Discussiones Mathematicae
Graph Theory 41 (2021) 531–558
doi:10.7151/dmgt.2208

ON $(p,1)$-TOTAL LABELLING OF SOME 1-PLANAR GRAPHS[1]

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Abstract

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is proved that the $(p,1)$-total labelling number ($p \geq 2$) of every 1-planar graph G is at most $\Delta(G)+2p-2$ provided that $\Delta(G) \geq 6p+7$ or $\Delta(G) \geq 4p+6$ and G is triangle-free.

Keywords: 1-planar graph, total coloring, $(p,1)$-total labelling, structural theorem.

2010 Mathematics Subject Classification: 05C15, 05C10.

1. Introduction

All graphs considered in this paper are simple and undirected. By $V(G), E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. A *planar graph* is a graph that can be drawn on a plane in such a way that no edges cross each other, and this drawing is a *plane graph*. By $F(G)$, we denote the face set of a plane graph G. The *degree of a vertex* v in G, denoted by $d_G(v)$, is the number of edges that are incident with v in G, and the *degree of a face* f in a plane graph G, denoted

[1]This work was partially supported by the Fundamental Research Funds for the Central Universities (No. JB170706), the Natural Science Basic Research Plan in Shaanxi Province of China (No. 2017JM1010), and the National Natural Science Foundation of China (Nos. 11871055, 11301410).

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by $d_G(f)$, is the the number of edges that are incident with f in G, where cut-edges are counted twice. A *k-, k^+-, or k^--vertex* (respectively *face*) is a vertex (respectively face) of degree exactly k, at least k, or at most k, respectively. The *length of a cycle* is the number of edges that are incident with it, and the *girth* of a graph G, denoted by $g(G)$, is the length of the shortest cycle in G.

In the channel assignment problems, we assign different channels to "close" transmitters and channels that are at least two channels apart to "very close" transmitters in order to avoiding interference and communication link failure. In the language of graph theory, the transmitters are represented by vertices of a graph, in which two vertices are adjacent (respectively of distance two) if and only if their representing transmitters are very close (respectively close). Hence the channel assignment problem is now to label the vertices of this graph G with nonnegative integers (representing the channels) so that $|L(u)-L(v)|$ is at least 2 if $uv \in E(G)$, and at least 1 if the distance between u and v is 2, where $L(u)$ and $L(v)$ denotes the labellings on u and v, respectively. Yeh [16] and then Griggs and Yeh [5] called it an *$L(2,1)$-labelling* of graphs.

In the network of the wireless communications, we may do some optimization to the efficiency by adding new transmitters between any two transmitters that are very close. This operation updates the structure of the network and we are to complete the channel assignment problem on this new network. Denote by G' and G the updated network/graph and the original network/graph, respectively. Comparing their structures with each other, we can easily find that G' is derived from G by replacing every edge xy of G with a path xzy (i.e., inserting a vertex z of degree 2 along every edge xy of G). In the words of graph theory, G' is called the *incidence graph* of G. It is clear to see that those new inserted vertices of degree 2 to G correspond to the edges of G and any two of them has distance 2 in G'. Therefore, the $L(2,1)$-labelling of G' is somehow a labelling φ of the vertices and edges of G so that $\varphi(u) \neq \varphi(v)$ if $uv \in E(G)$, $\varphi(e_1) \neq \varphi(e_2)$ if e_1 and e_2 are two adjacent edges in G, and $|\varphi(u) - \varphi(e)| \geq 2$ if a vertex u is incident with an edge e in G. Such kind of a labelling is called the *$(2,1)$-total labelling* of graphs, which was introduced by Havet and Yu [7, 8] at the beginning of the 2000', and generalized to the *$(p,1)$-total labelling* of graphs.

Formally, a *$(p,1)$-total k-labelling* of a graph G is a function f from $V(G) \cup E(G)$ to the color set $\{0,1,\ldots,k\}$ such that $|f(u) - f(v)| \geq 1$ if $uv \in E(G)$, $|f(e_1) - f(e_2)| \geq 1$ if e_1 and e_2 are two adjacent edges in G, and $|f(u) - f(e)| \geq p$ if a vertex u is incident with the edge e. The minimum k such that G has a $(p,1)$-total k-labelling, denoted by $\lambda_p^T(G)$, is the *$(p,1)$-total labelling number* of G. Note that the $(1,1)$-total $(k-1)$-labeling of G is equivalent to the total k-coloring of G, that is a function f from $V(G) \cup E(G)$ to $\{1,2,\ldots,k\}$ such that any two adjacent or incident elements from $V(G) \cup E(G)$ receive different colors. The minimum integer k so that G admits a total k-coloring is the *total chromatic* *number* of G, denoted by $\chi''(G)$. It is easy to see that

$$\lambda_1^T(G) + 1 = \chi''(G) \geq \Delta(G) + 1.$$

On the other side, the well-known *Total Coloring Conjecture* states that $\chi''(G) \leq \Delta(G)+2$, and this conjecture is still quite open. As a generalization of the Total Colouring Conjecture, Havet and Yu [8, 9] put forward the following so called $(p,1)$-*Total Labelling Conjecture*.

Conjecture 1. $\lambda_p^T(G) \leq \min\{\Delta(G) + 2p - 1, 2\Delta(G) + p - 1\}$.

Note that Havet [7] proved that $\lambda_p^T(G) \leq 2\Delta(G) + p - 1$ for any graph G. Hence Conjecture 1 can be rewritten as follows.

Conjecture 2. $\lambda_p^T(G) \leq \Delta(G) + 2p - 1$.

If $p \geq \Delta(G)$, then $2\Delta(G) + p - 1 \leq \Delta(G) + 2p - 1$. Therefore, Conjecture 2 is only interesting for $p < \Delta(G)$. For planar graphs G, Havet [7] showed that $\lambda_p^T(G) \leq \Delta(G)+p+3$ and $\lambda_p^T(G) \leq \Delta(G)+p+2$ if $\Delta(G) \geq 7$. This implies that Conjecture 2 with $p \geq 4$ holds for all planar graphs and Conjecture 2 with $p = 3$ holds for planar graphs with maximum degree at least 7.

We now move the attentions to a larger class of graphs than planar graphs. A graph is 1-*planar* if it can be drawn on a plane in such a way that one edge is crossed by at most one other edge, and this drawing is a 1-*plane graph*. The notion of 1-planarity was first introduced in 1965 by Ringle [13] while considering the proper vertex-face coloring of plane graphs, which can be translated into the proper vertex coloring of 1-plane graphs. Ringel [13] proved that 7 colors are sufficient to properly color the vertices of any 1-planar graph, and asked whether 6 colors are enough. In 1984, Borodin [3, 4] answered Ringel's question by proving that the chromatic number of any 1-planar graph is at most 6 (being sharp). Nowadays various research streams on 1-planar graphs, such as characterization and recognition, combinatorial properties, and geometric representations, are active. A survey contributed by Kobourov, Liotta and Montecchiani [11] in 2017 reviewed the current literatures covering those research streams on 1-planar graphs.

Concerning the $(p,1)$-total labelling, Bazzaro, Montassier and Raspaud [1] raised another interesting problem that is to answer when

$$\lambda_p^T(G) \leq \Delta(G) + 2p - 2 \tag{1}$$

holds. If $p = 1$, then this reduces to ask when it holds that $\lambda_1^T(G) = \Delta(G)$, that is, $\chi''(G) = \Delta(G) + 1$. Actually, this reduced problem on total coloring attracts interests from many researchers. For example, Kowalik, Sereni and Škrekovski [10] proved $\chi''(G) = \Delta(G)+1$ for planar graphs with maximum degree at least 9, and Zhang, Wu and Liu [20] showed $\chi''(G) = \Delta(G) + 1$ for 1-planar graphs with maximum degree at least 21 (this lower bound 21 has recently been improved to 18 by Zhang, Niu and Yu in an upcoming paper [18]).

For $p \geq 2$, Montassier and Raspaud [12] proved (1) for graphs with a given maximum average degree. Bazzaro, Montassier and Raspaud proved (1) for planar graphs with large girth and with high maximum degree — especially for planar graphs with maximum degree $\Delta(G) \geq 8p + 2$. Yu *et al.* [17] proved for planar graphs with maximum degree $\Delta(G) \geq 12$ that $\lambda_2^T(G) \leq \Delta(G) + 2$. Recently, Sun and Wu [14] proved that $\lambda_p^T(G) \leq \Delta(G) + 2p - 2$ for planar graphs with maximum degree $\Delta(G) \geq 4p + 4$ and $p \geq 2$, which improved both the result of Bazzaro, Montassier and Raspaud, and the result of Yu *et al.* mentioned above. The $(p, 1)$-total labelling of 1-planar graphs was first considered in 2011 by Zhang, Yu and Liu [21]. They proved the following result.

Theorem 3 [21], Theorem 1.5**.** *Let $p \geq 2$ be an integer and let G be a 1-planar graph with maximum degree Δ and girth g. If $\Delta \geq 8p + 4$ or $\Delta \geq 6p + 2$ and $g \geq 4$, then $\lambda_p^T(G) \leq \Delta + 2p - 2$.*

In the paper [21], the authors actually feel that the two lower bounds $8p+4$ and $6p + 2$ for Δ in Theorem 3 may not be sharp. This motivates us to ask whether those lower bounds can be improved and how much they can be lowered. In this paper, we are to prove the following Theorem 4, which pulls the first and the second lower bounds for Δ in Theorem 3 down to $6p + 7$ and $4p + 6$, respectively. Note that $6p + 7 < 8p + 4$ and $4p + 6 \leq 6p + 2$ if $p \geq 2$.

Theorem 4. *Let $p \geq 2$ be an integer and let G be a 1-planar graph with maximum degree Δ and girth g. If $\Delta \geq 6p + 7$ or $\Delta \geq 4p + 6$ and $g \geq 4$, then $\lambda_p^T(G) \leq \Delta + 2p - 2$.*

In the next sections, we are to give the detailed proof of the above theorem.

2. The Proof of Theorem 4

Alternatively, we set an integer M such that $\Delta \leq M$ and prove $\lambda_p^T(G) \leq M+2p-2$ while $M \geq 6p + 7$ or $M \geq 4p + 6$ and $g \geq 4$. This will imply Theorem 4.

We prove the required conclusion by contradiction. First, assume that there is a 1-planar graph G with maximum degree at most M such that

(1) G admits no $(p, 1)$-total $(M + 2p - 2)$-labelling, and

(2) any proper subgraph of G has a $(p, 1)$-total $(M + 2p - 2)$-labelling.

We call such a graph G an $(M + 2p - 2)$-*critical graph.* The following are two useful properties on the $(M + 2p - 2)$-critical graphs, which are given by Zhang, Yu and Liu [21].

Lemma 5 [21, Lemmas 2.1 and 2.2]**.** *Let G be an $(M+2p-2)$-critical graph. For any edge $uv \in E(G)$, if $\min\{d_G(u), d_G(v)\} \le \left\lfloor \frac{M+2p-2}{2p} \right\rfloor$, then $d_G(u) + d_G(v) \ge M+2$, and otherwise, $d_G(u) + d_G(v) \ge M - 2p + 3$.*

A *k-alternator* of G for some $2 \le k \le \left\lfloor \frac{\Delta}{2} \right\rfloor$ is a bipartite subgraph F of G with partite sets X, Y such that $d_F(x) = d_G(x) \le k$ for each $x \in X$, and $d_F(y) \ge d_G(y) + k - \Delta$ for each $y \in Y$.

Lemma 6 [21, Lemma 2.4]**.** *If G is an $(M+2p-2)$-critical graph, then there is no k-alternator in G for any integer $2 \le k \le \left\lfloor \frac{M+2p-2}{2p} \right\rfloor$.*

Using Lemmas 5 and 6, we conclude the following two corollaries.

Corollary 7. *If G is an $(M+2p-2)$-critical graph with $M \ge 4p+6$ and $p \ge 2$, then*

(a) *$d_G(x) + d_G(y) \ge 13$ for every edge $xy \in E(G)$;*

(b) *$d_G(x) + d_G(y) \ge 16$ for every edge $xy \in E(G)$ with $\min\{d_G(x), d_G(y)\} \le 3$;*

(c) *there is no 2-alternator or 3-alternator in G.*

Proof. By Lemma 5, $d_G(x) + d_G(y) \ge M - 2p + 3 \ge 2p + 9 \ge 13$ for every edge $xy \in E(G)$. Since $\left\lfloor \frac{M+2p-2}{2p} \right\rfloor \ge \left\lfloor \frac{6p+4}{2p} \right\rfloor \ge 3$, $d_G(x)+d_G(y) \ge M+2 \ge 4p+8 \ge 16$ for every edge $xy \in E(G)$ with $\min\{d_G(x), d_G(y)\} \le 3$ by Lemma 5, and there is no 2-alternator or 3-alternator in G by Lemma 6. ■

Corollary 8. *If G is an $(M+2p-2)$-critical graph with $M \ge 6p+7$ and $p \ge 2$, then*

(a) *$d_G(x) + d_G(y) \ge 18$ for every edge $xy \in E(G)$;*

(b) *$d_G(x) + d_G(y) \ge 21$ for every edge $xy \in E(G)$ with $\min\{d_G(x), d_G(y)\} \le 4$;*

(c) *there is no 2-alternator or 3-alternator or 4-alternator in G.*

Proof. By Lemma 5, $d_G(x) + d_G(y) \ge M - 2p + 3 \ge 4p + 10 \ge 18$ for every edge $xy \in E(G)$. Since $\left\lfloor \frac{M+2p-2}{2p} \right\rfloor \ge \left\lfloor \frac{8p+5}{2p} \right\rfloor \ge 4$, $d_G(x)+d_G(y) \ge M+2 \ge 6p+9 \ge 21$ for every edge $xy \in E(G)$ with $\min\{d_G(x), d_G(y)\} \le 4$ by Lemma 5, and there is no 2-alternator or 3-alternator or 4-alternator in G by Lemma 6. ■

The following Theorems 9 and 10 are two structural theorems on 1-planar graphs. We move their proofs to the next section.

Theorem 9. *If G is a triangle-free 1-planar graph with minimum degree at least 2, then G contains*

(a) *an edge xy with $d_G(x) + d_G(y) \le 12$, or*

(b) *an edge* xy *with* $\min\{d_G(x), d_G(y)\} \leq 3$ *and* $d_G(x) + d_G(y) \leq 14$*, or*
(c) *a* k*-alternator for some* $k \in \{2, 3\}$*.*

Proof. See Subsection 3.1. ■

Theorem 10. *If* G *is a* 1*-planar graph with minimum degree at least* 2*, then* G *contains*

(a) *an edge* xy *with* $d_G(x) + d_G(y) \leq 16$*, or*
(b) *an edge* xy *with* $\min\{d_G(x), d_G(y)\} \leq 4$ *and* $d_G(x) + d_G(y) \leq 20$*, or*
(c) *a* k*-alternator for some* $k \in \{2, 3, 4\}$*.*

Proof. See Subsection 3.2. ■

Combining Theorem 9 with Corollary 7, and Theorem 10 with Corollary 8, we conclude the following two corollaries, respectively.

Conjecture 11. *There is no* $(M + 2p - 2)$*-critical triangle-free* 1*-planar graph with* $M \geq 4p + 6$ *and* $p \geq 2$*.*

Proof. If G is an $(M + 2p - 2)$-critical triangle-free 1-planar graph, then by Lemma 5, $\delta(G) \geq 2$. By Theorem 9, G contains (i) an edge xy with $d_G(x) + d_G(y) \leq 12$, or (ii) an edge xy with $\min\{d_G(x), d_G(y)\} \leq 3$ and $d_G(x) + d_G(y) \leq 14$, or (iii) a k-alternator for some $k \in \{2, 3\}$. However, the local structures (i) and (ii) cannot appear in G by Corollaries 7(a) and 7(b), and the local structure (iii) is absent form G by Corollary 7(c). ■

Conjecture 12. *There is no* $(M+2p-2)$*-critical* 1*-planar graph with* $M \geq 6p+7$ *and* $p \geq 2$*.*

Proof. If G is an $(M+2p-2)$-critical 1-planar graph, then by Lemma 5, $\delta(G) \geq 2$. By Theorem 10, G contains (i) an edge xy with $d_G(x) + d_G(y) \leq 16$, or (ii) an edge xy with $\min\{d_G(x), d_G(y)\} \leq 4$ and $d_G(x) + d_G(y) \leq 20$, or (iii) a k-alternator for some $k \in \{2, 3, 4\}$. However, the local structures (i) and (ii) cannot appear in G by Corollaries 8(a) and 8(b), and the local structure (iii) is absent form G by Corollary 8(c). ■

Corollaries 11 and 12 imply that there is no counterexample to Theorem 4. Hence Theorem 4 has been proved.

3. The Proofs of Theorems 9 and 10

The *associated plane graph* $G^\times$ of a 1-plane graph G is the plane graph obtained from G by turning all crossings of G into new vertices of degree four. These new vertices are *false* vertices in $G^{\times}$, and the original vertices of G are *true* vertices in $G^{\times}$. A face in $G^{\times}$ is *false* if it is incident with at least one false vertex, and *true* otherwise.

We will mainly use the *discharging method* to prove Theorems 9 and 10. As we know, the discharging method is a technique used to prove theorems in structural graph theory. Discharging is most well-known for its central role in the proof of the Four Color Theorem. The discharging method is used to prove that every graph in a certain class contains some subgraph from a specified list (see the statements of Theorems 9 and 10 for examples). The presence of the desired subgraph is then often used to prove a coloring result, which likes what we have done in Section 2. Most commonly, discharging is applied to plane graphs, so in this paper the discharging will be applied to the associated plane graph $G^{\times}$ of G.

Lemma 13 [15, Lemma 2.4]**.** *Let* $2 \le k \le \lfloor \frac{\Delta}{2} \rfloor$ *be a fixed integer and let* G *be a graph without* k*-alternator. Let* $X_k = \{x \in V(G) \mid d_G(x) \le k\}$ *and* $Y_k = \bigcup_{x \in X_k} N_G(x)$*. If* $X_k \neq \emptyset$*, then there exists a bipartite subgraph* M_k *of* G *with partite sets* X_k, Y_k *such that* $d_{M_k}(x) = 1$ *for each* $x \in X_k$ *and* $d_{M_k}(y) \le k-1$ *for each* $y \in Y_k$*.*

Following Lemma 13, we call y the k*-master* of x if $xy \in M_k$ and $x \in X_k$.

***Sketch proofs of Theorems* 9 *and* 10.** Suppose, to the contrary, that G is a minimal counterexample to Theorem 9 (respectively Theorem 10) in terms of $|V(G)| + |E(G)|$. If $\Delta(G) \le 6$ (respectively $\Delta(G) \le 8$), then $d_G(x) + d_G(y) \le 2\Delta(G) \le 12$ (respectively $d_G(x)+d_G(y) \le 2\Delta(G) \le 16$) for any edge $xy \in E(G)$, and thus the configuration (a) occurs. Hence $\Delta(G) \ge 7$ (respectively $\Delta(G) \ge 9$).

We apply the discharging method to the associated plane graph $G^{\times}$ of G. Formally, for each vertex $v \in V(G^{\times})$, let $c(v) = d_{G^{\times}}(v) - 6$ be its initial charge, and for each face $f \in F(G^{\times})$, let $c(f) = 2d_{G^{\times}}(f) - 6$ be its initial charge. Clearly,

$$\begin{aligned}\sum_{x \in V(G^{\times}) \cup F(G^{\times})} c(x) &= \sum_{x \in V(G^{\times})} \big(d_{G^{\times}}(v) - 6\big) + \sum_{x \in F(G^{\times})} \big(2d_{G^{\times}}(f) - 6\big) \\ &= 6\big(|E(G^{\times})| - |V(G^{\times})| - |F(G^{\times})|\big) \\ &= -12 < 0\end{aligned}$$

by the well-known Euler's formula.

At this stage, we need the so-called *discharging rules* that only move charge around the vertices and faces of $G^{\times}$ so that $c'(x) \ge 0$ for every $x \in V(G^{\times}) \cup F(G^{\times})$, where $c'(x)$ denotes the charge of $x \in V(G^{\times}) \cup F(G^{\times})$ after applying the discharging rules. This therefore implies

$$\sum_{x \in V(G^{\times}) \cup F(G^{\times})} c'(x) = \sum_{x \in V(G^{\times}) \cup F(G^{\times})} c(x) < 0$$

a contradiction. ■

The idea to define the discharging rules. From the above sketch proofs, one can find that the most important and complicated task is to define the discharging rules. It is easy to see that $c(x) < 0$, where $x \in V(G^\times) \cup F(G^\times)$, only if x is a 5^--vertex of $G^\times$ (hence it may be a false 4-vertex). Hence 5^--vertices need charges, which may be obtained from elements from $V(G^\times) \cup F(G^\times)$ that have "rich" initial charge. Such rich elements includes 7^+-vertices and 4^+-faces.

If G is a minimal counterexample to Theorem 9, then 2-alternator or 3-alternator is absent from G. We conclude, by Lemma 13, that (i) every 3^--vertex of G has a 3-master, and (ii) every vertex of G can be 3-masters of at most two vertices. If a vertex u is a 3-master of a 3^--vertex v, then $uv \in E(G)$ and thus $d_{G^\times}(u) = d_G(u) \geq 12$, because otherwise $d_G(u) + d_G(v) \leq 3 + 11 = 14$ and the configuration (b) occurs. Hence it is natural to propose the following requests while defining the discharging rules:

(1) any 5^--vertex receives charge from rich elements including 7^+-vertices and 4^+-faces;

(2) any 3^--vertex receives additional charge from its adjacent 3-master, which is a 12^+-vertex.

If G is a minimal counterexample to Theorem 10, then G does not have 2-alternator or 3-alternator or 4-alternator. We conclude, by Lemma 13, that (i) every 2-vertex of G has a 2-master, a 3-master and a 4-master, (ii) every 3-vertex of G has a 3-master and a 4-master, (iii) every 4-vertex of G has a 4-master, and (iv) every vertex of G can be 2-master of at most one vertex, 3-masters of at most two vertices, and 4-masters of at most three vertices. If a vertex u is a 2-master of a 2-vertex v, then $uv \in E(G)$ and thus $d_{G^\times}(u) = d_G(u) \geq 19$, because otherwise $d_G(u)+d_G(v) \leq 2+18 = 20$ and the configuration (b) occurs. If a vertex u is a 3-master of a 3^--vertex v, then $uv \in E(G)$ and thus $d_{G^\times}(u) = d_G(u) \geq 18$, because otherwise $d_G(u)+d_G(v) \leq 3+17 = 20$ and the configuration (b) occurs. If a vertex u is a 4-master of a 4^--vertex v, then $uv \in E(G)$ and thus $d_{G^\times}(u) = d_G(u) \geq 17$, because otherwise $d_G(u) + d_G(v) \leq 4 + 16 = 20$ and the configuration (b) occurs. Hence it is natural to propose the following requests while defining the discharging rules:

(1) any 5^--vertex receives charge from rich elements including 7^+-vertices and 4^+-faces;

(2) any 4^--vertex receives additional charge from its adjacent 4-master, which is a 17^+-vertex;

(3) any 3^--vertex receives additional charge from its adjacent 3-master, which is a 18^+-vertex;

(4) any 2-vertex receives additional charge from its adjacent 2-master, which is a 19^+-vertex.

However, this is just the idea to define the discharging rules but it is not meticulous enough. Actually, in order to define the discharging rules precisely, careful analysis on the local structures of the associated plane graph $G^\times$ and the original graph G are necessary. For example, the following lemma will be used frequently latter.

Lemma 14 [19, Lemma 1]**.** *If G is a 1-plane graph, then*

(a) *false vertices in $G^\times$ are not adjacent;*

(b) *false 3-face in $G^\times$ is not incident with 2-vertex;*

(c) *if a 3-vertex v is incident with two 3-faces and adjacent to two false vertices in $G^\times$, then v is incident with a 5^+-face;*

(d) *there exists no edge uv in $G^\times$ such that $d_{G^\times}(u) = 3$, v is a false vertex, and uv is incident with two 3-faces.*

The detailed discharging rules will be released in the next subsections.

3.1. Detailed proof of Theorem 9

Let G be a minimal counterexample to Theorem 9 in terms of $|V(G)| + |E(G)|$. A true vertex v of $G^\times$ is *big* if $d_{G^\times}(v) \geq 8$, *middle* if $6 \leq d_{G^\times}(v) \leq 7$, and *small* if $d_{G^\times}(v) \leq 5$. By the absence of the configuration (a), any neighbor of a small vertex in $G^\times$ is a big vertex. For convenience, we use F, B, M and S to represent false vertex, big vertex, middle vertex and small vertex, respectively, and then use these notations to represent the structure of a face of $G^\times$. For example, we say that a face is an (F, M, B, S)-face if it is a 4-face with vertices u_1, u_2, u_3 and u_4 lying clockwise on the boundary of f such that u_1 is false, u_2 is middle, u_3 is big and u_4 is small. A face in $G^\times$ is *burdened* if it is incident with at least one small vertex.

A *special* burdened 4-face in $G^\times$ is a 4-face of type $(F, S, F, *)$, where $*$ represents either B or M or S, see Figure 1, where hollow vertices represent false vertices. A false vertex v in $G^\times$ is *BFS-incident* with a face f if one neighbor of v in the cycle induced by f is big, and the other is small. In this case, we also say that f is *BFS-incident* with a false vertex v. Similarly, we can define "BFB-incident", "SFM-incident", etc.

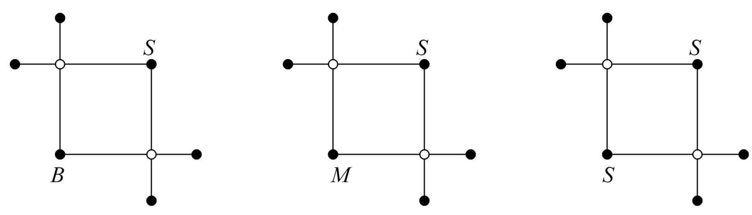

Figure 1. Three kinds of special burdened 4-face.

The promised discharging rules are defined as follows.

R1. Every big vertex of $G^{\times}$ sends $\frac{1}{4}$ to each of its incident faces.

R2. Let v be a false vertex incident with a 4^{+}-face f.

R2.1. If v is SFS-incident with f, then f sends $\frac{1}{4}$ to v.

R2.2. If v is BFB-incident with f, then f sends $\frac{5}{4}$ to v.

R2.3. If v is BFS-incident with f, then f sends $\frac{3}{4}$ to v.

R2.4. If v is BFM-incident with f, then f sends $\frac{3}{4}$ to v.

R2.5. If v is MFM-incident with f, then f sends 1 to v.

R2.6. If v is SFM-incident with f, then f sends $\frac{5}{8}$ to v.

R3. Every 3-face of $G^{\times}$ sends all of its received charge after applying R1 to its incident false vertex.

R4. Every 4^{+}-face of $G^{\times}$ redistributes it remaining charge after applying R1 and R2 equitably to each of its incident small vertices (if exists).

R5. Every 3^{-}-vertex of G receives $\frac{3}{2}$ from its 3-master.

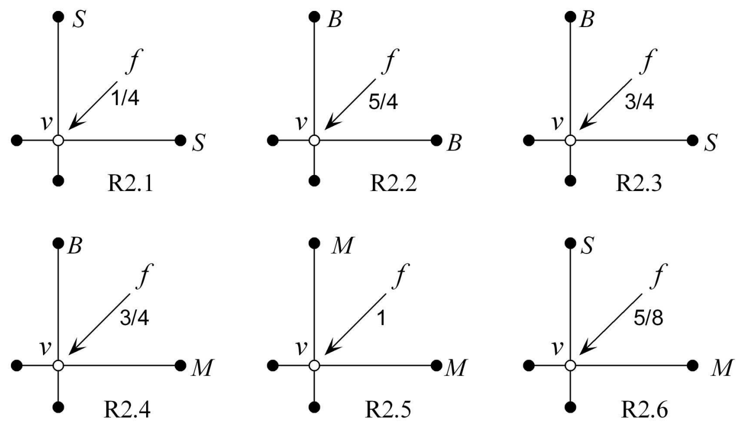

Figure 2. The discharging rule R2 for the proof of Theorem 9.

In the following, we check $c'(x) \geq 0$ for every $x \in V(G^{\times}) \cup F(G^{\times})$, and then complete the proof.

Since every 4^{+}-face f of $F(G^{\times})$ is incident with at most $\left\lfloor \frac{d_{G^{\times}}(f)}{2} \right\rfloor$ false vertices by Lemma 14(a), the remaining charge of f after applying R1 and R2 is at least

$$2d_{G^{\times}}(f) - 6 - \max\left\{\frac{1}{4}, \frac{3}{4}, 1, \frac{5}{8}\right\} \times \left\lfloor \frac{d_{G^{\times}}(f)}{2} \right\rfloor \geq 0$$

if R2.2 is not applied to f, and is at least

$$2d_{G^{\times}}(f) - 6 + 2 \times \frac{1}{4} - \frac{5}{4} \times \left\lfloor \frac{d_{G^{\times}}(f)}{2} \right\rfloor \geq 0$$

otherwise. Hence, R1–R4 guarantee that $c'(f) \geq 0$ for each $f \in F(G^{\times})$.

Before calculating the final charge of each vertex $v \in V(G^{\times})$, we count first how many charge v can receive from its incident faces if v is false or small.

By R1 and R3, it is easy to conclude the following two claims, respectively.

Claim 15. *Every (B, F, B)-face sends $\frac{1}{2}$ to its incident false vertex.*

Claim 16. *Every (B, F, M)-face or (B, F, S)-face sends $\frac{1}{4}$ to its incident false vertex.*

Now we consider burdened 4^{+}-faces.

Claim 17. *Every burdened 4-face sends to each of its incident small vertices $\frac{3}{4}$ if f is a special 4-face, 1 if f is an (F, S, B, S)-face, and at least $\frac{5}{4}$ otherwise.*

Proof. If f is an (F, S, F, S)-face, then by R2.1 and R4, f sends

$$\frac{1}{2} \times \left(2 \times 4 - 6 - 2 \times \frac{1}{4}\right) = \frac{3}{4}$$

to each of its incident small vertices. If f is an (F, S, F, M)-face, then by R2.6 and R4, f sends

$$2 \times 4 - 6 - 2 \times \frac{5}{8} = \frac{3}{4}$$

to its incident small vertex. If f is an (F, S, F, B)-face, then by R1, R2.3 and R4, f sends

$$2 \times 4 - 6 + \frac{1}{4} - 2 \times \frac{3}{4} = \frac{3}{4}$$

to its incident small vertex. If f is an (F, S, B, S)-face, then by R1, R2.1 and R4, f sends

$$\frac{1}{2} \times \left(2 \times 4 - 6 + \frac{1}{4} - \frac{1}{4}\right) = 1$$

to each of its incident small vertices.

By symmetry, f can be of another types among $(S, B, B, B), (S, B, S, B)$, $(F, S, B, M), (F, B, S, B), (S, B, M, B)$ and (F, S, B, B). In each case we can similarly calculate that f sends at least $\frac{5}{4}$ to each of its incident small vertices. □

Claim 18. *Every burdened 5-face sends at least $\frac{13}{8}$ to each of its incident small vertices.*

Proof. Note that f is incident with at most two false vertices and at most two small vertices.

If f is incident with exactly one small vertex, then f sends at least

$$2 \times 5 - 6 - 2 \times 1 = 2$$

to this small vertex by R2 and R4.

If f is incident with exactly two small vertices and at most one false vertex, then f sends at least

$$\frac{1}{2} \times \left(2 \times 5 - 6 + \frac{1}{4} - 1\right) = \frac{13}{8}$$

to each of its incident small vertices by R1, R2 and R4. Note that in this case f is incident with at least one big vertex.

If f is incident with exactly two small vertices and exactly two false vertices, then f shall be an (F, S, B, F, S)-face. Therefore, by R1, R2.1, R2.3 and R4, f sends at least

$$\frac{1}{2} \times \left(2 \times 5 - 6 + \frac{1}{4} - \frac{1}{4} - \frac{3}{4}\right) = \frac{13}{8}$$

to each of its incident small vertices. □

Claim 19. *Every burdened* 6^+*-face sends at least* $\frac{7}{4}$ *to each of its incident small vertices.*

Proof. First, suppose that f is not BFB-incident or MFM-incident with any false vertex. If f is incident with at most $\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor - 1$ small vertices, then f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \frac{3}{4}\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor}{\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor - 1} \geq \frac{15}{8} > \frac{7}{4}$$

to each of its incident small vertices by R2 and R4, for $d_{G^\times}(G) \geq 6$. If f is incident with $\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor$ small vertices and $d_{G^\times}(v)$ is even, then any false vertex on f is SFS-incident with f, thus f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \frac{1}{4}\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor}{\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor} \geq \frac{7}{4}$$

to each of its incident small vertices by R2.1 and R4, for $d_{G^\times}(G) \geq 6$. If f is incident with $\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor$ small vertices and $d_{G^\times}(v)$ is odd, that is, $d_{G^\times}(v) \geq 7$, then f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \frac{3}{4}\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor}{\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor} \geq \frac{7}{4}$$

to each of its incident small vertices by R2 and R4.

Second, suppose that f is BFB-incident with a false vertex. In this case, f is incident with at most $\left\lceil \frac{d_{G^\times}(v)-3}{2} \right\rceil$ small vertices, and thus f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \frac{5}{4}\left(\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor - 1\right) + 2 \times \frac{1}{4}}{\left\lceil \frac{d_{G^\times}(v)-3}{2} \right\rceil} \geq 2 > \frac{7}{4}$$

to each of its incident small vertices by R1, R2 and R4 provided that f is incident with at most $\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor - 1$ false vertices. If f is incident with exactly $\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor$ false vertices and at most $\left\lceil \frac{d_{G^\times}(v)-3}{2} \right\rceil - 1$ small vertices, then f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \frac{5}{4}\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor + 2 \times \frac{1}{4}}{\left\lceil \frac{d_{G^\times}(v)-3}{2} \right\rceil - 1} \geq \frac{11}{4} > \frac{7}{4}$$

to each of its incident small vertices by R1, R2 and R4. If f is incident with exactly $\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor$ false vertices and exactly $\left\lceil \frac{d_{G^\times}(v)-3}{2} \right\rceil$ small vertices, then

$$\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor + \left\lceil \frac{d_{G^\times}(v) - 3}{2} \right\rceil + 2 \leq d_{G^\times}(v)$$

implying that $d_{G^\times}(v) \geq 7$. Note that f is incident with at least two big vertices. Now, we conclude that f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \frac{5}{4}\left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor + 2 \times \frac{1}{4}}{\left\lceil \frac{d_{G^\times}(v)-3}{2} \right\rceil} \geq \frac{11}{6} > \frac{7}{4}$$

to each of its incident small vertices by R1, R2 and R4, for $d_{G^\times}(v) \geq 7$.

Third, suppose that f is MFM-incident with a false vertex and not BFB-incident with any false vertex. In this case, f is incident with at most $\left\lceil \frac{d_{G^\times}(v)-5}{2} \right\rceil$ small vertices (note that any neighbor of a middle vertex cannot be a small vertex), and thus f sends at least

$$\frac{2d_{G^\times}(f) - 6 - \left\lfloor \frac{d_{G^\times}(v)}{2} \right\rfloor}{\left\lceil \frac{d_{G^\times}(v)-5}{2} \right\rceil} \geq 3 > \frac{7}{4}$$

to each of its incident small vertices by R1, R2 and R4, for $d_{G^\times}(f) \geq 6$. □

Now we calculate the final charge of each vertex $v \in V(G^\times)$.

Case 1. If v is a false vertex, then v is incident with at most two 3-faces, because otherwise a triangle occurs in G.

(1.1) If v is incident only with 4^+-faces, then v is SFS-incident with at most one face, since small vertices are not adjacent in G. Therefore, by R2, we have

$$c'(v) \geq 4 - 6 + \frac{1}{4} + 3 \times \min\left\{\frac{5}{4}, \frac{3}{4}, 1, \frac{5}{8}\right\} > 0.$$

(1.2) If v is incident with one 3-face and three 4^+-faces, we distinguish two subcases.

If v is incident with a $(B, F, *)$-face, where $*$ represents B or M or S, then v is SFS-incident with at most one 4^+-face, since small vertices are not adjacent in G. If v is SFS-incident with exactly one 4^+-face, then either v is BFS-incident with a 4^+-face and BFB-incident with a 4^+-face, or BFS-incident with two 4^+-faces and incident with a (B, F, B)-face. In the former case, we have

$$c'(v) \geq 4 - 6 + \frac{3}{4} + \frac{5}{4} = 0$$

by R2.2 and R2.3, and in the latter case, we conclude

$$c'(v) \geq 4 - 6 + \frac{1}{2} + \frac{1}{4} + 2 \times \frac{3}{4} > 0$$

by Claim 15, R2.1 and R2.3. If v is not SFS-incident with any 4^+-face, then by R2.2–R2.6, we have

$$c'(v) \geq 4 - 6 + 2 \times \frac{5}{8} + \min\left\{\frac{5}{4}, \frac{3}{4}, 1\right\} = 0.$$

Note that in this case v is SFM-incident with at most two faces (otherwise two small vertices are adjacent in G).

If v is incident with an (M, F, M)-face, then v is not SFS-incident or SFM-incident with any 4^+-face. Hence by R2.2–R2.5,

$$c'(v) \geq 4 - 6 + 3 \times \min\left\{\frac{5}{4}, \frac{3}{4}, 1\right\} > 0.$$

(1.3) If v is incident with two 3-faces and two 4^+-faces, then the two 3-faces are not adjacent in $G^\times$, because otherwise G contains a triangle.

If v is adjacent to at least two big vertices in $G^\times$, then those big vertices lie on the 3-faces that are incident with v. If v is BFB-incident with a 4^+-face, then

$$c'(v) \geq 4 - 6 + 2 \times \frac{1}{4} + \frac{5}{4} + \frac{1}{4} = 0$$

by R1, R2.1, R2.2 and R3. If v is not BFB-incident with any 4^+-face, then v is BF*-incident with a 4^+-face and BF#-incident with another 4^+-face, where * and # represent M or S. Therefore,

$$c'(v) \geq 4 - 6 + 2 \times \frac{1}{4} + 2 \times \frac{3}{4} = 0$$

by R1, R2.1, R2.2 and R3.

If v is adjacent to at most one big vertex in $G^\times$, then v is not adjacent to any small vertex in $G^\times$, since the neighbor of any small vertex in $G^\times$ is a big vertex. Now, if v is not adjacent to any big vertex in $G^\times$, then v is MFM-incident with two 4^+-faces, which implies

$$c'(v) \geq 4 - 6 + 2 \times 1 = 0$$

by R2.5. If v is adjacent to exactly one big vertex in $G^\times$, then v is incident with a (B, F, M)-face, BFM-incident with one 4^+-face and MFM-incident with one 4^+-face. This implies, by Claim 16, R2.4 and R2.5, that

$$c'(v) \geq 4 - 6 + \frac{1}{4} + \frac{3}{4} + 1 = 0.$$

Case 2. If v is a 2-vertex, then v is not incident with any 3-face by Lemma 14(b) and by the fact that G is triangle-free.

(2.1) If v is incident only with 5^+-faces, then by Claims 18, 19 and R5, we have

$$c'(v) \geq 2 - 6 + \frac{13}{8} \times 2 + \frac{3}{2} > 0.$$

(2.2) If v is incident with one 4-face and one 6^+-face, then by Claims 17, 19 and R5, we have

$$c'(v) \geq 2 - 6 + \frac{3}{4} + \frac{7}{4} + \frac{3}{2} = 0.$$

(2.3) If v is incident with one 4-face and one 5-face, then the 4-face incident with v is a non-special 4-face (otherwise a triangle occurs in G). Therefore, by Claims 17, 18 and R5 we have

$$c'(v) \geq 2 - 6 + 1 + \frac{13}{8} + \frac{3}{2} > 0.$$

(2.4) If v is incident with two 4-faces, then none of the two 4-faces incident with v is a special 4-face or an (F, S, B, S)-face (otherwise a multi-edge or a triangle appears in G). This implies, by Claim 17 and R5, that

$$c'(v) \geq 2 - 6 + 2 \times \frac{5}{4} + \frac{3}{2} = 0.$$

Case 3. If v is a 3-vertex, then v is incident with at most two 3-faces since G is triangle-free. In other words, v is incident with at least one 4^+-faces in $G^\times$.

(3.1) If v is incident with exactly one 4^+-face, then by Lemmas 14(c) and 14(d), this 4^+-face shall be a 5^+-face. Hence

$$c'(v) \geq 3 - 6 + \frac{13}{8} + \frac{3}{2} > 0$$

by Claims 18, 19 and R5.

(3.2) If v is incident with at least two 4^+-faces in $G^\times$, then

$$c'(v) \geq 3 - 6 + 2 \times \frac{3}{4} + \frac{3}{2} = 0$$

by Claims 17, 18, 19 and R5.

Case 4. If v is a true 4-vertex, then v is incident with at most two 3-faces since G is triangle-free. In other words, v is incident with at least two 4^+-faces in $G^\times$.

(4.1) If v is incident with at least one 5^+-face, then

$$c'(v) \geq 4 - 6 + \frac{3}{4} + \frac{13}{8} > 0$$

by Claims 17, 18 and 19.

(4.2) If v is incident with at least three 4-faces, then

$$c'(v) \geq 4 - 6 + 3 \times \frac{3}{4} > 0$$

by Claim 17.

(4.3) If v is incident with exactly two 4-faces, then the two 4-faces incident with v cannot be both special 4-faces (otherwise a triangle appears in G).

If none of them is a special 4-face, then by Claim 17, we have

$$c'(v) \geq 4 - 6 + 2 \times 1 = 0.$$

If one of them is a special 4-face, then v is not incident with (F, S, B, S)-face (otherwise a triangle occurs in G or v is incident with three 4-faces). This implies, by Claim 17, that

$$c'(v) \geq 4 - 6 + \frac{3}{4} + \frac{5}{4} = 0.$$

Case 5. If v is a 5-vertex, then v is incident with at most three 3-faces in $G^\times$ since G is triangle-free. Therefore, v is incident with at least two 4^+-faces, and thus

$$c'(v) \geq 5 - 6 + 2 \times \frac{3}{4} > 0$$

by Claims 17, 18 and 19.

Case 6. If v is a vertex of degree between 6 and 11, then by the absence of the configuration (b), v cannot be a master of any 3^--vertex. If v is a 6-vertex or a 7-vertex, then v does not give out any charge by R1–R5, and thus

$$c'(v) = c(v) = d_{G^\times}(v) - 6 \geq 0.$$

If v is a big vertex, that is, $d_{G^\times}(v) \geq 8$, then by R1,

$$c'(v) \geq d_{G^\times}(v) - 6 - \frac{1}{4}d_{G^\times}(v) = \frac{1}{4} \times \big(3d_{G^\times}(v) - 24\big) \geq 0.$$

Case 7. If v is a 12^+-vertex, then v can be 3-masters of at most two vertices. By R1 and R5,

$$c'(v) \geq d_{G^\times}(v) - 6 - \frac{1}{4}d_{G^\times}(v) - 2 \times \frac{3}{2} = \frac{1}{4} \times \big(3d_{G^\times}(v) - 36\big) \geq 0.$$

3.2. Detailed proof of Theorem 10

Let G be a minimal counterexample to Theorem 10 in terms of $|V(G)| + |E(G)|$. A true vertex v of $G^\times$ is *big* if $d_{G^\times}(v) \geq 12$, is *middle* if $6 \leq d_{G^\times}(v) \leq 11$, and is *small* if $d_{G^\times}(v) \leq 5$. A middle vertex is an M^{9+}*-vertex* if $9 \leq d_{G^\times}(v) \leq 11$, and is an M^{8-}*-vertex* if $6 \leq d_{G^\times}(v) \leq 8$. By the absence of the configuration (a), any neighbor of a small vertex in $G^\times$ is a big vertex. For convenience, we also use F, B, M and S to represent false vertex, big vertex, middle vertex and small vertex, respectively, and then use these notations to represent the structure of a face of $G^\times$. For example, we say that a face is an (F, M, B, S)-face if it is a 4-face with vertices u_1, u_2, u_3 and u_4 lying clockwise on the boundary of f such that u_1 is false, u_2 is middle, u_3 is big and u_4 is small. A face in $G^\times$ is *burdened* if it is incident with at least one small vertex.

A *special* burdened 4-face in $G^\times$ is a 4-face of type $(F, S, F, *)$, where $*$ represents either B or M or S, see Figure 1. A false vertex v in $G^\times$ is *BFS-incident* with a face f if one neighbor of v in the cycle induced by f is big, and the other is small. In this case, we also say that f is *BFS-incident* with a false vertex v. Similarly, we can define “BFB-incident”, “SFM-incident”, etc.

The promised discharging rules are defined as follows.

R1. Every M^{9+}-vertex of $G^\times$ sends $\frac{1}{3}$ to each of its incident faces.

R2. Every big vertex of $G^\times$ sends $\frac{1}{2}$ to each of its incident faces.

R3. Let v be a false vertex incident with a 4^+-face f.

R3.1. If v is BFB-incident with f, then f sends $\frac{3}{2}$ to v.

R3.2. If v is BFS-incident with f, then f sends $\frac{1}{2}$ to v.

R3.3. If v is BFM-incident with f, then f sends $\frac{2}{3}$ to v.

R3.4. If v is MFM-incident with f, then f sends $\frac{2}{3}$ to v.

R3.5. If v is SFM-incident with f, then f sends $\frac{1}{2}$ to v.

R4. Every false 3-face of $G^\times$ sends all of its received charge after applying R1 and R2 to its incident false vertex.

R5. Every true 3-face of $G^\times$ sends all of its received charge after applying R2 to its incident small vertex (if exists).

R6. Every 4^+-face of $G^\times$ redistributes it remaining charge after applying R1, R2 and R3 equitably to each of its incident small vertices (if exists).

R7. Every 2-vertex of G receives $\frac{1}{2}, \frac{1}{4}$ and $\frac{3}{4}$ from its 2-master, 3-master, and 4-master, respectively.

R8. Every 3-vertex of G receives $\frac{1}{4}$ and $\frac{3}{4}$ from its 3-master and 4-master, respectively.

R9. Every 4-vertex of G receives $\frac{3}{4}$ from its 4-master.

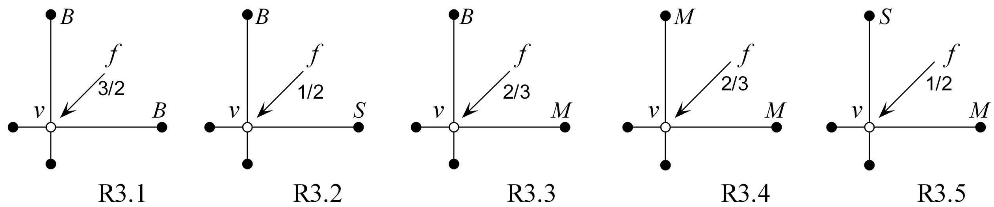

Figure 3. The discharging rule R3 for the proof of Theorem 10.

In the following, we check $c'(x) \geq 0$ for every $x \in V(G^\times) \cup F(G^\times)$, and then complete the proof.

Since every 4^+-face f of $F(G^\times)$ is incident with at most $\left\lfloor \frac{d_{G^\times}(f)}{2} \right\rfloor$ false vertices by Lemma 14(a), the charge of f after applying R2 is at least

$$2d_{G^\times}(f) - 6 - \max\left\{\frac{3}{2}, \frac{1}{2}, \frac{2}{3}\right\} \times \left\lfloor \frac{d_{G^\times}(f)}{2} \right\rfloor > 0$$

for $d_{G^\times}(f) \geq 5$.

If $d_{G^\times}(f) = 4$ and f is BFB-incident with a false vertex, then f is incident with at least two big vertices and thus

$$c'(f) \geq 2 \times 4 - 6 + 2 \times \frac{1}{2} - 2 \times \max\left\{\frac{3}{2}, \frac{1}{2}, \frac{2}{3}\right\} = 0$$

by R2 and R3.

If $d_{G^\times}(f) = 4$ and f is not BFB-incident with any false vertex, then

$$c'(f) \geq 2 \times 4 - 6 - 2 \times \max\left\{\frac{1}{2}, \frac{2}{3}\right\} > 0$$

by R3. Hence, R1–R6 guarantee that $c'(f) \geq 0$ for each $f \in F(G^\times)$.

Claim 20. *Let v be a false vertex on a face $f \in F(G^\times)$ so that the neighbors of v in the cycle induced by f are u and w (note that $uw \notin E(G)$).*

(1) *If at least one of u and w is a big vertex, then f sends at least $\frac{1}{2}$ to v.*

(2) *If both u and w are big vertices, then f sends at least 1 to v.*

(3) *If u is a middle vertex and w is a small vertex, then f sends $\frac{1}{2}$ to v.*

(4) *If u is an M^{9+}-vertex and w is a big vertex, then f sends at least $\frac{2}{3}$ to v.*

(5) *Let wy be a crossed edge of G so that v is a crossing on wy and let f' be the face of $G^\times$ incident with the path uvy. If u, w, y are middle vertices, then f and f' totally send at least 1 to v unless $uw, uy \in E(G)$, u is an M^{8-}-vertex, w and y are M^{9+}-vertices, in which case f and f' totally send $\frac{2}{3}$ to v.*

Proof. (1) If f is a 3-face, then f sends at least $\frac{1}{2}$ to v by R2 and R4. If f is a 4^+-face, then f sends at least $\frac{1}{2}$ to v by R3.1, R3.2 and R3.3.

(2) If f is a 3-face, then f sends $2 \times \frac{1}{2} = 1$ to v by R2 and R4. If f is a 4^+-face, then f sends $\frac{3}{2}$ to v by R3.1.

(3) Since small vertex is not adjacent to middle vertex in $G^\times$, f is a 4^+-face. By R3.5, f sends $\frac{1}{2}$ to v.

(4) If f is a 3-face, then f sends $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ to v by R1, R2 and R4. If f is a 4^+-face, then f sends $\frac{2}{3}$ to v by R3.3.

(5) If f and f' are 4^+-faces, then each of f and f' sends $\frac{2}{3}$ to v by R3.4, and thus they totally send $2 \times \frac{2}{3} = \frac{4}{3}$ to v. If f is a 4^+-face and f' is a 3-face (the case that f is a 3-face and f' is a 4^+-face can be similarly discussed), then $uy \in E(G)$, which implies that u or y is an M^{9+}-vertex, because 8^--vertices are not adjacent in G by the absence of the local configuration (a). By R1 and R4, f' sends at least $\frac{1}{3}$ to v, and f sends $\frac{2}{3}$ to v by R3.4. Hence f and f' totally send at least $\frac{1}{3} + \frac{2}{3} = 1$ to v. If f and f' are 3-faces, then uwy is a triangle in G. Since 8^--vertices are not adjacent in G, w or y, say w, is an M^{9+}-vertex. If u is an M^{9+}-vertex, then f sends $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ to v by R1 and R4, and f' sends at least $\frac{1}{3}$ to v by R1 and R4. Hence f and f' totally send at least $\frac{2}{3} + \frac{1}{3} = 1$ to v. If u is an M^{8-}-vertex, then y is an M^{9+}-vertex. By R1 and R4, each of f and f' sends $\frac{1}{3}$ to v. Hence f and f' totally send $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ to v. □

By R2 and R5, we immediately conclude the following result. Note that small vertices are adjacent only to big vertices in G.

Claim 21. *Every burdened true 3-face sends 1 to its incident small vertex.*

Now we consider burdened 4^+-faces.

Claim 22. *Every burdened* 4*-face sends to each of its incident small vertices* 1 *if f is an (F,S,F,S)-face or an (F,S,F,M)-face, $\frac{5}{4}$ if f is an (F,S,B,S)-face, and at least $\frac{3}{2}$ otherwise.*

Proof. If f is an (F,S,F,S)-face, then by R6, f sends

$$\frac{1}{2} \times \big(2 \times 4 - 6\big) = 1$$

to each of its incident small vertices. If f is an (F,S,F,M)-face, then by R3.5 and R6, f sends

$$2 \times 4 - 6 - 2 \times \frac{1}{2} = 1$$

to each of its incident small vertices. If f is an (F,S,B,S)-face, then by R6, f sends

$$\frac{1}{2} \times \left(2 \times 4 - 6 + \frac{1}{2}\right) = \frac{5}{4}$$

to each of its incident small vertices.

By symmetry, f can be of another types among $(S,B,B,B), (S,B,S,B)$, $(F,S,B,M), (F,B,S,B), (F,S,F,B), (S,B,M,B)$ and (F,S,B,B). In each case we can similarly calculate that f sends at least $\frac{3}{2}$ to each of its incident small vertices. □

Claim 23. *Every burdened* 5^+*-face sends at least* 2 *to each of its incident small vertices.*

Proof. Let f be a burdened 5^+-face with degree d that is BFB-incident or BFM-incident or MFM-incident with t false vertices. Under this condition, f is incident with at least $t+1$ big vertices or middle vertices. Since small vertices are not adjacent in G, f is incident with at most $\left\lceil \frac{d-2t-1}{2} \right\rceil$ small vertices. Since false vertices are not adjacent in $G^\times$, there are at most $\left\lfloor \frac{d}{2} \right\rfloor - t$ false vertices that are BFS-incident or SFM-incident or SFS-incident with f. Note that $t \le \left\lfloor \frac{d}{2} \right\rfloor - 1$ since f is burdened.

By R2, R3 and R6, f sends to each of its incident small vertices at least

$$\partial = \frac{2d - 6 - \frac{3}{2}t - \frac{1}{2}\left(\left\lfloor \frac{d}{2} \right\rfloor - t\right)}{\left\lceil \frac{d-2t-1}{2} \right\rceil}.$$

If d is odd, then

$$\partial = \frac{2d - 6 - \frac{3}{2}t - \frac{1}{2}\left(\frac{d-1}{2} - t\right)}{\frac{d-2t-1}{2}} = \frac{7d - 4t - 23}{2d - 4t - 2} = 2 + \frac{3d + 4t - 19}{2d - 4t - 2}.$$

If d is even, then

$$\partial = \frac{2d-6-\frac{3}{2}t-\frac{1}{2}\left(\frac{d}{2}-t\right)}{\frac{d-2t}{2}} = \frac{7d-4t-24}{2d-4t} = 2+\frac{3d+4t-24}{2d-4t}.$$

Hence if $d \geq 7$, or $d=5$ and $t \geq 1$, or $d=6$ and $t \geq 2$, then $\partial \geq 2$.

Suppose that $d=5$ and $t=0$. If f is incident with exactly one small vertex, then f sends to its incident small vertex at least

$$2 \times 5 - 6 - 2 \times \frac{1}{2} = 3 > 2$$

by R3 and R6. If f is incident with exactly two small vertices, then v is incident with at least one big vertex, and furthermore, v is incident with exactly one false vertex, or incident with exactly two false vertices, one of which is SFS-incident with f. In each case, f sends to each of its incident small vertices at least

$$\frac{1}{2} \times \left(2 \times 5 - 6 + \frac{1}{2} - \frac{1}{2}\right) = 2$$

by R2 and R3.

Suppose that $d=6$. If v is incident with exactly three small vertices, then any false vertex on f is SFS-incident with f. Therefore, by R3 and R6, f sends to each of its incident small vertices at least

$$\frac{1}{3} \times \left(2 \times 6 - 6\right) = 2.$$

Hence in the following we assume that v is incident with at most two small vertices.

If $t=0$, then f sends to each of its incident small vertices at least

$$\frac{1}{2} \times \left(2 \times 6 - 6 - 3 \times \frac{1}{2}\right) = \frac{9}{4} > 2$$

by R3 and R6.

If $t=1$ and f is incident with exactly one small vertex, then f sends to this incident small vertex at least

$$2 \times 6 - 6 - \frac{3}{2} - 2 \times \frac{1}{2} = \frac{7}{2} > 2$$

by R3 and R6.

If $t=1$ and f is incident with exactly two small vertices, then v is incident with at most two false vertices. Hence f sends to each of its incident small vertices at least

$$\frac{1}{2} \times \left(2 \times 6 - 6 - \frac{3}{2} - \frac{1}{2}\right) = 2$$

by R3 and R6. □

Now we calculate the final charge of each vertex $v \in V(G^{\times})$.

Case 1. Let v be a false vertex and let v_1, v_2, v_3, v_4 be neighbors of v in $G^{\times}$ lying on a clockwise order. Let f_i with $1 \le i \le 4$ be the face incident with the path $v_i v v_{i+1}$ (mod 4), see Figure 4.

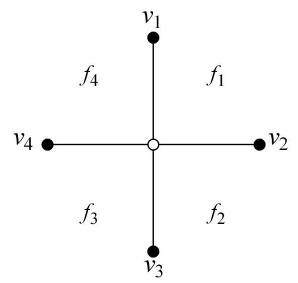

Figure 4. The structure around a false vertex v.

(1.1) If at least two of vertices among v_1, v_2, v_3, v_4 are big vertices, then we distinguish two subcases by symmetry. If v_1 and v_2 are big, then f_1 sends at least 1 to v by Claim 20(2), and each of f_2 and f_4 sends at least $\frac{1}{2}$ to v by Claim 20(1). This implies that

$$c'(v) \ge 4 - 6 + 1 + 2 \times \frac{1}{2} = 0.$$

If v_1 and v_3 are big, then every face incident with v sends at least $\frac{1}{2}$ to v by Claim 20(1), which implies that

$$c'(v) \ge 4 - 6 + 4 \times \frac{1}{2} = 0.$$

(1.2) If exactly one of the vertices among v_1, v_2, v_3, v_4, say v_1, is a big vertex, then v_2 and v_4 are middle vertices, because otherwise one vertex among v_2 and v_4 is a small vertex and the other is either middle or small, which is impossible since a small vertex is adjacent only to big vertices in G. By Claim 20(1), each of f_1 and f_4 sends at least $\frac{1}{2}$ to v. If v_3 is a small vertex, then each of f_2 and f_3 sends $\frac{1}{2}$ to v by Claim 20(3), which implies that

$$c'(v) \ge 4 - 6 + 2 \times \frac{1}{2} + 2 \times \frac{1}{2} = 0.$$

Now, suppose that v_3 is a middle vertex. If f_2 and f_3 are 3-faces so that v_3 is an M^{8-}-vertex, v_2 and v_4 are M^{9+}-vertices, then each of f_1 and f_4 sends at least $\frac{2}{3}$ to v by Claim 20(4), and f_2 and f_3 totally sends $\frac{2}{3}$ to v by Claim 20(5), which implies that

$$c'(v) \ge 4 - 6 + 2 \times \frac{2}{3} + \frac{2}{3} = 0.$$

Otherwise, by Claim 20(5), f_2 and f_3 totally sends at least 1 to v, and thus

$$c'(v) \geq 4 - 6 + 2 \times \frac{1}{2} + 1 = 0.$$

(1.3) If none of the vertices among v_1, v_2, v_3, v_4 is a big vertex, then they are all middle vertices. Note again that a small vertex is adjacent only to big vertices in G by the absence of the configuration (a). Since M^{8-}-vertices are not adjacent in G, there are at least two M^{9+}-vertices among v_1, v_2, v_3, v_4. If three of them, say v_1, v_2, v_3, are M^{9+}-vertices, then by Claim 20(5), f_1 and f_4 totally send 1 to v, and f_2 and f_3 totally send 1 to v. Hence

$$c'(v) \geq 4 - 6 + 1 + 1 = 0.$$

If exactly two vertices among v_1, v_2, v_3, v_4 are M^{9+}-vertices, then we distinguish two subcases: (i) v_1 and v_3 are M^{9+}-vertices, and (ii) v_1 and v_2 are M^{9+}-vertices, and v_3 and v_4 are M^{8-}-vertices. In each of the above two cases, we conclude by Claim 20(5) that f_1 and f_4 totally send 1 to v, and f_2 and f_3 totally send 1 to v. Therefore,

$$c'(v) \geq 4 - 6 + 1 + 1 = 0.$$

Case 2. If v is a 2-vertex, then v is not incident with a false 3-face by Lemma 14(b).

(2.1) If v is incident with a true 3-face, then v is adjacent to two big vertices in $G^{\times}$, and the other face f incident with v is either a 5^+-face, or a $(B, S, B, *)$-face, where $*$ stands for F or B or M or S. In either case, f sends at least $\frac{3}{2}$ to v by Claims 22 and 23. Hence by Claim 21 and R7, we have

$$c'(v) \geq 2 - 6 + 1 + \frac{3}{2} + \frac{1}{2} + \frac{1}{4} + \frac{3}{4} = 0.$$

(2.2) If v is incident with two 4^+-faces, one of which is a 5^+-face, then

$$c'(v) \geq 2 - 6 + 1 + 2 + \frac{1}{2} + \frac{1}{4} + \frac{3}{4} > 0$$

by Claims 22, 23 and R7.

(2.3) If v is incident with two 4-faces, then none of the two 4-faces incident with v is an (F, S, F, S)-face or an (F, S, F, M)-face (otherwise a multi-edge appears in G). This implies

$$c'(v) \geq 2 - 6 + 2 \times \frac{5}{4} + \frac{1}{2} + \frac{1}{4} + \frac{3}{4} = 0$$

by Claim 22 and R7.

Case 3. Suppose that v is a 3-vertex.

(3.1) If v is incident with at least one 5^+-face, then

$$c'(v) \geq 3 - 6 + 2 + \frac{1}{4} + \frac{3}{4} = 0$$

by Claim 23 and R8. Therefore we assume that v is incident only with 4^--faces.

(3.2) If v is incident with at least two 4-faces, then by Claim 22 and R8,

$$c'(v) \geq 3 - 6 + 2 \times 1 + \frac{1}{4} + \frac{3}{4} = 0.$$

(3.3) If v is incident with one 4-face and two 3-faces, then the 4-face incident with v is not of the (F, S, F, S)-type or the (F, S, F, M)-type (otherwise a multi-edge occurs in G). If v is incident with a true 3-face, then

$$c'(v) \geq 3 - 6 + 1 + \frac{5}{4} + \frac{1}{4} + \frac{3}{4} > 0$$

by Claims 21, 22 and R8. If v is incident with two false 3-faces, then v is adjacent to two false vertices and incident with a 5^+-face by Lemmas 14(c) and 14(d), which is impossible in this case.

(3.4) If v is incident with three 3-faces, then all of those 3-faces are true by Lemma 14(d). This implies

$$c'(v) \geq 3 - 6 + 3 \times 1 + \frac{1}{4} + \frac{3}{4} > 0$$

by Claim 21 and R8.

Case 4. Suppose that v is a true 4-vertex.

(4.1) If v is incident with at least one 5^+-face, then by Claim 23 and R9,

$$c'(v) \geq 4 - 6 + 2 + \frac{3}{4} > 0.$$

Therefore we assume that v is incident only with 4^--faces.

(4.2) If v is incident with four 3-faces, then at least two of them are true ones (otherwise two false vertices are adjacent in $G^\times$ or there exists a multi-edge in G). Hence by Claim 21 and R9,

$$c'(v) \geq 4 - 6 + 2 \times 1 + \frac{3}{4} > 0.$$

(4.3) If v is incident with at least two 4-faces, then by Claim 22 and R9, we have

$$c'(v) \geq 4 - 6 + 2 \times 1 + \frac{3}{4} > 0.$$

(4.4) If v is incident with exactly one 4-face and it is not of (F,S,F,S)-type or (F,S,F,M)-type, then by Claim 22 and R9, we have

$$c'(v) \geq 4 - 6 + \frac{5}{4} + \frac{3}{4} = 0.$$

(4.5) If v is incident with one $(F,S,F,*)$-face and three 3-faces, where $*$ stands for S or M, then v is incident with a true 3-face. This implies that

$$c'(v) \geq 4 - 6 + 1 + 1 + \frac{3}{4} > 0$$

by Claims 21, 22 and R9.

Case 5. Suppose that v is a 5-vertex.

(5.1) If v is incident with at least one 4^+-face, then by Claim 22,

$$c'(v) \geq 5 - 6 + 1 = 0.$$

(5.2) If v is incident with five 3-faces, then at least one of them is true, which implies, by Claim 21, that

$$c'(v) \geq 5 - 6 + 1 = 0.$$

Case 6. If v is a vertex of degree between 6 and 16, then v is adjacent only to 5^+-vertices in G by the absence of the configuration (b). Therefore, v cannot be a master of any vertex.

(6.1) If v is an M^{8-}-vertex, then v does not give out any charge by R1–R9, and thus

$$c'(v) = c(v) = d_{G^\times}(v) - 6 \geq 0.$$

(6.2) If v is an M^{9+}-vertex, then v sends $\frac{1}{3}$ to each of its incident faces, thus by R1,

$$c'(v) \geq d_{G^\times}(v) - 6 - \frac{1}{3}d_{G^\times}(v) = \frac{1}{3} \times \big(2d_{G^\times}(v) - 18\big) \geq 0.$$

(6.3) If v is a big vertex, that is, $d_{G^\times}(v) \geq 12$, then by R2, we have

$$c'(v) \geq d_{G^\times}(v) - 6 - \frac{1}{2}d_{G^\times}(v) = \frac{1}{2} \times \big(d_{G^\times}(v) - 12\big) \geq 0.$$

Case 7. If v is a 17-vertex, then by the absence of the configuration (b), v is adjacent only to 4^+-vertex in G. Therefore, v can be 4-masters of at most three vertices, and cannot be 3-master or 2-master of any vertex, thus by R2 and R9,

$$c'(v) \geq 17 - 6 - \frac{1}{2} \times 17 - 3 \times \frac{3}{4} > 0.$$

Case 8. If v is a 18-vertex, then by the absence of the configuration (b), v is adjacent only to 3^+-vertex in G. Therefore, v can be 4-masters of at most three vertices, 3-masters of at most two vertices, and cannot be 2-master of any vertex, thus by R2, R8 and R9,

$$c'(v) \geq 18 - 6 - \frac{1}{2} \times 18 - 3 \times \frac{3}{4} - 2 \times \frac{1}{4} > 0.$$

Case 9. If v is a 19^+-vertex, then v can be 4-masters of at most three vertices, 3-masters of at most two vertices, and 2-master of at most one vertex, thus by R2, R7, R8 and R9,

$$c'(v) \geq d_{G^\times}(v) - 6 - \frac{1}{2}d_{G^\times}(v) - 3 \times \frac{3}{4} - 2 \times \frac{1}{4} - \frac{1}{2} = \frac{1}{4} \times \big(2d_{G^\times}(v) - 37\big) > 0.$$

4. Open Problem

As we know, $\lambda_p^T(G) \leq \Delta(G) + 2p - 2$ for planar graphs with maximum degree $\Delta(G) \geq 4p + 4$ and $p \geq 2$, which is a result due to Sun and Wu [14]. In this paper, we prove $\lambda_p^T(G) \leq \Delta(G) + 2p - 2$ for 1-planar graphs with maximum degree $\Delta(G) \geq 6p + 7$ and $p \geq 2$.

A graph G is *d-degenerate* if every subgraph of G contains a vertex of degree at most d. It is well-known that planar graphs are 5-degenerate (see [2]) and 1-planar graphs are 7-degenerate (see [6]). Therefore, a natural question is whether $\lambda_p^T(G) \leq \Delta(G) + 2p - 2$ holds for every d-degenerate with large maximum degree and $p \geq 2$. Precisely, we propose the following problem.

Problem 24. Does there exist a constant k independent of G and p so that

$$\lambda_p^T(G) \leq \Delta(G) + 2p - 2$$

holds for every d-degenerate G with $\Delta(G) \geq kdp$?

An early result of Havet and Yu [9] says that $\lambda_p^T(G) \leq \chi'(G) + \chi(G) + p - 2$, where $\chi'(G)$ and $\chi(G)$ denote the chromatic index and chromatic number of G. The well-known Vizing's Theorem on edge coloring tells us that $\chi'(G) \leq \Delta(G)+1$ (see [2]), and it is easy to see that $\chi(G) \leq d + 1$ for every d-degenerate graph. Therefore, for every d-degenerate graph G, it holds that

$$\lambda_p^T(G) \leq \chi'(G) + \chi(G) + p - 2 \leq \Delta(G) + d + p.$$

If $p \geq d + 2$, then

$$\Delta(G) + d + p \leq \Delta(G) + 2p - 2.$$

Hence the above question is interesting when $p \leq d + 1$.

References

[1] F. Bazzaro, M. Montassier and A. Raspaud, *$(d,1)$-total labelling of planar graphs with large girth and high maximum degree*, Discrete Math. **307** (2007) 2141–2151. doi:10.1016/j.disc.2005.12.059

[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, New York, 1976).

[3] O.V. Borodin, *A new proof of the* 6 *color theorem*, J. Graph Theory **19** (1995) 507–521. doi:10.1002/jgt.3190190406

[4] O.V. Borodin, *Solution of Ringel's problems on the vertex-face coloring of plane graphs and on the coloring of* 1*-planar graphs*, Diskret. Analiz **41** (1984) 12–26.

[5] J.R. Griggs and R.K. Yeh, *Labeling graphs with a condition at distance two*, SIAM J. Discrete Math. **5** (1992) 586–595. doi:10.1137/0405048

[6] I. Fabrici and T. Madaras, *The structure of* 1*-planar graphs*, Discrete Math. **307** (2007) 854–865. doi:10.1016/j.disc.2005.11.056

[7] F. Havet, *$(d,1)$-total labelling of graphs*, in: Workshop on Graphs and Algorithms (Dijon, France, 2003).

[8] F. Havet and M.-L. Yu, $(d,1)$-total labelling of graphs, Technical Report, 4650 (INRIA, 2002).

[9] F. Havet and M.-L. Yu, *$(p,1)$-total labelling of graphs*, Discrete Math. **308** (2008) 496–513. doi:10.1016/j.disc.2007.03.034

[10] Ł. Kowalik, J.-S. Sereni and R. Škrekovski, *Total-coloring of plane graphs with maximum degree nine*, SIAM J. Discrete Math. **22** (2008) 1462–1479. doi:10.1137/070688389

[11] S.G. Kobourov, G. Liotta and F. Montecchiani, *An annotated bibliography on* 1*-planarity*, Comput. Sci. Rev. **25** (2017) 49–67. doi:10.1016/j.cosrev.2017.06.002

[12] M. Montassier and A. Raspaud, *$(d,1)$-total labeling of graphs with a given maximum average degree*, J. Graph Theory **51** (2006) 93–109. doi:10.1002/jgt.20124

[13] G. Ringel, *Ein Sechsfarbenproblem auf der Kugel*, Abh. Math. Semin. Univ. Hambg. **29** (1965) 107–117. doi:10.1007/BF02996313

[14] L. Sun and J.-L. Wu, *On $(p,1)$-total labelling of planar graphs*, J. Comb. Optim. **33** (2017) 317–325. doi:10.1007/s10878-015-9958-7

[15] J.-L. Wu and P. Wang, *List-edge and list-total colorings of graphs embedded on hyperbolic surfaces*, Discrete Math. **308** (2008) 6210–6215.
doi:10.1016/j.disc.2007.11.044

[16] R.K. Yeh, Labelling Graphs with a Condition at Distance Two, Ph.D. Thesis (Dept. of Math., Univ. of South Carolina, Columbia, SC, USA, 1990).

[17] Y. Yu, X. Zhang, G. Wang, G. Liu and J. Li, $(2,1)$*-total labelling of planar graphs with large maximum degree*, J. Discrete Math. Sci. Cryptogr. **20** (2017) 1625–1636.
doi:10.1080/09720529.2016.1139850

[18] X. Zhang, B. Niu and J. Yu, *A structure of* 1*-planar graph and its applications to coloring problems*, Graphs Combin. **35** (2019) 677–688.
doi:10.1007/s00373-019-02027-0

[19] X. Zhang and J.-L. Wu, *On edge colorings of* 1*-planar graphs*, Inform. Process. Lett. **111** (2011) 124–128.
doi:10.1016/j.ipl.2010.11.001

[20] X. Zhang, J.-L. Wu and G. Liu, *List edge and list total coloring of* 1*-planar graphs*, Front. Math. China **7** (2012) 1005–1018.
doi:10.1007/s11464-012-0184-7

[21] X. Zhang, Y. Yu and G. Liu, *On* $(p,1)$*-total labelling of* 1*-planar graphs*, Cent. Eur. J. Math. **9** (2011) 1424–1434.
doi:10.2478/s11533-011-0092-1

Received 16 March 2018
Revised 14 December 2018
Accepted 1 February 2019