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REMOVABLE EDGES ON A HAMILTON CYCLE OR OUTSIDE A CYCLE IN A 4-CONNECTED GRAPH[1]

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Abstract

Let G be a 4-connected graph. We call an edge e of G removable if the following sequence of operations results in a 4-connected graph: delete e from G; if there are vertices with degree 3 in $G-e$, then for each (of the at most two) such vertex x, delete x from $G-e$ and turn the three neighbors of x into a clique by adding any missing edges (avoiding multiple edges). In this paper, we continue the study on the distribution of removable edges in a 4-connected graph G, in particular outside a cycle of G or in a spanning tree or on a Hamilton cycle of G. We give examples to show that our results are in some sense best possible.

Keywords: 4-connected graph, removable edge, fragment, atom.

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1. Introduction

All graphs considered here are finite and simple. For notations and terminology not defined here, we refer the reader to [2].

We start off by repeating the definition of the central concept of this paper. Let G be a 4-connected graph and let e be an edge of G. Then e is called *removable* if the following operations result in a 4-connected graph: delete e from G; if there are vertices with degree 3 in $G-e$, then for every vertex x with degree 3 in $G-e$, delete x from $G-e$ and turn the three neighbors of x into a clique by adding any missing edges (avoiding multiple edges). We denote the resulting graph by $G \ominus e$. Note that there are at most two vertices with degree 3 in $G-e$, and that $G \ominus e$ is simply the graph $G-e$ if there are no such vertices. If e is not removable, we also call it *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$, whereas the set of all unremovable edges of G is denoted by $E_N(G)$. The number of removable edges of G is denoted by $e_R(G)$.

The concept of removable edges in 4-connected graphs has been introduced as a tool for alternative methods to construct 4-connected graphs [8], and for proving properties of 4-connected graphs. Slater [4] gave a method to construct 4-connected graphs by proving that any 4-connected graph can be obtained from K_5 by applying the following operations (that we will not specify here) repeatedly: (1) adding edges; (2) 4-soldering; (3) 4-point-splitting; (4) 4-line-splitting; (5) 3-fold-4-point-splitting. Later, Yin [8] gave an alternative method to construct 4-connected graphs by using the concepts of removable edges and contractible edges in 4-connected graphs. In particular, in [8] Yin proved that there always exists a removable edge in a 4-connected graph G, unless G is a so-called 2-cyclic graph with order 5 or 6, i.e., the square of a cycle on 5 or 6 vertices. Ando *et al.* [1] and Wu *et al.* [7] studied the number of contractible edges and removable edges of a 4-connected graph, respectively. In another paper [6], Wu *et al.* studied the distribution of removable edges. Here we continue this research by studying the distribution of removable edges outside a cycle or on a Hamilton cycle or a spanning tree of a 4-connected graph. Studying removable edges outside a given subgraph is motivated by the hope that (large) substructures of a 4-connected graph stay more or less unaffected after applying the operations involved in the definition of a removable edge.

In [5], the similar concept of removable edges in 3-connected graphs has been used to verify two special cases of a well-known conjecture of 1976 due to Thomassen stating that every longest cycle in a 3-connected graph contains a chord. In [5] it is proved that Thomassen's conjecture is true for two classes of 3-connected graphs that have a bounded number of removable edges on or off a longest cycle. There an edge e of a 3-connected graph G is said to be removable if $G-e$ is still 3-connected or a subdivision of a 3-connected (multi)graph.

Such results show that it is natural and can be useful to study the distribution of removable edges inside or outside a special subgraph. In Section 4 we present our main results on the distribution of removable edges outside a cycle in a 4-connected graph, or on a Hamilton cycle or a spanning tree of a 4-connected graph.

Before we can state our main results we have to introduce some additional terminology and notation. We also have to present some known graph classes, and we have to summarize several known results that we need for the proofs. This is done in Sections 2 and 3, respectively.

We complete this section with some general terminology. Without any specification, in the following G always denotes a 4-connected graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order and size of G are denoted by $|G| = |V(G)|$ and $|E(G)|$, respectively. For $x \in V(G)$, we simply write $x \in G$. For $x \in G$, the neighborhood of x is denoted by $\Gamma_G(x)$, and the degree of x is denoted by $d_G(x) = |\Gamma_G(x)|$. If no confusion can arise, we simply write $d(x)$ for $d_G(x)$. If x and y are the two vertices incident with an edge e, we write $e = xy$. For a nonempty subset F of $E(G)$, or N of $V(G)$, the induced subgraph by F or N in G is denoted by $[F]$ or $[N]$. For $V_1, V_2 \subset V(G)$ with $V_1 \neq \emptyset \neq V_2$ and $V_1 \cap V_2 = \emptyset$, we define $[V_1, V_2] = \{xy \in E(G) \mid x \in V_1, y \in V_2\}$. If H is a subgraph of G, we say that G contains H. For a proper subset S of $V(G)$, $G-S$ denotes the graph obtained by deleting all the vertices of S from G together with all the incident edges, so $G - S = [V(G) \setminus S]$. If $G - S$ is disconnected, we say that S is a vertex cut of G; if $|S| = s$ for such a vertex cut S, we say that S is an s-cut. A cycle of G with length ℓ is simply called an ℓ-cycle of G.

It is easy to check and folklore knowledge that for every edge e of a 4-connected graph G, the graph $G - e$ is at least 3-connected. Moreover, if in this case $G - e$ has a 3-cut S, then $G - e - S$ consists of precisely two components. If one of these components has only one vertex, this vertex has degree 3 in $G - e$ and will disappear in $G \ominus e$. This motivated the following definitions.

Let $xy = e \in E(G)$, and let $S \subset V(G)$ with $|S| = 3$. If $G - e - S$ has exactly two components, say A and B, such that $|A| \geq 2$, $|B| \geq 2$ and $x \in A, y \in B$, then we say that (e, S) is a *separating pair* and $(e, S; A, B)$ is a *separating group*; in that case, A and B are called the *fragments*; if, moreover $|A| = 2$, then A is called an *atom*.

Let A be an atom, and suppose $A = \{x, z\}, S = \{a, b, c\}$. If $ax, bx \in E(G), cx \notin E(G)$, then A is called a 1-*atom*; if $ax, bx, cx \in E(G)$, then A is called a 2-*atom*. It is easy to check that any atom is either a 1-atom or a 2-atom.

Let $E_0 \subset E_N(G)$ such that $E_0 \neq \emptyset$, and let $(xy, S; A, B)$ be a separating group of G with $x \in A$ and $y \in B$. If $xy \in E_0$, then A and B are called E_0-*fragments*. Similarly, if A is an E_0-fragment and $|A| = 2$, then A is called an E_0-*atom*. An E_0-fragment is called an E_0-*end-fragment* of G if it does not contain any other E_0-fragment of G as a proper subset. It is easy to see that any E_0-fragment of G contains such an E_0-end-fragment.

2. Special Subgraphs and Their Properties

The following definitions of several special families of subgraphs of a 4-connected graph can be found in [7]. However, since these subgraphs play a crucial role in the sequel, for convenience we repeat the definitions here.

Definition 2.1. Let G be a 4-connected graph, and let H be a subgraph of G with $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. The subgraph H is called a *helm* if it satisfies the following conditions:

(i) $d_G(a) = 4$ and $d_G(x_i) = 4$ for $i = 1, 2, 3, 4$,

(ii) $ax_1, ax_2, ax_3, ax_4 \in E_N(G)$ and $x_1x_2, x_2x_3, x_3x_4, x_4x_1 \in E_R(G)$.

The vertices a and x_i for $i = 1, 2, 3, 4$ of a helm H are called the *inner vertices* of H.

Definition 2.2. Let G be a 4-connected graph, and let H be a subgraph of G with $V(H) = \{a, b, x_1, x_2, \ldots, x_{l+3}\}$ and $E(H) = \{x_1x_2, x_2x_3, \ldots, x_{l+2}x_{l+3}, ax_2, ax_3, \ldots, ax_{l+2}, bx_2, bx_3, \ldots, bx_{l+2}\}$, where $l \geq 1$. The subgraph H is called an *l-bi-fan* if it satisfies the following conditions:

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2, \ldots, l+2$,

(ii) $ax_j, bx_j \in E_R(G)$ for $j = 2, 3, \ldots, l+2$,

(iii) $d_G(x_j) = 4$ for $j = 2, 3, \ldots, l+2$.

An l-bi-fan H is said to be *maximal* if $\Gamma_G(x_1) \neq \{a, b, x_2, u\}$ and $\Gamma_G(x_{l+3}) \neq \{a, b, x_{l+2}, v\}$ for any $u, v \in G$. The vertices of an l-bi-fan or a maximal l-bi-fan H satisfying condition (iii) are called the *inner vertices* of H.

Definition 2.3. Let G be a 4-connected graph, and let H be a subgraph of G with $V(H) = \{x_1, x_2, \ldots, x_{l+2}, y_1, y_2, \ldots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$, where $E_1(H) = \{x_1x_2, x_2x_3, \ldots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \ldots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \ldots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$. Then H is called an *l-belt* if the following conditions are satisfied:

(i) $E_1(H) \subset E_N(H)$ and $E_2(H) \subset E_R(H)$,

(ii) $d_G(x_i) = d_G(y_j) = 4$ for $i = 2, 3, \ldots, l+1$; $j = 2, 3, \ldots, l+1$.

An l-belt H is said to be *maximal* if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(x_{l+2}) \neq \{x_{l+1}, y_{l+1}, y_{l+2}, v\}$ for any $u, v \in G$. The vertices of an l-belt or a maximal l-belt H satisfying condition (ii) are called the *inner vertices* of H.

Definition 2.4. Let G be a 4-connected graph, and let H be a subgraph of G with $V(H) = \{x_1, x_2, \ldots, x_{l+2}, x_{l+3}, y_1, y_2, \ldots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$, where $E_1(H) = \{x_1x_2, x_2x_3, \ldots, x_{l+1}x_{l+2}, x_{l+2}x_{l+3}, y_1y_2, y_2y_3, \ldots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \ldots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}, x_{l+2}y_{l+2}\}$. Then H is called an l*-co-belt* if the following conditions are satisfied:

(i) $E_1(H) \subset E_N(H)$ and $E_2(H) \subset E_R(H)$,

(ii) $d_G(x_i) = d_G(y_j) = 4$ for $i = 2, 3, \ldots, l+1, l+2$; $j = 2, 3, \ldots, l+1$.

An l-co-belt H is said to be *maximal* if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(y_{l+2}) \neq \{x_{l+2}, y_{l+1}, x_{l+3}, v\}$ for any $u, v \in G$. The vertices of an l-co-belt or a maximal l-co-belt H satisfying condition (ii) are called the *inner vertices* of H.

Definition 2.5. Let G be a 4-connected graph, and let H be a subgraph of G with $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then H is called a W*-framework* if the following conditions are satisfied:

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2$,

(ii) $d_G(x_2) = d_G(y_2) = d_G(y_3) = 4$,

(iii) $y_2y_3, x_1y_2, x_2y_2, x_2y_3, x_3y_3 \in E_R(G)$.

The vertex x_2 of a W-framework H is called the *inner vertex* of H.

Definition 2.6. Let G be a 4-connected graph, and let H be a subgraph of G with $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then H is called a W'*-framework* if the following conditions are satisfied:

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2$,

(ii) $d_G(x_2) = d_G(x_3) = d_G(y_2) = d_G(y_3) = 4$ and $d_G(x_1) \geq 5$,

(iii) $y_2y_3, x_1y_2, x_2y_3, x_3y_3, x_1x_3 \in E_R(G)$, $x_2y_2 \in E_N(G)$.

The vertices x_2 and x_3 of a W'-framework H are called the *inner vertices* of H.

For convenience, we denote by $\Re(G)$ (or simply $\Re$ if no confusion can arise) the set of all helms, maximal l-bi-fans, maximal l-belts, maximal l-co-belts, W-frameworks and W'-frameworks of a 4-connected graph G.

3. Some Known Results

First of all, we list some known results on removable edges of 4-connected graphs from [6, 7, 8] that will be applied in the sequel.

Theorem 3.1 [8]**.** *Let G be a 4-connected graph with $|G| \geq 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S) (and hence a separating group $(e, S; A, B)$) in G.*

Theorem 3.2 [8]**.** *Let G be a 4-connected graph with $|G| \geq 8$, and let $(xy, S; A, B)$ be a separating group of G with $x \in A$, $y \in B$ and $|A| \geq 3$. Then every edge of $[\{x\}, S]$ is removable.*

Corollary 3.1 [8]**.** *Let G be a 4-connected graph with $|G| \geq 8$. Then every 3-cycle of G contains at least one removable edge.*

Theorem 3.3 [8]**.** *Let G be a 4-connected graph with $|G| \geq 8$. If xy is an unremovable edge with a separating group $(xy, S; A, B)$, then all the edges in $E([S])$ are removable.*

Lemma 3.1 [7]**.** *Let G be a 4-connected graph, and let $P = y_1y_2 \cdots y_k$ be a path of $[E_N(G)]$ with $k \geq 3$. Consider a nonempty subset D of $V(G)$. Suppose that $(y_1y_2, S_1; A_1, B_1)$ is a separating group of G such that $y_1 \in B_1, y_2 \in A_1$ and $D \cap B_1 \neq \emptyset$. For $i \in \{1, 2, \ldots, k\}$ we consider a separating group $(y_iy_{i+1}, S; A, B)$ such that $y_i \in B, y_{i+1} \in A, D \cap B \neq \emptyset$, and $|A|$ is as small as possible. If $i \leq k-2$, we consider another separating group $(y_{i+1}y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B'$, $y_{i+2} \in A'$, and $|A'|$ is as small as possible. Then one of the following conclusions holds:*

- (i) *$A \cap B' = \{y_{i+1}\}, A \cap A' = \{y_{i+2}\}, A \cap S' = \{a\}$, $B' \cap S = \{b\}, S \cap S' = \emptyset, y_i \in B \cap B'$, $|B \cap S'| = |A' \cap S| = 2$, $A' \cap S = \{u, v\}$, where $y_{i+2}u, y_{i+2}v, y_{i+2}a \in E_R(G)$ and $a, b, u, v \in G$.*
- (ii) *$A \cap A' = \{y_{i+2}\}, y_{i+1} \in A \cap B', S \cap S' = \emptyset = A' \cap B$, $B \cap S' = \{d\} = D \cap B, D \cap B' = \emptyset, A' \cap S = \{c\}$, $|B' \cap S| = |A \cap S'| = 2, y_i \in B \cap B'$, where $d, c \in G$.*
- (iii) *$A \cap A' = \{y_{i+2}\}$, $y_{i+1} \in A \cap B', S \cap S' = \{w\}$, $D \cap B = \{d\} = B \cap S', D \cap B' = \emptyset = B \cap A', A' \cap S = \{c\}$, $|B' \cap S| = |A \cap S'| = 1, y_i \in B \cap B'$, where $d, c \in G$.*
- (iv) *$\Re(G) \neq \emptyset$ and at least one inner vertex of one of the graphs of $\Re(G)$ is on P.*

Lemma 3.2 [7]**.** *Let G be a 4-connected graph, and let $(xy, S; A, B)$ be a separating group of G with $x \in B, y \in A$. If there exists an edge $yz \in E_N(G)$, we consider its separating group $(yz, S'; A', B')$ with $y \in A', z \in B'$. If the following conditions hold for the four vertices $a, b, u, v \in G$:*

- (i) *$A \cap A' = \{y\}, A \cap B' = \{z\}$, $A \cap S' = \{a\}, A' \cap S = \{b\}, B' \cap S = \{u, v\}$,*
- (ii) *$\{zu, zv\} \cap E_N(G) \neq \emptyset$,*

then au, av cannot both be edges of G.

A 2-cyclic graph G of order n is defined to be the square of the cycle C_n, C_n^2 is obtained from C_n by adding edges between all pairs of vertices of C_n which are at distance 2 in C_n.

Theorem 3.4 [7]**.** *Let G be a 4-connected graph of order at least 5. If G is neither C_5^2 nor C_6^2, then $e_R(G) \geq (4|G| + 16)/7$.*

Definition 3.1. Let C be a cycle of a 4-connected graph G, and let $H \in \Re(G) \neq \emptyset$. If C contains at least one inner vertex of H, then we say that C *passes through* H.

Theorem 3.5. [6]. *Let G be a 4-connected graph, and let C be a cycle of G. If C passes through only one graph of $\Re(G)$, then C contains at least one removable edge of G.*

Lemma 3.3 [6]. *Let G be a 4-connected graph, let $(xy, S; A, B)$ be a separating group of G, and let A be a 2-atom, say $A = \{x, z\}$ and $S = \{a, b, c\}$. Then, ax, $bx, cx, xz \in E_R(G)$.*

Theorem 3.6 [7]. *Let G be a 4-connected graph with $|G| \geq 8$, and let C be a cycle of G. If C does not contain any removable edges of G, then G has one of the following structures as its subgraph: l-belt, l-bi-fan ($l \geq 1$), W-framework, W'-framework or helm, such that at least one inner vertex of one of these graphs is on C.*

Corollary 3.2 [7]. *Let G be a 4-connected graph, and let $(xy, S; A, B)$ be a separating group of G with $x \in A, y \in B$, $S = \{a, b, c\}$. Let A be a 1-atom, say $A = \{x, z\}$. If $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$, then x is an inner vertex of one of the following subgraphs in G: helm, l-co-belt, l-belt, W'-framework, W-framework or l-bi-fan.*

In the following section we shall obtain lower bounds on the number of removable edges outside a cycle, in a spanning tree, and on a Hamilton cycle in a 4-connected graph (which is assumed to be Hamiltonian in the latter case).

4. MAIN RESULTS

Before we present and prove our main results, we first prove the following technical lemma.

Lemma 4.1. *Let G be a 4-connected graph, $E_0 \subset E_N(G)$ and $E_0 \neq \emptyset$. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}, xy \in E_0$. If A is an E_0-end-fragment of G, and $|A| \geq 3$, then one of the following conclusions* (i), (ii) *or* (iii) *holds.*

(i) $(E(A) \cup [A, S]) \cap E_0 = \emptyset$.

(ii) *There exists a separating group $(x'y', S'; A', B')$ of G such that $x' \in A', y' \in B' \cap A, x'y' \in E_0$, B' is a 1-atom, and $|A \cap B'| = |B' \cap S| = 1$.*

(iii) *There exists a separating group $(xy', S'; A', B')$ of G such that $x \in A', y' \in B', xy' \in E_0$, $A \cap A' = \{x\}, |A \cap S'| = 1 = |A' \cap S|, A \cap B' = \{y'\}, |B' \cap S| = 2$.*

Proof. Suppose conclusion (i) does not hold. Next we prove that one of the conclusions (ii) or (iii) holds. Now either $E(A) \cap E_0 \neq \emptyset$ or $[A, S] \cap E_0 \neq \emptyset$. We will distinguish these two cases to complete the proof.

Case 1. There exists an edge $uz \in E(A) \cap E_0$. We consider the separating group $(uz, T; C, D)$ such that $u \in C, z \in D$. Then we have that $u \in A \cap C, z \in A \cap D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T),$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (D \cap S),$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T),$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S).$$

Next we distinguish the subcases that $x \neq u$ and that $x = u$.

Subcase 1.1. $x \neq u$. Then we have that $x \in A \cap C$, $A \cap T$, or $A \cap D$. These subcases are treated separately.

(1) Let $x \in A \cap C$. Then we have that $y \in B \cap C$ or $B \cap T$. We again treat these subcases separately.

(1.1) Suppose $y \in B \cap C$. Since $A \cap D \neq \emptyset$, we have that X_2 is a vertex cut of $G - uz$. Since G is 4-connected, $|X_2| \geq 3$. By a similar argument, we can get that $|X_4| \geq 3$. Noting that $|X_2| + |X_4| = |S| + |T| = 6$, we have that $|X_2| = |X_4| = 3$, and so $|S \cap C| = |A \cap T|, |B \cap T| = |D \cap S|$. First, we claim that $A \cap D = \{z\}$; otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D, S_1 = X_2, B_1 = G - uz - S_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G. Since $uz \in E_0$, A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Hence $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph of G, $D \cap S \neq \emptyset$. Next we deal with all possibilities for $|D \cap S|$ as follows.

(1.1.1) $|D \cap S| = |B \cap T| = 3$. Noting that $|S| = |T| = 3$, it is easy to see that $|X_1| = 0$. Then $\{z, y\}$ would be a 2-cut of G, a contradiction.

(1.1.2) $|D \cap S| = |B \cap T| = 2$. Since X_1 is a vertex cut of $G - uz - xy$ and G is 4-connected, we have $|X_1| \geq 2$, which implies that $|S \cap C| = |A \cap T| = 1, |S \cap T| = 0$. Noting that $x, u \in A \cap C$, we have $|A \cap C| \geq 2$. Let $A_1 = A \cap C, S_1 = \{z\} \cup X_1$, $B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment.

(1.1.3) $|D \cap S| = |B \cap T| = 1$. Obviously, $|S \cap T| \leq 2$. We claim that $|S \cap T| \neq 2$; otherwise, if $|S \cap T| = 2$, then $|C \cap S| = |A \cap T| = 0$. Let $A_1 = A \cap C, S_1 = (S \cap T) \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Hence $|S \cap T| \neq 2$, i.e., $|S \cap T| \leq 1$. Then we have that $|X_3| \leq 3$, and so $B \cap D = \emptyset$. It is easy to see that D is a 1-atom, and $|A \cap D| = 1, |S \cap D| = 1$, $|B \cap T| = 1$. Let $D = B', T = S', C = A', u = x', z = y'$. Then conclusion (ii) holds.

(1.2) Suppose $y \in B \cap T$. Since $A \cap D \neq \emptyset$, X_2 is a vertex cut of $G - uz$. So $|X_2| \geq 3$, and hence $|D \cap S| \geq |B \cap T| \geq 1$. Since $|S| = 3$, we have $|C \cap S| \leq 2$. Noting that $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_4| \leq 3$. Since G is 4-connected, we have that $B \cap C = \emptyset$. If $C \cap S = \emptyset$, then $C = A \cap C$. It is easy to see that C is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Hence $C \cap S \neq \emptyset$. If $S \cap T \neq \emptyset$, then $|S \cap T| = 1$, and $|C \cap S| = |D \cap S| = 1$. Since $|D \cap S| \geq |B \cap T|$, we have $B \cap T = \{y\}$. Obviously, here we have that $|X_3| = 3$, and so $B \cap D = \emptyset$. Hence $B = B \cap T = \{y\}$, which contradicts $|B| \geq 2$, and so $S \cap T = \emptyset$. If $|C \cap S| = 2$, then $|D \cap S| = 1$, and so $|B \cap T| = 1$. Here we have that $|X_3| = 2$. So $B \cap D = \emptyset$, and hence $B = \{y\}$, which contradicts that $|B| \geq 2$. Hence, $|C \cap S| = 1$, and so $|S \cap D| = 2$. Since $|S \cap D| \geq |B \cap T| \geq 1$, we have $|B \cap T| = 1$ or 2. If $|B \cap T| = 1$, then we have $|X_3| = 3$. and so $B \cap D = \emptyset$. Hence $B = B \cap T = \{y\}$, which contradicts $|B| \geq 2$. Hence, $|B \cap T| = 2$. Then $|A \cap T| = 1$, and so $|X_1| = 2$. Noting that $|A \cap C| \geq 2$, we let $A_1 = A \cap C, S_1 = X_1 \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G. Since $xy \in E_0$, A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. So (1.2) does not occur.

(2) Let $x \in A \cap T$. From Theorem 3.3, we know that $y \notin B \cap T$. By symmetry, we may assume that $y \in B \cap C$. Since $A \cap D \neq \emptyset$, X_2 is a vertex cut of $G - uz$, and so $|X_2| \geq 3$. By a similar argument, we can get that $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |T| = 6$, we have that $|X_2| = |X_4| = 3$, and so $|S \cap C| = |A \cap T|, |B \cap T| = |D \cap S|$. We claim that $A \cap D = \{z\}$; otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D, S_1 = X_2, B_1 = G - uz - S_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G. Since $uz \in E_0$, A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Hence we have that $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph of G, we have that $D \cap S \neq \emptyset$. Since $A \cap C \neq \emptyset$, we have that X_1 is a vertex cut of $G - uz$. Then $|X_1| \geq 3$, and so $|S \cap C| \geq |B \cap T|, |A \cap T| \geq |D \cap S|$. Since $|X_1| + |X_3| = |S| + |T| = 6$, we have that $|X_3| \leq 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$. Noting that $|A \cap T| \geq |D \cap S|, |S \cap C| \geq |B \cap T|$, we have $|D \cap S| = |B \cap T| = 1$. Obviously, here D is a 1-atom. Let $D = B', T = S', C = A', u = x', z = y'$. Then conclusion (ii) holds.

(3) Let $x \in A \cap D$. By symmetry, analogous arguments as used in (1) lead to the conclusion.

Subcase 1.2. $x = u$. Then we have that $x \in A \cap C$, $y \in B \cap C$ or $B \cap T$. We distinguish these subcases separately.

(1) Let $y \in B \cap C$. Since $A \cap D \neq \emptyset$, X_2 is a vertex cut of $G - xz$. Since G is 4-connected, we have $|X_2| \geq 3$. By a similar argument, we can get that $|X_4| \geq 3$. Noting that $|X_2| + |X_4| = |S| + |T| = 6$, we have $|X_2| = |X_4| = 3$, and so $|S \cap C| = |A \cap T|, |B \cap T| = |D \cap S|$. First, we claim that $A \cap D = \{z\}$; otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D, S_1 = X_2, B_1 = G - xz - S_1 - A_1$. Then $(xz, S_1; A_1, B_1)$ is a separating group of G. Since $xz \in E_0$, A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Hence $A \cap D = \{z\}$. Since $|D| \geq 2$ and D is a connected subgraph, we have that $S \cap D \neq \emptyset$. If $|D \cap S| = |B \cap T| = 3$, then it is easy to see that $\{y, z\}$ would be a 2-cut of G, a contradiction. So, $|D \cap S| = |B \cap T| \leq 2$. We deal with the two possibilities separately.

(1.1) $|B \cap T| = |D \cap S| = 2$. Since X_1 is a vertex cut of $G - xy - xz$, we have that $|X_1| \geq 2$. Note that $|S| = |T| = 3$ only if $|S \cap C| = |A \cap T| = 1$, $S \cap T = \emptyset$. Here we claim that $A \cap C = \{x\}$; otherwise, $|A \cap C| \geq 2$. Then it is easy to see that $\{x\} \cup X_1$ would be a 3-cut of G, a contradiction. Let $z = y', C = A', T = S', D = B'$. Then conclusion (iii) holds.

(1.2) $|B \cap T| = |D \cap S| = 1$. If $|S \cap T| = 2$, then $|C \cap S| = |A \cap T| = 0$. We claim that $A \cap C = \{x\}$; otherwise, $|A \cap C| \geq 2$. Then it is easy to see that $\{x\} \cup X_1$ would be a 3-cut of G, a contradiction. Since $A \cap D = \{z\}$, here we would have that $|A| = 2$, which contradicts that $|A| \geq 3$. Hence, we get that $|S \cap T| \leq 1$. So $|X_3| \leq 3$, and hence $B \cap D = \emptyset$. Here D is a 1-atom, and $|A \cap D| = |D \cap S| = 1$. Let $x = x', z = y', C = A', T = S', D = B'$. Then conclusion (ii) holds.

(2) Let $y \in B \cap T$. From Theorem 3.2, we have that $|C| = 2$. We claim that $C \cap S \neq \emptyset$; otherwise, $S \cap C = \emptyset$. Since C is a connected subgraph, we have that $B \cap C = \emptyset$. Then $C = A \cap C$, and C would be an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. So, $|A \cap C| = |S \cap C| = 1$. Noting that $|S| = 3$, we have $|S \cap (D \cup T)| = 2$. If $B \cap T = \{y\}$, then we have that $|X_3| = 3$, and so $B \cap D = \emptyset$. Obviously, $B = \{y\}$, which contradicts $|B| \geq 2$. Hence $|B \cap T| \geq 2$. If $|B \cap T| = 3$, then $T \cap (A \cup S) = \emptyset$, and so $|X_1| = 1$. Then $X_1 \cup \{y, z\}$ would be a 3-cut of G, a contradiction. So, $|B \cap T| = 2$, and $|A \cap C| = |S \cap C| = 1$. Let $x = y', z = x', C = B', T = S', D = A'$. Then conclusion (ii) holds. This completes Case 1.

Case 2. There exists an edge $uz \in [A, S] \cap E_0$. Obviously, $u \neq x$; otherwise, $u = x$, and then from Theorem 3.2, we have that $|A| = 2$, which contradicts $|A| \geq 3$. Analogously, we consider the separating group $(uz, T; C, D)$ such that $u \in C, z \in D$. It is easy to see that $u \in A \cap C, z \in S \cap D$. Let X_1, X_2, X_3, X_4 be defined in the same way as in Case 1. We distinguish the following six subcases, according to the different locations for x and y, to complete the proof of the lemma.

Subcase 2.1. $x \in A \cap C, y \in B \cap C$. Since $B \cap C \neq \emptyset$, X_4 is a vertex cut of

$G-xy$, and so $|X_4| \geq 3$. Since $|X_2|+|X_4| = |S|+|T| = 6$, we have$|X_2| \leq 3$, and so $A \cap D = \emptyset$. First suppose $A \cap T = \emptyset$. Then $A = A \cap C$, and so $|A \cap C| \geq 3$. Since X_1 is a vertex cut of $G-uz-xy$, then $|X_1| \geq 2$. Note that $D \cap S \neq \emptyset$ only if $|X_1| = |S \cap (C \cup T)| = 2$. We let $A_1 = A - \{u\}, S_1 = X_1 \cup \{u\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. So, $A \cap T \neq \emptyset$, and hence $|T \cap (B \cup S)| \leq 2$. If $S \cap D = \{z\}$, then $|X_3| \leq 3$, and so $B \cap D = \emptyset$ and $D = \{z\}$, which contradicts $|D| \geq 2$. Hence, $|D \cap S| \geq 2$. Then $|S \cap (C \cup T)| \leq 1$. Noting that $|X_4| \geq 3$, we have $|B \cap T| \geq 2$, which implies that $|B \cap T| = 2, |A \cap T| = 1$, and we have $S \cap T = \emptyset$. Here we have that $|X_1| = 2$. Let $A_1 = A \cap C, S_1 = X_1 \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Therefore, Subcase 2.1 does not occur.

Subcase 2.2. $x \in A \cap C, y \in B \cap T$. Since X_1 is a vertex cut of $G-xy-uz$, we have $|X_1| \geq 2$. First, we show that $A \cap T = \emptyset$. If $A \cap T \neq \emptyset$, then we claim that $|X_1| \geq 3$. Otherwise, $|X_1| = 2$. Obviously, $|A \cap C| \geq 2$. Let $A_1 = A \cap C, S_1 = X_1 \cup \{z\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. So, $|X_1| \geq 3$, and $|C \cap S| \geq |B \cap T| \geq 1$, $|A \cap T| \geq |D \cap S| \geq 1$, which implies that $|B \cap T| = |D \cap S| = 1$. Since $|X_1|+|X_3| = 6$, we have $|X_3| \leq 3$, and so $B \cap D = \emptyset$. From $|D| \geq 2$, we know that $A \cap D \neq \emptyset$. Then $|X_2| \geq 4$, and so $|X_4| \leq 2$. Then $|B \cap C| = 0$, and $B = \{y\}$, which contradicts $|B| \geq 2$. Therefore, $A \cap T = \emptyset$. Since A is a connected subgraph, $A \cap D = \emptyset$, and so $|A| = |A \cap C| \geq 3$. Since $D \cap S \neq \emptyset$ and $|S| = 3$, we have that $|X_1| = |S \cap (C \cup T)| = 2$. We let $A_1 = A - u, S_1 = X_1 \cup \{u\}, B_1 = G - xy - S_1 - A_1$. Then $(xy, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Therefore, Subcase 2.2 does not occur.

Subcase 2.3. $x \in A \cap T, y \in B \cap C$. Since $B \cap C \neq \emptyset$, X_4 is a vertex cut of $G-xy$, and then $|X_4| \geq 3$. Since $|X_2|+|X_4| = |S|+|T| = 6$, we have$|X_2| \leq 3$, and so $A \cap D = \emptyset$. Analogously, since X_1 is a vertex cut of $G-uz$, we have $|X_1| \geq 3$. Noting that $|X_1|+|X_3| = 6$, we have that $|X_3| \leq 3$, and so $B \cap D = \emptyset$. Hence, $|D| = |D \cap S| \geq 2$. Noting that $|S| = 3$, we have $|S \cap (C \cup T)| \leq 1$. From $|X_4| \geq 3$, we can get that $|B \cap T| \geq 2$. Then it is easy to see that $|A \cap T| = 1, S \cap T = \emptyset$. Obviously, we have that $|X_1| \leq 2$, which contradicts that $|X_1| \geq 3$. So, Subcase 2.3 does not occur.

Subcase 2.4. $x \in A \cap T, y \in B \cap D$. Since X_1 is a vertex cut of $G-uz$, we have $|X_1| \geq 3$. Similarly, we have that $|X_3| \geq 3$. Since $|X_1|+|X_3| = |S|+|T| = 6$, we have that $|X_1| = |X_3| = 3$. Then we can get that $|A \cap T| = |D \cap S|, |C \cap S| = |B \cap T|$. First, we claim that $A \cap C = \{u\}$; otherwise, $|A \cap C| \geq 2$. Then we let $A_1 = A \cap C, S_1 = X_1, B_1 = G - uz - S_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. So, $A \cap C = \{u\}$. Since C is a connected subgraph and $|C| \geq 2$, we have that $|C \cap S| = |B \cap T| \geq 1$. If $|C \cap S| = |B \cap T| = 2$, then $S \cap T = \emptyset, |A \cap T| = |D \cap S| = 1$, and we have that $|X_2| = 2$. Then $A \cap D = \emptyset$. Here we have that $|A| = 2$, which contradicts $|A| \geq 3$. So $|S \cap C| = |B \cap T| = 1$, and we have that C is a 1-atom, and $|A \cap C| = |C \cap S| = 1$. Let $u = y', z = x', C = B', T = S', D = A'$. Then conclusion (ii) holds.

Subcase 2.5. $x \in A \cap D, y \in B \cap T$. Since X_2 is a vertex cut of $G - xy$, we have $|X_2| \geq 3$. From $|X_2| + |X_4| = |S| + |T| = 6$, we know that $|X_4| \leq 3$. Then, $B \cap C = \emptyset$. Since X_1 is a vertex cut of $G - uz$, we have $|X_1| \geq 3$. From $|X_1| + |X_3| = |S| + |T| = 6$, we know that $|X_3| \leq 3$. Then, we can get that $B \cap D = \emptyset$. Then we have that $|B| = |B \cap T| \geq 2$. Noting that A is a connected subgraph, we have $A \cap T \neq \emptyset$, which implies that $|A \cap T| = 1, |B \cap T| = 2$ and $S \cap T = \emptyset$. Since $|X_2| \geq 3$, we have that $|D \cap S| \geq 2$ and $|C \cap S| \leq 1$. Here we have that $|X_1| \leq 2$, which contradicts that X_1 is a vertex cut of $G - uz$. So, Subcase 2.5 does not occur.

Subcase 2.6. $x \in A \cap D, y \in B \cap D$. Since X_2 is a vertex cut of $G - xy$, we have $|X_2| \geq 3$. From $|X_2| + |X_4| = |S| + |T| = 6$, we know that $|X_4| \leq 3$, and so $B \cap C = \emptyset$. We claim that $C \cap S \neq \emptyset$; otherwise, it is easy to see that C is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. So, $C \cap S \neq \emptyset$. Noting that X_1 is a vertex cut of $G - uz$, we have $|X_1| \geq 3$. Similarly, we have that $|X_3| \geq 3$. From $|X_1| + |X_3| = |S| + |T| = 6$, we know that $|X_1| = |X_3| = 3$, and so $|C \cap S| = |B \cap T| \geq 1, |A \cap T| = |D \cap S| \geq 1$. If $|C \cap S| = 2$, then $|A \cap T| = |D \cap S| = 1$, and so $|X_2| = 2$, a contradiction. Therefore, $|C \cap S| = |B \cap T| = 1$. We claim that $A \cap C = \{u\}$; otherwise, if $|A \cap C| \geq 2$, we let $A_1 = A \cap C, S_1 = X_1, B_1 = G - uz - X_1 - A_1$. Then $(uz, S_1; A_1, B_1)$ is a separating group of G, and A_1 is an E_0-fragment, contradicting that A is an E_0-end-fragment. So, $A \cap C = \{u\}$. Let $z = x', u = y', C = B', T = S', D = A'$. Then conclusion (ii) holds.

This completes Case 2 and the proof of the lemma. ■

The following lemma will help us to show under which circumstances a 4-connected graph has removable edges in a given spanning tree.

Lemma 4.2. *Let G be a 4-connected graph with $\Re(G) = \emptyset$, and suppose $[E_N(G)]$ is a tree. Then $|[E_N(G)]| \leq |G| - 3$.*

Proof. By contradiction. Suppose that $|[E_N(G)]| \geq |G| - 2$. Let x be a vertex of degree 1 in the tree $[E_N(G)]$. Since $d_G(x) \geq 4$ and $|[E_N(G)]| \geq |G| - 2$, there is a vertex $y \in [E_N(G)]$ such that $xy \in E_R(G)$. Let P be the unique path connecting x and y in $[E_N(G)]$. Then $P + xy$ is a cycle of G that contains only one removable edge xy. We choose such a cycle $C = y_1 y_2 \cdots y_k y_1$ with $y_1 y_k \in E_R(G)$

and $E(C) - \{y_1y_k\} \subset E_N(G)$ such that the length of C is as small as possible in G.

Let $D = \{y_1\}$. Consider the path $P = y_1y_2 \cdots y_k$ in $[E_N(G)]$. Consider a separating group $(y_1y_2, S_1; A_1, B_1)$ such that $y_1 \in B_1, y_2 \in A_1$. Obviously, $D \cap B_1 \neq \emptyset$. We take the separating group $(y_iy_{i+1}, S; A, B)$ such that $y_i \in B, y_{i+1} \in A, i \in \{1, 2, \ldots, k-1\}$, $D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i + 1 \leq k - 1$; otherwise, $i + 1 = k$. Then $y_k \in A$. Since $y_1y_k \in E(G)$, $y_1 \in A \cup S$, which contradicts $D \cap B \neq \emptyset$. So, $i+1 \leq k-1$. We take the separating group $(y_{i+1}y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B', y_{i+2} \in A'$ and $|A'|$ is as small as possible. By Lemma 3.1, we have that one of the conclusions (i), (ii), (iii) or (iv) of Lemma 3.1 holds.

(1) If conclusion (i) holds, since $y_1 \in B$, we have $y_k \in B \cup S$. So y_{i+2} is not the end vertex of P, and so y_{i+2} is incident with at least two unremovable edges in G, which contradicts conclusion (i).

(2) If conclusion (ii) holds, then $d = y_1$. We let $C' = A', T' = A \cap S' \cup \{y_{i+1}\}, D' = G - cd - T' - C'$. Then $(cd, T'; C', D')$ is a separating group of G, and so $cd \in E_N(G)$. Since $y_1y_k \in E_R(G)$, we have $c \neq y_k$. Hence $y_k \in B' \cap (B \cup S)$. Let $A \cap S' = \{u, v\}$. Since y_{i+2} is not an end vertex of P, we have $\{cy_{i+2}, uy_{i+2}, vy_{i+2}\} \cap E_N(G) \neq \emptyset$. Then, from Corollary 3.2 we know that y_{i+2} is an inner vertex of some subgraph belonging to $\Re(G)$, which contradicts the assumptions of the lemma. Hence, conclusion (ii) does not occur.

(3) If conclusion (iii) holds, then let $C' = A', T' = (S' - \{d\}) \cup \{y_{i+1}\}, D' = G - cd - T' - C'$. Then $(cd, T'; C', D')$ is a separating group of G. So $cd \in E_N(G)$, and hence $c \neq y_k$. Obviously, y_{i+2} is not an end vertex of P. By analogous arguments as in (2), we get that conclusion (iii) does not occur.

(4) From the assumptions of the lemma, we immediately get that conclusion (iv) does not occur.

This completes the proof of the lemma. ■

4.1. Removable edges in spanning trees

Our first result shows under which circumstances a spanning tree of a 4-connected graph contains a removable edge. The result follows almost directly from the above lemma.

Theorem 4.1. *Let G be a 4-connected graph with $\Re(G) = \emptyset$. Then any spanning tree of G contains at least one removable edge.*

Proof. First of all, we claim that $[E_N(G)]$ does not contain any cycles; otherwise, using Theorem 3.6, we get that $\Re(G) \neq \emptyset$, contradicting the assumptions of the theorem.

If $[E_N(G)]$ is a tree, then by Lemma 4.2 we have $|[E_N(G)]| \leq |G| - 3$. Since $|E(T)| = |G| - 1$ for any spanning tree T of G, we have $|E(T) \cap E_R(G)| \geq 2$, and we are done.

It remains to deal with the case that $[E_N(G)]$ is a forest with at least two components. In that case, the statement of the theorem clearly holds. ■

We next present an example to show that the lower bound of the theorem is sharp, i.e., there exists a 4-connected graph G with $\Re(G) = \emptyset$ and a spanning tree of G containing precisely one removable edge.

Example 4.1. Let H be a helm as in Definition 2.1, such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. Let $L = H - \{v_1, v_2, v_3, v_4\}$, and let L' be a copy of L such that $V(L') = \{a', x'_1, x'_2, x'_3, x'_4\}$. We construct a graph G as follows. Let $V(G) = V(L) \cup V(L')$, and let $E(G) = E(L) \cup E(L') \cup \{x_1x_3, x'_2x'_4, x_1x'_1, x_2x'_2, x_3x'_3, x_4x'_4\}$. Obviously, G is a 4-connected graph with $\Re(G) = \emptyset$. It is easy to check that $(ax_2, \{x_1, x_3, x'_4\})$ is a separating pair of G, and so $ax_2 \in E_N(G)$. By symmetry, $ax_4, a'x'_1, a'x'_3 \in E_N(G)$. Similarly, $(x_1x'_1, \{x_2, x_3, x_4\})$ is a separating pair of G, and hence $x_1x'_1 \in E_N(G)$. By symmetry, we have $x_2x'_2, x_3x'_3, x_4x'_4 \in E_N(G)$. Let T be a spanning tree of G such that $E(T) = \{x_1x'_1, x_2x'_2, x_3x'_3, x_4x'_4, a'x'_2, a'x'_3, ax'_1, ax_2, ax_4\}$. Then it is easily checked that there is only one removable edge $a'x'_2$ in T.

4.2. Removable edges avoiding a fixed cycle

Our next results show under which circumstances a 4-connected graph G has removable edges outside a given cycle of G. The first of these results deals with arbitrary cycles avoiding l-belts and l-co-belts, in the following sense.

Theorem 4.2. *Let G be a 4-connected graph with $|G| \geq 7$ and let C be a cycle of G. If C does not pass through any maximal l-belt or l-co-belt, then there are at least two removable edges outside C.*

Proof. By contradiction. Assume that G and C are as in the theorem, and suppose there is at most one removable edge outside C. Let $F = (E(G) \setminus E(C)) \cap E_R(G)$, thus $|F| \leq 1$. We denote $(E(G) \setminus E(C)) \setminus F$ by E_0.

Case 1. First suppose that C does not pass through any subgraph H of G that belongs to $\Re(G)$. We consider a separating group $(uw, S'; A', B')$ such that $u \in A', w \in B', uw \in E_0$. From $|F| \leq 1$, we know that either $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [S', B']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-fragment, A' must contain an E_0-end-fragment as its subgraph, say A. Then we have that $(E(A) \cup [A, S]) \cap F = \emptyset$, and we take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$.

The following observations on $|A|$ are checked.

(1) $|A| = 2$. Then either A is a 1-atom or a 2-atom.

(1.1) If A is a 1-atom, let $A = \{x, z\}$, $S = \{a, b, c\}$. If $\{xz, xa, xb\} \cap E_N(G) \neq \emptyset$, from Corollary 3.2 we have that x is an inner vertex of some subgraph belonging to $\Re(G)$. Since C does not pass through any subgraph H of G that belongs to $\Re(G)$, we have $x \notin V(C)$. It is easily checked that x is associated with at least one removable edge, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$. If $xz, xa, xb \in E_R(G)$, since C is a cycle, $d_C(x) \leq 2$, we have that cycle C does not contain all three removable edges xz, xa, xb, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(1.2) If A is a 2-atom. From Lemma 3.3, we know $\{xa, xb, xc, xz\} \subset E_R(G)$. Since $d_C(x) \leq 2$, it is impossible that cycle C contains four removable edges xa, xb, xc, xz, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(2) $|A| \geq 3$. Since C is a cycle of G, We have $d_C(x) \leq 2$. Since $(E(A) \cup [A, S]) \cap F = \emptyset$, there exists $xz \in E_0 \cap (E(A) \cup [A, S])$. Obviously $z \notin S$; otherwise, from Theorem 3.2 we know $|A| = 2$, contradicting $|A| \geq 3$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then $x \in A \cap A_1, z \in A \cap B_1$. Using Lemma 4.1 we conclude that one of the three conclusions of Lemma 4.1 holds.

(2.1) Since $xz \in E_0 \cap (E(A) \cup [A, S])$, conclusion (i) does not occur.

(2.2) Suppose that conclusion (ii) holds. Then we have that B' is a 1-atom. If vertex z is associated with another unremovable edge except xz, from Corollary 3.2 we know that z is an inner vertex of subgraph H of G that belongs to $\Re(G)$. Since C does not pass through any subgraph H of G that belongs to $\Re(G)$, we know $z \notin V(C)$, and vertex z is associated with at least one removable edge, which contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$. If vertex z is associated with three removable edges, it is easily checked that contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$.

(2.3) Suppose that conclusion (iii) holds. Let $B' \cap S = \{x_1, x_2\}, A \cap S' = \{x_3\}$. Clearly, $A \cap B' = \{z\}$. First we claim that $zx_1, zx_2 \in E_R(G)$. Otherwise, $\{zx_1, zx_2\} \cap E_N(G) \neq \emptyset$, from Lemma 3.2 we know x_3x_1, x_3x_2 cannot both be edges of G. We may assume that $x_3x_2 \notin E(G)$, let $A'' = A - z, S'' = S \cup \{z\} - x_2, B'' = G - xz - S'' - A''$, then A'' is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Therefore, we have that $zx_1, zx_2 \in E_R(G)$. Since $|B'| \geq 3$, using Theorem 3.2, we obtain that $zx_3 \in E_R(G)$. Then vertex z is associated with at least three removable edges. Note that C is a cycle of G, $d_C(z) \leq 2$, then there exists at least one removable edge outside cycle C, which contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$.

Case 2. Suppose that C passes through a subgraph H of G that belongs to $\Re(G)$. Note that H is neither an l-belt nor an l-co-belt.

Subcase 2.1. Suppose that H is a maximal l-bi-fan ($l \geq 1$). From the assumption $|F| \leq 1$ we know only $l = 1$ holds. If $C \subset E(H)$, since $|F| \leq 1$, we have that $e_R(G) \leq 5$. Since $|G| \geq 7$, from Theorem 3.4 we have that $e_R(G) \geq$

$(4|G| + 16)/7 > 5$, a contradiction. So, according to the assumption, we have that $F \cap E(H) \neq \emptyset$. Since $|F| \leq 1$, we may assume that $ax_2, bx_2 \in E(C)$, then $x_1x_2 \in E_0$. By letting $S' = \{a, b, x_4\}, e = x_2x_1, B' = \{x_2, x_3\}, A' = G-e-S'-B'$, we get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the maximal l-bi-fan, and $F \cap (E(A') \cup [A', S']) = \emptyset$. Since A' is an E_0-fragment, A' must contain an E_0-end-fragment as its subgraph, say A. We take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$. Clearly, A does not contain any inner vertex of the maximal l-bi-fan and $(E(A) \cup [A, S]) \cap F = \emptyset$.

We make some observations on $|A|$ as follows.

(1) $|A| = 2$. Then either A is a 1-atom or a 2-atom.

(1.1) If A is a 1-atom, let $A = \{x, z\}, S = \{a, b, c\}$. If $\{xz, xa, xb\} \cap E_N(G) \neq \emptyset$, from Corollary 3.2 we have that x is an inner vertex of some subgraph belonging to $\Re(G)$. From $(E(A) \cup [A, S]) \cap F = \emptyset$, we know that all the removable edges in H are covered by cycle C. However, since H is neither l-belt nor l-co-belt, it is easily checked that no matter what subgraph H is, cycle C cannot cover all of removable edges in H, a contradiction. If $xz, xa, xb \in E_R(G)$, since C is a cycle, we have $d_C(x) \leq 2$, cycle C cannot contain three removable edges xa, xb, xz, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(1.2) If A is a 2-atom, then vertex x is associated with four removable edges. It is easily checked that $F \cap (E(A) \cup [A, S]) \neq \emptyset$, a contradiction.

(2) $|A| \geq 3$. Since C is a cycle of G, and $(E(A) \cup [A, S]) \cap F = \emptyset$, there exists $xz \in E_0 \cap (E(A) \cup [A, S])$. Obviously $z \notin S$; otherwise, from Theorem 3.2 we know $|A| = 2$, contradicting $|A| \geq 3$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then $x \in A \cap A_1, z \in A \cap B_1$. Using Lemma 4.1 we conclude that one of the three conclusions of Lemma 4.1 holds.

(2.1) Since $xz \in E_0 \cap (E(A) \cup [A, S])$, conclusion (i) does not occur.

(2.2) Suppose that conclusion (ii) holds. B' is a 1-atom, if z is associated with three removable edges, we have that there exists at least one removable edges outside cycle C, contradicting $F \cap (E(A) \cup [A, S]) = \emptyset$. If vertex z is associated with one unremovable edge except xz, from Corollary 3.2 we have that z is an inner vertex of some subgraph belonging to $\Re(G)$. Note that H is neither l-belt nor l-co-belt, it is easily checked that no matter which subgraph H is, cycle C cannot cover all of removable edges in H, a contradiction.

(2.3) Suppose that conclusion (iii) holds. Let $B' \cap S = \{x_1, x_2\}, A \cap S' = \{x_3\}$. Clearly, $A \cap B' = \{z\}$. First we claim that $zx_1, zx_2 \in E_R(G)$. Otherwise, $\{zx_1, zx_2\} \cap E_N(G) \neq \emptyset$, from Lemma 3.2 we know x_3x_1, x_3x_2 cannot both be edges of G. We may assume that $x_3x_2 \notin E(G)$, let $A'' = A - z, S'' = S \cup \{z\} - x_2, B'' = G-xz-S''-A''$, then A'' is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Therefore, we have that $zx_1, zx_2 \in E_R(G)$. Since $|B'| \geq 3$, using Theorem 3.2, we obtain that $zx_3 \in E_R(G)$. Note that C is a cycle of G, C cannot contain all three removable edges zx_1, zx_2, zx_3, which contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$.

Subcase 2.2. If H is a helm. If $E(C) \subset E(H)$, then according to the assumption $|F| \leq 1$, we have that $e_R(G) \leq 5$, obviously, $|G| \geq 9$. From Theorem 3.4 we have that $e_R(G) \geq (4|G| + 16)/7 > 5$, a contradiction. So, according to the assumption, we have that $F \cap E(H) \neq \emptyset$. Since $|F| \leq 1$, we may assume that $x_4x_1, x_1x_2 \in E(C)$, then $x_1v_1 \in E_0$. By letting $e = x_1v_1, S' = \{v_2, v_3, v_4\}, B' = \{a, x_1, x_2, x_3, x_4\}$, $A' = G - e - S' - B'$, we get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the helm H, and $F \cap (E(A') \cup [A', S']) = \emptyset$. Since A' is an E_0-fragment, A' must contain an E_0-end-fragment as its subgraph, say A. Then we take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$. Obviously, we have that A does not contain any inner vertex of the helm H and $(E(A) \cup [A, S]) \cap F = \emptyset$.

Similarly, we will make some observations on $|A|$ as used in Subcase 2.1.

(1) $|A| = 2$. Then either A is a 1-atom or a 2-atom.

(1.1) If A is a 1-atom, let $A = \{x, z\}, S = \{a, b, c\}$. If $\{xz, xa, xb\} \cap E_N(G) \neq \emptyset$, from Corollary 3.2 we have that x is an inner vertex of a subgraph H belonging to $\Re(G)$. Note that H is neither l-belt nor l-co-belt, it is impossible for cycle C to cover all of removable edges in H, which contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$. If $xz, xa, xb \in E_R(G)$, cycle C cannot contain three removable edges xa, xb, xz, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(1.2) If A is a 2-atom, then vertex x is associated with four removable edges. It is impossible that $F \cap (E(A) \cup [A, S]) = \emptyset$, a contradiction.

(2) $|A| \geq 3$. Since C is a cycle of G, and $(E(A) \cup [A, S]) \cap F = \emptyset$, there exists $xz \in E_0 \cap (E(A) \cup [A, S])$. Since $|A| \geq 3$, from Theorem 3.2 we know $z \notin S$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then $x \in A \cap A_1$, $z \in A \cap B_1$. Using Lemma 4.1 we conclude that one of the three conclusions of Lemma 4.1 holds.

(2.1) Since $xz \in E_0 \cap (E(A) \cup [A, S])$, conclusion (i) does not occur.

(2.2) Suppose that conclusion (ii) holds. Since B' is a 1-atom, we can use a similar argument as used in (2.2) of Subcase 2.1 to get two possible conclusions: (i) vertex z is associated with three removable edges, then there exists at least one removable edge outside cycle C, contradicting $F \cap (E(A) \cup [A, S]) = \emptyset$; (ii) vertex z is is an inner vertex of some subgraph H belonging to $\Re(G)$. In this case cycle C cannot cover all of removable edges in H, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(2.3) Suppose that conclusion (iii) holds. Let $B' \cap S = \{x_1, x_2\}, A \cap S' = \{x_3\}$. Clearly, $A \cap B' = \{z\}$. We can use a similar argument as used in (2.3) of Subcase 2.1 to get $zx_1, zx_2 \in E_R(G)$. Since $|B'| \geq 3$, we have that $zx_3 \in E_R(G)$. Clearly, it is impossible that C contains all three removable edges zx_1, zx_2, zx_3, which contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$.

Subcase 2.3. If H is a W-framework, then according to the assumption, we must have that $F = \{y_2y_3\}$, and $x_1x_2 \in E_0$. In this case, by letting $e = x_1x_2, S' = \{x_3, y_4, y_2\}, B' = \{x_2, y_3\}, A' = G - e - S' - B'$, we get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the W-framework, and $F \cap (E(A') \cup [A', S']) = \emptyset$. Since A' is an E_0-fragment, A' must contain an E_0-end-fragment as its subgraph, say A. Then we have that $(E(A) \cup [A, S]) \cap F = \emptyset$, and we take the corresponding separating group $(xy, S; A, B)$ such that $x \in A$, $y \in B$.

Similarly, we will make some observations on $|A|$ as used in Subcase 2.1.

(1) $|A| = 2$. Then either A is a 1-atom or a 2-atom.

(1.1) If A is a 1-atom, let $A = \{x, z\}, S = \{a, b, c\}$. We use a similar argument as used in (1.1) of Subcase 2.1 to get the following two possible conclusions: (i) vertex x is an inner vertex of a subgraph H belonging to $\Re(G)$; (ii) vertex x is associated with three removable edges xz, xa, xb. From the argument used in (1.1) of Subcase 2.1, we know that no matter which conclusion is true, it will contradict $(E(A) \cup [A, S]) \cap F = \emptyset$.

(1.2) If A is a 2-atom, then vertex x is associated with four removable edges. It is impossible that $F \cap (E(A) \cup [A, S]) \neq \emptyset$, a contradiction.

(2) $|A| \geq 3$. We use a similar argument as used in (2) of Subcase 2.1 to get that there exists $xz \in E_0 \cap (E(A) \cup [A, S])$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then $x \in A \cap A_1, z \in A \cap B_1$. Since $|A| \geq 3$, we have $z \notin S$. Using Lemma 4.1 we conclude that one of the three conclusions of Lemma 4.1 holds.

(2.1) Since $xz \in E_0 \cap (E(A) \cup [A, S])$, conclusion (i) does not occur.

(2.2) Suppose that conclusion (ii) holds. Since B' is a 1-atom, similarly, we have two possible conclusions hold: (i) vertex z is associated with three removable edges, then there exists at least one removable edges outside cycle C, contradicting $F \cap (E(A) \cup [A, S]) = \emptyset$; (ii) vertex z is an inner vertex of some a subgraph H belonging to $\Re(G)$. In this case, cycle C cannot cover all of removable edges in H, which contradicts $(E(A) \cup [A, S]) \cap F = \emptyset$.

(2.3) Suppose that conclusion (iii) holds. Let $B' \cap S = \{x_1, x_2\}, A \cap S' = \{x_3\}$. We can use a similar argument as used in (2.3) of Subcase 2.1 to get $zx_1, zx_2, zx_3 \in E_R(G)$. Clearly, it is impossible that C contains all three removable edges zx_1, zx_2, zx_3, which contradicts $F \cap (E(A) \cup [A, S]) = \emptyset$.

Subcase 2.4. If H is a W'-framework, according to the assumption, we have $E(C) \subset E(H)$ and $F \subset E(H)$. Then $E_R(G) = 5$. However, since $|G| \geq 7$, from Theorem 3.4 we have that $e_R(G) \geq (4|G| + 16)/7 > 5$, a contradiction.

This completes the proof of the last case and hence of the theorem. ∎

Next we present an example to show that the lower bound on the number of removable edges given in the conclusion of Theorem 4.2 is sharp.

Example 4.2. Let H be a helm as in Definition 2.1, such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. Let $L = H - \{v_1, v_2, v_3, v_4\}$, and let L' be a copy of L such that $V(L') = \{a', x_1', x_2', x_3', x_4'\}$. We construct a graph G as follows. Let $V(G) = V(L) \cup V(L')$, $E(G) = E(L) \cup E(L') \cup \{x_1x_1', x_2x_2', x_3x_3', x_4x_4'\}$. Obviously, G is a 4-connected graph. It is easy to see that $(ax_2, \{x_1, x_3, x_4'\})$ is a separating pair of G, and so $ax_2 \in E_N(G)$. By symmetry, $ax_4, ax_1, ax_3, a'x_1', a'x_2', a'x_3', a'x_4' \in E_N(G)$. Similarly, $(x_1x_1', \{x_2, x_3, x_4\})$ is a separating pair of G, and hence $x_1x_1' \in E_N(G)$. By symmetry, we have $x_2x_2', x_3x_3', x_4x_4' \in E_N(G)$. Consider the cycle $C = x_1x_4x_3x_2x_2'x_3'x_4'x_1'x_1$ of G. Then C does not pass through any l-belt or l-co-belt. Moreover, it is easy to check that outside C there are exactly two removable edges $x_1x_2, x_1'x_2'$.

In the next result we allow a cycle to pass through (at most) one l-belt or l-co-belt, and show that we can still guarantee the existence of at least one removable edge outside the cycle, but not two, as shown by examples following the proof of the next theorem.

Theorem 4.3. *Let G be a 4-connected graph with $|G| \geq 7$ and let C be a cycle of G. If C passes through exactly one of the maximal l-belt or l-co-belt ($l \geq 1$), then there is at least one removable edge outside C.*

Proof. By contradiction. Suppose that there is no removable edge outside C. Let $E_0 = E(G) - E(C)$. We may assume that H is either an maximal l-belt or a maximal l-co-belt. If H is a maximal l-belt as in Definition 2.3, then from the assumptions it is easy to conclude that $E_2 \subset E(C)$, and $x_2x_1 \in E_0$. By letting $S' = \{y_{l+2}, x_{l+2}, y_1\}, e = x_2x_1, B' = \{x_2, \ldots, x_{l+1}, y_2, \ldots, y_{l+1}\}, A' = G - e - S' - B'$, Since H is a maximal l-belt, we have $x_1y_1 \notin E(G)$. Since $d(x_1) \geq 4$, there exits a vertex $u \notin B'$ such that $x_1u \in E(G)$, we have $|A'| \geq 2$. We get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the maximal l-belt ($l \geq 1$). Since $x_2x_1 \in E_0$, A' is an E_0-fragment; If H is a maximal l-co-belt, similarly, we have that $x_1x_2 \in E_0$. By letting $S' = \{y_{l+2}, x_{l+3}, y_1\}, e = x_2x_1, B' = \{x_2, \ldots, x_{l+2}, y_2, \ldots, y_{l+1}\}, A' = G - e - S' - B'$. Since H is a maximal l-co-belt, we have $x_1y_1 \notin E(G)$. Since $d(x_1) \geq 4$, there exits a vertex $v \notin B'$ such that $x_1v \in E(G)$, we have $|A'| \geq 2$. We get that $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of the maximal l-co-belt ($l \geq 1$), and A' is an E_0-fragment.

Since A' is an E_0-fragment, A' must contain an E_0-end-fragment as its subgraph, say A. Then we have that $(E(A) \cup [A,S]) \cap F = \emptyset$, and we take the corresponding separating group $(xy, S; A, B)$ such that $x \in A$, $y \in B$.

The following observations on $|A|$ are easy to check.

(1) $|A| = 2$. Then either A is a 1-atom or a 2-atom.

(1.1) If A is a 1-atom, let $A = \{x, z\}, S = \{a, b, c\}$. If $\{xz, xa, xb\} \cap E_N(G) \neq \emptyset$, from Corollary 3.2 we have that x is an inner vertex of some subgraph H of G belonging to $\Re(G)$. Noting that H is neither maximal l-belt nor maximal l-co-belt. It is easily checked that regardless of which subgraph H is, it is impossible for cycle C to contain all the removable edges of H, then there exists at least one removable edge outside cycle C, a contradiction. If $xz, xa, xb \in E_R(G)$, since C is a cycle, it is impossible for cycle C to contain all the three removable edges xa, xb, xz, a contradiction.

(1.2) If A is a 2-atom, it is easily checked that there exist at least two removable edges outside cycle C, a contradiction.

(2) $|A| \geq 3$. Since C is a cycle of G, and $(E(A) \cup [A,S]) \cap F = \emptyset$, there exists $xz \in E_0 \cap (E(A) \cup [A,S])$. Obviously $z \notin S$; otherwise, $|A| = 2$, contradicting $|A| \geq 3$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then $x \in A \cap A_1, z \in A \cap B_1$. Using Lemma 4.1, we conclude that one of the three conclusions of Lemma 4.1 holds.

(2.1) Since $xz \in E_0 \cap (E(A) \cup [A,S])$, conclusion (i) does not occur.

(2.2) Suppose that conclusion (ii) holds. Since B' is a 1-atom. Using a similar argument as used in (2.2) of Theorem 4.2, we have two possible conclusions hold: (i) vertex z is associated with three removable edges, then there exists at least one removable edges outside cycle C, contradicting $F \cap (E(A) \cup [A,S]) = \emptyset$; (ii) vertex z is an inner vertex of some subgraph H belonging to $\Re(G)$. In this case, since H is neither l-belt nor l-co-belt, cycle C cannot cover all of removable edges in H, which contradicts $(E(A) \cup [A,S]) \cap F = \emptyset$.

(2.3) Suppose that conclusion (iii) holds. Let $B' \cap S = \{x_1, x_2\}, A \cap S' = \{x_3\}$. Clearly, $A \cap B' = \{z\}$. We claim that $zx_1, zx_2 \in E_R(G)$. Otherwise, $\{zx_1, zx_2\} \cap E_N(G) \neq \emptyset$. From Lemma 3.2, we know x_3x_1, x_3x_2 cannot both be edges of G. We may assume that $x_3x_2 \notin E(G)$, let $A'' = A - z, S'' = S \cup \{z\} - x_2, B'' = G - xz - S'' - A''$, then A'' is an E_0-fragment contained in A, contradicting that A is an E_0-end-fragment. Therefore, we have that $zx_1, zx_2 \in E_R(G)$. Since $|B'| \geq 3$, using Theorem 3.2 we obtain that $zx_3 \in E_R(G)$. Note that C is a cycle of G, we have $d_C(z) \leq 2$. Then cycle C does not contain all the removable edges associated with vertex z, which contradicts $(E(A) \cup [A,S]) \cap F = \emptyset$.

This completes the proof of the last case and hence of the theorem. ∎

We next present two examples to show that if a cycle of a 4-connected graph passes through two l-belts or l-co-belts, then we cannot guarantee the existence of a removable edge outside the cycle. So, in this sense the conclusion of the above theorem cannot be strengthened.

Example 4.3. Let H be an l-belt as in Definition 2.3, and let H' be a copy of H such that $V(H') = \{x'_1, x'_2, \ldots, x'_{l+2}, y'_1, y'_2, \ldots, y'_{l+2}\}$ and $E(H') = E_1(H') \cup E_2(H')$, where $E_1(H') = \{x'_1x'_2, x'_2x'_3, \ldots, x'_{l+1}x'_{l+2}, y'_1y'_2, y'_2y'_3, \ldots, y'_{l+1}y'_{l+2}\}$ and $E_2(H') = \{y'_1x'_2,\ x'_2y'_2,\ y'_2x'_3, \ldots, y'_lx'_{l+1}, x'_{l+1}y'_{l+1}, y'_{l+1}x'_{l+2}\}$. Identify vertex x_1 with y'_1, vertex y_1 with x'_1, vertex y_{l+2} with x'_{l+2}, and vertex x_{l+2} with y'_{l+2}, respectively. Join vertex x_{l+2} and y'_1 and vertex x'_{l+2} and y_1 by an edge, respectively. Denote the resulting graph by G. It is straightforward to check that G is a 4-connected graph, and that $(x_2y'_1, \{y_1, x_3, y_3\})$ is a separating pair of G, so $x_2y'_1 \in E_N(G)$. Similarly, we can show that $\{y_1x'_2, y_{l+1}x'_{l+2}, y'_{l+1}x_{l+2}\} \subset E_N(G)$. Let C be the following cycle of G: $C = y_1x_2y_2x_3 \cdots x_{l+1}y_{l+2}x_{l+2}y'_1x'_2y'_2 \cdots x'_{l+1}y'_{l+1}x'_{l+2}y_1$. Then it is easy to check that there is no removable edge outside C.

Example 4.4. Let H be an l-co-belt as in Definition 2.4, and let H' be a copy of H such that $V(H') = \{x'_1, x'_2, \ldots, x'_{l+2}, x'_{l+3}, y'_1, y'_2, \ldots, y'_{l+2}\}$ and $E(H') = E_1(H') \cup E_2(H')$, where $E_1(H') = \{x'_1x'_2, x'_2x'_3, \ldots, x'_{l+1}x'_{l+2}, x'_{l+2}x'_{l+3}, y'_1y'_2, y'_2y'_3, \ldots, y'_{l+1}y'_{l+2}\}$ and $E_2(H') = \{y'_1x'_2,\ x'_2y'_2,\ y'_2x'_3, \ldots, y'_lx'_{l+1}, x'_{l+1}y'_{l+1}, y'_{l+1}x'_{l+2}, x'_{l+2}y'_{l+2}\}$. First, delete the vertices x_1, x'_1, x_{l+3}, x'_{l+3} from H and H', respectively. Then, join vertex x_{l+2} and y'_{l+2}, vertex y_1 and y'_1, vertex x'_{l+2} and y_{l+2}, vertex x_2 and x'_2, vertex y_1 and y'_{l+2}, and vertex y'_1 and y_{l+2} by an edge, respectively. Denote the resulting graph by G. It is straightforward to check that G is a 4-connected graph, and that $(y_1y'_1, \{y_{l+2}, y'_{l+2}, x'_2\})$ is a separating pair of G, so $y_1y'_1 \in E_N(G)$. Similarly, $(x_{l+2}y'_{l+2}, \{y_l, x_{l+1}, y_{l+2}\})$, $(x_2x'_2, \{y_1, x_3, y_3\})$ and $(y_{l+2}x'_{l+2}, \{x'_{l+1}, y'_l, y'_{l+2}\})$ are separating pairs of G, and so $\{x_{l+2}y'_{l+2}, x_2x'_2, y_{l+2}x'_{l+2}\} \subset E_N(G)$. Let C be the following cycle of G: $C = y_1x_2y_2x_3 \cdots x_{l+1}y_{l+1}x_{l+2}y_{l+2}y'_1x'_2y'_2 \cdots x'_{l+1}y'_{l+1}x'_{l+2}y'_{l+2}y_1$. Then it is easy to see that there is no removable edge outside C.

4.3. Removable edges on a fixed (Hamilton) cycle

The next results deal with circumstances that guarantee the existence of removable edges on cycles of 4-connected graphs, in particular on Hamilton cycles. Before we present these results, we first introduce a definition and prove an auxiliary result.

Definition 4.1. Let G be a 4-connected graph, let C be a cycle of G, and let $(xy, S; A, B)$ be a separating group of G such that A is an atom. We say that C *passes* through this atom if $x, y \in V(C)$.

The following useful lemma deals with removable edges on a cycle that does not pass through any atom.

Lemma 4.2. *Let G be a 4-connected graph with $|G| \geq 7$, and let C be a cycle that does not pass through any atom. Then there are at least two removable edges on C.*

Proof. By contradiction. Suppose that C does not pass through any atom of G, and suppose there is at most one removable edge of G in C. Let $F = E(C) \cap E_R(G)$. Then $|F| \leq 1$. Denote $E(C) - F$ by E_0. We take the separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$ and $uw \in E_0$. From $|F| \leq 1$ we know that $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [S', B']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-fragment, A' contains an E_0-end-fragment as its subgraph, say A. We take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. Clearly, we have $(E(A) \cup [A, S]) \cap F = \emptyset$. Since C does not pass through any atom, we have $|A| \geq 3$. Using Lemma 4.1, we know that one of the three conclusions of Lemma 4.1 holds. Here we discuss them as follows. Since $(E(A) \cup [A, S]) \cap F = \emptyset$, we have that conclusion (i) does not hold. Since C does not pass through any atom, conclusion (ii) of Lemma 4.1 does not hold either. So conclusion (iii) of Lemma 4.1 holds. Let $A \cap S' = \{w\}, B' \cap S = \{u, v\}, \Gamma_G(y') = \{w, u, v, x\}$. Since $|B'| \geq 3$, using Theorem 3.2, we conclude that $y'w \in E_R(G)$. Noticing that C is a cycle and $(E(A) \cup [A, S]) \cap F = \emptyset$, we have $\{y'u, y'v\} \cap E_0 \neq \emptyset$. Using Lemma 3.2, we have that wu, wv cannot belong to $E(G)$ simultaneously. Without loss of generality, we may assume $wu \notin E(G)$. Let $A_0 = A - \{y'\}, S_0 = S \cup \{y'\} - u, B_0 = G - xy - S_0 - A_0$. Then A_0 is an E_0-fragment contained in A, which contradicts that A is an E_0-end-fragment. So, conclusion (iii) does not hold either.

This completes the proof of Lemma 4.2. ∎

In the remainder we will mainly deal with the existence of removable edges on Hamilton cycles in 4-connected Hamiltonian graphs. Our first result shows that Hamilton cycles in 4-connected graphs without atoms contain at least six removable edges.

Theorem 4.4. *Let G be a 4-connected Hamiltonian graph with $|G| \geq 7$, and suppose that G does not contain any atom. Then any Hamilton cycle C of G contains at least six removable edges.*

Proof. Let $F = E(C) \cap E_R(G)$, $E_0 = E(C) \cap E_N(G)$. If $E_0 = \emptyset$, then C contains at least seven removable edges, we have theorem holds. So in what follows we may assume $E_0 \neq \emptyset$. We consider a separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$, $uw \in E_0$. By symmetry, we may assume that $|(E(A') \cup [A', S']) \cap F| \leq |(E(B') \cup [S', B']) \cap F|$. Since A' is an E_0-fragment, A' must contain an E_0-end-fragment as its subgraph, say A, and we take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ and $xy \in E_0$. Note that $|(E(A) \cup [A, S]) \cap F| \leq |(E(A') \cup [A', S']) \cap F|$. Since C does not pass through any atom, we have $|A| \geq 3$. From Lemma 4.1 we know that there are three possible conclusions (i), (ii) or (iii).

First, we suppose that $E_0 \cap (E(A) \cup [A, S]) \neq \emptyset$. From Lemma 4.1 we have two possible conclusions (ii) or (iii). Since C does not pass through any atom, conclusion (ii) does not hold. Suppose conclusion (iii) holds. We consider a separating group of G as in conclusion (iii) of Lemma 4.1. Let $B' \cap S = \{b, c\}$, $A' \cap S = \{a\}, A \cap S' = \{d\}$. Let $A_1 = \{d, y_1\}, S_1 = \{b, c, x\}, B_1 = G - ad - S_1 - A_1$, then $(ad, S_1; A_1, B_1)$ is a separating group of G. However, A_1 is an atom, which contradicts that G does not contain any atom. So conclusion (iii) does not hold.

From the arguments above, we have that only conclusion (i) holds. Since $E_0 \cap (E(A) \cup [A, S]) = \emptyset$, it is easily checked that $|(E(A) \cup [A, S]) \cap F| \geq 3$. Since $|(E(A) \cup [A, S]) \cap F| \leq |(E(A') \cup [A', S']) \cap F|$, we have that $|(E(A') \cup [A', S']) \cap F| \geq 3$. Hence $|E(C) \cap E_R(G)| = |F| \geq 6$. The proof is complete. ∎

Before we present and prove our next result about removable edges on Hamilton cycles, we first prove the following auxiliary result.

Lemma 4.3. *Let G be a 4-connected graph with $|G| \geq 7$, and let C be a cycle of G passing through exactly one inner vertex of some maximal l-bi-fan H of G, and not passing through any other subgraph belonging to $\Re(G)$. Then there are at least two removable edges on C.*

Proof. Suppose that there is at most one removable edge on C. Using Theorem 3.5, we conclude that there is exactly one removable edge on C. Let $E(C) \cap E_R(G) = \{e\} = F$. Let H be a maximal l-bi-fan defined as in Definition 2.2. From the assumption $|V(C) \cap \{x_2, x_3, \ldots, x_{l+2}\}| = 1$ and $|E(C) \cap E_R(G)| = 1$, it can be checked easily that either $x_2 \in V(C)$ or $x_{l+2} \in V(C)$. Without loss of generality, we may assume $x_2 \in V(C)$ and $e = ax_2$. Letting $S' = \{a, b, x_{l+3}\}, e' = x_2x_1, B' = \{x_2, \ldots, x_{l+2}\}, A' = G - e' - S' - B'$, $(e', S'; A', B')$ is a separating group of G such that $A' \cap V(C)$ does not contain any inner vertex of the maximal l-bi-fan. Let $E_0 = E(C) - \{ax_2\}$. Then $x_1x_2 \in E_0$ and A' is an E_0-fragment. Clearly, A' contains an E_0-end-fragment, say A. It is easily checked that $A \cap V(C)$ does not contain any inner vertex of H, and that $(E(A) \cup [A, S]) \cap F = \emptyset$. We consider a corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$.

Next, we distinguish a number of cases and subcases.

(1) $|A| = 2$. Then A is a 1-atom or a 2-atom, say $A = \{x, z\}$. Let $S = \{a, b, c\}$.

(1.1) A is a 2-atom. Since $xy \in E(C)$ and C is a cycle of G, we have $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$. From Lemma 3.3 we know that $\{xa, xb, xc, xz\} \subset E_R(G)$, contradicting $(E(A) \cup [A, S]) \cap F = \emptyset$.

(1.2) A is a 1-atom. Noting that C is a cycle of G and $x \in V(C)$, we know that $\{xa, xb, xz\} \cap E(C) \neq \emptyset$. From the assumption $(E(A) \cup [A, S]) \cap F = \emptyset$, we have

$\{xa, xb, xz\} \cap E_0 \neq \emptyset$. From Corollary 3.2 we know that x is an inner vertex of one of the graphs of $\Re(G)$, which contradicts $A \cap V(C)$ does not contain any inner vertex of H.

(2) $|A| \geq 3$. Using Lemma 4.1 we have three possible conclusions (i), (ii) or (iii).

(2.1) Conclusion (i) holds. Since C is a cycle, we have $(E(A) \cup [A, S]) \cap E(C) \neq \emptyset$. However, $(E(A) \cup [A, S]) \cap E_0 = \emptyset$, it is impossible that $(E(A) \cup [A, S]) \cap F = \emptyset$, a contradiction.

(2.2) Conclusion (ii) holds. Let $B' = \{y', z'\}, A \cap B' = \{y'\}, \Gamma_G(y') = \{x', a', b', z'\}$. Noting that C is a cycle, and $(E(A) \cup [A, S]) \cap F = \emptyset$, we have $\{a'y', b'y', y'z'\} \cap E_0 \neq \emptyset$. From Corollary 3.2 we know that y' is an inner vertex of one of the graphs of $\Re(G)$. Since $y' \in V(C)$, contradicting that $A \cap V(C)$ does not contain any inner vertex of H.

(2.3) Conclusion (iii) holds. Let $B' \cap S = \{u, v\}, A \cap S' = \{a\}$, $A \cap B' = \{y'\}$. Since $|B'| \geq 3$, from Theorem 3.2 we know $ay' \in E_R(G)$. Noting that C is a cycle and $(E(A) \cup [A, S]) \cap F = \emptyset$, we have $\{y'u, y'v\} \cap E(C) \neq \emptyset$, and so $\{y'u, y'v\} \cap E_0 \neq \emptyset$. By Lemma 3.2, we know that au, av cannot both be edges of G. Without loss of generality, we may assume that $au \notin E(G)$, let $A'' = A - y', S'' = S \cup \{y'\} - u, B'' = G - xy' - S'' - A''$. Then A'' is an E_0-fragment contained in A, contradicting the choice of A.

This completes the proof of the lemma. ∎

Theorem 4.5. *Let G be a 4-connected Hamiltonian graph with $|G| \geq 7$, and let C be any Hamilton cycle of G. Then, if G contains only one subgraph H of G that belongs to $\Re(G)$, but not any maximal l-belt or l-co-belt, then there are at least two removable edges on C.*

Proof. By contradiction. Suppose that G contains only one subgraph H of G that belongs to $\Re(G)$, but not any maximal l-belt or l-co-belt, and suppose there is at most one removable edge on C. Let $E_0 = E(C) \cap E_N(G), F = E(C) \cap E_R(G)$, then $|F| \leq 1$.

Since H is not any maximal l-belt or l-co-belt, then H is one of the following four graphs: helm, maximal l-bi-fan, W-fragment, W'-fragment. Note that H is the only subgraph in $\Re(G)$ that C passes through. Next we will discuss them separately.

Case 1. H is one of the following three subgraphs: helm, W-fragment, W'-fragment.

(1) C passes through a helm H. Let H be defined as in Definition 2.1. Since C is a Hamilton cycle, $E(H) \cap F \neq \emptyset$. From the assumption $|F| \leq 1$, we know that there is exactly one removable edge on C, then $|F| = 1$. Without loss of generality, we may assume $F = \{x_3x_4\}$. Since C is a Hamilton cycle, it is easily checked that $x_1v_1 \in E(C)$. According to the assumptions, we have $E(C) - x_3x_4 = E_0$. By

letting $e = x_1v_1, S' = \{v_2, v_3, v_4\}, B' = \{a, x_1, x_2, x_3, x_4\}, A' = G - e - S' - B'$, $(e, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of H and $(E(A') \cup [A', S']) \cap F = \emptyset$. Since $x_1v_1 \in E_0$, A' is an E_0-fragment of G. Since A' contains an E_0-end-fragment, say A. clearly, A does not contain any inner vertex of H, and $(E(A) \cup [A, S]) \cap F = \emptyset$. We take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$.

(2) C passes through a W'-framework H. Let H be defined as in Definition 2.6. Since C is a Hamilton cycle, we have $F \neq \emptyset$, from the assumption, we have $|F| = 1$. It can be checked easily that $F = \{y_2y_3\}$ and $y_3y_4 \in E(C)$. By letting $S' = \{x_1, x_3, y_1\}, B' = \{x_2, y_2, y_3\}, A' = G - y_3y_4 - S' - B'$, then $(y_3y_4, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of H and $(E(A') \cup [A', S']) \cap F = \emptyset$. Since $y_3y_4 \in E_0$, we have that A' is an E_0-fragment. Since A' contains an E_0-end-fragment, say A. Clearly, A does not contain any inner vertex of H and $(E(A) \cup [A, S]) \cap F = \emptyset$. We take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$.

(3) C passes through a W-framework H. Let H be defined as in Definition 2.5. Since C is a Hamilton cycle, it is easy to see that $y_1y_2 \in E_0$ and $F = \{y_2y_3\}$. By letting $S' = \{x_1, x_3, y_4\}, B' = \{x_2, y_2, y_3\}, A' = G - y_1y_2 - S' - B'$, $(y_1y_2, S'; A', B')$ is a separating group of G such that A' does not contain any inner vertex of H, and $(E(A) \cup [A, S]) \cap F = \emptyset$. Since $y_1y_2 \in E_0$, we have that A' is an E_0-fragment. Since A' contains an E_0-end-fragment, say A, as its subgraph. Clearly, A does not contain any inner vertex of H and $(E(A) \cup [A, S]) \cap F = \emptyset$. We take the corresponding separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$.

From the above arguments we can see that no matter which of the three subgraphs H is, we always can take the separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. A is an E_0-end-fragment such that A does not contain any inner vertex of H, and $(E(A) \cup [A, S]) \cap F = \emptyset$.

Next we will distinguish a number subcases to discuss.

Subcase 1.1. $|A| = 2$. Then A is a 1-atom or a 2-atom, say $A = \{x, z\}$. Let $S = \{a, b, c\}$.

(1) A is a 2-atom. Since $xy \in E(C)$ and C is a cycle of G, we have $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$. From Lemma 3.3 we know that $\{xa, xb, xc, xz\} \subset E_R(G)$, contradicting $(E(A) \cup [A, S]) \cap F = \emptyset$.

(2) A is a 1-atom. Noting that C is a cycle of G and $(E(A) \cup [A, S]) \cap F = \emptyset$, we have $\{xa, xb, xz\} \cap E_0 \neq \emptyset$. From Corollary 3.2 we know that x is an inner vertex of one of the graphs in $\Re(G)$, contradicting that A does not contain any inner vertex of H.

Subcase 1.2. $|A| \geq 3$. Using Lemma 4.1 we have three possible conclusions (i), (ii) or (iii).

(1) Conclusion (i) holds. Since C is a Hamilton cycle, we have $(E(A) \cup [A,S]) \cap E(C) \neq \emptyset$. However, $(E(A) \cup [A,S]) \cap E_0 = \emptyset$, and it is impossible that $(E(A) \cup [A,S]) \cap F = \emptyset$, a contradiction.

(2) Conclusion (ii) holds. Let $B' = \{y', z'\}, A \cap B' = \{y'\}, \Gamma_G(y') = \{x', a', b', z'\}$. Noting that C is a Hamilton cycle, and $(E(A) \cup [A,S]) \cap F = \emptyset$, we have $\{a'y', b'y', y'z'\} \cap E_N(G) \neq \emptyset$. From Corollary 3.2 we know that y' is an inner vertex of one of the subgraphs of $\Re(G)$, contradicting that A does not contain any inner vertex of H.

(3) Conclusion (iii) holds. Let $B' \cap S = \{u, v\}, A \cap S' = \{a\}$, $A \cap B' = \{y'\}$. Since $|B'| \geq 3$, from Theorem 3.2 we know $ay' \in E_R(G)$. Noting that C is a cycle and $(E(A) \cup [A,S]) \cap F = \emptyset$, we have $ay' \notin E(C)$ and $\{y'u, y'v\} \cap E(C) \neq \emptyset$, and so $\{y'u, y'v\} \cap E_0 \neq \emptyset$. By Lemma 3.2, we know that au, av cannot both be edges of G. Without loss of generality, we may assume that $au \notin E(G)$, let $A'' = A - y', S'' = S \cup \{y'\} - u, B'' = G - xy' - S'' - A''$. Then A'' is an E_0-fragment contained in A, contradicting the choice of A.

Case 2. C passes through a maximal l-bi-fan H. Let H be defined as in Definition 2.2. By assumption we have $|E(C) \cap \{ax_2, ax_3, \ldots, ax_{l+2}, bx_2, bx_3, \ldots, bx_{l+2}\}| \leq 1$. Next we distinguish the following two subcases.

Subcase 2.1. $\{x_1x_2, x_2x_3, \ldots, x_{l+2}x_{l+3}\} \subset E(C)$. Let $C = x_1x_2 \cdots x_{l+2}x_{l+3} \cdots vau \cdots x_1$. We let P_1 denote the path from a to x_2 on C passing through u, and P_2 the path from x_{l+2} to a on C passing through v. Then both C_1 formed by P_1 and ax_2, and C_2 formed by P_2 and ax_{l+2} are cycles containing just one inner vertex of H and not passing through any other graph of $\Re(G)$. Using Lemma 4.3 we get that both C_1 and C_2 contain at least two removable edges, so there are at least two removable edges on C.

Subcase 2.2. For some $i \in \{2, \ldots, l+2\}$, either $\{ax_i, x_ix_{i+1}, \ldots, x_{l+2}x_{l+3}\} \subset E(C)$ or $\{bx_i, x_ix_{i+1}, \ldots, x_{l+2}x_{l+3}\} \subset E(C)$. Without loss of generality, we assume $\{ax_i, x_ix_{i+1}, \ldots, x_{l+2}x_{l+3}\} \subset E(C)$. From $|E(C) \cap \{ax_2, \ldots, ax_{l+2}\}| \leq 1$, we get that $(\{ax_2, \ldots, ax_{i+2}\} - \{ax_i\}) \cap E(C) = \emptyset$. It can be checked easily that only $i = 2$ is possible. Let $C = ax_2x_3 \cdots x_{l+2}x_{l+3} \cdots ua$. Let P denote the path from x_{l+2} to a on C passing through u. Then the cycle C_1 formed by P and ax_{l+2} passes through just one inner vertex of H and does not pass through any other graph of $\Re(G)$. Using Lemma 4.3 we get that C_1 contains at least two removable edges, and so P contains at least one removable edge. Since $ax_2 \in E(C) - E(C_1)$, there are at least two removable edges on C.

This completes the proof of Theorem 4.5. ■

Next we present examples in order to show that in each of the cases in the above proof of Theorem 4.5 we cannot improve the lower bound on the number of removable edges. We start with an example to show that in (1) of Case 1 of (the proof of) Theorem 4.5 the lower bound is sharp.

Example 4.5. Let H be a helm with $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3,\ x_3x_4, x_4x_1,\ x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. Let $L = H - \{v_1, v_2, v_3, v_4\}$, and let L' be a copy of L such that $V(L') = \{a', x'_1, x'_2, x'_3, x'_4\}$. We construct a graph G as follows. $V(G) = V(L) \cup V(L')$, and we join vertices x_1 and x'_1, x_2 and x'_2, x_3 and x'_3, x_4 and x'_4, x'_2 and x'_4 by an edge, respectively. Now $(x_1x'_1, \{x_2, x_3, x_4\})$ is a separating pair of G, hence $x_1x'_1 \in E_N(G)$. By symmetry, $x_2x'_2, x_3x'_3, x_4x'_4 \in E_N(G)$. It is easy to check that $(a'x'_1, \{x'_2, x'_4, x_3\})$ and $(a'x'_3, \{x'_2, x'_4, x_1\})$ are also separating pairs of G, so we conclude that $a'x'_1, a'x'_3 \in E_N(G)$. Let $C = x_1x'_1a'x'_3x_3x_4x'_4x'_2x_2ax_1$. Then C is a Hamilton cycle passing through precisely one helm (and no other graphs of $\Re(G)$) and containing just two removable edges $x_3x_4, x'_2x'_4$.

Our next example shows the sharpness of the lower bound in (2) of Case 1.

Example 4.6. Let H be a W'-framework defined as in Definition 2.6, with $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$. Let L' be the graph as defined in Example 4.5, with $V(L') = \{a', x'_1, x'_2, x'_3, x'_4\}$. We construct a graph G as follows. Let $V(G) = V(H) - \{y_1, y_4\} \cup V(L')$, and let $E(G) = E(H) - \{y_1y_2, y_3y_4\} \cup E(L') \cup \{x_1x'_1, y_2x'_4, y_3x'_3, x'_2x_3,\ x_1x'_3, x'_1x'_3\}$. It is easy to check that G is 4-connected. Now $(x_1x'_1, \{x'_2, x'_3, x'_4\})$, $(x'_2x_3, \{x'_1, x'_3, x'_4\})$, $(y_2x'_4, \{x'_1, x'_2, x'_3\})$, $(y_3x'_3, \{x_1, x_3, y_2\})$, $(a'x'_4, \{x'_1, x'_3, x_3\})$, and $(a'x'_2, \{y_2, x'_1, x'_3\})$ are separating pairs of G, so $x_1x'_1, x'_2x_3, y_2x'_4, y_3x'_3, a'x'_4, a'x'_2 \in E_N(G)$. Let $C = x_1x_2x_3x'_2a'x'_4y_2y_3x'_3x'_1x_1$. Then C is a Hamilton cycle which passes through only one W'-framework and contains two removable edges $y_2y_3, x'_1x'_3$.

Next we give an example to show the sharpness in (3) of Case 1.

Example 4.7. Let H be a W-framework defined as in Definition 2.5, and let L be a graph such that $V(L) = \{a', x'_1, x'_2, x'_3, x'_4\}$ and $E(L) = \{a'x'_1, a'x'_2, a'x'_3, a'x'_4, x'_1x'_2, x'_2x'_3, x'_3x'_4, x'_4x'_1, x'_1x'_3\}$. We construct a graph G as follows. Let $V(G) = V(L) \cup V(H) - \{y_1, y_2\}$ and $E(G) = E(L) \cup E(H) - \{y_1y_2, y_3y_4\} \cup \{x_1x'_1, x'_2x_3, y_3x'_3, y_2x'_4, x_1x'_3, x'_1x_3\}$. It is easy to check that G is 4-connected. Now $(x'_2x_3, \{x'_1, x'_3, x'_4\})$, $(y_3x'_3, \{x_1, x_3, y_2\})$, $(y_2x'_4, \{x'_1, x'_2, x'_3\})$, $(x'_1x_1, \{y_2, x_3, x'_3\})$, $(a'x'_4, \{x'_1, x'_3, x_3\})$, and $(a'x'_2, \{y_2, x'_1, x'_3\})$ are separating pairs of G. Let $C = x_1x_2x_3x'_2a'x'_4 y_2y_3x'_3x'_1x_1$. Clearly, C is a Hamilton cycle which contains two removable edges $x'_1x'_3, y_2y_3$.

The next example discusses the sharpness in Case 2.

Example 4.8. We give an example to show that in Case 4 of the proof the lower bound cannot be improved. Let H be an l-bi-fan ($l \geq 2$) defined as in Definition 2.2, and let L be a graph such that $V(L) = \{a', x'_1, x'_2, x'_3, x'_4\}$ and $E(L) = \{a'x'_1, a'x'_2, a'x'_3, a'x'_4, x'_1x'_2, x'_2x'_3, x'_3x'_4, x'_4x'_1, x'_1x'_3\}$. We construct a graph G as follows. First, we identify the vertex x_1 with x'_1, and x_{l+3} with x'_4, respectively.

Then we join the vertices a and x'_2, b and x'_3, and a and b by an edge. Obviously, G is a 4-connected graph. Similar arguments as used in Example 4.2 yield that $bx'_3, ax'_2, x_{l+2}x'_4, x_2x'_1, a'x'_4, a'x'_2 \in E_N(G)$. Let $C = ax'_2a'x'_4x_{l+2}x_{l+1}\cdots x_2x'_1x'_3ba$. Then C is a Hamilton cycle which passes through two removable edges $x'_1x'_3$ and ab.

Note that in Theorem 4.5, we exclude the case that C passes through only one maximal l-belt or maximal l-co-belt. In fact, for a Hamilton cycle C that passes through only one maximal l-belt or maximal l-co-belt, the conclusion of Theorem 4.5 does not hold in general. We present the following two examples to show that.

Example 4.9. (1) Let H be a maximal l-belt as in Definition 2.3, and let L be a graph with $V(L) = \{a', x'_1, x'_2, x'_3, x'_4\}$ and $E(L) = \{a'x'_1, a'x'_2, a'x'_3, a'x'_4, x'_1x'_2, x'_2x'_3, x'_3x'_4, x'_4x'_1, x'_1x'_3\}$. Now we construct a graph G as follows. First, we identify the vertices x_1 and x'_1, and the vertices y_{l+2} and x'_3, respectively. Then we join the vertices y_1 and x'_2, x_{l+2} and x'_4, and y_1 and x_{l+2} by an edge, respectively. It is easy to check that G is a 4-connected graph. Similar arguments as before yield that $x'_1x_2, x'_2y_1, x'_3y_{l+1}, x'_4x_{l+2} \in E_N(G)$. Let $C = x'_1x_2x_3\cdots x_{l+2}x'_4a'x'_2y_1y_2\cdots y_{l+1}x'_3x'_1$. Then C is a Hamilton cycle containing only one removable edge $x'_1x'_3$.

(2) Let H be a maximal l-co-belt defined as in Definition 2.4, and let L be defined as in (1). Now we construct the graph G as follows. First, we identify the vertices x_{l+3} and x'_4, and x_1 and x'_1, respectively. Then we join the vertices y_1 and x'_2, y_1 and y_{l+2}, and x'_3 and y_{l+2} by an edge, respectively. It is easy to check that G is a 4-connected graph. Similar arguments as in (1) can be used to show that $C = x'_1x_2x_3\cdots x_{l+2}x'_4a'x'_2y_1y_2\cdots y_{l+2}x'_3x'_1$ is a Hamilton cycle containing only one removable edge $x'_1x'_3$.

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