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CONFLICT-FREE VERTEX CONNECTION NUMBER AT MOST 3 AND SIZE OF GRAPHS

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Abstract

A path in a vertex-coloured graph is called *conflict-free* if there is a colour used on exactly one of its vertices. A vertex-coloured graph is said to be *conflict-free vertex-connected* if any two distinct vertices of the graph are connected by a conflict-free vertex-path. The *conflict-free vertex-connection number*, denoted by $vcfc(G)$, is the smallest number of colours needed in order to make G conflict-free vertex-connected. Clearly, $vcfc(G) \geq 2$ for every connected graph on $n \geq 2$ vertices.

Our main result of this paper is the following. Let G be a connected graph of order n. If $|E(G)| \geq \binom{n-6}{2} + 7$, then $vcfc(G) \leq 3$. We also show that $vcfc(G) \leq k + 3 - t$ for every connected graph G with k cut-vertices and t being the maximum number of cut-vertices belonging to a block of G.

Keywords: vertex-colouring, conflict-free vertex-connection number, size of graph.

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1. Introduction

We use [23] for terminology and notation not defined here and consider simple, finite and undirected graphs only. Let G be a graph, we denote by $V(G), E(G), n, m$ the vertex set, the edge set, the number of vertices, and the number of edges, respectively. A block is an *end-block* if it has only one cut-vertex. We abbreviate the set $\{1, \ldots, k\}$ by $[k]$.

A path P in an edge-coloured graph G is called a *rainbow path* if its edges have distinct colours. An edge-coloured graph G is *rainbow connected* if every two vertices are connected by at least one rainbow path in G. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is defined as the smallest number of colours required to make it rainbow connected. The concept of rainbow connection number was first introduced by Chartrand *et al.* [6]. Readers who are interested in this topic are referred to [20, 21].

Motivated by the proper colouring and rainbow connection, Borozan *et al.* [3] and Andrews *et al.* [2], independently introduced the concept of *proper connection.* A path P in an edge-coloured graph G is a *proper path* if any two consecutive edges receive distinct colours. An edge-coloured graph G is *properly connected* if every two vertices are connected by at least one proper path in G. For a connected graph G, the *proper connection number* of G, denoted by $pc(G)$, is defined as the smallest number of colours required to make it properly connected. Since then, a lot of results on this concept have been obtained; see [17] for a survey.

Recently, Czap *et al.* [10] introduced the concept of *conflict-free connection.* A path in an edge-coloured graph G is called *conflict-free* if there is a colour used on exactly one of its edges. An edge-coloured graph G is said to be *conflict-free connected* if any two vertices are connected by at least one conflict-free path. The *conflict-free connection number* of a connected graph G, denoted by $cfc(G)$, is defined as the smallest number of colours in order to make it conflict-free connected. For more results, the readers are referred to [4, 5, 7, 10].

As a natural counterpart of the concepts of rainbow connection, proper connection and conflict-free connection, the concepts of *rainbow vertex-connection, proper vertex-connection*, and *conflict-free vertex-connection* were introduced, respectively.

The concept of *rainbow vertex-connection* was first introduced by Krivelevich *et al.* [15]. A path in a vertex-coloured graph G is called a *vertex rainbow path* if its internal vertices have distinct colours. A vertex-coloured graph G is *rainbow vertex-connected* if any two vertices are connected by at least one vertex rainbow path. For a connected graph G, the *rainbow vertex-connection number*, denoted by $rvc(G)$, is the smallest number of colours that are needed in order to make G rainbow vertex-connected. Recently, a lot of results on this topic have been obtained, see [8, 16, 18, 19].

Similarly, Jiang *et al.* [13] and Chizmar *et al.* [9], independently introduced the concept of *proper vertex-connection.* A path P in a vertex-coloured graph G is a *proper vertex-path* if any two internal adjacent vertices differ in colour. A vertex-coloured graph G is called *properly vertex-connected* if any two vertices are connected by at least one proper vertex-path. For a connected graph G, the *proper vertex-connection number*, denoted by $pvc(G)$, is the smallest number of colours that are needed in order to make G properly vertex-connected.

Motivated by the above mentioned concepts, Li *et al.* [11] introduced the concept of *conflict-free vertex-connection.* A path P in a vertex-coloured graph G is said to be a *conflict-free vertex-path* if there is a colour used on exactly one of its vertices. A vertex-coloured graph G is said to be *conflict-free vertex-connected* if any two vertices of the graph are connected by a conflict-free vertex-path. The *conflict-free vertex-connection number* of a connected graph G, denoted by $vcfc(G)$, is defined as the smallest number of colours in order to make G conflict-free vertex-connected. Recently, some results on this topic have been shown in [12, 22].

Our research was motivated by the following results for the proper connection number and the rainbow connection number of graphs depending on their size.

Theorem 1 (Kemnitz *et al.* [14])**.** *Let G be a connected graph of order n and size m. If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

Theorem 2 (Aardt *et al.* [1])**.** *Let $k \geq 3$ be an integer and G be a connected graph of order n. If $|E(G)| \geq \binom{n-k-1}{2} + k + 2$, then $pc(G) \leq k$.*

In [1], the authors also considered the case $k = 2$. Let $G_1 = K_1 \vee (2K_1 + K_2)$ and $G_2 = K_1 \vee (K_1 + 2K_2)$, where $G + H = (V_G \cup V_H, E_G \cup E_H)$ is the disjoint union and $G \vee H = (V_G \cup V_H, E_G \cup E_H \cup \{uv : u \in V_G, v \in V_H\})$ is the join of the graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$.

Theorem 3 (Aardt *et al.* [1])**.** *Let G be a connected graph of order n. If $|E(G)| \geq \binom{n-3}{2} + 4$, then $pc(G) \leq 2$ unless $G \in \{G_1, G_2\}$.*

2. Auxiliary Results

In this section, we state some fundamental results, which will be used throughout later in the proofs of our results. First note that for a connected graph G of order $n \geq 2$ we can easily observe that $vcfc(G) \geq 2$.

The conflict-free vertex-connection number of a path has been computed by Li *et al.* [11].

Theorem 4 (Li *et al.* [11]). *If P_n is a path of order n, then $vcfc(P_n) = \lceil log_2(n+1) \rceil$.*

The proof of Theorem 4 is similar to that of Theorem 3 in [10]. The following result, which is very important to determine the existence of a conflict-free path of a subgraph or a 2-connected graph, was proved by the authors in [11].

Theorem 5 (Li *et al.* [11]). *If G is a 2-connected graph and w is a vertex of G, then for any two vertices u and v in G, there is a $u-v$ path containing the vertex w.*

Next, Li *et al.* [11] used the result of Theorem 5, as a basic tool, to determine the necessary and the sufficient conditions of connected graphs having conflict-free connection number 2.

Theorem 6 (Li *et al.* [11]). *Let G be a connected graph of order at least 3, then $vcfc(G) = 2$ if and only if G is 2-connected or G has only one cut-vertex.*

Since $vcfc(G) = 2$ if and only if G is 2-connected or G has only one cut-vertex, it is natural to determine the conflict-free vertex-connection number of a graph having at least two cut-vertices. Hence, the following corollary is immediately obtained by the authors in [11].

Corollary 7 (Li *et al.* [11]). *Let G be a connected graph. Then $vcfc(G) \geq 3$ if and only if G has at least two cut-vertices.*

The next lemma provides a combinatorial equality and an inequality, which will be used several times in our later proofs.

Lemma 8. (i) *For every positive integer a it holds*

$$\binom{a+1}{2} = \binom{a}{2} + a.$$

(ii) *For every three positive integers n, t, a such that $t \geq a+1$ it holds*

$$\binom{n-t}{2} \leq \binom{n-a}{2} - n + a + 1.$$

Proof. Case (i) is immediately obtained after some manipulations.

Using case (i) and note that $t \geq a+1$, we obtain case (ii) as follows

$$\binom{n-t}{2} \leq \binom{n-(a+1)}{2} = \binom{n-a}{2} - (n-(a+1))$$
$$= \binom{n-a}{2} - n + a + 1.$$

∎

Lemma 9. *Let $k \geq 2$ be an integer. If $a_1, \ldots, a_k$ are integers and all of them are greater than one, then*

$$\sum_{i=1}^{k} \binom{a_i}{2} < \binom{\sum_{i=1}^{k} a_i - (k-1)}{2}.$$

Proof. We prove this lemma by induction on k. Let $k = 2$. Clearly, $(a_1 - 1)(a_2 - 1) > 0$ since $a_i \geq 2$ for every $i \in [k]$. After some manipulations we get

$$\binom{a_1}{2} + \binom{a_2}{2} < \binom{a_1 + a_2 - 1}{2}.$$

We may assume that the inequality is true for some $k = t \geq 2$. Hence,

$$\sum_{i=1}^{t} \binom{a_i}{2} < \binom{\sum_{i=1}^{t} a_i - (t-1)}{2}.$$

By the induction hypothesis, we deduce that

$$\sum_{i=1}^{t} \binom{a_i}{2} + \binom{a_{t+1}}{2} < \binom{\sum_{i=1}^{t} a_i - (t-1)}{2} + \binom{a_{t+1}}{2} < \binom{\sum_{i=1}^{t+1} a_i - t}{2}.$$

The result is obtained. ∎

Lemma 10. *Let $k \geq 2$ be an integer. If $a_1, \ldots, a_k$ are integers and all of them are greater than one, then*

$$\sum_{i=1}^{k} \binom{a_i}{2} \leq \binom{\sum_{i=1}^{k} a_i - 2(k-1)}{2} + (k-1).$$

Proof. We prove this lemma by induction on k. Let $k = 2$. Clearly, $(a_1 - 2)(a_2 - 2) \geq 0$ since $a_i \geq 2$ for every $i \in [k]$. After some manipulations we get

$$\binom{a_1}{2} + \binom{a_2}{2} \leq \binom{a_1 + a_2 - 2}{2} + 1.$$

Using the induction on k, in the same way as in the proof of Lemma 9, the result is obtained. ∎

By Lemma 10, an upper bound for the size of a graph G having exactly k blocks is attained by the following corollary.

Corollary 11. *Let G be a graph and $k \geq 2$ be an integer. If G has exactly k blocks B_i of order n_i, then*

$$|E(G)| \leq \binom{\sum_{i=1}^{k} n_i - 2(k-1)}{2} + (k-1).$$

Now we consider some results on the number of cut-vertices of a connected graph. By using Lemma 9 and Lemma 10, one can readily obtain the next result, which is the vertex version of the result in [1]. Moreover, this result is very important in the proof of our main result.

Lemma 12. *If a connected graph G on n vertices has t cut-vertices, then*

$$|E(G)| \leq \binom{n-t}{2} + t.$$

Proof. We prove the lemma by induction on t. Clearly, the result immediately holds if $t = 0$. We may assume, now, $t \geq 1$. Let v be a cut-vertex of G such that $v \in V(B_i)$, where all B_i are end-blocks of G. Now for every end-block B_i containing v, we delete $V(B_i - v)$ and all edges incident to $V(B_i - v)$. Call the resulting graph G'.

By our assumption, it can be readily seen that G' has exactly $t-1$ cut-vertices which are distinct from v. Let n_{B_i} be the order of B_i for every i and $n_{G'}$ be the order of G'. Hence, by the induction hypothesis, $|E(G')| \leq \binom{n_{G'}-(t-1)}{2} + t - 1$. Let k be the number of end-blocks B_i of G incident with v. Since all blocks B_i and the component G' have the common cut-vertex v,

$$n = \sum_{i=1}^{k}(n_{B_i} - 1) + n_{G'} - 1 + 1 = \sum_{i=1}^{k} n_{B_i} + n_{G'} - k.$$

Hence,

$$|E(G)| = \sum_{i=1}^{k} |E(B_i)| + |E_{G'}| \leq \sum_{i=1}^{k} \binom{n_{B_i}}{2} + |E_{G'}|.$$

Using Lemma 9 for k blocks B_i and the induction hypothesis on G', we deduce that

$$|E(G)| \leq \binom{\sum_{i=1}^{k} n_{B_i} - (k-1)}{2} + \binom{n_{G'} - (t-1)}{2} + t - 1.$$

After using Lemma 10 and the order of G, the result is obtained

$$|E(G)| \leq \binom{\sum_{i=1}^{k} n_{B_i} - (k-1) + n_{G'} - (t-1) - 2}{2} + 1 + t - 1 = \binom{n-t}{2} + t.$$

This completes our proof. ■

Lemma 13. *Let G be a connected graph and $t \geq 3$ be an integer. If G has t cut-vertices, then there always exists a subset of at least 3 cut-vertices that are connected by a path.*

Proof. If a block of G has at least three cut-vertices, then the statement follows from Theorem 5. Now assume that a block contains exactly two cut-vertices u and v. Let w be a third cut-vertex in G. Let P_1 be a path connecting u and v and P_2 be a path connecting w and u. If P_2 contains v, then P_2 contains three cut-vertices, otherwise the path wP_2uP_1v has the required property. This finishes our proof. ■

The following example shows that there are connected graphs having no path connecting at least 4 cut-vertices. Let $k \geq 3$ be an integer and $K_{1,k}$ be a star on $k+1$ vertices. Let G be a graph obtained from $K_{1,k}$ by subdividing each of its edges. Clearly, every path in G contains at most 3 cut-vertices.

3. Main Result

In this section, we study graphs with conflict-free vertex-connection number at most 3 depending on their size. Before presenting our main result, we prove several useful results on the conflict-free vertex-connection number of a connected graph. First of all, we consider the upper bound of the conflict-free vertex-connection number of graphs.

3.1. The upper bound

By Theorem 6, the conflict-free vertex-connection number equals 2 for every connected graph having no cut-vertex or only one cut-vertex. Consequently, the conflict-free vertex-connection number is at least 3, if G has at least two cut-vertices. Motivated by these results, we consider the upper bound of the conflict-free vertex-connection number depending on the number of cut-vertices of a connected graph. Let G be a connected graph having at least 2 cut-vertices. For every block B_i of G, let x_i denote the number of cut-vertices of G belonging to B_i. Hence, $x_i \geq 1$, for $1 \leq i \leq l$, where l is the number of (trivial or non-trivial) blocks in G. Let $t = \max\{x_i \mid 1 \leq i \leq l\}$. Hence, t is the maximum number of cut-vertices belonging to a block of G. The following theorem gives a tight upper bound for the conflict-free vertex-connection number of a connected graph.

Theorem 14. *Let G be a connected graph. If k is the number of cut-vertices of G and t is the maximum number of cut-vertices belonging to a block of G, then $vcfc(G) \leq k+3-t$. Moreover, this bound is sharp.*

Proof. Renaming blocks if necessary, assume that B_1 is a block of G having the maximum number of cut-vertices. Trivially, if $t = 1$, then $k = 1$. It follows that G has only one cut-vertex. By Theorem 6, the result is obtained. Hence, $t \geq 2$. Let $S = \{v_1, v_2, \ldots, v_t\}$ be the set of cut-vertices of G belonging to B_1. Now, we assign colour 2 to the vertex v_1, colour 3 to all the vertices in $S \setminus \{v_1\}$. Every cut-vertex of G that is not in S is assigned a different colour from $\{4, 5, \ldots, k+3-t\}$. Next, we colour all the remaining vertices of G by colour 1. In such a way we obtain a vertex-coloring c of G. For every block B_i from G, it can be readily seen that there always exists at least one cut-vertex having the colour which is different from the colour of the remaining vertices of B_i. It follows that there is a conflict-free vertex-path between any two arbitrary vertices belonging to the same block by Theorem 5.

Suppose now that two arbitrary vertices, say x and y, belong to two different blocks, say B_x and B_y, respectively. Since G is connected, there is a path, say P, connecting x and y. Let $v_x \in V(P)$ and $v_y \in V(P)$ be the cut-vertices of B_x and B_y, respectively. If $v_x \equiv v_y$ or $c(v_x) \neq c(v_y)$, then P is the conflict-free vertex-path with the unique colour, say $c(v_x)$ or $c(v_y)$. If $c(v_x) = c(v_y)$ and $v_x \neq v_y$, then $v_x, v_y \in V(B_1)$. Hence, B_1 is the nontrivial block. By Theorem 5, there is a $v_x - v_y$ path in B_1, say P', containing v_1. Now, $xPv_xP'v_yPy$ is a conflict-free vertex-path. Hence, there always exists at least one conflict-free vertex-path connecting any two arbitrary vertices of G.

The result is obtained. ■

The sharpness examples for Theorem 14 are given as follows. Let G be a connected graph having at least two cut-vertices and let all the cut-vertices belong to the same block, i.e., $k = t \geq 2$. By Corollary 7, $vcfc(G) \geq 3$. On the other hand, by Theorem 14, $vcfc \leq 3$. Hence, $vcfc(G) = 3 = k + 3 - t$.

Clearly, if G has exactly two cut-vertices, then they are in the same block. Hence, the following corollary is immediately obtained.

Corollary 15. *Let G be a connected graph. If G has exactly two cut-vertices, then $vcfc(G) = 3$.*

Now, we consider the conflict-free vertex-connection number of a connected graph having many cut-vertices in the same block.

Lemma 16. *Let $k \geq 3$ be an integer and G be a connected graph of k cut-vertices. If at least $k-1$ cut-vertices belong to an unique block, then $vcfc(G) = 3$.*

Proof. By Corollary 7, $vcfc(G) \geq 3$ since G has at least 3 cut-vertices. Let $v_1, \ldots, v_k$ be the cut-vertices of G. By Theorem 14 it suffices to consider the case when $k-1$ cut-vertices belong to the same block. Hence we may assume that $v_1, \ldots, v_{k-1} \in V(B)$ and $v_k \notin V(B)$ for some block B.

Now, we show that we are able to assign three colours to all the vertices of G in order to make it conflict-free vertex-connected. Since G is connected, there is a path, say P, connecting v_k and $V(B)$. Clearly, the end-vertex of P is a cut-vertex of G. Otherwise, G contains at least $k+1$ cut-vertices since v_k does not belong to the block B. Renaming vertices if necessary, we may assume, that the end-vertex of P is v_1. Now, let B' be a block containing v_1 and v_k. Clearly, $V(B) \cap V(B') = \{v_1\}$. If v_1v_k is a bridge of G, then B' is trivial. Otherwise, B' is non-trivial. We assign the colour 1 to the vertex v_1, the colour 2 to all the vertices in $V(B) \cup V(B') \setminus \{v_1\}$, the colour 3 to all the remaining uncoloured vertices of G. By simple case to case analysis, G is conflict-free vertex-connected. Hence, $vcfc(G) \leq 3$. We deduce that $vcfc(G) = 3$.

The proof is obtained. ∎

Let G be a connected graph. The *eccentricity* $\epsilon_G(v)$ of a vertex $v \in V(G)$ is the maximum value among the distance between v and the other vertices in G. The *radius* $rad(G)$ of G is the minimum eccentricity among all the vertices of G. In [11], the authors proved an upper bound for the conflict-free vertex-connection of a connected graph depending on the radius $rad(G)$ as follows.

Theorem 17 (Li *et al.* [11])**.** *If T is a tree with radius $rad(T)$, then $vcfc(T) \leq rad(T) + 1$. Moreover, the bound is sharp.*

Corollary 18 (Li *et al.* [11])**.** *If G is a connected graph, then $vcfc(G) \leq rad(G) + 1$.*

Let G be a connected graph with $k \geq 2$ cut-vertices and let t be the maximum number of cut-vertices belonging to a block, say B, of G. Clearly, $t \geq 2$. Now, there are $k - t$ cut-vertices of G not belonging to B. Let $S = \{v_1, v_2, \ldots, v_t, v_{t+1}, \ldots, v_k\}$ be the cut-vertex set of G such that $v_1, \ldots, v_t$ are in B and $v_{t+1}, \ldots, v_k$ are not in B. Two blocks are *neighbours* if they have a common cut-vertex. We construct the tree $T^*_{v_i}$, where $i \in [t]$, as follows.

1. Every $v_i \in V(B)$ is the root of $T^*_{v_i}$.
2. We consider all the neighbour blocks of B containing v_i, say B_j, having the cut-vertices of G. We denote these cut-vertices, say $v_j \in V(B_j)$, by $v_j^{T^*}$ and add them to $T^*_{v_i}$ by adding an edge $v_iv_j^{T^*}$. Next, we continue to consider the neighbour block of B_j and repeat these processes to the end-blocks of G.

Now applying steps 1 and 2 above, we construct the tree T^* by identifying all v_i of $T^*_{v_i}$, where $i \in [t]$, by a vertex, say v. Hence, $|V(T^*)| = k - t + 1$ and v is the root of T^*. An example of T^* is depicted in Figure 1. By Theorem 17, the following result is obtained.

Theorem 19. *If G is a connected graph with at least two cut-vertices, then $vcfc(G) \leq rad(T^*) + 4$.*

Proof. By Theorem 17, T^* is conflict-free vertex-connected with $rad(T^*) + 1$ colours. We assign $rad(T^*) + 1$ colours from $\{4, 5, \ldots, rad(T^*) + 4\}$ to make it conflict-free vertex-connected. Since T^* is a tree, every its two vertices are

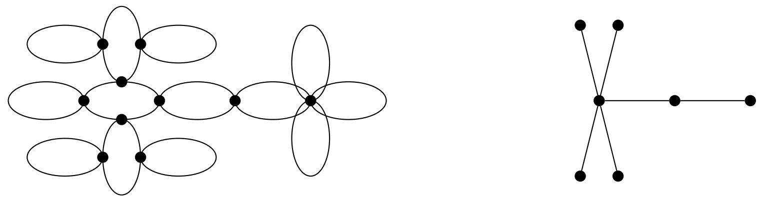


Figure 1. Graph G and T^*.

connected by only one path. We colour all the vertices of G as following: $c(v_j) = c(v_j^{T^*})$, for every $j \in [k]\backslash[t]$. We assign colour 2 to the vertex v_1, colour 3 to all the $t-1$ remaining cut-vertices in B. Hence, all the cut-vertices of G are coloured. We colour all the remaining vertices of G by colour 1. It can be readily seen that there always exists at least one conflict-free vertex-path between any two cut-vertices of G since T^* is conflict-free vertex-connected and B is conflict-free vertex-connected with 3 colours from [3]. Moreover, every block has at least one cut-vertex having different colour to all its remaining vertices. As in the proof of Theorem 14, there always exists at least one conflict-free vertex-path between any two arbitrary vertices of G. ■

3.2. Conflict-free vertex-connection number at most 3

In this subsection, we consider our main result as follows.

Theorem 20. *Let G be a connected graph of order $n \geq 8$. If $|E(G)| \geq \binom{n-6}{2} + 7$, then $vcfc(G) \leq 3$.*

Before starting to prove Theorem 20, we show that the bound for the size of the graph is sharp by the following proposition.

Proposition 21. *There exists a connected graph of order n and $|E(G)| = \binom{n-6}{2} + 6$ such that $vcfc(G) \geq 4$.*

Proposition 21 can be extended for any integer $k \geq 2$ by the following theorem.

Theorem 22. *Let $k \geq 2$ be an integer. There exists a connected graph of order n and $|E(G)| = \binom{n-(2^k-2)}{2} + 2^k - 2$ such that $vcfc(G) \geq k+1$.*

Proof. We construct a connected graph G as follows. Take a complete graph $K_{n-(2^k-2)}$ and a path $P_{2^k-1} = w_1 \cdots w_{2^k-1}$. Note that $|V(P_{2^k-1})|$ is odd. Let u be an arbitrary vertex of $K_{n-(2^k-2)}$. Now, we identify the vertex u with the vertex w_1. It can be readily observed that the resulting graph G has order n, and size $|E(G)| = \binom{n-(2^k-2)}{2} + 2^k - 2$. Next, we prove that $vcfc(G) \geq k+1$.

Note that, by our construction above, there is an unique path between any two vertices w_i, w_j, where $w_i, w_j \in V\big(P_{2^k-1}\big)$. It follows that P_{2^k-1} is conflict-free vertex-connected. By Theorem 4, we must use at least k colours in order to make P_{2^k-1} conflict-free vertex-connected since $\left|V\left(P_{2^k-1}\right)\right| = 2^k - 1$. Hence, $vcfc(G) \geq k$. Now, suppose to the contrary, that we are able to use exactly k colours to colour all the vertices of G in order to make it conflict-free vertex-connected. By the concept of conflict-free vertex-connection, there must be an unique colour on P_{2^k-1}, say k, that is assigned to a vertex w_i. We show that $c\big(w_{2^{k-1}}\big) = k$. Otherwise, without loss of generality, we may assume that $c(w_i) = k$, where $i \in \big[2^k - 1\big] \setminus \big[2^{k-1}\big]$ since P_{2^k-1} is symmetry by $w_{2^{k-1}}$. By Theorem 4, $vcfc\big(w_1 P_{2^k-1} w_{2^{k-1}}\big) = k$. Hence, the colour k appears at least two times on the path P_{2^k-1}, a contradiction. Thus, $c\big(w_{2^{k-1}}\big) = k$ and all vertices in $V\big(w_1 P_{2^k-1} w_{2^{k-1}-1}\big)$ can receive the colours from $[k-1]$. On the other hand, by Theorem 4, $vcfc\big(w_1 P_{2^k-1} w_{2^{k-1}-1}\big) = k-1$. Similarly, a unique colour, say $k-1$, on the path $w_1 P_{2^k-1} w_{2^{k-1}-1}$ must be assigned to $w_{2^{k-2}}$. We continue these steps by decreasing k to 2. Hence, by Theorem 4, $vcfc\big(w_1 P_{2^k-1} w_3\big) = 2$. It follows that we can use two colours from [2] to colour all the vertices of the subgraph $H = G - \{w_4, w_5, \ldots, w_{2^k-1}\}$ to make it conflict-free vertex-connected. Note that H has two cut-vertices. By Corollary 15, $vcfc(H) = 3$, a contradiction. Hence, k is not enough to make G conflict-free vertex-connected. Therefore, $vcfc(G) \geq k + 1$. This completes our proof. ■

Clearly, when $k = 3$ we immediately obtain Proposition 21.

Now we are able to prove our main result. Recall its statement here.

Theorem 20. *Let G be a connected graph of order $n \geq 8$. If $|E(G)| \geq \binom{n-6}{2} + 7$, then $vcfc(G) \leq 3$.*

Proof. We prove our theorem by several claims as follows. Let G be a connected graph of order n and $|E(G)| \geq \binom{n-6}{2} + 7$.

Claim 23. *G has at most 5 cut-vertices.*

Proof. Let t be the number of cut-vertices of G. By Lemma 12, $|E(G)| \leq \binom{n-t}{2} + t$. If $t = 6$, then $|E(G)| \leq \binom{n-6}{2} + 6$, a contradiction.

If $t \geq 7$, then by Lemma 8

$$|E(G)| \leq \binom{n-t}{2} + t \leq \binom{n-6}{2} - n + 7 + t.$$

Note that $n \geq t + 2$ since G has order n and t cut-vertices. Hence, $|E(G)| \leq \binom{n-6}{2} + 5$, a contradiction.

Therefore, we deduce that $t \leq 5$. □

Let $B_1, \ldots, B_l$ be the blocks of G which contain at least two cut-vertices of G, and let $Q_1, \ldots, Q_k$ be the other blocks (the end-blocks) of G.

By Theorem 6, Corollary 15 and Lemma 16, now we consider that G has at least 4 cut-vertices.

Claim 24. *If G has 4 cut-vertices, then $vcfc(G) = 3$.*

Proof. Let $S = \{v_1, v_2, v_3, v_4\}$ be the set of cut-vertices of G. By Lemma 16, we can assume that there are at most two vertices of S in the same block. By Lemma 13, there always exist a subset of at least three cut-vertices that are connected by a path. Hence, two cases are considered as follows. Let B_i, where $i \in [3]$, be three blocks containing exactly two cut-vertices of G.

Case 1. At most three cut-vertices are connected by a path. Renaming vertices if necessary, we may assume that $v_1, v_2 \in B_1$, $v_2, v_3 \in B_2$ and $v_2, v_4 \in B_3$, see Figure 2. Clearly, G must contain several other blocks. We assign colour 1 to the vertex v_2, colour 2 to all three vertices v_1, v_3, v_4 and colour 3 to all uncoloured vertices of G. By simple case to case analysis, it can be observed that G is conflict-free vertex-connected with three colours. Hence, $vcfc(G) \leq 3$.

At most 3 cut-vertices in a path. All 4 cut-vertices in a path.

Figure 2. Graph G has 4 cut-vertices.

Case 2. All four cut-vetices are connected by a path. Renaming vertices if necessary, we may assume that $v_i, v_{i+1} \in B_i$, where $i \in [3]$, see Figure 2. Since v_1, v_4 are cut-vertices of G, there are at least two end-blocks Q_i, i.e., $k \geq 2$. Let v_1, v_4 belong to Q_1, Q_2, respectively. Hence,

$$n = \sum_{i=1}^{k}(n_{Q_i} - 1) + \sum_{i=1}^{3}(n_{B_i} - 2) + 4 = \sum_{i=1}^{k} n_{Q_i} + \sum_{i=1}^{3} n_{B_i} - k - 2.$$

Now, $|E(G)| = \sum_{i=1}^{k} |E(Q_i)| + \sum_{i=1}^{3} |E(B_i)|$. By Lemma 10, after some manipulations we get

$$|E(G)| \leq \binom{n-(k+2)}{2} + k + 2.$$

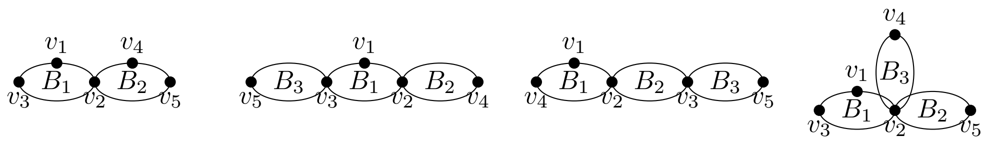
By Lemma 8 and $n \geq k+4$, we conclude that $k \leq 3$. By the symmetry of induced subgraph $G[V(G) \setminus V(Q_3)]$, when $k = 3$, renaming vertices if necessary, there are two following cases: $v_1 \in Q_3$ or $v_2 \in Q_3$. Now we assign colour 1 to the vertex v_2, colour 2 to two vertices v_1, v_4, and colour 3 to all uncoloured vertices of G. By simple case to case analysis, it can be observed that G is conflict-free vertex-connected with three colours. Hence, $vcfc(G) \leq 3$.

On the other hand, by Corollary 7, $vcfc(G) \geq 3$. Therefore, $vcfc(G) = 3$. This completes our proof. □

Claim 25. *If G has 5 cut-vertices, then $vcfc(G) = 3$.*

Proof. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ be the set of cut-vertices of G. Again by Lemma 16, we can assume there are at most three vertices of S in the same block. Now, we consider two following cases.

Case 1. If three vertices of S are in the same block, then there are three subcases, see Figure 3.



(a) Two blocks contain 3 cut-vertices.

(b)–(d) One block contains 3 cut-vertices.

Figure 3. Graph G has 5 cut-vertices, where 3 cut-vertices are in the same block.

For subcase 1(a), $4 \leq k \leq n-5$ since v_1, v_3, v_4, v_5 are cut-vertices and $l = 2$. Hence, $n = \sum_{i=1}^{k} n_{Q_i} + n_{B_1} + n_{B_2} - k - 1$. Now, $|E(G)| \leq \binom{n-(k+1)}{2} + k + 1$. As in the proof of Claim 24, one can easily see that $k \leq 4$. Hence, $k = 4$ and $v_2 \notin Q_i$ for all $i \in [4]$. We assign colour 1 to the vertex v_2, colour 2 to all uncoloured vertices of B_1 and B_2, and colour 3 to all uncoloured vertices of G.

For subcases 1(b) and 1(c), $3 \leq k \leq n-5$ since v_1, v_4, v_5 are cut-vertices and $l = 3$. Hence, $n = \sum_{i=1}^{k} n_{Q_i} + \sum_{i=1}^{3} n_{B_i} - k - 2$. Now $|E(G)| \leq \binom{n-(k+2)}{2} + k + 2$. As in the proof of Claim 24, one can easily see that $k \leq 3$. Hence, $k = 3$ and $v_2, v_3 \notin Q_i$ for all $i \in [3]$. Renaming vertices if necessary, we may assume that $v_1 \in Q_1$, $v_4 \in Q_2$ and $v_5 \in Q_3$. We assign colour 1 to the vertex v_2, colour 2 to the vertex v_5, all uncoloured vertices in $V(Q_2) \setminus \{v_4\}$ and $V(Q_1) \setminus v_1$, and colour 3 to all remaining uncoloured vertices of G.

For subcase 1(d), $4 \leq k \leq n-5$ since v_1, v_3, v_4, v_5 are cut-vertices and $l = 3$. Hence, $n = \sum_{i=1}^{k} n_{Q_i} + n_{B_1} + n_{B_2} + n_{B_3} - k - 2$. Now, $|E(G)| \leq \binom{n-(k+2)}{2} + k + 2$. As in the proof of Claim 24, one can easily see that $k \leq 3$, a contradiction.

By simple case to case analysis, it can be readily observed that G is conflict-free vertex-connected. Hence, $vcfc(G) \leq 3$.

Case 2. Now we consider the last case that at most two vertices of S are in the same block. It can be readily observed that there are 4 blocks B_i containing exactly two cut-vertices of G. Hence, $n = \sum_{i=1}^{k} Q_i + \sum_{i=1}^{4} B_i - k - 3$, where $2 \leq k \leq n-5$. Now $|E(G)| \leq \binom{n-(k+3)}{2} + k + 3$. As in the proof of Claim 24, one can easily see that $k \leq 2$. Hence, $k = 2$, i.e., there are only two blocks Q_i such that Q_i contains only one cut-vertex of G, where $i \in [2]$. Therefore, there always exist a path in G that connects all five cut-vertives of G, see Figure 4. We assign colour 1 to the vertex v_3, colour 2 to two vertices v_1, v_5 and colour 3 to all remaining uncoloured vertices of G.

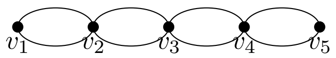


Figure 4. Graph G has 5 cut-vertices, where at most 2 cut-vertices are in the same block.

By simple case to case analysis, it can be readily observed that G is conflict-free vertex-connected. Hence, $vcfc(G) \leq 3$.

On the other hand, by Corollary 7, $vcfc(G) \geq 3$. Therefore, $vcfc(G) = 3$. This completes our proof. □

The proof is obtained. ■

By Theorem 22 and Theorem 20, we pose the following conjecture.

Conjecture 26. *Let $k \geq 3$ be an integer, and G be a connected graph of order n. If $|E(G)| \geq \binom{n-(2^k-2)}{2} + 2^k - 1$, then $vcfc(G) \leq k$.*

Clearly, Conjecture 26 is true for $k = 3$ by Theorem 20.

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