is a $T T_{3}$-free oriented graph, $N^{-}(u) \cap N^{-}[X] \subseteq\left\{x_{2}\right\}$ and $u \notin Y$. Observe that, by symmetry, we can conclude the same for $v$ and $z$, obtaining again a contradiction.

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