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OUTPATHS OF ARCS IN REGULAR 3-PARTITE TOURNAMENTS

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Abstract

Guo [*Outpaths in semicomplete multipartite digraphs*, Discrete Appl. Math. 95 (1999) 273–277] proposed the concept of the outpath in digraphs. An outpath of a vertex x (an arc xy, respectively) in a digraph is a directed path starting at x (an arc xy, respectively) such that x does not dominate the end vertex of this directed path. A k-outpath is an outpath of length k. The outpath is a generalization of the directed cycle. A c-partite tournament is an orientation of a complete c-partite graph.

In this paper, we investigate outpaths of arcs in regular 3-partite tournaments. We prove that every arc of an r-regular 3-partite tournament has 2- (when $r \geq 1$), 3- (when $r \geq 2$), and 5-, 6-outpaths (when $r \geq 3$). We also give the structure of an r-regular 3-partite tournament D with $r \geq 2$ that contains arcs which have no 4-outpaths. Based on these results, we conjecture that for all $k \in \{1, 2, \ldots, r-1\}$, every arc of r-regular 3-partite tournaments with $r \geq 2$ has $(3k-1)$- and $3k$-outpaths, and it has a $(3k+1)$-outpath except an r-regular 3-partite tournament.

Keywords: multipartite tournament, regular 3-partite tournament, outpaths.

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1. Introduction

Throughout the paper, we use the terminology and notation of [1]. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. A digraph D is said to be *strongly connected*, if for all $x, y \in V(D)$, there is a directed path from x to y. A digraph D is *r-regular*, if there is an integer r such that $d^+(x) = d^-(x) = r$ holds for every $x \in V(D)$. A digraph obtained by replacing each edge of a complete c-partite graph with exactly one arc is called a *c-partite tournament* or a *multipartite tournament*. If D is a multipartite tournament and $x \in V(D)$, we denote by $V(x)$ the partite set of D to which x belongs.

An *l-outpath* of an arc x_1x_2 in a digraph D is a directed path $P = x_1x_2 \cdots x_{l+1}$ with length l starting at x_1x_2 such that x_1 does not dominate the end vertex x_{l+1} of this directed path P. Note that if D is a tournament, an l-outpath $P = x_1x_2 \cdots x_{l+1}$ of an arc x_1x_2 corresponds in fact to an $(l+1)$-cycle $C = x_1x_2 \cdots x_{l+1}x_1$ through x_1x_2, so the concept of the outpath is a generalization of the directed cycle. If D is a multipartite tournament, then $x_1x_2 \cdots x_{l+1}$ is an l-outpath of the arc x_1x_2 if and only if $x_{l+1} \in V(x_1)$ or $x_{l+1} \to x_1$ holds.

There are lots of results in multipartite tournaments, see for example [5]. However, the results on 3-partite tournaments are still very few. In 1999, Guo proposed the concept of the outpath in digraphs. At present, outpaths in multipartite tournaments have also been studied by some scholars, see for example [2, 3, 4, 7]. The earliest results are the following two theorems.

Theorem 1 (Guo). *Let D be a strongly connected c-partite tournament with $c \geq 3$. Then every vertex v of D has a $(k-1)$-outpath for each $k \in \{3, 4, \ldots, c\}$.*

Theorem 2 (Guo). *Let D be a regular c-partite tournament with $c \geq 3$. Then every arc of D has a $(k-1)$-outpath for each $k \in \{3, 4, \ldots, c\}$.*

As a generalization of Theorem 2, Cui and the first author proved in [3] that every arc of a regular c-partite tournament D with $c \geq 5$ has a $(k-1)$-outpath for each $k \in \{3, 4, \ldots, |V(D)|\}$. In this paper, we investigate outpaths of arcs in regular 3-partite tournaments. However, the following example will show that there exists an infinite family of regular 3-partite tournaments D such that not every arc of D has a k-outpath for all $k \in \{3, 4, \ldots, |V(D)|\}$.

Example 3. Let D be an r-regular 3-partite tournament with $r \geq 2$ and let V_1, V_2, V_3 be three partite sets of D satisfying that $V_1 \to V_2 \to V_3 \to V_1$. (Note that $V_1 \to V_2 \to V_3 \to V_1$ was defined below firstly.) Then it is easy to check that every arc of D has no $(3k+1)$-outpaths for all $k \in \{1, 2, \ldots, r-1\}$.

In this paper, we prove that every arc of an r-regular 3-partite tournament has 2- (when $r \geq 1$), 3- (when $r \geq 2$), and 5-, 6-outpaths (when $r \geq 3$). We also give a characterization of regular 3-partite tournaments with at least six vertices whose arcs have no 4-outpaths. We prove that an r-regular 3-partite tournament D with $r \geq 2$ contains arcs which have no 4-outpaths if and only if D is the digraph in Example 3. Based on the above results, we conjecture that for all $k \in \{1, 2, \ldots, r-1\}$, every arc of an r-regular 3-partite tournament D with $r \geq 2$ has $(3k-1)$- and $3k$-outpaths, and every arc of D has a $(3k+1)$-outpath unless D is the digraph in Example 3.

2. Preliminaries

Let D be a digraph. If xy is an arc of D, then we say x *dominates* y and write $x \to y$. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B and denote it by $A \to B$. Otherwise, we denote it by $A \nrightarrow B$. Let X be a subset of $V(D)$. We use $|X|$ to stand for the number of the vertices of X. Let D' be a subdigraph or a vertex set of D. The *outset* $N^+_{D'}(x)$ of a vertex x is the set of vertices of D' dominated by x and the *inset* $N^-_{D'}(x)$ is the set of vertices of D' dominating x. We call the numbers $d^+_{D'}(x) = |N^+_{D'}(x)|$ and $d^-_{D'}(x) = |N^-_{D'}(x)|$ the *out-degree* and *in-degree* of x in D', respectively. When $D' = D$, we usually use $N^+(x)$, $N^-(x)$, $d^+(x)$ and $d^-(x)$ instead of $N^+_{D'}(x)$, $N^-_{D'}(x)$, $d^+_{D'}(x)$ and $d^-_{D'}(x)$, respectively.

The following three lemmas are important to prove our main results.

Lemma 3. *If D is an r-regular 3-partite tournament with partite sets V_1, V_2, V_3 and v is a vertex of D, then $|V_1| = |V_2| = |V_3| = r$ and $d^+(v) = d^-(v) = r$.*

Lemma 4 (Xu, Li, Guo and Li)**.** *If D is an r-regular 3-partite tournament with partite sets V_1, V_2, V_3 and v is a vertex of V_1, then $d^+_{V_2}(v) = d^-_{V_3}(v)$ and $d^-_{V_2}(v) = d^+_{V_3}(v)$.*

Lemma 5. *Let D be an r-regular 3-partite tournament with $r \geq 2$ and partite sets V_1, V_2, V_3. Let ab be an arc of D such that $a \in V_1$ and $b \in V_2$ and $V_3 \nrightarrow a \nrightarrow V_2$. We divide V_2 and V_3 into two nonempty parts V_2^+, V_2^- and V_3^+, V_3^- respectively, such that $V_2^- \to a \to V_2^+$ and $V_3^- \to a \to V_3^+$. Let $V' = V_2^+ \cup V_3^+$ and $V'' = V_2^- \cup V_3^-$. Then the following hold.*

(1) *$N^+(a) = V'$, $N^-(a) = V''$ and $|V'| = |V''| = r$.*

(2) *$|V_2^+| = |V_3^-|$ and $|V_2^-| = |V_3^+|$.*

(3) *$d^+_{V'}(y) = d^-_{V''}(y)$ and $d^-_{V'}(y) = d^+_{V''}(y)$ for each vertex $y \in V_1$.*

Proof. Observe $N^+(a) = V'$ and $N^-(a) = V''$. By Lemma 3, we have $d^+(a) = d^-(a) = r$. Therefore, $|V'| = |V''| = r$ holds. This proves (1). By Lemma 3, we get $|V_2| = |V_2^+| + |V_2^-| = r$, $|V_3| = |V_3^+| + |V_3^-| = r$ and $d^+(a) = |V_2^+| + |V_3^+| = r$. Therefore, we have $|V_2^-| = |V_3^+|$ and $|V_2^+| = |V_3^-|$. So (2) is proved. For each vertex $y \in V_1$, by Lemma 3, we have $d^+_{V''}(y) + d^-_{V''}(y) = |V''| = r$, $d^+_{V'}(y) + d^+_{V''}(y) = d^+(y) = r$ and $d^-_{V'}(y) + d^-_{V''}(y) = d^-(y) = r$. So $d^-_{V'}(y) = d^+_{V''}(y)$ and $d^+_{V'}(y) = d^-_{V''}(y)$. This completes the proof of (3). ■

3. Main Results

By Theorem 2, it is easy to get the following Theorem 6.

Theorem 6. *If D is an r-regular 3-partite tournament with $r \geq 1$ and ab is an arc of D, then ab has a 2-outpath.*

Theorem 7. *If D is an r-regular 3-partite tournament with $r \geq 2$ and ab is an arc of D, then ab has a 3-outpath.*

Proof. Let V_1, V_2, V_3 be three partite sets of D. By Lemma 3, we have $|V_1| = |V_2| = |V_3| = r \geq 2$. Without loss of generality, we suppose $a \in V_1$ and $b \in V_2$.

Suppose first that $V_3 \to a \to V_2$. If $V_1 \not\to b$, then there exists a vertex $y \in V_1 - \{a\}$ such that $b \to y$. By Lemma 4, there is a vertex $x \in V_3$ such that $y \to x$. Then $x \to a$ and $abyx$ is a 3-outpath of ab. Assume $V_1 \to b$. Then $b \to V_3$. Let $u \in V_2 - \{b\}$. Then $a \to u$. By Lemma 4, there exists a vertex $x \in V_3$ such that $u \to x$. Obviously, we also have $b \to x$. By $\{b, u\} \to x$ and Lemma 4, there exists a vertex $y \in V_1 - \{a\}$ such that $x \to y$. Then $y \in V(a)$ and $abxy$ is a 3-outpath of ab.

Suppose now that $V_3 \not\to a \not\to V_2$. We divide the partite set V_2 into two nonempty parts V_2^+, V_2^- such that $V_2^- \to a \to V_2^+$. By $a \to b$ and Lemma 4, there exists a vertex $x \in V_3$ such that $b \to x$. If $V_2^- \not\to x$, then there is an arc xu for some $u \in V_2^-$. Then $u \to a$ and $abxu$ is a 3-outpath of ab. Assume $V_2^- \to x$. By $V_2^- \cup \{b\} \to x$ and Lemma 4, there exists a vertex $y \in V_1 - \{a\}$ such that $x \to y$. Then $y \in V(a)$ and $abxy$ is a 3-outpath of ab. ∎

Theorem 8. *Let D be an r-regular 3-partite tournament with $r \geq 2$ and partite sets V_1, V_2, V_3. If ab is an arc of D, then ab has no 4-outpaths if and only if $V_1 \to V_2 \to V_3 \to V_1$.*

Proof. By Lemma 3, we have $|V_1| = |V_2| = |V_3| = r \geq 2$. Suppose, without loss of generality, that $a \in V_1$ and $b \in V_2$. By Example 3, sufficiency is obvious.

Now, we prove the necessity. Suppose that ab has no 4-outpaths. We consider the following two cases.

Case 1. $V_3 \to a \to V_2$. By $a \to b$ and Lemma 4, there exists a vertex $x \in V_3$ such that $b \to x$. If $V_2 \not\to x$, then there is a vertex $u \in V_2 - \{b\}$ such that $x \to u$. By $x \to u$ and Lemma 4, there exists a vertex $y \in V_1$ such that $u \to y$. Obviously, $a \to u$ and $y \neq a$. Then $y \in V(a)$ and $abxuy$ is a 4-outpath of ab, a contradiction. So $V_2 \to x$ and $x \to V_1$.

If $V_1 \not\to b$, then there exists a vertex $y \in V_1$ such that $b \to y$. Obviously, $y \neq a$ and $x \to y$. By $b \to y$ and Lemma 4, there is a vertex $w \in V_3$ such that $y \to w$. Obviously, $w \neq x$ and $w \to a$. Then $abxyw$ is a 4-outpath of ab, a contradiction. So $V_1 \to b$ and $b \to V_3$.

If $(V_2 - \{b\}) \not\to (V_3 - \{x\})$, then there exists an arc $x'u'$ for some $x' \in V_3 - \{x\}$ and $u' \in V_2 - \{b\}$. Obviously, $b \to x'$ and $u' \to x \to a$. Thus, $abx'u'x$ is a 4-outpath of ab, a contradiction. Therefore, we get $(V_2 - \{b\}) \to (V_3 - \{x\})$. Since $b \to V_3$ and $V_2 \to x$, we have $V_2 \to V_3$. So $V_3 \to V_1$ and $V_1 \to V_2$ hold.

Case 2. $V_3 \nrightarrow a \nrightarrow V_2$. In this case, we prove that ab always has a 4-outpath, which contradicts our assumption. We divide the partite set V_2 into two nonempty parts V_2^+, V_2^- such that $V_2^- \to a \to V_2^+$. Similarly, the partite set V_3 is also divided into two nonempty parts V_3^+, V_3^- such that $V_3^- \to a \to V_3^+$. Let $V' = V_2^+ \cup V_3^+$ and $V'' = V_2^- \cup V_3^-$. By Lemma 5(1), we have $N^+(a) = V'$ and $N^-(a) = V''$.

If $V_3^+ \nrightarrow b$, then there is an arc bx for some $x \in V_3^+$. By $b \to x$ and Lemma 4, there exists a vertex $y \in V_1$ such that $x \to y$. Obviously, $a \to x$ and $y \neq a$. By $x \in V'$ and Lemma 5(3), there exists a vertex $z \in V''$ such that $y \to z$. Then $z \to a$ and $abxyz$ is a 4-outpath of ab, a contradiction. So $V_3^+ \to b$. By Lemma 4, there exists a vertex $y \in V_1 - \{a\}$ such that $b \to y$. By $a \to b$ and Lemma 4, there is a vertex $v \in V_3$ such that $b \to v$. It is easy to see that $v \in V_3^-$.

If $y \to v$, then Lemma 4 implies that there is a vertex $u \in V_2$ such that $v \to u$. Obviously, $u \neq b$. When $u \in V_2^-$, we get that $u \to a$ and $abyvu$ is a 4-outpath of ab. When $u \in V_2^+$, we have $a \to u$. By $v \to u$ and Lemma 4, there exists a vertex $y' \in V_1$ (y' may be equal to y) such that $u \to y'$. Since $a \to u$, we get $y' \neq a$. Then $y' \in V(a)$ and $abvuy'$ is a 4-outpath of ab, a contradiction. Assume $v \to y$. By $b \to y$, $b \in V'$ and Lemma 5(3), there exists a vertex $z \in V''$ such that $y \to z$. Obviously, $z \neq v$ and $z \to a$. Then $abvyz$ is a 4-outpath of ab, a contradiction.

Therefore, we have shown that if ab has no 4-outpaths, then $V_1 \to V_2 \to V_3 \to V_1$, and the proof is complete. ■

Theorem 9. *If D is an r-regular 3-partite tournament with $r \geq 3$ and ab is an arc of D, then ab has a 5-outpath and a 6-outpath.*

Proof. Let V_1, V_2, V_3 be three partite sets of D. By Lemma 3, we have $|V_1| = |V_2| = |V_3| = r$ and $d^+(v) = d^-(v) = r$ for each vertex v of D. Without loss of generality, suppose $a \in V_1$ and $b \in V_2$. We distinguish the following two cases.

Case 1. $V_3 \to a \to V_2$. By $a \to b$ and Lemma 4, there exists a vertex $x \in V_3$ such that $b \to x$.

Case 1.1. $(V_2 - \{b\}) \nrightarrow x$. By the hypothesis, there is a vertex $u \in V_2 - \{b\}$ such that $x \to u$. By Lemma 4, there is a vertex $y \in V_1$ such that $u \to y$. Since $a \to V_2$, we have $a \to u$ and $y \neq a$. By $a \to u$ and Lemma 4, there exists a vertex $v \in V_3$ such that $u \to v$. Obviously, $v \neq x$ and $v \to a$. Then $abxuvy$ (when $v \to y$) or $abxuyv$ (when $y \to v$) is a 5-outpath of ab. We will prove that ab has a 6-outpath.

Subcase 1.1.1. $v \to y$. If $(V_3 - \{x, v\}) \nrightarrow y$, then there exists a vertex $w \in V_3 - \{x, v\}$ such that $y \to w$. Thus, $w \to a$ and $abxuvyw$ is a 6-outpath of ab. Assume $(V_3 - \{x, v\}) \to y$. Note that $\{u, v\} \to y$. We have $N^-(y) = (V_3 - \{x\}) \cup \{u\}$ and $N^+(y) = (V_2 - \{u\}) \cup \{x\}$. Let $u' \in V_2 - \{b, u\}$. Then

$y \to u'$. If $(V_1 - \{a, y\}) \nrightarrow u'$, then there is an arc $u'y'$ for some $y' \in V_1 - \{a, y\}$. Thus, $y' \in V(a)$ and ab has a 6-outpath $abxuyu'y'$. Assume $(V_1 - \{a, y\}) \to u'$. Note $\{a, y\} \to u'$. We get $V_1 \to u'$ and $u' \to V_3$. Then $u' \to v \to a$ and $abxuyu'v$ is a 6-outpath of ab.

Subcase 1.1.2. $y \to v$. If $(V_1 - \{a, y\}) \nrightarrow v$, then there exists a vertex $y' \in V_1 - \{a, y\}$ such that $v \to y'$. Thus, $y' \in V(a)$ and $abxuyvy'$ is a 6-outpath of ab. Assume $(V_1 - \{a, y\}) \to v$. Note that $\{u, y\} \to v$. We have $N^-(v) = (V_1 - \{a\}) \cup \{u\}$ and $N^+(v) = \{a\} \cup (V_2 - \{u\})$. Let $u' \in V_2 - \{b, u\}$. Then $v \to u'$. If $(V_3 - \{x, v\}) \nrightarrow u'$, then there is an arc $u'v'$ for some $v' \in V_3 - \{x, v\}$. Thus, $v' \to a$ and ab has a 6-outpath $abxuvu'v'$. Assume $(V_3 - \{x, v\}) \to u'$. Since $\{a, v\} \to u'$, we get $N^-(u') = \{a\} \cup (V_3 - \{x\})$ and $N^+(u') = (V_1 - \{a\}) \cup \{x\}$. Then $u' \to y$ and $abxuvu'y$ is a 6-outpath of ab.

Case 1.2. $(V_2 - \{b\}) \to x$. In this case, we have $V_2 \to x \to V_1$ since $b \to x$.

Subcase 1.2.1. $(V_1 - \{a\}) \nrightarrow b$. By the hypothesis, there exists a vertex $y \in V_1 - \{a\}$ such that $b \to y$. By Lemma 4, there is a vertex $w \in V_3 - \{x\}$ such that $y \to w$. Obviously, we have $x \to y$. Then Lemma 4 implies that there is a vertex $u \in V_2$ such that $y \to u$. It is easy to see $u \neq b$ and $u \to x$. Note $\{x, w\} \to a$. Then $abxyuw$ (when $u \to w$) or $abywux$ (when $w \to u$) is a 5-outpath of ab. We will prove that ab has a 6-outpath. Let $y' \in V_1 - \{a, y\}$. Then $y' \in V(a)$.

Suppose first that $u \to w$. If $w \to y'$, then $abxyuwy'$ is a 6-outpath of ab. Assume $y' \to w$. By $\{y, y'\} \to w$ and Lemma 4, there exists a vertex $v \in V_2 - \{b\}$ such that $w \to v$. Since $u \to w$, we have $v \neq u$. Obviously, $v \to x \to a$ and $abyuwvx$ is a 6-outpath of ab.

Suppose now that $w \to u$. If $u \to y'$, then $abxywuy'$ is a 6-outpath of ab. Assume $y' \to u$. By $\{a, y, y'\} \to u$ and Lemma 4, there exists a vertex $w' \in V_3 - \{x, w\}$ such that $u \to w'$. Then $w' \to a$ and $abxywuw'$ is a 6-outpath of ab.

Subcase 1.2.2. $(V_1 - \{a\}) \to b$. Since $a \to b$, we have $V_1 \to b$ and $b \to V_3$. Let $w \in V_3 - \{x\}$. Then $b \to w$.

Suppose first that $(V_2 - \{b\}) \nrightarrow w$. Then there is a vertex $u \in V_2 - \{b\}$ such that $w \to u$. By Lemma 4, there exists a vertex $y \in V_1$ such that $u \to y$. Obviously, $a \to u$, $y \neq a$ and $y \in V(a)$. Recalling that $V_2 \to x \to V_1$, we get $u \to x \to y$. Then $abwuxy$ is a 5-outpath of ab. Let $w' \in V_3 - \{x, v\}$. Then $w' \to a$. If $y \to w'$, then $abwuxyw'$ is a 6-outpath of ab. Assume $w' \to y$. By $\{x, w'\} \to y$ and Lemma 4, there is a vertex $u' \in V_2 - \{b\}$ such that $y \to u'$. Then $u' \to x \to a$. Since $u \to y$, we have that $u' \neq u$ and $abwuyu'x$ is a 6-outpath of ab.

Suppose now that $(V_2 - \{b\}) \to w$. Since $b \to w$, we have $V_2 \to w$ and $w \to V_1$. Then $w \to a$. Let $y \in V_1 - \{a\}$. Then $\{x, w\} \to y$. By Lemma 4, there is a vertex $z \in V_2 - \{b\}$ such that $y \to z$. Obviously, $z \to w$ and ab has a 5-outpath $abxyzw$. Let $y' \in V_1 - \{a, y\}$. Then $y' \in V(a)$ and $w \to y'$. Now, ab has a 6-outpath $abxyzwy'$.

Subcase 2. $V_3 \not\twoheadrightarrow a \not\twoheadrightarrow V_2$. We divide the partite set V_2 into two nonempty parts V_2^+, V_2^- such that $V_2^- \to a \to V_2^+$. Similarly, the partite set V_3 is divided into two nonempty parts V_3^+, V_3^- such that $V_3^- \to a \to V_3^+$. Let $V' = V_2^+ \cup V_3^+$ and $V'' = V_2^- \cup V_3^-$. By Lemma 5(1), we have $N^+(a) = V'$, $N^-(a) = V''$ and $|V'| = |V''| = r$.

Subcase 2.1. $V_3^+ \not\twoheadrightarrow b$. By the hypothesis, there is a vertex $x \in V_3^+$ such that $b \to x$. By Lemma 4, there is a vertex $y \in V_1 - \{a\}$ such that $x \to y$. Obviously, $y \in V(a)$. Similarly, by $a \to x$ and Lemma 4, there is a vertex $u \in V_2 - \{b\}$ such that $x \to u$.

We first show that ab has a 5-outpath. If $u \to y$, then by $x \in V'$, $x \to y$ and Lemma 5(3), there is a vertex $z \in V''$ such that $y \to z$. Obviously, $z \neq u$ and $z \to a$. Then ab has a 5-outpath $abxuyz$. Assume $y \to u$. Then $(V_1 - \{a, y\}) \cup V_3^- \not\twoheadrightarrow u$ (as otherwise, we have $(V_1 - \{a, y\}) \cup V_3^- \cup \{x, y\} \subseteq N^-(u)$ and $d^-(u) \geq r + 1$, a contradiction). Therefore, there exists a vertex $z' \in V_1 - \{a, y\} \cup V_3^-$ such that $u \to z'$. Then $z' \in V(a)$ or $z' \to a$. Now, ab has a 5-outpath $abxyuz'$.

Next, we will prove that ab has a 6-outpath. We discuss the following two subcases.

Subcase 2.1.1. $|V_2^+| = 1$. By Lemma 5(2), we have $|V_2^+| = |V_3^-| = 1$, $|V_2^-| = |V_3^+| = r - 1 \geq 2$. Obviously, $V_2^+ = \{b\}$ and $V_2^- = V_2 - \{b\}$. Let $V_3^- = \{v\}$. Then $v \to a$ and $V_3^+ = V_3 - \{v\}$.

Suppose first that $y \to u$. If $V_3^+ \to u$, then $N^-(u) = V_3^+ \cup \{y\} = (V_3 - \{v\}) \cup \{y\}$ and $N^+(u) = (V_1 - \{y\}) \cup \{v\}$. Let $y' \in V_1 - \{a, y\}$. Then $y' \in V(a)$ and $u \to y'$. Thus, $abxyuy'v$ (when $y' \to v$) or $abxyuvy'$ (when $v \to y'$) is a 6-outpath of ab. Assume $V_3^+ \not\twoheadrightarrow u$. Then there exists a vertex $w \in V_3^+$ such that $u \to w$. Obviously, $w \neq x$ and $a \to w$. If $(V_1 - \{a, y\}) \not\twoheadrightarrow w$, then there is a vertex $y' \in V_1 - \{a, y\}$ such that $w \to y'$. Then $y' \in V(a)$ and ab has a 6-outpath $abxyuwy'$. Assume $(V_1 - \{a, y\}) \to w$. Since $\{a, u\} \to w$, we have $N^-(w) = (V_1 - \{y\}) \cup \{u\}$ and $N^+(w) = \{y\} \cup (V_2 - \{u\})$. Let $u' \in V_2^- - \{u\}$. Then $w \to u' \to a$ and ab has a 6-outpath $abxyuwu'$.

Suppose now that $u \to y$. If $V_2^- \to y$, then $N^-(y) = V_2^- \cup \{x\} = (V_2 - \{b\}) \cup \{x\}$ and $N^+(y) = \{b\} \cup (V_3 - \{x\})$. Let $w \in V_3^+ - \{x\}$. Then $\{a, y\} \to w$. When $(V_1 - \{a, y\}) \not\twoheadrightarrow w$, there is a vertex $y' \in V_1 - \{a, y\}$ such that $w \to y'$. Then $y' \in V(a)$ and $abxuywy'$ is a 6-outpath of ab. When $(V_1 - \{a, y\}) \to w$, we have $V_1 \to w \to V_2$ since $\{a, y\} \to w$. Let $u' \in V_2^- - \{u\}$. Then $w \to u' \to a$ and $abxuywu'$ is a 6-outpath of ab.

Assume $V_2^- \not\twoheadrightarrow y$. Then there is a vertex $z \in V_2^-$ such that $y \to z$. Clearly, $z \neq u$. If $(V_1 - \{a, y\}) \not\twoheadrightarrow z$, then there exists a vertex $y_0 \in V_1 - \{a, y\}$ such that $z \to y_0$. Thus, $y_0 \in V(a)$ and $abxuyzy_0$ is a 6-outpath of ab. Assume $(V_1 - \{a,y\}) \to z$. Since $y \to z$, we get $(V_1 - \{y\}) \to z$ and $d^-_{V_1}(z) \geq r-1 \geq 2$. By Lemma 4, there is a vertex $w \in V_3 - \{x\}$ such that $z \to w$. When $w = v$, we know that $w \to a$ and $abxuyzw$ is a 6-outpath of ab. When $w \neq v$, we have $w \in V_3^+ - \{x\}$ and $a \to w$. If $w \to u$, then $abxyzwu$ is a 6-outpath of ab. If $u \to w$, then by $\{u,z\} \to w$ and Lemma 4, there exists a vertex $y_1 \in V_1 - \{a,y\}$ such that $w \to y_1$. Thus, $y_1 \in V(a)$ and ab has a 6-outpath $abxyzwy_1$.

Subcase 2.1.2. $2 \leq |V_2^+| \leq r-1$. By Lemma 5(2), we have $2 \leq |V_2^+| = |V_3^-| \leq r-1$, $1 \leq |V_2^-| = |V_3^+| \leq r-2$.

Suppose first that $(V' - \{b\}) \nrightarrow y$. Then there exists a vertex $v \in V' - \{b\}$ such that $y \to v$. Obviously, $v \in V_3^+$ or $v \in V_2^+ - \{b\}$.

When $v \in V_3^+$, by $\{a,y\} \to v$ and Lemma 4, there exists a vertex $w \in V_2 - \{b\}$ such that $v \to w$. If $V_3^- \nrightarrow w$, then there is an arc wz for some vertex $z \in V_3^-$ and $abxyvwz$ is a 6-outpath of ab. Assume $V_3^- \to w$. By $V_3^- \cup \{v\} \to w$ and Lemma 4, there is a vertex $y' \in V_1 - \{a,y\}$ such that $w \to y'$. Now, $y' \in V(a)$ and $abxyvwy'$ is a 6-outpath of ab.

When $v \in V_2^+ - \{b\}$, by $\{a,y\} \to v$ and Lemma 4, there exists a vertex $v' \in V_3 - \{x\}$ such that $v \to v'$. If $V_2^- \nrightarrow v'$, then there is an arc $v'w'$ for some vertex $w' \in V_2^-$ and $abxyvv'w'$ is a 6-outpath of ab. Assume $V_2^- \to v'$. If $(V_1 - \{a,y\}) \nrightarrow v'$, then there is a vertex $y' \in V_1 - \{a,y\}$ such that $v' \to y'$. Now, $y' \in V(a)$ and $abxyvv'y'$ is a 6-outpath of ab. Assume $(V_1 - \{a,y\}) \to v'$. Then $(V_1 - \{a,y\}) \cup V_2^- \cup \{v\} \to v'$. Note $d^-(v') = r$. We get $|V_2^-| = |V_3^+| = 1$. So $V_3^+ = \{x\}$ and $v' \in V_3^-$. Let $V_2^- = \{w'\}$. Then $N^-(v') = (V_1 - \{a,y\}) \cup \{v, w'\}$ and $N^+(v') = \{a,y\} \cup (V_2 - \{v,w'\})$.

If $(V_1 - \{a,y\}) \nrightarrow v$, then there exists an arc vy_0 for some $y_0 \in V_1 - \{a,y\}$. Note $y_0 \to v' \to a$. Then $abxyvy_0v'$ is a 6-outpath of ab. Assume $(V_1 - \{a,y\}) \to v$. Since $\{a,y\} \to v$, we get $V_1 \to v \to V_3$. Then $v \to x$. By $\{b,v\} \to x$ and Lemma 4, there exists a vertex $y_1 \in V_1 - \{y\}$ such that $x \to y_1$. Since $a \to x$, we get $y_1 \neq a$ and $y_1 \in V_1 - \{a,y\}$. Note $y_1 \to v$ and $y \in V(a)$. Then $abxy_1vv'y$ is a 6-outpath of ab.

Suppose now that $(V' - \{b\}) \to y$. If $V_2^- \to y$, then $N^-(y) = (V_2 - \{b\}) \cup V_3^+$. So we have $|V_3^+| = |V_2^-| = 1$ and $V_3^+ = \{x\}$. Thus, $N^+(y) = \{b\} \cup V_3^-$. Let $V_2^- = \{w\}$. By $w \to \{a,y\}$ and Lemma 4, there exists a vertex $z \in V_3^-$ such that $z \to w$. Clearly, we have $y \to z$. Let $u' \in V_2^+ - \{b\}$. Then $u' \to y$. If $x \to u'$, then $abxu'yzw$ is a 6-outpath of ab. Assume $u' \to x$. By $\{b,u'\} \to x$ and Lemma 4, there exists a vertex $y' \in V_1 - \{y\}$ such that $x \to y'$. Obviously, $y' \neq a$, $y' \in V(a)$ and $z \to a$. Then $abxy'wyz$ (when $y' \to w$) or $abxyzwy'$ (when $w \to y'$) is a 6-outpath of ab.

Assume $V_2^- \nrightarrow y$. Then there is an arc yw for some vertex $w \in V_2^-$. By $(V_2^+ - \{b\}) \to y$ and Lemma 4, there exists a vertex $v_1 \in V_3$ such that $y \to v_1$. Since $V_3^+ \to y$, we get $v_1 \in V_3^-$ and $v_1 \to a$.

If $v_1 \to w$, then $(V_1 - \{a, y\}) \cup (V_3^- - \{v_1\}) \nrightarrow w$ (as otherwise, we get $(V_1 - \{a, y\}) \cup (V_3^- - \{v_1\}) \cup \{y, v_1\} \subseteq N^-(w)$ and $d^-(w) \geq r+1$, a contradiction). So there is a vertex z_1 in $(V_1 - \{a, y\}) \cup (V_3^- - \{v_1\})$ such that $w \to z_1$. Note $z_1 \in V(a)$ or $z_1 \to a$. Then $abxyv_1wz_1$ is a 6-outpath of ab.

Assume $w \to v_1$. If $(V_1 - \{a, y\}) \nrightarrow v_1$, then there exists a vertex $y' \in V_1 - \{a, y\}$ such that $v_1 \to y'$. Then $y' \in V(a)$ and $abxywv_1y'$ is a 6-outpath of ab. Assume $(V_1 - \{a, y\}) \to v_1$. Since $\{y, w\} \to v_1$, we have $N^-(v_1) = (V_1 - \{a, y\}) \cup \{y, w\} = (V_1 - \{a\}) \cup \{w\}$ and $N^+(v_1) = \{a\} \cup (V_2 - \{w\})$. Let $u_1 \in V_2^+ - \{b\}$. Then $v_1 \to u_1 \to y$. If $x \to w$, then $abxwv_1u_1y$ is a 6-outpath of ab. Assume $w \to x$. By $\{b, w\} \to x$ and Lemma 4, there exists a vertex $y_1 \in V_1 - \{y\}$ such that $x \to y_1$. Obviously, $y_1 \neq a$ and $y_1 \to v_1$. Then $abxy_1v_1u_1y$ is a 6-outpath of ab.

Subcase 2.2. $V_3^+ \to b$.

Subcase 2.2.1. $|V_2^+| = 1$. By Lemma 5(2), we have $|V_2^+| = |V_3^-| = 1$ and $|V_2^-| = |V_3^+| = r - 1 \geq 2$. Obviously, we have $V_2^+ = \{b\}$. Let $V_3^- = \{v\}$. Then $v \to a$, $V_2^- = V_2 - \{b\}$ and $V_3^+ = V_3 - \{v\}$. Since $V_3^+ \to b$, we get $N^-(b) = V_3^+ \cup \{a\}$ and $N^+(b) = (V_1 - \{a\}) \cup \{v\}$.

If $V_3^+ \to (V_1 - \{a\})$, then we have $V_3^+ \cup \{b\} \to (V_1 - \{a\}) \to (V_2 - \{b\}) \cup \{v\}$, $V_3^+ \to (V_1 - \{a\}) \cup \{b\}$ and $\{a\} \cup (V_2 - \{b\}) \to V_3^+$. Let y, y' be two distinct vertices in $V_1 - \{a\}$ and let u and x be two arbitrary vertices in $V_2 - \{b\}$ and V_3^+, respectively. Then $y' \in V(a)$ and ab has a 5-outpath $abyuxy'$ and a 6-outpath $abyuxy'v$.

Assume $V_3^+ \nrightarrow (V_1 - \{a\})$. Then there is an arc yx for some $y \in V_1 - \{a\}$ and $x \in V_3^+$. Clearly, we get $b \to y$.

If $(V_1 - \{a, y\}) \nrightarrow x$, then there exists a vertex $y' \in V_1 - \{a, y\}$ such that $x \to y'$. By Lemma 4, there is a vertex $u \in V_2$ such that $y' \to u$. Note $b \to y'$. We have $u \neq b$ and $u \in V_2^-$. Then $u \to a$ and $abyxy'u$ is a 5-outpath of ab. We will seek for a 6-outpath of ab. If $y' \to v$, then $abyxy'uv$ (when $u \to v$) or $abyxy'vu$ (when $v \to u$) is a 6-outpath of ab. Assume $v \to y'$. By $b \to y'$ and Lemma 4, there exists a vertex $w \in V_3$ such that $y' \to w$. Since $\{x, v\} \to y'$, we have $w \neq x$ and $w \neq v$. Then $w \in V_3^+ - \{x\}$ and $a \to w$. By $\{a, y'\} \to w$ and Lemma 4, there exists a vertex $u' \in V_2 - \{b\}$ (u' may be equal to u) such that $w \to u'$. Then $u' \to a$ and $abyxy'wu'$ is a 6-outpath of ab.

If $(V_1 - \{a, y\}) \to x$, we have $V_1 \to x \to V_2$ since $\{a, y\} \to x$.

In the case when $(V_1 - \{a\}) \to (V_2 - \{b\})$, we have $(V_1 - \{a\}) \to (V_2 - \{b\}) \cup \{x\}$ and $\{b\} \cup (V_3 - \{x\}) \to (V_1 - \{a\})$. In addition, we also have $(V_1 - \{a\}) \cup \{x\} \to (V_2 - \{b\}) \to \{a\} \cup (V_3 - \{x\})$. Let y' and u be two arbitrary vertices in $V_1 - \{a, y\}$ and $V_2 - \{b\}$, respectively. Then $x \to u \to v \to y'$. Note $y' \in V(a)$ and $v \to a$. We have that ab has a 5-outpath $abyxuv$ and a 6-outpath $abyxuvy'$.

In the other case when $(V_1 - \{a\}) \nrightarrow (V_2 - \{b\})$, there exists an arc uy' from $V_2 - \{b\}$ to $V_1 - \{a\}$ (y' may be equal to y). Let $y_1 \in V_1 - \{a, y'\}$ (when $y' \neq y$,

y_1 may be equal to y). Then $b \to y_1 \to x \to u$. By $b \to y'$ and Lemma 5(3), there exists a vertex $z \in V''$ such that $y' \to z$. Since $u \to y'$, we have $z \neq u$. Note $y' \in V(a)$ and $z \to a$. Then ab has a 5-outpath aby_1xuy' and a 6-outpath $aby_1xuy'z$.

Subcase 2.2.2. $2 \leq |V_2^+| \leq r-1$. By Lemma 5(2), we have $2 \leq |V_2^+| = |V_3^-| \leq r-1$, $1 \leq |V_2^-| = |V_3^+| \leq r-2$. By $V_3^+ \to b$ and Lemma 4, there is an arc by for some $y \in V_1 - \{a\}$. Obviously, $y \in V(a)$.

Subcase 2.2.2.1. $V_3^+ \nrightarrow y$. By the hypothesis, there is a vertex $x \in V_3^+$ such that $y \to x$.

Suppose first that $(V_2^+ - \{b\}) \nrightarrow x$. Then there exists a vertex $u \in V_2^+ - \{b\}$ such that $x \to u$. If $(V_1 - \{a, y\}) \nrightarrow u$, then there is a vertex $y' \in V_1 - \{a, y\}$ such that $u \to y'$. By $u \in V'$ and Lemma 5(3), there exists a vertex $w \in V''$ such that $y' \to w$. Note $y' \in V(a)$ and $w \to a$. Then ab has a 5-outpath $abyxuy'$ and a 6-outpath $abyxuy'w$.

Assume $(V_1 - \{a, y\}) \to u$. Since $\{a, x\} \to u$, we get $N^-(u) = (V_1 - \{y\}) \cup \{x\}$ and $N^+(u) = \{y\} \cup (V_3 - \{x\})$. Let $z \in V_3^-$. Then $u \to z \to a$ and $abyxuz$ is a 5-outpath of ab. We will seek for a 6-outpath of ab. Let $w \in V_2^-$ be arbitrary. If $(V_1 - \{a, y\}) \cup \{w\} \nrightarrow z$, then there is an arc zy' or zw for some $y' \in V_1 - \{a, y\}$. Note $y' \in V(a)$ and $w \to a$. Then $abyxuzy'$ or $abyxuzw$ is a 6-outpath of ab. Assume $(V_1 - \{a, y\}) \cup \{w\} \to z$. Then it is easy to see that $N^-(z) = (V_1 - \{a, y\}) \cup \{u, w\}$ and $N^+(z) = \{a, y\} \cup (V_2 - \{u, w\})$. So $z \to \{b, y\}$. By $\{x, z\} \to b$ and Lemma 4, there exists a vertex $y_0 \in V_1 - \{y\}$ such that $b \to y_0$. Obviously, $y_0 \neq a$ and $y_0 \to u$. Then aby_0xuzy (when $y_0 \to x$) or $abyxy_0uz$ (when $x \to y_0$) is a 6-outpath of ab.

Suppose now that $(V_2^+ - \{b\}) \to x$. By Lemma 4, there exists a vertex $y' \in V_1$ such that $x \to y'$. Since $\{a, y\} \to x$, we have $y' \neq a$ and $y' \neq y$. By $x \in V'$, $x \to y'$ and Lemma 5(3), there is a vertex $z \in V''$ such that $y' \to z$. Then $z \to a$ and $abyxy'z$ is a 5-outpath of ab. We will prove that ab has a 6-outpath.

By $\{a, y\} \to x$ and Lemma 4, there exists a vertex $v \in V_2 - \{b\}$ such that $x \to v$. Since $(V_2^+ - \{b\}) \to x$, we get $v \in V_2^-$. When $v \to y'$, we have that $v \neq z$ and $abyxvy'z$ is a 6-outpath of ab. When $y' \to v$, it is easy to see that $(V_1 - \{a, y, y'\}) \cup V_3^- \nrightarrow v$ (as otherwise, $(V_1 - \{a, y, y'\}) \cup V_3^- \cup \{y', x\} \subseteq N^-(v)$ and $d^-(v) \geq r+1$, a contradiction). So there is a vertex $v' \in (V_1 - \{a, y, y'\}) \cup V_3^-$ such that $v \to v'$. Note $v' \in V(a)$ or $v' \to a$. Then $abyxy'vv'$ is a 6-outpath of ab.

Subcase 2.2.2.2. $(V_2^+ - \{b\}) \nrightarrow y$. By the hypothesis, there is an arc yu for some vertex $u \in V_2^+ - \{b\}$.

Suppose first that $V_3^+ \nrightarrow u$. Then there exists a vertex $x \in V_3^+$ such that $u \to x$. If $(V_1 - \{a, y\}) \nrightarrow x$, then there is a vertex $y' \in V_1 - \{a, y\}$ such that $x \to y'$. By $x \in V'$ and Lemma 5(3), there exists a vertex $w \in V''$ such that $y' \to w$. Note $y' \in V(a)$ and $w \to a$. Then ab has a 5-outpath $abyuxy'$ and a 6-

outpath $abyuxy'w$. Assume $(V_1-\{a,y\}) \to x$. Since $\{a,u\} \to x$, we get $N^-(x) = (V_1-\{y\}) \cup \{u\}$ and $N^+(x) = \{y\} \cup (V_2-\{u\})$. Let $z \in V_2^-$. Then $x \to z \to a$ and $abyuxz$ is a 5-outpath of ab. In addition, we also have $(V_1-\{a,y\}) \cup V_3^- \nrightarrow z$ (as otherwise, $(V_1-\{a,y\}) \cup V_3^- \cup \{x\} \subseteq N^-(z)$ and $d^-(z) \geq r+1$, a contradiction). So there is an arc zy_1 or zv for some $y_1 \in V_1-\{a,y\}$ and $v \in V_3^-$. Note $y_1 \in V(a)$ and $v \to a$. Then $abyuxzy_1$ or $abyuxzv$ is a 6-outpath of ab.

Suppose now that $V_3^+ \to u$. By Lemma 4, there exists a vertex $y' \in V_1$ such that $u \to y'$. Since $\{a,y\} \to u$, we have $y' \neq a$ and $y' \neq y$. By $u \in V'$, $u \to y'$ and Lemma 5(3), there is a vertex $z' \in V''$ such that $y' \to z'$. Then $z' \to a$ and $abyuy'z'$ is a 5-outpath of ab. We will prove that ab has a 6-outpath.

By $\{a,y\} \to u$ and Lemma 4, there exist two distinct vertices $v, v' \in V_3$ such that $u \to \{v,v'\}$. Since $V_3^+ \to u$, we get $v, v' \in V_3^-$. If $v \to y'$, then $v \neq z'$ and $abyxvy'z'$ is a 6-outpath of ab. Assume $y' \to v$. If $(V_1-\{a,y,y'\}) \cup V_2^- \nrightarrow v$, then there is a vertex $w \in (V_1-\{a,y,y'\}) \cup V_2^-$ such that $v \to w$. Note $w \in V(a)$ or $w \to a$. Then $abyuy'vw$ is a 6-outpath of ab. Assume $(V_1-\{a,y,y'\}) \cup V_2^- \to v$. Note $(V_1-\{a,y,y'\}) \cup V_2^- \cup \{y',u\} \to v$ and $d^-(v) = r$. We have $|V_2^-| = 1$ and $N^+(v) = \{a,y\} \cup (V_2^+ - \{u\})$. Then $v \to b$. By $V_3^+ \cup \{v\} \to b$ and Lemma 4, there is a vertex $y_0 \in V_1-\{y\}$ (y_0 may be equal to y') such that $b \to y_0$. Obviously, $y_0 \neq a$ and $y_0 \to v$. Thus, aby_0vyuv' is a 6-outpath of ab.

Subcase 2.2.2.3. $(V_2^+ - \{b\}) \cup V_3^+ \to y$. In this case, we have $V' \to y \to V''$ since $b \to y$. By $a \to b$ and Lemma 4, there exists a vertex $c \in V_3$ such that $b \to c$. Note $V_3^+ \to b$. We get $c \in V_3^-$. Let $x \in V_3^+$ be arbitrary. Note that $\{b,y\} \to c \to a$ and $a \to x \to \{b,y\}$. Then there exists a vertex $u \in (V_1-\{a,y\}) \cup (V_2-\{b\})$ such that $c \to u \to x$ (as otherwise, we have $d^+(c) < d^+(x)$, this is impossible). Then ab has a 5-outpath $abcuxy$. Let $v \in V_3^- - \{c\}$. Then $y \to v$ and ab has a 6-outpath $abcuxyv$.

The proof of Theorem 9 is complete. ∎

Theorems 6–9 give support to the following conjecture.

Conjecture 10. *Let D be an r-regular 3-partite tournament with $r \geq 2$ and partite sets V_1, V_2, V_3. If ab is an arc of D, then the following hold for all $k \in \{1,2,\ldots,r-1\}$.*

(1) *ab has a $(3k-1)$-outpath.*

(2) *ab has a $3k$-outpath.*

(3) *ab has a $(3k+1)$-outpath unless $V_1 \to V_2 \to V_3 \to V_1$.*

Note that the length of the longest path in an r-regular 3-partite tournament is at most $3r-1$. So the value of k cannot exceed $r-1$ in (2) and (3) of Conjecture 10. However, the following example show that (1) of Conjecture 10 is not always true when $k = r$.

Example 11. Let $V_1 = \{a, y\}, V_2 = \{b, u\}$ and $V_3 = \{x, v\}$ be the partite sets of a 3-partite tournament D such that $\{u, v\} \to a \to \{b, x\}$, $\{a, x\} \to b \to \{y, v\}$, $\{y, b\} \to v \to \{a, u\}$ $V_2 \to y \to V_3$, $V_3 \to u \to V_1$ and $V_1 \to x \to V_2$. Then D is 2-regular, but the arc ab has no 5-outpath since there is only one path $abvuyx$ of length 5 starting from ab, which is not an outpath of ab.

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