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TRIAMETER OF GRAPHS

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Abstract

In this paper, we study a new distance parameter *triameter* of a connected graph G, which is defined as $\max\{d(u,v)+d(v,w)+d(u,w) : u,v,w \in V\}$ and is denoted by $tr(G)$. We find various upper and lower bounds on $tr(G)$ in terms of order, girth, domination parameters etc., and characterize the graphs attaining those bounds. In the process, we provide some lower bounds of (connected, total) domination numbers of a connected graph in terms of its triameter. The lower bound on total domination number was proved earlier by Henning and Yeo. We provide a shorter proof of that. Moreover, we prove Nordhaus-Gaddum type bounds on $tr(G)$ and find $tr(G)$ for some specific family of graphs.

Keywords: distance, radio k-coloring, Nordhaus-Gaddum bounds.

2010 Mathematics Subject Classification: 05C12.

1. Introduction

The channel assignment problem is the problem of assigning frequencies to the transmitters in some optimal manner and with no interferences. Keeping this problem in mind, Chartrand *et al.* in [1] introduced the concept of *radio k-coloring* of a simple connected graph. As finding the radio k-chromatic number of graphs is highly non-trivial and therefore is known for very few graphs, determining good and sharp bounds is an interesting problem and has been studied by many authors [8, 10–13] etc. In [8, 10, 11], authors provides some sharp lower bounds on radio k-chromatic number of connected graphs in terms of a newly defined parameter called triameter of a graph (it was denoted as M-value of a graph in [11]). Apart from this, the concept of triameter also finds application in metric polytopes [9]. Recently, in [7], Henning and Yeo proved a graffiti conjecture on lower bound of total domination number of a connected graph in terms of its triameter. Keeping these as motivation, in this paper, we formally study triameter of connected graphs and various bounds associated with it. In fact, in the process, we provide a shorter proof of the main result in [7].

2. Preliminaries

In this section, for convenience of the reader and also for later use, we recall some definitions, notations and results concerning elementary graph theory. For undefined terms and concepts the reader is referred to [14].

By a graph $G = (V, E)$, we mean a non-empty set V and a symmetric binary relation (possibly empty) E on V. If two vertices u, v are adjacent in G, either we write $(u, v) \in E$ or $u \sim v$ in G. The complement of a graph G, denoted by G^c, is defined to be a graph on same vertex set as that of G and two vertices are adjacent in G^c if and only if they are non-adjacent in G. The distance $d_G(u, v)$ or $d(u, v)$ between two vertices $u, v \in V$ is the length of the shortest path joining u and v in G. The *eccentricity* of a vertex v is defined as $\max\{d(u, v) : u \in V\}$ and is denoted by $ecc(v)$. The *radius*, *diameter* and *center* of a connected graph G are defined as $rad(G) = \min\{ecc(v) : v \in V\}$, $diam(G) = \max\{ecc(v) : v \in V\}$ and $center(G) = \{v \in V : ecc(v) = rad(G)\}$ respectively. The *Wiener index* $\sigma(G)$ is defined as $\sum_{\{u,v\}\subset V} d(u, v)$. A graph G is said to be *vertex transitive* if $\mathsf{Aut}(G)$, the automorphism group of G, acts transitively on G. The length of a cycle, if it exists, of smallest length is said to be the girth $g(G)$ of G. A graph G is said to be *Hamiltonian* if there exists a cycle containing all the vertices of G as a subgraph of G. A graph G is said to be *strongly regular* with parameters (n, k, λ, μ) if it is a k-regular n-vertex graph in which any two adjacent vertices have λ common neighbours and any two non-adjacent vertices have μ common neighbours. A graph is said to be a *bistar* if it is obtained by joining the root vertices of two stars K_{1,n_1} and K_{1,n_2}. We denote this graph by $K^{n_1}_{n_2}$ and it is a graph on $n_1 + n_2 + 2$ vertices. For other definitions, e.g., domination number γ, total domination number γ_t, connected domination number γ_c of a graph, readers are referred to [5].

3. Triameter of a Graph and its Bounds

In what follows, even if not mentioned, G denotes a finite simple connected undirected graph with at least 3 vertices. We start by defining triameter of a connected graph.

Definition. Let $G = (V, E)$ be a connected graph on $n \geq 3$ vertices. The *triameter* of G is defined as $\max\{d(u, v) + d(v, w) + d(u, w) : u, v, w \in V\}$ and is denoted by $tr(G)$.

From the definition, it follows that $tr(G)$ is always greater than or equal to 3. However, triameter of a graph on n vertices can be as large as $2n-2$, as evident from the following results proved in [8]: $tr(P_n) = 2(n-1)$ and $tr(C_n) = n$.

If G and H are two connected graphs on same vertex set with $E(H) \subseteq E(G)$, then by definition of triameter, we have $tr(G) \leq tr(H)$. For any three vertices u, v, w, let us denote by $d(u, v, w)$, the sum $d(u, v) + d(v, w) + d(u, w)$. Now, we investigate other bounds on $tr(G)$.

Theorem 3.1. *For any connected graph G, $2 \cdot diam(G) \leq tr(G) \leq 3 \cdot diam(G)$, and the bounds are tight.*

Proof. The upper bound follows from the definition of diameter and triameter of a connected graph. For the lower bound, let $d(u, v) = diam(G)$. Choose $w \in V \setminus \{u, v\}$. Then $d(u, v) \leq d(v, w) + d(w, u)$, implying that $2 \cdot diam(G) = 2d(u, v) \leq d(u, v) + d(v, w) + d(w, u) \leq tr(G)$.

The tightness of the bounds follows from the following examples. For $n \geq 3$, $tr(P_n) = 2 \cdot diam(P_n)$. For Petersen graph P, $tr(P) = 3 \cdot diam(P)$. ■

Corollary 3.2. *Let G be a connected graph on n vertices such that $\delta(G) \geq \frac{n}{2}$. Then $tr(G) \leq 6$.*

Proof. It follows from Theorem 3.1 and the fact that $\delta(G) \geq \frac{n}{2}$ implies $diam(G) \leq 2$. ■

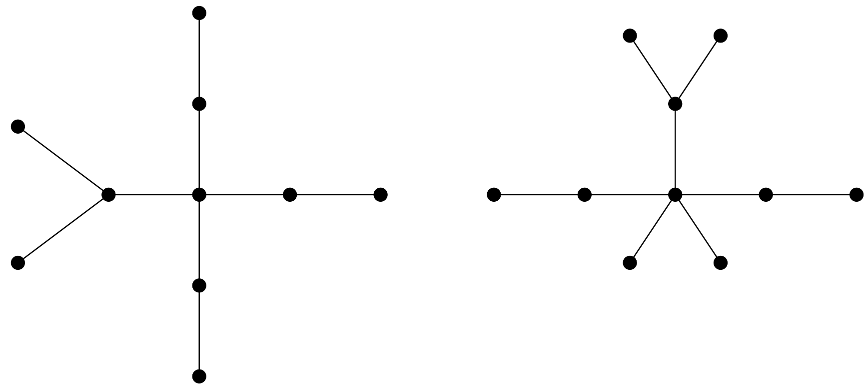
Corollary 3.3. *For any connected graph G, $2 \cdot rad(G) \leq tr(G) \leq 6 \cdot rad(G)$, and the bounds are tight.*

Proof. As for any connected graph G, $rad(G) \leq diam(G) \leq 2 \cdot rad(G)$, we have $2 \cdot rad(G) \leq tr(G) \leq 6 \cdot rad(G)$. For the tightness of the lower bound, take $G = C_{2n}$ where $tr(G) = 2n = 2 \cdot rad(G)$, and for upper bound, take $G = K_{1,3}$ where $tr(G) = 6$ and $rad(G) = 1$. ■

Remark 3.4. Some other examples demonstrating the tightness of the upper bounds are shown in Figure 1. The bound in Corollary 3.3 can be substantially tightened in case of vertex transitive graphs; see Theorem 4.6.

Corollary 3.5. *For any tree T, $4 \cdot rad(T) - 2 \leq tr(T) \leq 6 \cdot rad(T)$, and the bounds are tight.*

Proof. We first recall a result on tree. A tree T has either $|center(T)| = 1$ or $|center(T)| = 2$, and $diam(T) = 2 \cdot rad(T)$ or $2 \cdot rad(T) - 1$ according as $|center(T)| = 1$ or $|center(T)| = 2$. Hence the corollary follows from Theorem 3.1. Tightness of the upper bound and lower bound follows respectively from $K_{1,3}$ and P_{2n}. ■



A Tree with $tr(G) = 3 \cdot diam(G)$. A Tree with $tr(G) = 6 \cdot rad(G)$.

Figure 1. Trees achieving the upper bounds.

The lower bound in Corollary 3.5 can be improved for trees with more than 2 leaves.

Theorem 3.6. *For any tree T with more than 2 leaves, $tr(T) \geq 4 \cdot rad(T)$, and the bound is tight.*

Proof. If T is central, i.e., $|center(T)| = 1$, let x_1 be the center of T and $rad(T) = r$ and u, v be two diametrical opposite vertices. Then $diam(T) = d(u, v) = 2r$ and $d(u, x_1) = d(v, x_1) = r$. Let w be another leaf of T other than u and v and k be the shortest distance of w from the vertices lying on the path joining u and v. Then $d(u, v, w) = 4r + 2k$. Since w is a leaf, $k \geq 1$. Hence $tr(T) \geq d(u, v, w) \geq 4r + 2 > 4rad(T)$.

If T is bicentral, i.e., $|center(T)| = 2$, let x_1, x_2 be the center of T and $rad(T) = r$ and u, v be two diametrical opposite vertices. Then $diam(T) = d(u, v) = 2r - 1$. Let w be another leaf of T other than u and v and k be the shortest distance of w from the vertices lying on the path joining u and v. Then $d(u, v, w) = 4r + 2k - 2$. Since w is a leaf, $k \geq 1$. Hence $tr(T) \geq d(u, v, w) \geq 4r = 4rad(T)$.

Hence, the theorem holds. The tightness follows from the tree obtained by subdividing one edge of $K_{1,3}$, where radius is 2 and triameter is 8. ∎

3.1. Upper bounds

Theorem 3.7. *For any connected graph G with $n \geq 3$ vertices, $tr(G) \leq 2n - 2$. Moreover $tr(G) = 2n - 2$ if and only if G is a tree with 2 or 3 leaves.*

Proof. It suffices to prove the bound for trees, as for any connected graph G and any spanning tree T of G, $tr(G) \leq tr(T)$ holds. Let T be a tree. Let u, v and w be three vertices of T such that $tr(T) = d(u, v, w)$. Suppose that P_1, P_2 and P_3 are three shortest paths from u to v, u to w, and v to w, respectively. Let $E(P_1 \cup P_2 \cup P_3)$ be a set of all edges in the paths P_1, P_2 and P_3. It is not hard to see that each edge in $E(P_1 \cup P_2 \cup P_3)$ is considered twice for computing $tr(T)$. Therefore, we have

$$tr(T) = 2|E(P_1 \cup P_2 \cup P_3)| \leq 2|E(T)| = 2|V(T)| - 2 = 2n - 2,$$

where $V(T)$ and $E(T)$ are the vertex set and edge set of T, respectively.

If $tr(T) = 2|V(T)| - 2$, then we have $2|V(T)| - 2 = 2|E(P_1 \cup P_2 \cup P_3)|$. Hence, $|V(T)| - 1 = |E(P_1 \cup P_2 \cup P_3)|$. This implies that $|E(T)| = |E(P_1 \cup P_2 \cup P_3)|$. Thus, $E(T) = E(P_1 \cup P_2 \cup P_3)$. It shows that T has either exactly 3 leaves u, v and w or two leaves u, v and w is another vertex on T.

Next we show that G can not be a connected graph which is not a tree with $tr(G) = 2n - 2$. Let G be a connected graph with $tr(G) = 2n - 2$ where $n = |V(G)|$. We note that G has a spanning tree T. Hence, $2n - 2 = tr(G) \leq tr(T) \leq 2n - 2$. Thus, $tr(T) = 2n - 2$. Therefore, T is a tree with 3 leaves. If $e \in E(G)$ and $e \notin E(T)$, then $tr(G) \leq tr(T+e) < tr(T) = 2n-2$, a contradiction. Therefore, $E(G) \subseteq E(T)$. Hence G is a tree with exactly two or three leaves. ∎

Theorem 3.8. *Let T be a tree on $n \geq 3$ vertices and $l \geq 4$ leaves. Then $tr(T) \leq 2n - 2l + 4$.*

Proof. Let $tr(T) = d(u^*, v^*, w^*)$ for three leaves u^*, v^*, w^* of T. Let T' be the tree on $n - (l - 3)$ vertices obtained by deleting the remaining $l - 3$ leaves from T. Thus $tr(T) \leq tr(T') = 2(n - l + 3) - 2 = 2n - 2l + 4$, by Theorem 3.7. ∎

Corollary 3.9. *Let T be a tree on $n \geq 3$ vertices such that $tr(T) = 2n - 4$, then T has exactly 4 leaves.*

Proof. From Theorem 3.8, we get $2n - 4 = tr(T) \leq 2n - 2l + 4$, i.e., $l \leq 4$. If $l = 2$ or $l = 3$, then $tr(T) = 2n - 2 \neq 2n - 4$. Thus $l = 4$. ∎

It is to be noted that the converse of the above corollary is not true; see Figure 2.

Corollary 3.10. *Let G be a connected graph on n vertices with connected domination number γ_c. Then $tr(G) \leq 2\gamma_c + 4$.*

Proof. Let T be a spanning tree of G with maximum number of leaves l. Then $l + \gamma_c = n$ (see [6]). Now, if $l \geq 4$, $tr(G) \leq tr(T) \leq 2(n - l) + 4 = 2\gamma_c + 4$. If $l = 2$ or $l = 3$, by Theorem 3.7, $tr(G) = 2n - 2$ and $\gamma_c = n - 2$ or $n - 3$. In this case also, $tr(G) \leq 2\gamma_c + 4$ holds. ∎

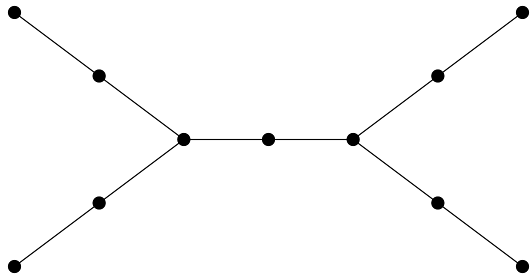


Figure 2. Example of a tree T with $l(T) = 4, n(T) = 11$ and $tr(T) = 16 < 18 = 2 \cdot 11 - 4$.

Corollary 3.11. *Let G be a connected graph with domination number $\gamma(G)$. Then $tr(G) \leq 6\gamma(G)$, and the bound is tight.*

Proof. It follows from the fact that $tr(G) \leq 2\gamma_c(G) + 4$ and $\gamma_c(G) \leq 3\gamma(G) - 2$ (see [3]). The bound is achieved by $K_{1,n}$. ■

Corollary 3.12. *Let $G = (V, E)$ be a connected graph with total domination number $\gamma_t(G)$. Then $tr(G) \leq 4\gamma_t(G)$.*

Proof. In [4], it was shown that $\gamma_c(G) \leq 2\gamma_t(G) - 2$. Thus from Corollary 3.10, we get $tr(G) \leq 2\gamma_c + 4 \leq 2(2\gamma_t(G) - 2) + 4 \leq 4\gamma_t(G)$. ■

Remark 3.13. Corollary 3.12 was also proved in [7]. However, here we provide a shorter proof of $tr(G) \leq 4\gamma_t(G)$ using Theorems 3.7 and 3.8 and Corollaries 3.10 and 3.12.

In the next proposition, we show that the upper bound proved in Theorem 3.7 can be substantially tightened if the vertex connectivity κ of G increases.

Proposition 3.14. *Let G be a graph on n vertices with vertex connectivity κ. Then $tr(G) \leq \frac{3(n-2)}{\kappa} + 3$.*

Proof. The proof follows from the result that $n \geq \kappa(diam(G) - 1) + 2$ (see p. 174, 4.2.22, [14]) and $tr(G) \leq 3 \cdot diam(G)$. ■

Theorem 3.15. *For a connected graph G on $n \geq 3$ vertices with chromatic number $\chi(G)$, $tr(G) + \chi(G) \leq 2n$, and the bound is tight.*

Proof. We first observe that the result holds for odd cycles and complete graphs, i.e., for $G = C_n$ with odd $n \geq 3$, $tr(G) = n, \chi(G) = 3$ and for $G = K_n$, $tr(G) = 3, \chi(G) = n$, and in both cases $tr(G) + \chi(G) = n + 3 \leq 2n$. Thus, we assume that G is neither an odd cycle nor a complete graph. Let T be a spanning tree of G with maximum degree $\Delta(T) = \Delta(G)$. Also, the number of leaves $l(T)$ of T satisfies $\Delta(T) \leq l(T)$. Therefore, by Brooks' Theorem, we have

$$(1) \qquad tr(G) + \chi(G) \leq tr(G) + \Delta(G) \leq tr(T) + \Delta(T) \leq tr(T) + l(T).$$

Now, if $l(T) \geq 4$, then by Theorem 3.8, $tr(T) + l(T) \leq 2n - l + 4 \leq 2n$. If $l(T) = 2$, then by Theorem 3.7, $tr(T) + l(T) = 2n - 2 + 2 = 2n$.

Thus the only case left is $l(T) = 3$. If $G = T$, then $tr(G)+\chi(G) = 2n-2+2 = 2n$. If G has at least one edge more than T, then $tr(G) \leq 2n - 3$ and hence by Brooks' Theorem, $tr(G)+\chi(G) \leq tr(G)+\Delta(G) = tr(G)+\Delta(T) \leq tr(G)+l(T) \leq 2n - 3 + 3 = 2n$.

The tightness of the bound follows from taking $G = P_n$ or any tree with 3 leaves. ■

3.2. Lower bounds

It is known that in a connected graph G that contains a cycle, $g(G) \leq 2 \cdot diam(G) + 1$. Thus, it trivially follows from Theorem 3.1 that $g(G) \leq tr(G) + 1$. In the next theorem, we prove a stronger inequality involving girth and triameter.

Theorem 3.16. *If G is a connected graph that contains a cycle, then $g(G) \leq tr(G)$.*

Proof. Let C be a cycle of length $g(G) = g$. Since C is a smallest cycle in G, there exists two vertices u and v on C such that $d(u, v) = \lfloor g/2 \rfloor$. Choose w on C such that $w \sim v$ and $d(u, w) = \lfloor (g-1)/2 \rfloor$. Again, existence of such w is guaranteed as C is a smallest cycle in G. Now, $d(u, v) + d(u, w) + d(v, w) = \lfloor g/2 \rfloor + \lfloor (g-1)/2 \rfloor + 1 = (g - 1) + 1 = g$ and hence the bound follows. ■

Theorem 3.17. *In a connected graph G, $g(G) = tr(G)$ holds if and only if G is a complete graph or a cycle.*

Proof. It is clear that if G is a cycle, then $tr(G) = g(G)$ and if G is a complete graph K_n with $n \geq 3$, then $tr(G) = g(G) = 3$. Conversely, let $tr(G) = g(G)$ holds for a graph G. If $tr(G) = g(G) = 3$, then $d(u, v) = 1$ for all vertices u, v in G, i.e., G is a complete graph K_n. Also, as $g(G) = 3$, we have $n \geq 3$. Thus let $tr(G) = g(G) > 3$ and C be a cycle of length $g = g(G)$ in G. Since C is a smallest cycle, C is a chordless induced cycle in G. If $G = C$, then the proof is over. If not, let v be a vertex in G, but not in C, which is adjacent to some vertex u in C, i.e., $d(u, v) = 1$.

Case 1. g is odd, say $g = 2k + 1 > 3$, i.e., $k > 1$. Then there exist two vertices x and y in C such that $d(u, x) = k = d(u, y)$ and $d(x, y) = 1$. Since the girth is $2k + 1$, $d(v, x)$ and $d(v, y)$ are greater or equal to k, otherwise we get a cycle of length less than g. If any one of them is greater than k, say $d(v,x) > k$, then we get $d(u,v)+d(u,x)+d(v,x) > 1+k+k = 2k+1 = g$, i.e., $tr(G) > g$, a contradiction. Thus, let both $d(v,x) = d(v,y) = k$. Since C is a cycle, there are two vertices in C which are adjacent to x, one being y. Let the other vertex in C which is adjacent to x be z. Thus $d(y,z) \leq 2$ via a path through x. However, as C is chordless, $d(y,z) \neq 1$. Thus $d(y,z) = 2$. Also $d(u,z) = k-1$ as $d(u,x) = k$. Now if $d(v,z) < k$, then we get a cycle of length less than $2k+1$ passing through u, v and z. Thus $d(v,z) = k$ via a path through u. Hence,

$$tr(G) \geq d(v,z) + d(v,y) + d(y,z) = k + k + 2 > 2k + 1 = g(G),$$

a contradiction.

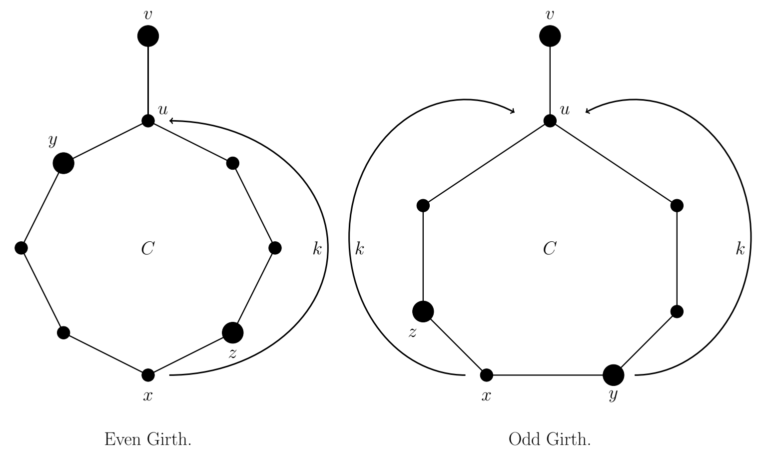


Figure 3. Schematic diagram of the proof of Theorem 3.17.

Case 2. g is even, say $g = 2k > 3$, i.e., $k > 1$. Then there exist a unique vertex x in C such that $d(u,x) = k$. Let y be a vertex in C adjacent to u. Since $k > 1$, $y \neq x$. Similarly let z be the unique vertex in C such that $d(y,z) = k$. Note that as C is a smallest cycle in G, $d(x,z) = 1$ and $d(u,z) = k-1$. Again, $d(v,z) \geq k-1$, because if $d(v,z) < k-1$, we get a cycle of length less than k through v, u and z in G, a contradiction. Also, $d(y,v) = 2$. Hence

$$tr(G) \geq d(v,z) + d(v,y) + d(y,z) \geq (k-1) + 2 + k = 2k + 1 > g(G),$$

a contradiction.

Thus, combining both the cases, there does not exist any vertex v in G which is not in C. Moreoer, as C is an induced chordless cycle in G, we have $G = C$, i.e., G is a cycle. ∎

Theorem 3.18. *Let T be a tree on n vertices with $l \geq 3$ leaves. Then $tr(T) \geq \left\lceil \frac{4(n-1)}{(l-1)} \right\rceil$ and the bound is tight.*

Proof. For $l = 3$, its an equality. So we assume that $l > 3$. Let $tr(T) = d(u, v, w)$ for three leaves u, v, w in T. Let P_1, P_2, P_3 be the unique shortest path joining $u - v$, $v - w$ and $w - u$, respectively. Let $T' = \langle P_1 \cup P_2 \cup P_3 \rangle$ be the subtree of T induced by the union of P_1, P_2 and P_3. Note that T' is a tree of with three leaves u, v, w and $tr(T') = tr(T)$. As T' is a tree with 3 leaves, it is obtained by subdividing edges of $K_{1,3}$. Let y be the root vertex in T'. Let $d(u, y) = k_1, d(v, y) = k_2$ and $d(w, y) = k_3$. Then $tr(T) = tr(T') = 2(k_1+k_2+k_3)$.

Since $l > 3$, let x be another leaf in T apart from u, v, w and $d(x, T') = k$, i.e., there exists $z \in T'$ such that $d(x, z) = k$ and $d(x, t) > k$ for all $t \in T' \setminus \{z\}$. Without loss of generality, let z lie on the path joining u and y, see Figure 4. Here the black vertices denote the vertices of T' and the blue vertex is x.

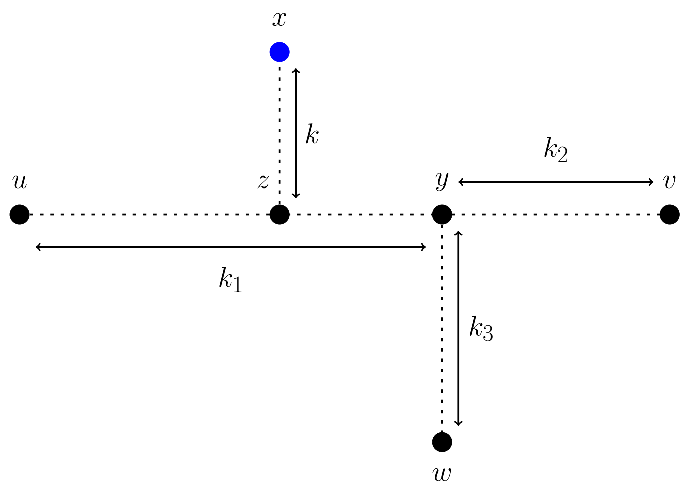


Figure 4. Schematic diagram for the proof of Theorem 3.18.

Claim 1. $d(u, z) \geq d(x, z) = k$.

Proof. If possible, let $d(u, z) < d(x, z)$, then

$$d(u, v) = d(u, z) + d(z, v) < d(x, z) + d(z, v) = d(x, v) \text{ and}$$

$$d(u, w) = d(u, z) + d(y, z) + d(y, w) < d(x, z) + d(y, z) + d(y, w) = d(x, w).$$

Thus

$$\begin{aligned} tr(T) &= d(u, v, w) = d(u, v) + d(u, w) + d(v, w) \\ &< d(x, v) + d(x, w) + d(v, w) = d(x, v, w), \end{aligned}$$

a contradiction. □

Claim 2. *Either* $d(v,z) \geq d(x,z)$ *or* $d(w,z) \geq d(x,z)$*.*

Proof. If possible, let $d(v,z) < d(x,z)$ or $d(w,z) < d(x,z)$. Without loss of generality, let $k_2 \leq k_3$. Then

$$\begin{aligned}
&d(u,v,w)\\
&= d(u,v) + d(w,u) + d(v,w) = (d(u,z) + d(z,v)) + (k_2 + k_3) + d(w,u)\\
&< d(u,y) + d(x,z) + (k_2 + k_3) + d(w,u) \text{ [as, } d(u,z) \leq d(u,y); d(v,z) < d(x,z)]\\
&= (d(u,z) + d(y,z)) + d(x,z) + (k_2 + k_3) + d(w,u)\\
&= (d(u,z) + d(x,z)) + (d(y,z) + k_3) + k_2 + d(w,u)\\
&= d(u,x) + d(w,z) + k_2 + d(w,u)\\
&< d(u,x) + d(x,z) + k_2 + d(w,u) \qquad \text{[as, } d(w,z) < d(x,z)]\\
&\leq d(u,x) + (d(x,y) + k_3) + d(w,u) \qquad \text{[as, } d(x,z) \leq d(x,y); k_2 \leq k_3]\\
&= d(u,x) + d(x,w) + d(w,u) = d(u,x,w),
\end{aligned}$$

a contradiction. □

As $d(x,z) = k$, from the above two claims, we have $d(u,z) \geq k$ and either $d(v,z)$ or $d(w,z) \geq k$. Thus adding them, we get $d(u,z) + d(v,z) \geq 2k$ or $d(u,z) + d(w,z) \geq 2k$, i.e., $d(u,y) + d(v,y) = k_1 + k_2 \geq 2k$ or $d(u,y) + d(w,y) = k_1 + k_3 \geq 2k$. In any case, $2k \leq k_1 + k_2 + k_3$, i.e.,

$$k \leq \frac{k_1 + k_2 + k_3}{2} \leq \frac{tr(T')}{4} = \frac{tr(T)}{4}. \tag{2}$$

Let n' be the number of vertices in T'. Then

$$n' = (k_1 + 1) + (k_2 + 1) + (k_3 + 1) - 2 = k_1 + k_2 + k_3 + 1 = \frac{tr(T)}{2} + 1.$$

From (2), we note that while deleting vertices from T to get T', we have deleted at most $\frac{tr(T)}{4}(l-3)$ vertices, i.e.,

$$\frac{tr(T)}{2} + 1 = n' \geq n - \frac{tr(T)}{4}(l-3),$$

i.e.,

$$2tr(T) + 4 \geq 4n - (l-3)tr(T),$$

i.e.,

$$(l-1)tr(T) \geq 4(n-1) \Rightarrow tr(T) \geq \left\lceil \frac{4(n-1)}{(l-1)} \right\rceil.$$

The lower bound is achieved by any tree with 3 leaves. ■

Theorem 3.19. *Let $G = (V, E)$ be a connected graph on n vertices with Wiener index σ. Then $tr(G) \geq \frac{6\sigma}{n(n-1)}$, and the bound is tight.*

Proof. Observe that for any pair of vertices $u, v \in V$, $d(u, v)$ appears $\binom{n-2}{1}$ times in the sum $\sum_{\{u,v,w\}\subset V} d(u, v, w)$. Thus, we get

$$\binom{n}{3} \cdot tr(G) \geq \sum_{\{u,v,w\}\subset V} d(u, v, w) = \binom{n-2}{1} \sum_{\{u,v\}\subset V} d(u, v) = (n-2)\sigma,$$

and hence the theorem follows. The tightness of the bound follows by taking $G = C_4$, the cycle on 4 vertices for which $\sigma = 8, tr(G) = 4$. ■

4. Triameter of Some Graph Families

In this section, we find the triameter of some important families of graphs. We start by recalling a result from [8].

Proposition 4.1 [8]**.** *For any two connected graphs G and H, $tr(G \square H) = tr(G) + tr(H)$.*

Corollary 4.2. *Let G be a $m \times n$ rectangular grid graph. Then $tr(G) = 2(m + n - 2)$.*

Proof. Since G is a $m \times n$ rectangular grid graph, $G \cong P_m \square P_n$. Thus $tr(G) = tr(P_m) + tr(P_n) = (2m - 2) + (2n - 2) = 2(m + n - 2)$. ■

Theorem 4.3. *Let G be a connected bipartite graph. Then $tr(G)$ is even.*

Proof. Let $V(G) = V_1 \cup V_2$ be the bipartition and u, v, w be 3 vertices in $V(G)$ such that $tr(G) = d(u, v, w)$. If $u, v, w \in V_i$ for same i, then each of $d(u, v), d(v, w)$ and $d(w, u)$ are even and hence $tr(G)$ is even. Thus, without loss of generality, let $u, v \in V_1$ and $w \in V_2$. Then $d(u, w)$ and $d(v, w)$ are odd and $d(u, v)$ is even and as a result, $tr(G)$ is even. ■

Theorem 4.4. *Let T be a tree on $n \geq 4$ vertices which is not a star. Then*

$$tr(T^c) = \begin{cases} 6, & \text{if } T \text{ is a bistar,} \\ 5, & \text{if } T \text{ is not a bistar.} \end{cases}$$

Proof. If T is neither a star nor bistar, then $diam(T) \geq 4$. Hence, $diam(T^c) \leq 2$. By Theorem 3.1, $tr(T^c) \leq 6$. We claim that $tr(T^c) < 6$. Suppose to the contrary that $tr(T^c) = 6$. Let $d_{T^c}(u, v, w) = 6$. We have $d_{T^c}(u, v) = 2$, $d_{T^c}(u, w) = 2$ and $d_{T^c}(v, w) = 2$, since $diam(T^c) \leq 2$. Hence, there is a triangle in T with vertices

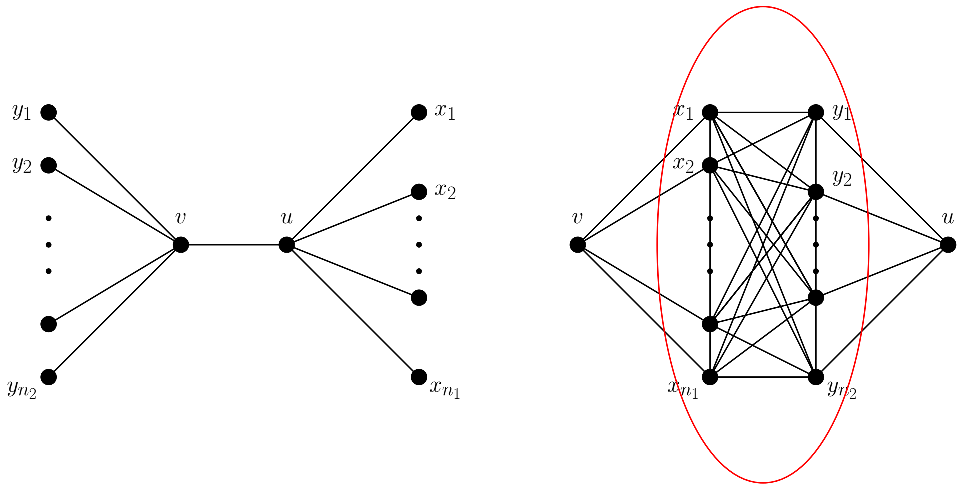


Figure 5. Bistar and its complement.

u, v and w, a contradiction. Thus, $tr(T^c) < 6$. Using a path of length two in T, one can prove that $tr(T^c) \geq 5$. Therefore, $tr(T^c) = 5$. For the other part, let $T = K_{n_2}^{n_1}$ be a bistar as in Figure 5 (left). Then its complement is as in Figure 5 (right).

Note that the complement consists of a clique induced by $x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2}$ (indicated in red) and v being adjacent to all the x_i's and u being adjacent to all the y_i's. Thus, for triameter of T^c, we need two of the vertices as u and v and the other to be any one of x_i's or y_i's. Hence, $tr(T^c) = 3+1+2 = 6$. ■

Proposition 4.5. *If G is a Hamiltonian graph on n vertices, then $tr(G) \leq n$.*

Proof. Since G is a Hamiltonian graph on n vertices, G contains C_n as a subgraph and hence $tr(G) \leq tr(C_n) = n$. ■

Theorem 4.6. *If $G = (V, E)$ is a connected vertex transitive graph, then*

$$2 \cdot rad(G) \leq tr(G) \leq 3 \cdot rad(G).$$

Proof. As G is vertex transitive, $V = center(G) = \{x \in V : ecc(x) = rad(G)\}$. Thus, for $u, v, w \in V$, $d(u, v) + d(v, w) + d(w, u) \leq ecc(u) + ecc(v) + ecc(w) = 3 \cdot rad(G)$. Hence the upper bound follows. The lower bound follows from Corollary 3.3. The tightness of lower and upper bounds follows by taking G as C_{2n} and Petersen graph, respectively. ■

Theorem 4.7. *If G is a connected strongly regular graph, then*

$$tr(G) = \begin{cases} 5, & \text{if } G^c \text{ is triangle-free,} \\ 6, & \text{otherwise.} \end{cases}$$

Proof. Let G be strongly regular with parameters (n, k, λ, μ). Since G is connected, $\mu \neq 0$ and G is not a complete graph. As a connected strongly regular graph has diameter 2, $tr(G) \leq 6$. Moreover G^c is again a strongly regular graph with parameter $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$. Let u, v be two non-adjacent vertices in G, i.e., $d(u, v) = 2$. If there exists a vertex w such that $d(u, w) = d(v, w) = 2$, then choosing u, v, w as the three vertices we get $tr(G) = 6$. If there does not exist such vertices in G, then all vertices other than u and v are either adjacent to u or v or both. Thus, counting the vertices in G, we get $n = (k-\mu) + (k-\mu) + \mu + 2$, i.e., $n - 2k + \mu - 2 = 0$, i.e., G^c is triangle free. In this case, choosing any $w \in N(v) \setminus N(u)$ in G, we get $d(u, w) = 2$ and $d(v, w) = 1$. Then $tr(G) = 5$. ∎

5. NORDHAUS-GADDUM BOUNDS

In this section, we prove some Nordhaus-Gaddum type bounds on triameter of a graph and its complement.

Lemma 5.1. *Let G be a connected graph such that G^c is connected. Then $tr(G) \geq 7$ implies $tr(G^c) \leq 12$.*

Proof. Since $diam(G) \geq tr(G)/3 > 2$, it follows that $\gamma(G^c) = 2$. Thus $tr(G^c) \leq 6\gamma(G^c) = 12$. ∎

Lemma 5.2. *Let $G = (V, E)$ be a graph such that G and G^c is connected. If $tr(G) > 9$, then $tr(G^c) \leq 6$.*

Proof. If possible, let $tr(G) > 9$ and $tr(G^c) \geq 7$. Let u, v, w be three arbitrary vertices in V.

Case 1. If at least one of $d_{G^c}(u, v), d_{G^c}(v, w), d_{G^c}(w, u)$, say $d_{G^c}(w, u)$ is greater than 1, then $d_G(w, u) = 1$. If $d_G(u, v)$ or $d_G(v, w)$ is greater than 3, then $diam(G) > 3$ implies $diam(G^c) \leq 2$ and $tr(G^c) \leq 6$, a contradiction. Thus $d_G(u, v), d_G(v, w) \leq 3$, i.e., $d_G(u, v) + d_G(v, w) + d_G(w, u) \leq 7 \leq 9$.

Case 2. If $d_{G^c}(u, v) = d_{G^c}(v, w) = d_{G^c}(w, u) = 1$, then $2 \leq d_G(u, v), d_G(v, w), d_G(w, u) \leq 3$ and hence $d_G(u, v) + d_G(v, w) + d_G(w, u) \leq 9$.

Combining the two cases we get $tr(G) \leq 9$, which is a contradiction to the assumption and hence the lemma holds. ∎

Theorem 5.3. *Let $G = (V, E)$ be a graph with $n \geq 4$ vertices such that G and G^c is connected. Then*

- $10 \leq tr(G) + tr(G^c) \leq 2n + 4$,

- $25 \leq tr(G) \cdot tr(G^c) \leq 12(n-1)$ *except for a finite family of graphs* $\mathcal{F}$,

and the bounds are tight.

Proof. If $tr(G) > 9$, $tr(G^c) \leq 6$. Also, by Theorem 3.7, $tr(G) \leq 2n-2$ and hence $tr(G) + tr(G^c) \leq 2n+4$ and $tr(G) \cdot tr(G^c) \leq 12(n-1)$. Let $tr(G) \leq 6$ and if possible, let $tr(G) + tr(G^c) > 2n+4$ or $tr(G) \cdot tr(G^c) > 12(n-1)$. Then $tr(G^c) > 2n-2$, a contradiction to Theorem 3.7. Thus, if $tr(G) > 9$ or $tr(G) \leq 6$, the both the upper bounds hold. Similarly, if $tr(G^c) > 9$ or $tr(G^c) \leq 6$, both the upper bounds hold.

So the only cases left are when $tr(G), tr(G^c) \in \{7, 8, 9\}$. Thus by Theorem 3.1, $diam(G)$, $diam(G^c) \in \{3, 4\}$. However, if $diam(G)$ or $diam(G^c)$ equals 4, then $diam(G^c)$ or $diam(G)$ is less than or equal to 2, a contradiction. Thus $diam(G) = diam(G^c) = 3$.

However, in this cases, for $n \geq 7$, $tr(G)+tr(G^c) \leq 18 \leq 2n+4$ and for $n \geq 8$, $tr(G) \cdot tr(G^c) \leq 81 \leq 12(n-1)$.

In [2], authors provide a complete list of 112 connected graphs on 6 vertices. Similarly, there are exactly 5 non-isomorphic graphs (see [15]) on 5 vertices for which both the graph and its complement is connected. Finally, P_4 is the only connected graph on 4 vertices whose complement is also connected. An exhaustive check (using Sage [16]) on these graphs revealed that the additive upper bound holds for $n = 4, 5, 6$, and hence the additive upper bound holds for all $n \geq 4$. Also note that for P_4, the multiplicative upper bound is an equality.

For the multiplicative upper bound in case of $n = 5, 6, 7$, let us define a family of graphs $\mathcal{F}$ as follows:

$$\mathcal{F} = \{G : |G| \in \{5, 6, 7\}; diam(G) = diam(G^c) = 3; tr(G), tr(G^c) \in \{7, 8, 9\}\}.$$

From the above discussions, it follows that the multiplicative upper bound holds for all graphs G not in $\mathcal{F}$.

For the lower bounds, observe that as $diam(G) = 1$ implies G^c is disconnected, we have $diam(G), diam(G^c) \geq 2$, and hence by Theorem 3.1, $tr(G), tr(G^c) \geq 4$. If possible, let $tr(G) = 4$, then there exists $u, v, w \in W$, such that $d(u,v)+d(v,w)+d(w,u) = 4$. Without loss of generality, let us assume $d(u,v) = 2$ and $d(v,w) = d(w,u) = 1$. If G is a graph on 3 vertices, then P_3 is the only choice for G satisfying the condition. However, complement of P_3 is not connected. Thus we assume that order of G is greater than 3. Note that for all $z \in V \setminus \{u, v\}$, we have $d(u,z) = d(v,z) = 1$ in G. But this implies that G^c is disconnected with u, v as one of the components. Thus, to ensure connectedness of G and G^c, we have $tr(G), tr(G^c) \geq 5$ and hence the additive and multiplicative lower bounds follows.

If $G = P_4$, path on 4 vertices, then $tr(G) = tr(G^c) = 6$ and hence the upper bounds are tight. If $G = C_5$, cycle on 5 vertices, then $tr(G) = tr(G^c) = 5$ and hence the lower bounds are tight. ∎

Remark 5.4. The multiplicative upper bound may not hold for graphs in $\mathcal{F}$. We demonstrate it in Figure 6. Here $n = 6, diam(G) = diam(G^c) = 3, tr(G) = tr(G^c) = 8$. Thus $tr(G) \cdot tr(G^c) = 64 > 12(6 - 1)$.

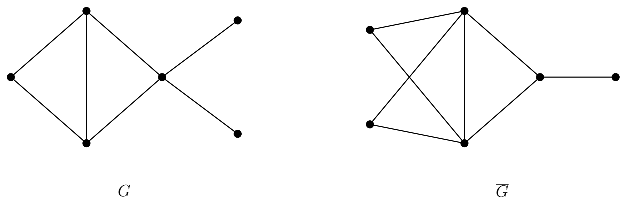


Figure 6. $G, G^c \in \mathcal{F}$.

6. Conclusion and Open Problems

In this paper, motivated by a lower bound on radio k-coloring in graphs, we formally introduce the idea of triameter in graphs and provide various bounds of various types with respect to other graph parameters. We also provide a shorter proof of a result in [7]. We conclude with some possible directions of further research and some open questions.

- Theorem 3.18 provides a lower bound of $tr(T)$ in terms of its order n and number of leaves $l \geq 3$. Though the bound is tight for $l = 3$, the bound loosens as l increases. To find a tighter bound can be an interesting topic of research.
- The only lower bound for connected graphs G (not necessarily trees) is in terms of girth (see Theorem 3.16). However, we believe that a better bound is possible in terms of the maximum $\Delta(G)$ and minimum degree $\delta(G)$.
- Let T be a tree with at least 3 leaves and u_1, u_2, and u_3 be three vertices of T such that $d(u_1, u_2, u_3) = tr(T)$. Is it true that there exist i and j where $i, j \in \{1, 2, 3\}$ and $i \neq j$ such that $d(u_i, u_j) = diam(T)$?
- Let T be a tree with at least 3 leaves and u, v be two vertices such that $d(u, v) = diam(T)$. Is it true that there exists a vertex w such that $d(u, v, w) = tr(T)$?

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