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ON SEMISYMMETRIC CUBIC GRAPHS OF ORDER $20p^2$, p PRIME

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Abstract

A simple graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive. Let p be an arbitrary prime. Folkman proved [*Regular line-symmetric graphs*, J. Combin. Theory 3 (1967) 215–232] that there is no semisymmetric graph of order $2p$ or $2p^2$. In this paper an extension of his result in the case of cubic graphs of order $20p^2$ is given. We prove that there is no connected cubic semisymmetric graph of order $20p^2$ or, equivalently, that every connected cubic edge-transitive graph of order $20p^2$ is necessarily symmetric.

Keywords: edge-transitive graph, vertex-transitive graph, semisymmetric graph, order of a graph, classification of cubic semisymmetric graphs.

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1. Introduction

In this paper all graphs are finite, undirected and simple, i.e., without loops or multiple edges. A graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive.

The class of semisymmetric graphs was first studied by Folkman [9], who found several infinite families of such graphs and posed eight open problems.

An interesting research problem is to classify connected cubic semisymmetric graphs of various types of orders. Among these, graphs of orders kp^i for prime p

and small k and i have been a target of much research. In [9], Folkman proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for any prime p. In [2] it is proved that there is no connected cubic semisymmetric graph of order $4p^2$ and also in [1] the authors proved that there is no connected cubic semisymmetric graph of order $8p^2$ for any prime p.

For prime p, cubic semisymmetric graphs of order $2p^3$ were investigated in [17], in which they proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray graph.

Connected cubic semisymmetric graphs of orders $4p^3$, $6p^2$, $6p^3$, $8p^3$, $10p^3$, $18p^n$ have been classified in [3, 8, 11, 13] and [22].

In this paper we investigate connected cubic semisymmetric graphs of order $20p^2$ for all primes p. During our trial for classifying such graphs, we managed to prove that no such graph can really exist. In fact we first assume the existence of such a graph and obtain a very useful structure for its group of automorphisms. Then we use this structure to prove that the existence of the graph will actually result in a contradiction. By [23] this result is equivalent to saying that any connected cubic edge-transitive graph of order $20p^2$ is symmetric.

2. Preliminaries

In this paper, the cardinality of a finite set A is denoted by $|A|$. A function f acts on its argument from the left, i.e., we write $f(x)$. The composition, fg, of two functions f and g, is defined as $(fg)(x) = f(g(x))$.

The symmetric and alternating groups of degree n, the dihedral group of order $2n$ and the cyclic group of order n are respectively denoted by $\mathbb{S}_n$, $\mathbb{A}_n$, $\mathbb{D}_{2n}$, $\mathbb{Z}_n$. If G is a group and $H \leq G$, then $\mathrm{Aut}(G)$, G', $Z(G)$, $C_G(H)$ and $N_G(H)$ denote respectively the group of automorphisms of G, the commutator subgroup of G, the center of G, the centralizer and the normalizer of H in G. We also write $H \trianglelefteq^c G$ to denote H is a characteristic subgroup of G. If $H \trianglelefteq^c K \trianglelefteq G$, then $H \trianglelefteq G$. For a prime p dividing the order of finite G, $O_p(G)$ will denote the largest normal p-subgroup of G. It is easy to verify that $O_p(G) \trianglelefteq^c G$.

For a group G and a nonempty set Ω, an *action* of G on Ω is a function $(g, \omega) \to g.\omega$ from $G \times \Omega$ to Ω, where $1.\omega = \omega$ and $g.(h.\omega) = (gh).\omega$, for every $g, h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$, if there is no fear of ambiguity. For $\omega \in \Omega$, the *stabilizer* of ω in G is defined as $G_\omega = \{g \in G : g\omega = \omega\}$. The action is called *semiregular* if the stabilizer of each element in Ω is trivial; it is called *regular* if it is semiregular and transitive.

For any two groups G and H and any homomorphism $\varphi : H \to \mathrm{Aut}(G)$ the *external semidirect product* $G \rtimes_\varphi H$ is defined as the group whose underlying

set is the Cartesian product $G \times H$ and whose binary operation is defined as $(g_1, h_1)(g_2, h_2) = (g_1\varphi(h_1)(g_2), h_1h_2)$. If $\varphi(h) = 1$ for each $h \in H$, then the semidirect product will coincide with the usual direct product. If $G = NK$ where $N \trianglelefteq G$, $K \leq G$ and $N \cap K = 1$, then G is said to be the *internal semidirect product* of N and K. These two concepts are in fact equivalent in the sense that there is some homomorphism $\varphi : K \to \mathrm{Aut}(N)$ where $G \cong N \rtimes_\varphi K$.

The dihedral group $\mathbb{D}_{2n}$ is defined as

$$\mathbb{D}_{2n} = \langle a, b \,|\, a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

So $\mathbb{D}_{2n} = \{a^i \,|\, i = 0, \ldots, n-1\} \cup \{ba^i \,|\, i = 0, \ldots, n-1\}$. All the elements of the form ba^i are of order 2.

Consider the dihedral group of order 12. According to Sylow theorems, the number of Sylow 2-subgroups of $\mathbb{D}_{12} = \{a^i \,|\, i = 0, \ldots, 5\} \cup \{ba^i \,|\, i = 0, \ldots, 5\}$ divides 3. Also the following three subgroups, all isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, are Sylow 2-subgroups of $\mathbb{D}_{12}$:

$$P_1 = \{1, a^3, b, ba^3\},\ P_2 = \{1, a^3, ba, ba^4\},\ P_3 = \{1, a^3, ba^2, ba^5\}.$$

Therefore $\mathbb{D}_{12}$ has exactly three Sylow 2-subgroups which are P_1, P_2 and P_3.

Let Γ be a graph. For two vertices u and v, we write $u \sim v$ to denote u is adjacent to v. If $u \sim v$, then each of the ordered pairs (u, v) and (v, u) is called an *arc*. The set of all vertices adjacent to a vertex u is denoted by $\Gamma(u)$. The *degree* or *valency* of u is $|\Gamma(u)|$. We call Γ *regular* if all of its vertices have the same valency. The vertex set, the edge set, the arc set and the set of all automorphisms of Γ are respectively denoted by $V(\Gamma)$, $E(\Gamma)$, $\mathrm{Arc}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$. If Γ is a graph and $N \trianglelefteq \mathrm{Aut}(\Gamma)$, then Γ_N will denote a simple undirected graph whose vertices are the orbits of N in its action on $V(\Gamma)$, and where two vertices Nu and Nv are adjacent if and only if $u \sim nv$ in Γ, for some $n \in N$.

Let Γ_c and Γ be two graphs. Then Γ_c is said to be a *covering graph* for Γ if there is a surjection $f : V(\Gamma_c) \to V(\Gamma)$ which preserves adjacency and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. f is called a *covering projection*. Clearly, if Γ is bipartite, then so is Γ_c. For each $u \in V(\Gamma)$, the *fibre* on u is defined as $fib_u = f^{-1}(u)$. The following important set is a subgroup of $\mathrm{Aut}(\Gamma_c)$ and is called the *group of covering transformations* for f:

$$CT(f) = \{\sigma \in \mathrm{Aut}(\Gamma_c) \,|\, \forall u \in V(\Gamma), \sigma(fib_u) = fib_u\}.$$

It is known that $K = CT(f)$ acts semiregularly on each fibre [14]. If this action is regular, then Γ_c is said to be a *regular K-cover* of Γ.

Let $X \leq \mathrm{Aut}(\Gamma)$. Then Γ is said to be *X-vertex-transitive*, *X-edge-transitive* or *X-arc-transitive* if X acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $\mathrm{Arc}(\Gamma)$ respectively.

The graph Γ is called *X-semisymmetric* if it is regular and X-edge-transitive but not X-vertex-transitive. Also Γ is called *X-symmetric* if it is X-vertex-transitive and X-arc-transitive. For $X = \mathrm{Aut}(\Gamma)$, we omit X and simply talk about Γ being edge-transitive, vertex-transitive, symmetric or semisymmetric. An X-edge-transitive but not X-vertex-transitive graph is necessarily bipartite, where the two partites are the orbits of the action of X on $V(\Gamma)$. If Γ is regular, then the two partite sets have equal cardinality. So an X-semisymmetric graph is bipartite such that X is transitive on each partite but X carries no vertex from one partite set to the other.

There are only two symmetric cubic graphs of order 20 which are denoted by F20A and F20B. Only F20B is bipartite [7].

Any minimal normal subgroup of a finite group, is the internal direct product of isomorphic copies of a simple group.

A finite simple group G is called a K_n-group if its order has exactly n distinct prime divisors, where $n \in \mathbb{N}$. The following two results determine all simple K_3-groups and K_4-groups [4, 12, 20, 25].

Theorem 2.1. (i) *If G is a simple K_3-group, then G is one of the following groups: $\mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, $L_2(2^3)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.*

(ii) *If G is a simple K_4-group, then G is one of the following groups:*

(1) $\mathbb{A}_7$, $\mathbb{A}_8$, $\mathbb{A}_9$, $\mathbb{A}_{10}$, M_{11}, M_{12}, J_2, $L_2(2^4)$, $L_2(5^2)$, $L_2(7^2)$, $L_2(3^4)$, $L_2(97)$, $L_2(3^5)$, $L_2(577)$, $L_3(2^2)$, $L_3(5)$, $L_3(7)$, $L_3(2^3)$, $L_3(17)$, $L_4(3)$, $U_3(2^2)$, $U_3(5)$, $U_3(7)$, $U_3(2^3)$, $U_3(3^2)$, $U_4(3)$, $U_5(2)$, $S_4(2^2)$, $S_4(5)$, $S_4(7)$, $S_4(3^2)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $Sz(2^3)$, $Sz(2^5)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) $L_2(r)$ *where r is a prime, $r^2 - 1 = 2^a \cdot 3^b \cdot s$, $s > 3$ is a prime, $a, b \in \mathbb{N}$;*

(3) $L_2(2^m)$ *where m, $2^m - 1$, $\frac{2^m+1}{3}$ are primes greater than 3;*

(4) $L_2(3^m)$ *where m, $\frac{3^m+1}{4}$ and $\frac{3^m-1}{2}$ are odd primes.*

Theorem 2.2 [21]**.** *If H is a subgroup of a group G, then $C_G(H) \trianglelefteq N_G(H)$ and $\frac{N_G(H)}{C_G(H)}$ is isomorphic to a subgroup of $Aut(H)$.*

Theorem 2.3 [19]**.** *Let G be a finite group and p a prime. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.*

Theorem 2.4 ([18], Theorem 9.1.2)**.** *Let G be a finite group and $N \trianglelefteq G$. If $|N|$ and $\left|\frac{G}{N}\right|$ are relatively prime, then G has a subgroup H such that $G = NH$ and $N \cap H = 1$ (therefore G is the internal semidirect product of N and H).*

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

Theorem 2.5 [21]**.** *For any two distinct primes p and q and any two nonnegative integers a and b, every finite group of order $p^a q^b$ is solvable.*

In the following proposition, the inverse of a pair (a, b) is meant to be (b, a). Also for each i, A_i, B_i, C_i and D_i are noncyclic groups of order i with known structures. We will not need their structures.

Theorem 2.6 [10]. *If Γ is a connected cubic X-semisymmetric graph, then the order of the stabilizer of any vertex is of the form $2^r \cdot 3$ for some $0 \leq r \leq 7$. More precisely, if $\{u, v\}$ is any edge of Γ, then the pair (X_u, X_v) can only be one of the following fifteen pairs or their inverses:*
$(\mathbb{Z}_3, \mathbb{Z}_3)$, $(\mathbb{S}_3, \mathbb{S}_3)$, $(\mathbb{S}_3, \mathbb{Z}_6)$, $(\mathbb{D}_{12}, \mathbb{D}_{12})$, $(\mathbb{D}_{12}, \mathbb{A}_4)$, $(\mathbb{S}_4, \mathbb{D}_{24})$, $(\mathbb{S}_4, \mathbb{Z}_3 \rtimes \mathbb{D}_8)$, $(\mathbb{A}_4 \times \mathbb{Z}_2, \mathbb{D}_{12} \times \mathbb{Z}_2)$, $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{D}_8 \times \mathbb{S}_3)$, $(\mathbb{S}_4, \mathbb{S}_4)$, $(\mathbb{S}_4 \times \mathbb{Z}_2, \mathbb{S}_4 \times \mathbb{Z}_2)$, (A_{96}, B_{96}), (A_{192}, B_{192}), (C_{192}, D_{192}), (A_{384}, B_{384}).

Proposition 2.7 [24]. *Let Γ be a connected cubic X-semisymmetric graph for some $X \leq Aut(\Gamma)$ and let $N \trianglelefteq X$. If $\left|\frac{X}{N}\right|$ is not divisible by 3, then Γ is also N-semisymmetric.*

Proposition 2.8 [17]. *Let Γ be a connected cubic X-semisymmetric graph for some $X \leq Aut(\Gamma)$; then either $\Gamma \cong K_{3,3}$, the complete bipartite graph on 6 vertices, or X acts faithfully on each of the bipartition sets of Γ.*

Theorem 2.9 [15]. *Let Γ be a connected cubic X-semisymmetric graph. Let $\{U, W\}$ be a bipartition for Γ and assume $N \trianglelefteq X$. If the actions of N on both U and W are intransitive, then N acts semiregularly on both U and W, Γ_N is $\frac{X}{N}$-semisymmetric, and Γ is a regular N-covering of Γ_N.*

This theorem has a nice result. For every normal subgroup $N \trianglelefteq X$ either N is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of N is divisible by $|U| = |W|$. In the latter case, according to Theorem 2.9, the induced action of N on both U and W is semiregular and hence the order of N divides $|U| = |W|$. So we have the following handy corollary.

Corollary 2.10. *If Γ is a connected cubic X-semisymmetric graph with $\{U, W\}$ as a bipartition and $N \trianglelefteq X$, then either $|N|$ divides $|U|$ or $|U|$ divides $|N|$.*

Following [10] (see also [16]) *the coset graph* $C(G; H_0, H_1)$ of a group G with respect to finite subgroups H_0 and H_1 is a bipartite graph with $\{H_0 g \mid g \in G\}$ and $\{H_1 g \mid g \in G\}$ as its bipartition sets of vertices where $H_0 g$ is adjacent to $H_1 g'$ whenever $H_0 g \cap H_1 g' \neq \emptyset$. The following proposition may be extracted from [10].

Proposition 2.11. *Let G be a finite group and $H_0, H_1 \leq G$. The coset graph $C(G; H_0, H_1)$ has the following properties:*

(i) *$C(G; H_0, H_1)$ is regular of valency d if and only if $H_0 \cap H_1$ has index d in both H_0 and H_1.*

(ii) *$C(G; H_0, H_1)$ is connected if and only if $G = \langle H_0, H_1 \rangle$.*

(iii) *G acts on $C(G; H_0, H_1)$ by right multiplication. Moreover, this action is faithful if and only if $Core_G(H_0 \cap H_1) = 1$.*

(iv) *In the case when the action of G is faithful, the coset graph $C(G; H_0, H_1)$ is G-semisymmetric.*

Proposition 2.12 [16]**.** *Let Γ be a regular graph and $G \leq Aut(\Gamma)$. If Γ is G-semisymmetric, then Γ is isomorphic to the coset graph $C(G; G_u, G_v)$ where u and v are adjacent vertices.*

3. Main Result

In this section, our goal is to prove the following important result.

Theorem 3.1. *There is no connected cubic semisymmetric graph of order $20p^2$ for any prime p.*

In the remainder we first state and prove some lemmas and then prove the main theorem.

Lemma 3.2. *Let G be a finite group and $H \trianglelefteq G$ such that $\frac{G}{H}$ is nonabelian simple. If $H \cong \mathbb{Z}_p$ or $\mathbb{Z}_{2p}$ for an odd prime p, then $H = Z(G)$.*

Proof. Since $\frac{G}{H}$ is nonabelian simple, G is nonabelian and H is a maximal normal subgroup of G. Also $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$ and $\mathrm{Aut}(\mathbb{Z}_{2p}) \cong \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_p) \cong \mathrm{Aut}(\mathbb{Z}_2) \times \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$. Because H is abelian, we have $H \leq C_G(H) \trianglelefteq N_G(H) = G$. So either $C_G(H) = H$ or $C_G(H) = G$. In the former case, according to Theorem 2.2, we should have $\frac{G}{H} \leq \mathrm{Aut}(H) \cong \mathbb{Z}_{p-1}$ which is impossible. On the other hand, $C_G(H) = G$ implies $H \leq Z(G)$ which again results in $Z(G) = H$ or $Z(G) = G$ since H is a maximal normal subgroup of G. As G is not abelian, the latter is impossible and so $Z(G) = H$. ■

Lemma 3.3. (i) *For any odd prime number p, there are only two groups of order $2p$: $\mathbb{Z}_{2p}$, $\mathbb{D}_{2p}$. The cardinalities of $Z(\mathbb{D}_{2p})$ and $Aut(\mathbb{D}_{2p})$ are 2 and $p(p-1)$ respectively.*

(ii) *There are only four groups of order 70 : $\mathbb{Z}_{70}$, $\mathbb{D}_{70}$, X_{70} and Y_{70}, where*

$$X_{70} = \left\langle a, b \,|\, a^{35} = b^2 = 1, b^{-1}ab = a^6 \right\rangle$$

$$Y_{70} = \left\langle a, b \,|\, a^{35} = b^2 = 1, b^{-1}ab = a^{-6} \right\rangle.$$

Moreover,

$$Z(\mathbb{D}_{70}) \cong \mathbb{Z}_2,\ Z(X_{70}) \cong \mathbb{Z}_5,\ Z(Y_{70}) \cong \mathbb{Z}_7,$$

$$|Aut(\mathbb{D}_{70})| = 24 \cdot 35,\ |Aut(X_{70})| = 24 \cdot 7 \text{ and } |Aut(Y_{70})| = 24 \cdot 5.$$

Proof. We prove part (ii). The proof for part (i) is similar. Let G be a group of order 70. Using the Sylow theorems, it is easily verified that the number of Sylow 5-subgroups as well as the number of Sylow 7-subgroups of G is 1. So if P and Q are the Sylow 5-subgroup and the Sylow 7-subgroup of G respectively, then $P, Q \trianglelefteq G$ and hence $N = PQ \trianglelefteq G$. Now $N \cong P \times Q \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$ is cyclic. Let $N = \langle a \rangle$ and take $b \in G$ to be an element of order 2. There is some $1 \leq i < 35$ for which $b^{-1}ab = a^i$. Therefore $a^{i^2} = b^{-1}(b^{-1}ab)b = a$ and so $i^2 \equiv 1 \pmod{35}$. This congruence has only four solutions $i = 1, 34, 6, 29$ which respectively correspond to $\mathbb{Z}_{70}$, $\mathbb{D}_{70}$, X_{70} and Y_{70}.

Now consider X_{70}. Each element equals a^i or a^ib for some i. It can be easily verified that no element of the form a^ib belongs to the center, and that $a^i \in Z(X_{70})$ if and only if $i \equiv 0 \pmod 7$. So $Z(X_{70}) = \langle a^7 \rangle$.

Every automorphism f of X_{70} is uniquely characterized by the two values $f(a)$ and $f(b)$. By an order argument, we find out that a group of order 70 has only one subgroup of order 35. So f takes $\langle a \rangle$ to $\langle a \rangle$ and hence $f(a) = a^i$ for some i coprime to 35. Therefore there are $\varphi(35)$ possibilities for $f(a)$, where φ is the Euler function. Also $f(b)$ must be an element of order 2 and has 7 possibilities, since the elements of order 2 in X_{70} are of the form a^ib where $i \equiv 0 \pmod 5$. We conclude that $|\mathrm{Aut}(X_{70})| = \varphi(35) \times 7 = 24 \times 7$. The corresponding results for $\mathbb{D}_{70}$ and Y_{70} follow quite similarly. ■

Lemma 3.4. *There are only three simple K_4-groups whose orders are of the form $2^i \cdot 3 \cdot 5 \cdot p$ for some prime $p > 5$ and some $1 \leq i \leq 8$: $L_2(2^4)$, $L_2(11)$ and $L_2(31)$.*

Proof. Considering the powers of primes, there is no possibility for such a group in sub-item (4) of item (ii) of Theorem 2.1. By inspecting orders of groups in sub-item (1), the only group of the desired form is $L_2(2^4)$. As for sub-item (3), let $L_2(2^m)$ be a group of order $2^i \cdot 3 \cdot 5 \cdot p$; then

$$2^m \cdot 3 \cdot (2^m - 1) \cdot \left(\tfrac{2^m+1}{3}\right) = 2^i \cdot 3 \cdot 5 \cdot p,$$

where m, $2^m - 1$ and $\frac{2^m+1}{3}$ are all primes according to Theorem 2.1. This equation has no answer as neither $2^m - 1$ nor $\frac{2^m+1}{3}$ could be equal to 5. Finally, consider groups $L_2(r)$ in sub-item (2). If for odd prime r and for prime $s > 3$ we have $r^2 - 1 = 2^a \cdot 3^b \cdot s$ and

$$2^{a-1} \cdot 3^b \cdot s \cdot r = 2^i \cdot 3 \cdot 5 \cdot p,$$

then $b = 1$, $a - 1 = i$ and either $s = 5$ or $r = 5$. The equality $r = 5$ is not possible, since $L_2(5)$ is not a K_4-group. Also if $s = 5$, then the equation $r^2 - 1 = 2^a \cdot 3 \cdot 5$ gives us only two solutions $r = 11, 31$ when a spans integers $2, 3, \ldots, 9$. ■

Lemma 3.5. *Suppose Γ is a semisymmetric cubic graph of order $20p^2$ where $p > 5$ is a prime. Let $A = Aut(\Gamma)$. Then*

(i) *If $O_p(A) = 1$, then A does not have a normal subgroup of order 10.*

(ii) *If $|O_p(A)| = p$, then A does not have a normal subgroup of order $10p$.*

Proof. Let $\{U, W\}$ be the bipartition for Γ. Then $|U| = |W| = 10p^2$. Also if $u \in U$ is an arbitrary vertex, according to Theorem 2.6, $|A_u| = 2^r \cdot 3$ for some $0 \leq r \leq 7$. Due to transitivity of A on U, the equality $[A : A_u] = |U|$ holds which yields $|A| = 2^{r+1} \cdot 3 \cdot 5 \cdot p^2$.

Let M be a normal subgroup of A of order 10 or $10p$. Then M is intransitive on the partite sets and according to Theorem 2.9 the quotient graph Γ_M is $\frac{A}{M}$-semisymmetric with a bipartition $\{U_M, W_M\}$. We prove that the combinations $(|O_p(A)|, |M|) = (1, 10)$ or $(p, 10p)$ lead to contradictions.

To prove (i), let $O_p(A) = 1$ and $|M| = 10$. Then $|U_M| = |W_M| = p^2$ and $\left|\frac{A}{M}\right| = 2^r \cdot 3 \cdot p^2$. Let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{A}{M}$. If $\frac{K}{M}$ is unsolvable, it must be a simple group of order $2^i \cdot 3 \cdot p^2$ for some i. But there is no simple K_3-group of such order. So $\frac{K}{M}$ is solvable and hence elementary abelian. Whether it is intransitive or transitive on the partite sets, its order must be p or p^2. Therefore $|K| = 10p^i$ for $i = 1$ or 2. The Sylow p-subgroup of K is normal in K. So it is characteristic in K and hence normal in A, contradicting the assumption that $O_p(A) = 1$.

Now consider part (ii) and suppose $|O_p(A)| = p$ and $|M| = 10p$. In this case $|U_M| = |W_M| = p$ and $\left|\frac{A}{M}\right| = 2^r \cdot 3 \cdot p$. Again let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{A}{M}$. If $\frac{K}{M}$ is unsolvable, it must be a simple group of order $2^i \cdot 3 \cdot p$ for some i and $p > 5$. So $\frac{K}{M} \cong L_2(7)$ and $p = 7$. Since 3 does not divide the order of $\frac{A}{K} \cong \frac{\frac{A}{M}}{\frac{K}{M}}$, we conclude that Γ is K-semisymmetric according to Proposition 2.7. Because $\frac{K}{M}$ is nonabelian simple, M is a maximal normal subgroup of K. Also note that $C_K(M) \trianglelefteq N_K(M) = K$. So $M \leq MC_K(M) \trianglelefteq K$ and hence either $MC_K(M) = M$ or $MC_K(M) = K$. If $MC_K(M) = M$, then $C_K(M) \leq M$ and so $C_K(M) = C_K(M) \cap M = Z(M)$. Now according to Theorem 2.2 the order of $\frac{K}{Z(M)}$ must divide $|\mathrm{Aut}(M)|$. The order of M is 70 and so according to Lemma 3.3, $M \cong \mathbb{Z}_{70}$, $\mathbb{D}_{70}$, X_{70} and Y_{70}. We have $|K| = |L_2(7)| \times |M| = 2^4 \cdot 3 \cdot 5 \cdot 7^2$. If $M \cong \mathbb{Z}_{70}$, then $\left|\frac{K}{Z(M)}\right| = 2^3 \cdot 3 \cdot 7$ does not divide $|\mathrm{Aut}(M)| = \varphi(70) = 24$. Also for the three remaining cases of M, the orders of $Z(M)$ and $\mathrm{Aut}(M)$ are known according to Lemma 3.3, and in each case one can make sure that $\left|\frac{K}{Z(M)}\right|$ does not divide $|\mathrm{Aut}(M)|$. So the equality $MC_K(M) = M$ could not be possible.

On the other hand if $MC_K(M) = K$, then $|K| = \frac{|M||C_K(M)|}{|M \cap C_K(M)|} = \frac{|M||C_K(M)|}{|Z(M)|}$. From this equation, in each of the four possibilities for M one can obtain $|C_K(M)|$, since the order of $Z(M)$ is known in each case. As Γ is K-semisymmetric and $C_K(M) \trianglelefteq K$, according to Corollary 2.10 either $|U| = 10 \cdot 7^2$ divides $|C_K(M)|$ or

$|C_K(M)|$ divides $10 \cdot 7^2$. In none of the four cases, these are possible. For example if $M \cong \mathbb{D}_{70}$, then from $|K| = \frac{|M||C_K(M)|}{|Z(M)|}$ we obtain $|C_K(M)| = 2^4 \cdot 3 \cdot 7$ which is neither divisible by $10 \cdot 7^2$ nor it divides $10 \cdot 7^2$. So $\frac{K}{M}$ cannot be unsolvable.

If $\frac{K}{M}$ is solvable, it is elementary abelian. Whether it is intransitive or transitive on the partite sets, its order must equal p. Therefore $|K| = 10p^2$. The Sylow p-subgroup of K is normal in K. So it is characteristic in K and hence normal in A, contradicting the assumption that $|O_p(A)| = p$. ■

Lemma 3.6. *If $p > 5$ is a prime and Γ is a connected cubic semisymmetric graph of order $20p^2$, then $Aut(\Gamma)$ has a normal Sylow p-subgroup.*

Proof. Take $\{U, W\}$ to be a bipartition for Γ and let $A =$Aut(Γ). Then $|U| = |W| = 10p^2$ and $|A| = 2^{r+1} \cdot 3 \cdot 5 \cdot p^2$ for some $0 \leq r \leq 7$. Let $N \cong T^k$ be a minimal normal subgroup of A, where T is simple.

If T is nonabelian, then $k = 1$ and $N = T$ since the powers of 3 and 5 in $|A|$ equal 1. According to Corollary 2.10 either $|N|$ divides $|U| = 10p^2$ or $10p^2$ divides $|N|$. If $|N|$ divides $10p^2$, then since $|N|$ is divisible by at least three distinct primes (Theorem 2.5), we must have $|N| = 2 \cdot 5 \cdot p^i$ for $i = 1$ or 2, and so N is a simple K_3-group. But the order of every simple K_3-group, listed in Theorem 2.1, is divisible by 3, a contradiction. Therefore $|N|$ is divisible by $10p^2$. Again since the order of every simple K_3-group is divisible by 3, N must be a simple K_4-group whose order is of the form $2^i \cdot 3 \cdot 5 \cdot p^2$. But no such simple K_4-group exists.

We conclude that N is elementary abelian and hence it follows from Corollary 2.10 that $|N|$ divides $10p^2$. Therefore $N \cong \mathbb{Z}_2$, $\mathbb{Z}_5$, $\mathbb{Z}_p$ or $\mathbb{Z}_p^2$. In each case Γ_N would itself be a connected cubic $\frac{A}{N}$-semisymmetric graph of order $\frac{20p^2}{|N|}$. We claim $|O_p(A)| = p^2$ by showing that $|O_p(A)| < p^2$ will result in a contradiction as follows.

Case 1. $O_p(A) = 1$. If this case happens, then the minimal normal subgroup of A is $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_5$ and Γ_N is $\frac{A}{N}$-semisymmetric of order $10p^2$ or $4p^2$ respectively. Take $\{U_N, W_N\}$ to be the bipartition for Γ_N. Also let $\frac{M}{N}$ be a minimal normal subgroup of $\frac{A}{N}$.

If $N \cong \mathbb{Z}_2$, then $\left|\frac{A}{N}\right| = 2^r \cdot 3 \cdot 5 \cdot p^2$ and $|U_N| = |W_N| = 5p^2$. If $\frac{M}{N}$ is unsolvable, then it must be a simple K_4-group whose order is of the form $2^i \cdot 3 \cdot 5 \cdot p^2$. But there is no such simple K_4-group. If $\frac{M}{N}$ is solvable, it is elementary abelian and hence according to Corollary 2.10 its order divides $5p^2$ which yields $\frac{M}{N} \cong \mathbb{Z}_5$ or $\mathbb{Z}_p^i$ for $i = 1$ or 2. It follows from part (i) of Lemma 3.5 that $\frac{M}{N}$ cannot be isomorphic to $\mathbb{Z}_5$. Accordingly, $\frac{M}{N} \cong \mathbb{Z}_p^i$ and so $|M| = 2p^i$ for $i = 1$ or 2. If P is a Sylow p-subgroup of M, then $P \trianglelefteq^c M \trianglelefteq A$ which implies $P \trianglelefteq A$, contradicting our assumption.

If $N \cong \mathbb{Z}_5$, then $\left|\frac{A}{N}\right| = 2^{r+1} \cdot 3 \cdot p^2$ and $|U_N| = |W_N| = 2p^2$. Now if $\frac{M}{N}$ is unsolvable, then it must be a simple group whose order is of the form $2^i \cdot 3 \cdot p^2$ for some $i \geq 1$. But there is no such simple K_3-group. If $\frac{M}{N}$ is solvable, it is elementary abelian and according to Corollary 2.10 its order divides $|U_N| = 2p^2$ which yields $\frac{M}{N} \cong \mathbb{Z}_2$ or $\mathbb{Z}_p^i$ for $i = 1$ or 2. Again it follows from Lemma 3.5 that $\frac{M}{N}$ is not isomorphic to $\mathbb{Z}_2$ and so $\frac{M}{N} \cong \mathbb{Z}_p^i$ which results in $|M| = 5p^i$ for $i = 1$ or 2. Again a Sylow p-subgroup of M is normal in A, contradicting our assumption on $O_p(A)$.

Case 2. $|O_p(A)| = p$. Let $M = O_p(A)$. According to Theorem 2.9, Γ_M is connected cubic $\frac{A}{M}$-semisymmetric with the bipartition $\{U_M, W_M\}$, where $|U_M| = |W_M| = 10p$. We have $\left|\frac{A}{M}\right| = 2^{r+1} \cdot 3 \cdot 5 \cdot p$. Let $\frac{L}{M}$ be a minimal normal subgroup of $\frac{A}{M}$ and consider the following two subcases based on solvability or unsolvability of $\frac{L}{M}$.

(a) $\frac{L}{M}$ is unsolvable. In this case $\frac{L}{M}$ should be of an order divisible by $|U_M| = 10p$ and so it is a simple K_4-group whose order equals $2^i \cdot 3 \cdot 5 \cdot p$ for some $1 \leq i \leq 8$. According to Lemma 3.4, $\frac{L}{M}$ is isomorphic to either $L_2(2^4)$ of order $2^4 \cdot 3 \cdot 5 \cdot 17$, $L_2(11)$ of order $2^2 \cdot 3 \cdot 5 \cdot 11$ or $L_2(31)$ of order $2^5 \cdot 3 \cdot 5 \cdot 31$. These three groups correspond to $p = 17$, 11 and 31 respectively. In each case the order of $\frac{A}{L} \cong \frac{\frac{A}{M}}{\frac{L}{M}}$ is not divisible by 3. So Γ would also be L-semisymmetric according to Proposition 2.7. Also since in all the three cases $\frac{L}{M}$ is nonabelian simple and $M \cong \mathbb{Z}_p$, according to Lemma 3.2 $Z(L) = M$. Now $L' \cap Z(L) \leq Z(L)$ and p does not divide the order of $L' \cap Z(L)$ according to Theorem 2.3. Therefore $L' \cap Z(L) = 1$. The relations $Z(L) \leq L'Z(L) \trianglelefteq L$ imply $L'Z(L) = Z(L)$ or $L'Z(L) = L$. If $L'Z(L) = Z(L)$, then $L' \leq Z(L)$ and so $L' = L' \cap Z(L) = 1$ which is not possible as L is not abelian. On the other hand if $L'Z(L) = L$, then $|L| = |L'||Z(L)|$ and so $|L'| = 2^i \cdot 3 \cdot 5 \cdot p$, where i depends on p (e.g. $i = 2$ for $p = 11$). Since Γ is L-semisymmetric, according to Corollary 2.10, $|L'|$ either divides $|U| = 10p^2$ or is divisible by $10p^2$. With the order that we just obtained for L', none of these divisibilities hold.

(b) $\frac{L}{M}$ is solvable. In this case, $\frac{L}{M}$ is elementary abelian and hence intransitive on both U_M and W_M. So $\frac{L}{M} \cong \mathbb{Z}_2$, $\mathbb{Z}_5$ or $\mathbb{Z}_p$. The isomorphism $\frac{L}{M} \cong \mathbb{Z}_p$ could not hold as it would lead to $|L| = p^2$ which contradicts the assumption that $|O_p(A)| = p$. In the following we discuss the two remaining cases.

(b1) $\frac{L}{M} \cong \mathbb{Z}_2$. This yields $|L| = 2p$. Consider the graph Γ_L which is connected cubic $\frac{A}{L}$-semisymmetric (Theorem 2.9) with the bipartition $\{U_L, W_L\}$, where $|U_L| = |W_L| = 5p$. Let $\frac{T}{L}$ be a minimal normal subgroup of $\frac{A}{L}$. It is either solvable or unsolvable. In the following we discuss that both cases lead to contradictions.

Suppose $\frac{T}{L}$ is solvable. Then $\frac{T}{L} \cong \mathbb{Z}_5$ or $\mathbb{Z}_p$. The case $\frac{T}{L} \cong \mathbb{Z}_5$ is not possible

since according to Lemma 3.5, A does not have any normal subgroup of order $10p$. Also the case $\frac{T}{L} \cong \mathbb{Z}_p$ yields $|T| = 2p^2$ and a Sylow p-subgroup of T would be normal in A, contradicting our assumption that $|O_p(A)| = p$.

Now suppose $\frac{T}{L}$ is unsolvable. So it should be a simple group of order $2^i \cdot 3 \cdot 5 \cdot p$ for some $1 \leq i \leq 7$. Again according to Lemma 3.4, $\frac{T}{L} \cong L_2(2^4)$, $L_2(11)$ or $L_2(31)$ which respectively correspond to $p = 17, 11, 31$. In each case Γ is T-semisymmetric according to Proposition 2.7, since the order of $\frac{A}{T} \cong \frac{\frac{A}{L}}{\frac{T}{L}}$ is not divisible by 3. As $|L| = 2p$ and p is odd prime, according to Lemma 3.3 we have $L \cong \mathbb{Z}_{2p}$ or $\mathbb{D}_{2p}$.

First suppose $L \cong \mathbb{Z}_{2p}$. Then according to Lemma 3.2, $Z(T) = L$. Now $T' \cap Z(T) \leq Z(T) \cong \mathbb{Z}_{2p}$ and p does not divide the order of $T' \cap Z(T)$ according to Theorem 2.3. Therefore $|T' \cap L| = |T' \cap Z(T)| = 1$ or 2. Since $\frac{T}{L}$ is nonabelian simple, $\frac{T'L}{L} = \left(\frac{T}{L}\right)' = \frac{T}{L}$ and so $T'L = T$. So $|T| = |T'| \cdot \frac{|L|}{|T' \cap L|}$. Because $|T| = 2^i \cdot 3 \cdot 5 \cdot p^2$ for some $i > 1$ and $\frac{|L|}{|T' \cap L|} = p$ or $2p$, we obtain $|T'| = 2^j \cdot 3 \cdot 5 \cdot p$ for some j. The graph Γ is T-semisymmetric and so according to Corollary 2.10 either $|T'|$ divides $|U| = 10p^2$ or $|T'|$ is divisible by $10p^2$. Both these cases are inconsistent with the order that we just obtained for T'.

Next assume $L \cong \mathbb{D}_{2p}$. The relations $L \leq LC_T(L) \trianglelefteq T$ imply $LC_T(L) = L$ or $LC_T(L) = T$. If $LC_T(L) = T$, then $|T| = \frac{|L||C_T(L)|}{|L \cap C_T(L)|}$. Since $|L \cap C_T(L)| = |Z(L)| = |Z(\mathbb{D}_{2p})| = 2$, we will have $|T| = \frac{|L||C_T(L)|}{2}$ and so $|C_T(L)| = 2^j \cdot 3 \cdot 5 \cdot p$ for some j. With this order, the normal subgroup $C_T(L) \trianglelefteq T$ does not satisfy Corollary 2.10. If $LC_T(L) = L$, then $C_T(L) \leq L$ and hence $C_T(L) = C_T(L) \cap L = Z(L) \cong Z(\mathbb{D}_{2p}) \cong \mathbb{Z}_2$. According to Theorem 2.2, $\frac{T}{C_T(L)} \leq \mathrm{Aut}(L) \cong \mathrm{Aut}(\mathbb{D}_{2p})$ and so $\left|\frac{T}{C_T(L)}\right| = \frac{2^i \cdot 3 \cdot 5 \cdot p^2}{2}$ divides $|\mathrm{Aut}(\mathbb{D}_{2p})| = p(p-1)$ which is impossible.

(b2) $\frac{L}{M} \cong \mathbb{Z}_5$. In this case $|L| = 5p$. The graph Γ_L is connected cubic $\frac{A}{L}$-semisymmetric (Theorem 2.9) with the bipartition $\{U_L, W_L\}$, where $|U_L| = |W_L| = 2p$. Let $\frac{T}{L}$ be a minimal normal subgroup of $\frac{A}{L}$.

If $\frac{T}{L}$ is solvable, then $\frac{T}{L} \cong \mathbb{Z}_2$ or $\mathbb{Z}_p$. The case $\frac{T}{L} \cong \mathbb{Z}_2$ is ruled out, as a result of part (ii) of Lemma 3.5. Also the case $\frac{T}{L} \cong \mathbb{Z}_p$ yields $|T| = 5p^2$ and a Sylow p-subgroup of T would be normal in A, contradicting our assumption that $|O_p(A)| = p$.

If $\frac{T}{L}$ is unsolvable, then it is a simple group of order $2^i \cdot 3 \cdot p$ for some $1 \leq i \leq 8$. So $\frac{T}{L} \cong \mathbb{A}_5, L_2(7)$ according to Theorem 2.1. But since $p > 5$, we may only have $\frac{T}{L} \cong L_2(7)$ and $p = 7$. The order of $\frac{\frac{A}{L}}{\frac{T}{L}}$ is not divisible by 3. So Γ_L is also G-semisymmetric where $G = \frac{T}{L}$. Therefore G is transitive on both U_L and W_L, each with $2p = 14$ points. So for any pair of vertices $u \in U_L$ and $w \in W_L$, the stabilizers G_u and G_w are of order 12. For any prime power q all subgroups of the group $L_2(q)$ have been characterized (see [21], Chapter 3). The only subgroup of

$L_2(7)$ of order 12 is $\mathbb{A}_4$. So $G_u \cong G_w \cong \mathbb{A}_4$. But the pair $(G_u, G_w) = (\mathbb{A}_4, \mathbb{A}_4)$ is not possible for an edge $\{u, w\}$ of a cubic G-semisymmetric graph according to Theorem 2.6. ■

Lemma 3.7. *For any prime p, the group $GL_2(p)$ does not have a subgroup isomorphic to $\mathbb{A}_5$.*

Proof. For $p = 2, 3$ the claim follows by comparing the orders. Let $p > 3$ and suppose on the contrary that $\mathbb{A}_5 \cong H \leq GL_2(p)$. Then $H = H' \leq (GL_2(p))' = SL_2(p)$. It is a well-known fact that $SL_2(p)$ has only one involution for odd prime p, whereas $\mathbb{A}_5$ has more that one element of order 2, a contradiction. ■

Lemma 3.8. *Suppose $p > 5$ is a prime and Γ is a connected cubic semisymmetric graph of order $20p^2$. Let $A = Aut(\Gamma)$ and take M to be the (normal) Sylow p-subgroup of A. For $G = \mathbb{A}_5$ or $\mathbb{S}_5$ if $\frac{A}{M} \cong G$, then*

(1) *For each vertex u the stabilizer A_u is isomorphic to a subgroup of G.*

(2) *$A \cong M \rtimes_\varphi G$ for some homomorphism $\varphi : G \to Aut(M)$.*

Proof. For each vertex u of Γ, $MA_u \leq A$. Therefore $A_u \cong \frac{A_u}{M \cap A_u} \cong \frac{MA_u}{M} \leq \frac{A}{M}$. This proves (1). Also it follows from Theorem 2.4 that $A = MH$ for some subgroup $H \leq A$ where $M \cap H = 1$. So A is the internal semidirect product of M and H and hence it is isomorphic to the external semidirect product of M and H; i.e., $A \cong M \rtimes_\psi H$ for some $\psi : H \to \mathrm{Aut}(M)$. Since $G \cong \frac{A}{M} = \frac{MH}{M} \cong \frac{H}{M \cap H} \cong H$, we can write $A \cong M \rtimes_\varphi G$ for some $\varphi : G \to \mathrm{Aut}(M)$. ■

Lemma 3.9. *Suppose $p > 5$ is a prime and Γ is a connected cubic semisymmetric graph of order $20p^2$. Let $A = Aut(\Gamma)$ and take M to be the Sylow p-subgroup of A. Then $\frac{A}{M}$ cannot be isomorphic to $\mathbb{A}_5$.*

Proof. Suppose on the contrary, that $\frac{A}{M} \cong \mathbb{A}_5$. Then for any vertex u from the equality $[A : A_u] = 10p^2$ we obtain $|A_u| = 6$ and hence $A_u \cong \mathbb{Z}_6$ or $\mathbb{S}_3$. By Lemma 3.8 $A_u \leq \mathbb{A}_5$. Since $\mathbb{A}_5$ does not have elements of order 6, we conclude that $A_u \cong \mathbb{S}_3$. Also according to Lemma 3.8, $A \cong M \rtimes_\varphi \mathbb{A}_5$. There are only two possibilities for the kernel of $\varphi : \mathbb{A}_5 \to \mathrm{Aut}(M)$.

(a) If $ker(\varphi) = 1$, then $\mathbb{A}_5$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$. Since M of order p^2 is abelian, either $M \cong \mathbb{Z}_{p^2}$ or $M \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In the first case $\mathrm{Aut}(M)$ is abelian and does not have a subgroup isomorphic to $\mathbb{A}_5$ and in the second case $\mathrm{Aut}(M) \cong GL_2(p)$ which again does not have a subgroup isomorphic to $\mathbb{A}_5$ according to Lemma 3.7.

(b) If $ker(\varphi) = \mathbb{A}_5$, then φ is the trivial homomorphism and so $A \cong M \times \mathbb{A}_5$. Since Γ is semisymmetric, according to Proposition 2.12 Γ is isomorphic to $C(A; A_u, A_v)$ where u and v are two adjacent vertices in Γ. As Γ is connected, according to Proposition 2.11 we must have $A = \langle A_u, A_v \rangle$. In view of $A_u \cong A_v \cong$

$\mathbb{S}_3$, this means that $M \times \mathbb{A}_5$ is generated by two of its subgroups, say H and K, both of them isomorphic to $\mathbb{S}_3$. Now for each element $(m, a) \in H$ we have $(m, a)^6 = 1$ which means $m^6 = 1$ in M. As $|M| = p^2$ and $p > 5$, this results in $m = 1$. Therefore the first component of each element of H (and similarly for K) equals 1. Consequently the first component of each element in $M \times \mathbb{A}_5 = \langle H, K\rangle$ equals 1 which is a contradiction. ■

Lemma 3.10. *Suppose $p > 5$ is a prime and Γ is a connected cubic semisymmetric graph of order $20p^2$. Let $A = Aut(\Gamma)$ and take M to be the Sylow p-subgroup of A. Then $\frac{A}{M}$ cannot be isomorphic to $\mathbb{S}_5$.*

Proof. If $\frac{A}{M} \cong \mathbb{S}_5$, then according to Lemma 3.8, $A \cong M \rtimes_\varphi \mathbb{S}_5$ for some homomorphism $\varphi : \mathbb{S}_5 \to \mathrm{Aut}(M)$. Moreover, according to the same Lemma for each vertex u, A_u is isomorphic to a subgroup of $\mathbb{S}_5$. From $[A : A_u] = 10p^2$ we have $|A_u| = 12$. Subgroups of $\mathbb{S}_5$ of order 12 are $\mathbb{A}_4$ and $\mathbb{D}_{12}$. So $A_u \cong \mathbb{D}_{12}$ or $\mathbb{A}_4$ for any vertex u. Now there are three possibilities for the kernel of φ.

(a) If $ker(\varphi) = 1$, then $\mathbb{S}_5$ and hence $\mathbb{A}_5$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$. But as we showed in part (a) of the proof of Lemma 3.9, $\mathbb{A}_5$ cannot be a subgroup of $\mathrm{Aut}(M)$. So this case cannot happen.

(b) If $ker(\varphi) = \mathbb{S}_5$, then φ is the trivial homomorphism and so $A \cong M \times \mathbb{S}_5$. As in part (b) of the proof of Lemma 3.9 we conclude that $A = \langle A_u, A_v\rangle$ where u and v are two adjacent vertices. Since $|A_u| = |A_v| = 12$, this means that $M \times \mathbb{S}_5$ is generated by two of its subgroups, say H and K, each of order 12. For each element $(m, a) \in H$ we have $(m, a)^{12} = 1$ which means $m^{12} = 1$. As $|M| = p^2$ and $p > 5$, it follows that $m = 1$. Therefore the first component of each element of H (and similarly for K) equals 1. Consequently the first component of each element in $M \times \mathbb{S}_5 = \langle H, K\rangle$ equals 1 which is a contradiction.

(c) If $ker(\varphi) = \mathbb{A}_5$, then the image of φ is isomorphic to $\frac{\mathbb{S}_5}{\mathbb{A}_5} \cong \mathbb{Z}_2$. So there is some $T \in \mathrm{Aut}(M)$ of order 2 for which $\varphi(x) = 1$ for all $x \in \mathbb{A}_5$ and $\varphi(x) = T$ for any $x \notin \mathbb{A}_5$. For any two elements (x, g) and (y, h) from $M \rtimes_\varphi \mathbb{S}_5$ the multiplication $(x, g)(y, h)$ equals (xy, gh) if $g \in \mathbb{A}_5$ and equals $(xT(y), gh)$ if $g \notin \mathbb{A}_5$. Now it is easy to see that in the group $M \rtimes_\varphi \mathbb{S}_5$ for any positive integer n if $g \in \mathbb{A}_5$, then $(x, g)^n = (x^n, g^n)$ for all $x \in M$, and if $g \notin \mathbb{A}_5$, then $(x, g)^{2n} = (x^n T(x^n), g^{2n})$ and $(x, g)^{2n+1} = (x^{n+1} T(x^n), g^{2n+1})$ for all $x \in M$.

Take $\{u, v\}$ to be a fixed edge in Γ. Then according to Proposition 2.12, $\Gamma \cong C(A; A_u, A_v)$. Now according to part (ii) of Proposition 2.11, $A = \langle A_u, A_v\rangle$ and also according to part (i) of Proposition 2.11, $|A_u \cap A_v| = 4$, i.e., $A_u \cap A_v$ is a common Sylow 2-subgroup of both A_u and A_v. Since $A \cong M \rtimes_\varphi \mathbb{S}_5$, a similar result holds for $M \rtimes_\varphi \mathbb{S}_5$; i.e., there are two subgroups $U, V \leq M \rtimes_\varphi \mathbb{S}_5$ where
(1) $U, V \in \{\mathbb{D}_{12}, \mathbb{A}_4\}$; and
(2) $M \rtimes_\varphi \mathbb{S}_5 = \langle U, V\rangle$; and
(3) $U \cap V$ is a common Sylow 2-subgroup of both U and V.

We will show however, that the existence of U and V with the above specifications will lead to a contradiction.

Let $H \leq M \rtimes_\varphi \mathbb{S}_5$ and $H \cong \mathbb{D}_{12}$ or $\mathbb{A}_4$. We say H is of type 1 if all elements of H have their second component in $\mathbb{A}_5$. We also call H of type 2 if there is at least one element in H whose second component is not in $\mathbb{A}_5$.

Define $K := \{(x, g) \in H \,|\, g \in \mathbb{A}_5\}$. Then $K \leq H$. Define $f : K \to M$ by $f(x, g) = x$. Then f is a homomorphism and hence $\frac{K}{ker(f)} \cong Im(f)$. Consequently $\left|\frac{K}{ker(f)}\right|$ divides both 12 and p^2 and hence $K = ker(f)$. This means that if $(x, g) \in H$ and $g \in \mathbb{A}_5$, then $x = 1$.

Suppose H is of type 1. Then the first component of each element of H equals 1 and so H is isomorphic to a subgroup of $\mathbb{A}_5$. Since $\mathbb{A}_5$ does not have any element of order 6, H cannot be isomorphic to $\mathbb{D}_{12}$. Hence $H \cong \mathbb{A}_4$.

Now suppose H is of type 2. If $(x, g) \in H$ is an arbitrary element with $g \notin \mathbb{A}_5$, then $(xT(x), g^2) = (x, g)^2 \in H$. Since $g^2 \in \mathbb{A}_5$, we conclude $xT(x) = 1$ and so $T(x) = x^{-1}$. Also if $(y, h) \in H$ is another element with $h \notin \mathbb{A}_5$, then $(yT(x), hg) = (y, h)(x, g) \in H$. Again $hg \in \mathbb{A}_5$ implies that $yT(x) = 1$ or $T(x) = y^{-1}$. Therefore $x^{-1} = y^{-1}$ and so $x = y$. In other words, for any pair of elements $(x, g) \in H$ and $(y, h) \in H$ with $g \notin \mathbb{A}_5$ and $h \notin \mathbb{A}_5$ we must have $x = y$. Certainly there are always elements in H whose second component lies in $\mathbb{A}_5$ (and hence their first component is 1).

Therefore we can write

$$H = \{(x, g_1), (x, g_2), \ldots, (x, g_n), (1, h_1), \ldots, (1, h_m)\},$$

where $n + m = 12$ and where $g_1, \ldots, g_n \notin \mathbb{A}_5$ and $h_1, \ldots, h_m \in \mathbb{A}_5$. If $\overline{H} = \{(1, h_1), \ldots, (1, h_m)\}$ then $\overline{H} \leq H$, and if $H_1 = \{h_1, \ldots, h_m\}$ then $H_1 \leq \mathbb{A}_5$ and $H_1 \cong \overline{H}$. Multiplying all the elements of H by (x, g_t), t arbitrary, we again obtain H. Therefore

$$H = \{(1, g_t g_1), (1, g_t g_2), \ldots, (1, g_t g_n), (x, g_t h_1), \ldots, (x, g_t h_m)\}.$$

Comparing the two equalities for H and by taking into account that $g_t g_i \in \mathbb{A}_5$ for $i = 1, \ldots, n$ and $g_t h_j \notin \mathbb{A}_5$ for $j = 1, \ldots, m$, we obtain

$$\{g_t g_1, g_t g_2, \ldots, g_t g_n\} = \{h_1, \ldots, h_m\}$$

and

$$\{g_t h_1, \ldots, g_t h_m\} = \{g_1, \ldots, g_n\}.$$

This results in $m = n = 6$. So $|\overline{H}| = 6$. Since $\mathbb{A}_4$ does not have a subgroup of order 6, the first conclusion is that H cannot be isomorphic to $\mathbb{A}_4$ and hence $H \cong \mathbb{D}_{12}$. The group $\mathbb{D}_{12}$ has only two subgroups of order 6, namely $\mathbb{Z}_6$ and $\mathbb{S}_3$. If $\overline{H} \cong \mathbb{Z}_6$, then $\mathbb{Z}_6 \cong H_1 \leq \mathbb{A}_5$. But $\mathbb{A}_5$ does not have elements of order 6. Therefore $\overline{H}$ cannot be isomorphic to $\mathbb{Z}_6$ and hence $\overline{H} \cong \mathbb{S}_3$. Since $H \cong \mathbb{D}_{12}$,

we can write $H = \{a^i \mid i = 0, \dots, 5\} \cup \{ba^i \mid i = 0, \dots, 5\}$. As $\overline{H} \cong \mathbb{S}_3$ does not have any element of order 6, a must be of the form (x, g) for some $g \notin \mathbb{A}_5$. As for b, there are two possible cases; either $b = (1, h) \in \overline{H}$ in which case we call H of (sub)type 2.1 or $b = (x, g') \in H - \overline{H}$ in which case we call H of (sub)type 2.2.

If H is of type 2.1 and $b = (1, h)$, $h \in \mathbb{A}_5$, then

$$\begin{aligned} H^1(x, g, h) &= \big\{(x, g)^i \mid i = 0, \dots, 5\} \cup \{(1, h)(x, g)^i \mid i = 0, \dots, 5\big\} \\ &= \big\{(1, 1), (x, g), \big(1, g^2\big), \big(x, g^3\big), \big(1, g^4\big), \big(x, g^5\big), \\ &\quad (1, h), (x, hg), \big(1, hg^2\big), \big(x, hg^3\big), \big(1, hg^4\big), \big(x, hg^5\big)\big\}. \end{aligned}$$

In this case all the Sylow 2-subgroups of H are as follows:

$$\begin{aligned} HP_1^1(x, g, h) &= \big\{(1, 1), \big(x, g^3\big), (1, h), \big(x, hg^3\big)\big\}, \\ HP_2^1(x, g, h) &= \big\{(1, 1), \big(x, g^3\big), (x, hg), \big(1, hg^4\big)\big\}, \\ HP_3^1(x, g, h) &= \big\{(1, 1), \big(x, g^3\big), \big(1, hg^2\big), \big(x, hg^5\big)\big\}. \end{aligned}$$

Also if H is of type 2.2 and $b = (x, g')$, $g' \notin \mathbb{A}_5$, then

$$\begin{aligned} H^2(x, g, g') = \big\{&(1, 1), (x, g), \big(1, g^2\big), \big(x, g^3\big), \big(1, g^4\big), \big(x, g^5\big), \\ &\big(x, g'\big), \big(1, g'g\big), \big(x, g'g^2\big), \big(1, g'g^3\big), \big(x, g'g^4\big), \big(1, g'g^5\big)\big\} \end{aligned}$$

and all the Sylow 2-subgroups of H are as follows:

$$\begin{aligned} HP_1^2(x, g, g') &= \big\{(1, 1), \big(x, g^3\big), \big(x, g'\big), \big(1, g'g^3\big)\big\}, \\ HP_2^2(x, g, g') &= \big\{(1, 1), \big(x, g^3\big), \big(1, g'g\big), \big(x, g'g^4\big)\big\}, \\ HP_3^2(x, g, g') &= \big\{(1, 1), \big(x, g^3\big), \big(x, g'g^2\big), \big(1, g'g^5\big)\big\}. \end{aligned}$$

In the above notations, the superindex $j = 1, 2$ refers to type 2.j of H.

Now we go back and consider U and V. There are only three possible cases for their types.

Case c1. Both U and V are of type 1. This means $(A_u, A_v) = (\mathbb{A}_4, \mathbb{A}_4)$ which is not possible according to Theorem 2.6.

Case c2. One of U and V is of type 1 and the other has type 2. Without loss of generality assume U has type 1 and V is of type 2. So every element of U has its first component equal to 1 and so will be every element of $U \cap V$ which is a Sylow 2-subgroup of both U and V. No matter what the subtype of V is, this says that the first component of every element of V should be 1. For example, if

$$U \cap V = VP_2^2(x, g, g') = \big\{(1, 1), \big(x, g^3\big), \big(1, g'g\big), \big(x, g'g^4\big)\big\}$$

for some $x \in M$, $g, g' \notin \mathbb{A}_5$, then $x = 1$ and hence the first component of every element of $V = V^2(x, g, g')$ is 1. Now U and V cannot both generate $M \rtimes_\varphi \mathbb{S}_5$ since every element of $\langle U, V\rangle$ is an alternating product of elements from U and V which will have its first component equal to 1 because $(1, t)(1, s) = (1, ts)$ for any $t, s \in \mathbb{S}_5$.

Case c3. Both U and V are of type 2. In this case, there are four possible cases for the subtypes of U and V. Suppose $U = U^i(x, a, b)$ and $V = V^j(y, \alpha, \beta)$ for some $x, y \in M$ and some suitable $a, b, \alpha, \beta \in \mathbb{S}_5$. Since $U \cap V$ is a Sylow 2-subgroup of both U and V, we must have $UP^i_{i'}(x, a, b) = VP^j_{j'}(y, \alpha, \beta)$ for some $i', j' \in \{1, 2, 3\}$. We compare the first components of non-identity elements from the two sides of this equality. If $x = 1$, then inevitably $y = 1$. If $x \neq 1$, then $x = y$. Therefore $x = y$ in any case.

Define $M_x = (\{1\}\times\mathbb{A}_5)\cup(\{x\}\times(\mathbb{S}_5-\mathbb{A}_5))$. It is easy to check that $M_x \leq M \rtimes_\varphi \mathbb{S}_5$. Obviously $U \cup V \subset M_x$, and so $\langle U, V\rangle \leq M_x$. Therefore $\langle U, V\rangle \neq M \rtimes_\varphi \mathbb{S}_5$. ■

***Proof of Theorem* 3.1.** According to [6] there is no connected cubic semisymmetric graph of order $20p^2$ for $p = 2$, 3 and 5. So let $p > 5$ and suppose on the contrary that Γ is a connected cubic semisymmetric graph of order $20p^2$. Let $\{U, W\}$ be a bipartition for Γ and $A = \mathrm{Aut}(\Gamma)$. Then $|U| = |W| = 10p^2$ and $|A| = 2^{r+1} \cdot 3 \cdot 5 \cdot p^2$ for some $0 \leq r \leq 7$. Also assume M is a Sylow p-subgroup of A. According to Lemma 3.6, $M \trianglelefteq A$. Due to its order, M is intransitive on both U and W and so according to Theorem 2.9, Γ_M is a connected cubic G-semisymmetric graph of order 20 with the bipartition $\{U_M, W_M\}$, where $G = \frac{A}{M}$ and $|U_M| = |W_M| = 10$.

Therefore Γ_M is G-edge-transitive and hence edge-transitive. Now if Γ_M is not vertex-transitive, then it must be semisymmetric, but there is no semisymmetric cubic graph of order 20 according to [6]. So Γ_M should be vertex-transitive and hence symmetric since according to [23] a cubic vertex and edge-transitive graph is necessarily symmetric. According to [7] there are only two symmetric cubic graphs of order 20: F20A and F20B. Since Γ_M is bipartite and F20A is not bipartite, we should have $\Gamma_M \cong$ F20B.

The automorphism group of F20B has 240 elements [7] and $G = \frac{A}{M}$ is isomorphic to a subgroup of Aut(F20B) of order $|G| = 2^{r+1} \cdot 3 \cdot 5$. The equality is not possible since G is not transitive on V(F20B) whereas Aut(F20B) is. So $|G| < 240$ and hence $1 \leq r+1 \leq 3$. Also G is transitive on both U_M and W_M and according to Proposition 2.8 the action of G on each of U_M and W_M is faithful. Therefore G is a transitive permutation group of degree 10. Transitive permutation groups of degree 10 have been completely classified in [5]. There are 45 such groups up to isomorphism which are denoted by $T1, T2, \ldots, T45$ in [5]. Among these, the only groups whose orders are of the form $2^i \cdot 3 \cdot 5$ for $1 \leq i \leq 3$, are $T7 \cong \mathbb{A}_5$ of order 60, and $T11$, $T12$ and $T13 \cong \mathbb{S}_5$ of order 120. We first argue

that G could not be isomorphic to $T11$ or $T12$.

In [5] all the transitive groups of degree 10 have been defined with a set of generating permutations on ten points. If

$$a = (1,2,3,4,5),\ b = (6,7,8,9,10),$$
$$e = (1,5)(2,3),\ f = (6,10)(7,8),\ g = (1,2),\ h = (6,7) \text{ and}$$
$$i = (1,6)(2,7)(3,8)(4,9)(5,10),$$

then

$$T11 = \langle ab, ef, i\rangle,\ T12 = \langle ab, ef, ghi\rangle.$$

Using the GAP software it is easy to verify that $H = \langle i\rangle \trianglelefteq T11$. Obviously $|H| = 2$.

If $G \cong T11$, then according to Theorem 2.9 the quotient graph of Γ_M with respect to H which we denote by $(\Gamma_M)_H$, would be R-semisymmetric of order 10, where $R = \frac{T11}{H}$. This implies that R is transitive on each partite set and by Proposition 2.8, R would be a transitive permutation group of degree 5. Again according to [5] the only transitive permutation group of degree 5 and of order 60 is $\mathbb{A}_5$. So we should have $R \cong \mathbb{A}_5$. Now the stabilizer of any vertex of $(\Gamma_M)_H$ under the action of R has $\frac{|R|}{5} = 12$ points and the only subgroup of $\mathbb{A}_5$ of order 12 is isomorphic to $\mathbb{A}_4$. So for an edge $\{u, w\}$ of the cubic R-semisymmetric graph $(\Gamma_M)_H$, we have $(R_u, R_w) = (\mathbb{A}_4, \mathbb{A}_4)$ which is not possible according to Theorem 2.6. Therefore the assumption that $G \cong T11$, leads to a contradiction.

Now suppose $G \cong T12$. Since G is transitive on either of the two partite sets, the stabilizers of any vertex in one partite set are isomorphic. Consider one vertex, e.g. 1. Using GAP software, we easily obtain the stabilizer of 1 under $T12$ as

$$\langle (2,4)(3,5)(7,9)(8,10), (3,5,4)(8,10,9)\rangle.$$

Using GAP one finds out that this group is nonabelian of order 12 which has the following group of order 4 as a normal subgroup:

$$\langle (2,3)(4,5)(7,8)(9,10), (2,4)(3,5)(7,9)(8,10)\rangle.$$

There are only 3 nonabelian groups of order 12 up to isomorphism: $\mathbb{A}_4$, $\mathbb{D}_{12}$ and the dicyclic group of order 12. Among these, the only one having a normal subgroup of order 4, is $\mathbb{A}_4$. So the stabilizer of any vertex in Γ_M would be isomorphic to $\mathbb{A}_4$ which is not possible according to Theorem 2.6.

Finally $\frac{A}{M} \cong T7 \cong \mathbb{A}_5$ is not possible according to Lemma 3.9 and $\frac{A}{M} \cong T13 \cong \mathbb{S}_5$ is not possible according to Lemma 3.10.

Since every case for $\frac{A}{M}$ is contradictory, we conclude that there is no connected cubic semisymmetric graph of order $20p^2$. ■

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