# CONVEX AND WEAKLY CONVEX DOMINATION IN PRISM GRAPHS 

Monika Rosicka<br>Faculty of Mathematics, Physics and Informatics<br>University of Gdańsk, 80-952 Gdańsk, Poland<br>Institute of Theoretical Physics and Astrophysics and<br>National Quantum Information Centre in Gdańsk 81-824 Sopot, Poland<br>e-mail: mrosicka@inf.ug.edu.pl


#### Abstract

For a given graph $G=(V, E)$ and permutation $\pi: V \mapsto V$ the prism $\pi G$ of $G$ is defined as follows: $V(\pi G)=V(G) \cup V\left(G^{\prime}\right)$, where $G^{\prime}$ is a copy of $G$, and $E(\pi G)=E(G) \cup E\left(G^{\prime}\right) \cup M_{\pi}$, where $M_{\pi}=\left\{u v^{\prime}: u \in V(G), v=\pi(u)\right\}$ and $v^{\prime}$ denotes the copy of $v$ in $G^{\prime}$.

We study and compare the properties of convex and weakly convex dominating sets in prism graphs. In particular, we characterize prism $\gamma_{c o n}$-fixers and -doublers. We also show that the differences $\gamma_{w c o n}(G)-\gamma_{w c o n}(\pi G)$ and $\gamma_{w c o n}(\pi G)-2 \gamma_{w c o n}(G)$ can be arbitrarily large, and that the convex domination number of $\pi G$ cannot be bounded in terms of $\gamma_{c o n}(G)$.


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## 1. Introduction

Let $G=\left(V_{G}, E_{G}\right)$ be an undirected graph, let $\pi: V_{G} \mapsto V_{G}$ be a permutation of its vertex set and let $G^{\prime}$ be a vertex-disjoint copy of $G$. We denote the copy of a vertex $v \in V_{G}$ in $G^{\prime}$ by $v^{\prime}$. If $S$ is a set of vertices of $G$, then $S^{\prime}$ denotes the copy of $S$ in $G^{\prime}$, i.e., the set $\left\{v^{\prime}: v \in S\right\}$. By $V$ and $V^{\prime}$ we denote the vertex sets of $G$ and $G^{\prime}$, respectively. The prism graph $\pi G$ is a graph with vertex set $V \cup V^{\prime}$ and edge set $E_{G} \cup E_{G^{\prime}} \cup M_{\pi}$, where $M_{\pi}=\left\{u \pi(u)^{\prime}: u \in V_{G}\right\}$.

The open neighborhood of a vertex $v \in V_{G}$, denoted by $N_{G}(v)$, is the set of all vertices adjacent to $v$ in $G$. The closed neighborhood, denoted by $N_{G}[v]$, is
the set $N_{G}(v) \cup\{v\}$. If $S$ is a subset of $V_{G}$, then by $N_{S}(v)$ and $N_{S}[v]$ we denote $N_{G}(v) \cap S$ and $N_{G}[v] \cap S$, respectively. The neighborhood $N_{G}[S]$ of a set $S$ is the set $\bigcup_{v \in S} N_{G}[v]$. If $A$ and $B$ are disjoint sets of vertices of a graph $G$, then by $E(A, B)$ we denote the set of all edges of $G$ joining a vertex from $A$ with a vertex from $B$. The distance between two vertices $u$ and $v$ in a graph $G$, denoted by $d_{G}(u, v)$, is the minimum length of a $u-v$ path in $G$. The diameter $\operatorname{diam}(G)$ of a graph $G$ is the maximum distance between two vertices of $G$.

We say that a set $A \subseteq V_{G}$ dominates $B \subseteq V_{G}$ if $B \subseteq N_{G}[A]$. We denote this by $A \succ B$. A set $A \subseteq V_{G}$ is called a dominating set of the graph $G$ if it dominates $V_{G}$, i.e., if $N_{G}[A]=V_{G}$. The minimum cardinality of a dominating set in $G$ is called the domination number of $G$ and denoted $\gamma(G)$. A minimum dominating set of a graph is sometimes called a $\gamma$-set.

Prism graphs were first defined in [2] and the problem of domination in prism graphs was first studied by Burger, Mynhardt and Weakley [1].

In a connected graph $G$ a set of vertices $A \subseteq V_{G}$ is said to be convex if for every pair of vertices $u, v \in A$ the set $A$ contains all vertices of every shortest $u-v$ path.

A convex dominating set of $G$ is a dominating set of $G$ which is convex. The minimum cardinality of a convex dominating set of $G$ is called the convex domination number of $G$ and denoted as $\gamma_{c o n}(G)$. A $\gamma_{c o n}$-set of $G$ is a convex dominating set of cardinality $\gamma_{c o n}(G)$.

A set $A$ of vertices in a connected graph $G$ is said to be weakly convex if for every pair of vertices $u, v \in A$ it contains all vertices of at least one shortest $u-v$ path.

A weakly convex dominating set $G$ is a dominating set of $G$ which is weakly convex. The minimum cardinality of a weakly convex dominating set of $G$ is called the weakly convex domination number and denoted as $\gamma_{w c o n}(G)$. A $\gamma_{w c o n}$-set of $G$ is a minimum weakly convex dominating set.

A set $A \subseteq V_{G}$ is said to be connected if it induces a connected subgraph, i.e., if for every pair of vertices $u, v \in A$ there exists a $u-v$ path contained entirely in $A$.

A connected dominating set of $G$ is a dominating set which is connected. The connected domination number of $G$, denoted as $\gamma_{c}(G)$, is the minimum cardinality of a connected dominating set of $G$. A $\gamma_{c}$-set of $G$ is a connected dominating set of cardinality $\gamma_{c}(G)$.

In this paper we compare the properties of convex and weakly convex sets in prism graphs, particularly convex and weakly convex dominating sets. We also generalize some known properties of convex domination in prism graphs to weakly convex and connected domination.

## 2. Connected, Convex and Weakly Convex Domination

It is clear that the notions of convex, weakly convex and connected domination are closely related. Since every convex set is weakly convex and every weakly convex set is connected, it is easy to see that $\gamma(G) \leq \gamma_{c}(G) \leq \gamma_{w c o n}(G) \leq \gamma_{c o n}(G)$. Many, but not all, properties of connected sets can be extended to both convex and weakly convex sets.

Lemma 2.1. If $G \neq K_{1}$ is a connected graph and $\pi$ is a permutation of $V_{G}$, then $\gamma_{c}(\pi G) \geq \gamma(G)+1$.
Proof. Obviously, if $\gamma(\pi G) \geq \gamma(G)+1$, then $\gamma_{c}(\pi G) \geq \gamma(G)+1$.
Now let $\gamma(\pi G)=\gamma(G)$ and let $D$ be a $\gamma$-set of $\pi G$. We denote $D_{1}=D \cap V$ and $D_{2}^{\prime}=D \cap V^{\prime}$. Note that neither of these sets is empty, as $|D|=\gamma(G)<\left|V_{G}\right|$ and a set of fewer than $\left|V_{G}\right|$ vertices in $V$ or $V^{\prime}$ cannot dominate all of $\pi G$. Since the only vertices in $V^{\prime}$ dominated by $D_{1}$ are in $\pi\left(D_{1}\right)^{\prime}$, it follows that $B=\pi\left(D_{1}\right) \cup D_{2} \succ V$ and therefore $|B|=|D|=\gamma(G)$. Thus, $\pi\left(D_{1}\right) \cap D_{2}=\emptyset$. Since $N_{V^{\prime}}\left(D_{1}\right)=\pi\left(D_{1}\right)^{\prime}$, it follows that $E\left(D_{1}, D_{2}^{\prime}\right)=\emptyset$ and therefore $D$ is not connected. Thus, every connected dominating set of $\pi G$ has cardinality at least $\gamma(G)+1$.

Since $\gamma_{c}(G) \leq \gamma_{w c o n}(G) \leq \gamma_{c o n}(G)$, Lemma 2.1 also applies to convex and weakly convex domination, that is, for any graph $G$ and permutation $\pi$ of $V_{G}$ we have $\gamma_{\text {con }}(\pi G) \geq \gamma(G)+1$ and $\gamma_{\text {wcon }}(\pi G) \geq \gamma(G)+1$.

Lemma 2.2. For any connected graph $G$ and permutation $\pi$ of $V_{G}$, if $G$ has a dominating set $A$ which can be partitioned into three nonempty disjoint sets $A_{1}, A_{2}, A_{3}$ such that
(1) $A_{1} \cup A_{2} \succ V_{G}-A_{3}$,
(2) $A_{1} \cup A_{2}$ is connected,
(3) $\pi\left(A_{2} \cup A_{3}\right)$ is connected,
(4) $\pi\left(A_{2} \cup A_{3}\right) \succ V_{G}-\pi\left(A_{1}\right)$,
then $D=A_{1} \cup A_{2} \cup\left(\pi\left(A_{2} \cup A_{3}\right)\right)^{\prime}$ is a connected dominating set of size $|A|+\left|A_{2}\right|$ in $\pi G$.

Proof. Since $A_{1} \cup A_{2} \succ\left(\pi\left(A_{1}\right)\right)^{\prime} \cup\left(V-A_{3}\right)$ and $\left(\pi\left(A_{2} \cup A_{3}\right)\right)^{\prime} \succ A_{3} \cup\left(V^{\prime}-\right.$ $\left.\left(\pi\left(A_{1}\right)\right)^{\prime}\right)$, it is clear that $D=A_{1} \cup A_{2} \cup\left(\pi\left(A_{2} \cup A_{3}\right)\right)^{\prime} \succ\left(\left(\pi\left(A_{1}\right)\right)^{\prime} \cup\left(V-A_{3}\right)\right) \cup$ $\left(A_{3} \cup\left(V^{\prime}-\left(\pi\left(A_{1}\right)\right)^{\prime}\right)\right)=V \cup V^{\prime}$, that is, $D$ is a dominating set in $\pi G$. The sets $D_{1}=A_{1} \cup A_{2}$ and $D_{2}^{\prime}=\left(\pi\left(A_{2} \cup A_{3}\right)\right)^{\prime}$ are connected and $E\left(D_{1}, D_{2}^{\prime}\right) \neq \emptyset$, and it follows that $D$ is connected. Hence $D$ is a connected dominating set of $\pi G$.

Note that this particular result cannot be extended to convex domination as, for example, if $\pi=\mathrm{Id}$, then for any set $A=A_{1} \cup A_{2} \cup A_{3}$, where $A_{1}, A_{2}$ and
$A_{3}$ are disjoint and not empty, the set $A_{1} \cup A_{2} \cup\left(A_{2} \cup A_{3}\right)^{\prime}$ is not convex. For connected domination, however, we can use it to show exactly when the bound from Lemma 2.1 is achieved.

Proposition 2.3. Let $G \notin\left\{K_{1}, K_{2}\right\}$ be a connected graph and let $\pi$ be a permutation of $V_{G}$. Then $\gamma_{c}(\pi G)=\gamma(G)+1$ if and only if there exists a $\gamma$-set $A$ which can be partitioned into two disjoint subsets $A_{1}, A_{2}$, and a vertex $v \in A_{1}$ such that
(1) $A_{1} \succ V_{G}-A_{2}$,
(2) $A_{1}$ is connected,
(3) $\pi\left(A_{2} \cup\{v\}\right) \succ V_{G}-\pi\left(A_{1}\right)$,
(4) $\pi\left(A_{2} \cup\{v\}\right)$ is connected.

Proof. Let $D=A_{1} \cup\left(\pi\left(A_{2} \cup\{v\}\right)\right)^{\prime}$ be a set of vertices in $\pi G$. It follows from Lemma 2.2 that $D$ is a connected dominating set of size $\gamma(G)+1$ in $\pi G$, so $\gamma_{c}(\pi G) \leq \gamma(G)+1$. By Lemma 2.1, $\gamma_{c}(\pi(G)) \geq \gamma(G)+1$. Thus, $\gamma_{c}(\pi G)=$ $\gamma(G)+1$.

Now let $D$ be a connected dominating set of size $\gamma(G)+1$ in $\pi G$ and let $D_{1}=D \cap V$ and $D_{2}^{\prime}=D \cap V^{\prime}$. Both sets $D_{1}, D_{2}$ are nonempty, as $\gamma_{c}(\pi G)<|V|$ and a set of fewer than $|V|$ vertices in $V$ or $V^{\prime}$ cannot dominate all of $\pi G$.

Since $D$ is a dominating set of $\pi G$, we have $D_{2} \succ V-\pi\left(D_{1}\right)$ and $D_{1} \succ V-$ $\pi^{-1}\left(D_{2}\right)$, and thus $D_{1} \cup \pi^{-1}\left(D_{2}\right) \succ V$. This implies that $\left|D_{1} \cup \pi^{-1}\left(D_{2}\right)\right| \geq \gamma(G)$. Furthermore, since $D$ is connected, it is necessary that $D_{1} \cap \pi^{-1}\left(D_{2}\right) \neq \emptyset$, but $\left|D_{1} \cap \pi^{-1}\left(D_{2}\right)\right| \leq 1$, as $\left|D_{1} \cup \pi^{-1}\left(D_{2}\right)\right| \geq \gamma(G)$ and $|D|=\gamma(G)+1$. Thus, there exists exactly one vertex $v \in D_{1}$ such that $\pi(v) \in D_{2}$. Finally, the fact that $D$ is connected and every path connecting a vertex from $D_{1}$ with a vertex from $D_{2}$ contains $v$ and $\pi(v)^{\prime}$ implies that both sets $D_{1}$ and $D_{2}$ are connected.

We now define $A_{1}=D_{1}, A_{2}=\pi^{-1}\left(D_{2}\right)-\{v\}$. The set $A=A_{1} \cup A_{2}$ is clearly a $\gamma$-set of $G$ satisfying conditions (1)-(4).

Again, the above is not true for convex domination, however, in Section 4 we prove a related property of weakly convex dominating sets.

## 3. Generalizing Some Properties of Convex Domination

Convex domination in prism graphs was studied by Lemańska and Zuazua in [3], where they prove the following theorem.
Theorem 3.1 [3]. Let $G$ be a connected graph. If $\gamma_{c o n}(G)=\left|V_{G}\right|$ and $\operatorname{diam}(G) \leq$ 2 , then $\gamma_{c o n}(G)=\gamma_{c o n}(\pi G)$ for every permutation $\pi$ of $V_{G}$.

The proof of Theorem 3.1 relies on some properties of convex dominating sets, which they prove in the same paper. We will now compare these properties with those of weakly convex dominating sets.

Theorem 3.2 [3]. For any connected graph $G$.
(1) If $\operatorname{diam}(G) \leq 2$, then $V$ and $V^{\prime}$ are both convex dominating sets of $\pi G$ for any permutation $\pi$.
(2) If $\operatorname{diam}(G) \geq 3$, then there exist permutations $\pi_{1}$ and $\pi_{2}$ such that $V$ is not a convex dominating set of $\pi_{1} G$ and $V^{\prime}$ is not a convex dominating set of $\pi_{2} G$.

It follows that if $\operatorname{diam}(G) \leq 2$, then $\gamma_{c o n}(\pi G) \leq\left|V_{G}\right|$ for any $\pi$. Weakly convex domination number has a similar property.

Theorem 3.3. Every connected graph $G$ has the following properties.
(1) If $\operatorname{diam}(G) \leq 3$, then $V$ and $V^{\prime}$ are weakly convex dominating sets of $\pi G$ for every permutation $\pi$.
(2) If $\operatorname{diam}(G)>3$, then there exist permutations $\pi_{1}$ and $\pi_{2}$ such that $V$ is not a weakly convex dominating set of $\pi_{1} G$ and $V^{\prime}$ is not a weakly convex dominating set of $\pi_{2} G$.

Proof. Obviously, for any permutation $\pi, V$ and $V^{\prime}$ are dominating sets of $\pi G$. For any pair of vertices $u, v \in V$, the shortest $u-v$ path containing at least one vertex from $V^{\prime}$ has length at least 3 . Thus, if $\operatorname{diam}(G) \leq 3$, then $d_{G}(u, v)=$ $d_{\pi G}(u, v)$ and $V$ contains at least one shortest $u-v$ path in $\pi G$. Similarly, if $\operatorname{diam}(G) \leq 3$, then $V^{\prime}$ must contain a shortest $u^{\prime}-v^{\prime}$ path in $\pi G$ for every $u^{\prime}, v^{\prime} \in$ $V^{\prime}$.

Now, let $\operatorname{diam}(G)$ be at least 4 and let $u, v \in V$ be a pair of vertices such that $d_{G}(u, v) \geq 4$. If $\pi_{1}$ is a permutation such that $\pi_{1}(u)$ and $\pi_{1}(v)$ are adjacent, then $d_{\pi G}(u, v)=3<d_{G}(u, v)$. Thus, $V$ is not a weakly convex dominating set in $\pi_{1} G$. Similarly, for $\pi_{2}=\pi_{1}^{-1}$, the set $V^{\prime}$ is not weakly convex.

The first part of Theorem 3.3 implies that $\gamma_{w c o n}(\pi G) \leq\left|V_{G}\right|$ for any graph $G$ with diameter at most 3 .

In this section as well as the next one, when $D$ is a convex or weakly convex dominating set of $\pi G$, we will denote $D_{1}=D \cap V$ and $D_{2}^{\prime}=D \cap V^{\prime}$. We will denote the equivalent of $D_{2}^{\prime}$ in $V$ by $D_{2}$.

Proposition 3.4 [3]. For a connected graph $G$ and permutation $\pi$ of $V_{G}$, let $D$ be a convex dominating set of $\pi G$. Then $D$ has the following properties.
(1) If $|D|<\left|V_{G}\right|$, then $D_{1} \neq \emptyset$ and $D_{2}^{\prime} \neq \emptyset$.
(2) If $D_{1} \neq \emptyset$ and $D_{2}^{\prime} \neq \emptyset$, then there exists at least one $x \in D_{1}$ such that $\pi(x) \in D_{2}$.

The above can be generalized to weakly convex and connected domination.
Proposition 3.5. For a connected graph $G$ and permutation $\pi$ of $V_{G}$, let $D$ be a connected dominating set of $\pi G$. Then $D$ has the following properties.
(1) If $|D|<\left|V_{G}\right|$, then $D_{1} \neq \emptyset$ and $D_{2}^{\prime} \neq \emptyset$.
(2) If $D_{1} \neq \emptyset$ and $D_{2}^{\prime} \neq \emptyset$, then there exists at least one $x \in D_{1}$ such that $\pi(x) \in D_{2}$.

Proof. If $D_{1}=\emptyset$, then $D=D_{2}^{\prime}$. But every vertex in $D_{2}^{\prime}$ only dominates one vertex in $V$. It follows that if $D$ is a dominating set, then $D_{1}=\emptyset$ implies $|D|=|V|$. The same reasoning applies if $D_{2}^{\prime}=\emptyset$. Thus, if $|D|<|V|$, then the subsets $D_{1}, D_{2}^{\prime}$ are not empty.

If both sets $D_{1}$ and $D_{2}$ are nonempty, then $D$ contains a pair of vertices $v_{1} \in D_{1}, v_{2}^{\prime} \in D_{2}^{\prime}$. Since the set $D$ is connected, there exists a $v_{1}-v_{2}^{\prime}$ path contained entirely in $D$. Each vertex $v \in V$ has only one neighbor in $V^{\prime}$, namely $\pi(v)^{\prime}$. Hence, every path connecting $v_{1} \in D_{1}$ with $v_{2}^{\prime} \in D_{2}^{\prime}$ contains a pair of vertices $x \in V$ and $\pi(x)^{\prime} \in V^{\prime}$. This shows that $D_{1}$ contains a vertex $x$ such that $\pi(x) \in D_{2}$.

Since every weakly convex dominating set is a connected dominating set, we also have the following.

Corollary 3.6. For a connected graph $G$ and permutation $\pi$ of $V_{G}$, let $D$ be a weakly convex dominating set of $\pi G$. Then $D$ has the following properties.
(1) If $|D|<\left|V_{G}\right|$, then $D_{1} \neq \emptyset$ and $D_{2}^{\prime} \neq \emptyset$.
(2) If $D_{1} \neq \emptyset$ and $D_{2}^{\prime} \neq \emptyset$, then there exists at least one $x \in D_{1}$ such that $\pi(x) \in D_{2}$.

Lemma 3.7 [3]. Let $G$ be a connected graph in which $\operatorname{diam}(G) \leq 2$. Let $D$ be a convex dominating set of $\pi G$. Then the set $D$ has the following properties.
(1) If $\pi\left(D_{1}\right) \subseteq D_{2}$, then $D_{2}$ is a convex dominating set of $G$.
(2) If $\pi^{-1}\left(D_{2}\right) \subseteq D_{1}$, then $D_{1}$ is a convex dominating set of $G$.

Again, we can prove a similar property for weakly convex domination.
Lemma 3.8. Let $G$ be a connected graph in which $\operatorname{diam}(G) \leq 2$. Let $D$ be a weakly convex dominating set of $\pi G$. Then the set $D$ has the following properties.
(1) If $\pi\left(D_{1}\right) \subseteq D_{2}$, then $D_{2}$ is a weakly convex dominating set of $G$.
(2) If $\pi^{-1}\left(D_{2}\right) \subseteq D_{1}$, then $D_{1}$ is a weakly convex dominating set of $G$.

Proof. If $\pi\left(D_{1}\right) \subseteq D_{2}$, then clearly $D_{2}^{\prime}$ dominates $V^{\prime}$, as $D_{1}$ does not dominate any part of $V^{\prime}-D_{2}^{\prime}$. It follows that $D_{2}$ is a dominating set of $G$. For any two vertices $u^{\prime}, v^{\prime} \in D_{2}^{\prime}$ a shortest possible $u^{\prime}-v^{\prime}$-path in $\pi G$ containing a vertex from $V$ has length at least 3 . If $\operatorname{diam}(G) \leq 2$, then any shortest $u^{\prime}-v^{\prime}$-path must be contained in $D_{2}^{\prime}$. Thus, for every $u, v \in D_{2}$ the set $D_{2}$ contains a shortest $u-v$ path. Hence, $D_{2}$ is a weakly convex dominating set of $G$.

Similarly, if $\pi^{-1}\left(D_{2}\right) \subseteq D_{1}$, then $D_{1}$ is a dominating set of $G$ because $D_{2}^{\prime}$ does not dominate any part of $V-D_{1} . D_{1}$ is also convex, because for any $u, v \in D_{1}$ no shortest $u-v$ path passes through $D_{2}^{\prime}$.

Note that, unlike Theorem 3.3, the above does not hold for every $G$ such that $\operatorname{diam}(G) \leq 3$. For example, if $G=P_{4}=(\{1,2,3,4\},\{12,23,34\})$ and $\pi=(12)(34)$, the set $D=\left\{1^{\prime}, 2,3,4^{\prime}\right\}$ is a weakly convex dominating set of $\pi G$. The set $\{1,4\}$ contains $\pi(\{2,3\})$ and it is not a weakly convex set.

For weakly convex domination an exact analogue of Theorem 3.1 makes no sense, as there is no graph with $\operatorname{diam}(G)=2$ and $\gamma_{w c o n}(G)=\left|V_{G}\right|$. Since Lemańska and Zuazua's proof relies on Lemma 3.7, whose weakly convex analogue does not hold for graphs with diameter 3 , as well as some properties of convex sets which weakly convex sets do not have, it seems unlikely that all graphs with diameter 3 would have such a property. Indeed, the cycle $C_{7}$ has diameter 3 and $\gamma_{w c o n}\left(C_{7}\right)=7$, yet a permutation $\pi$, defined in Section 5 , exists such that $\gamma_{\text {wcon }}\left(\pi C_{7}\right)=6$.

## 4. Convex and Weakly Convex Domination in Id $G$

We now consider the special case where the permutation is $\pi=\mathrm{Id}$. A graph $G$ is called a prism fixer if $\gamma(\operatorname{Id} G)=\gamma(G)$ and a prism doubler if $\gamma(\operatorname{Id} G)=2 \gamma(G)$. A graph $G$ is called a universal fixer if $\gamma(\pi G)=\gamma(G)$ for every permutation $\pi$ of $V_{G}$ and a universal doubler if $\gamma(\pi G)=2 \gamma(G)$ for every $\pi$. Prism fixers are characterized in [4] and universal fixers in [5] and [6]. Prism doublers and universal doublers are studied in [1].

Similarly, a graph $G$ such that $\gamma_{c o n}(\operatorname{Id} G)=\gamma_{c o n}(G)$ is called a prism $\gamma_{c o n}{ }^{-}$ fixer and a graph with $\gamma_{c o n}(\operatorname{Id} G)=2 \gamma_{c o n}(G)$ is called a prism $\gamma_{c o n}$-doubler. A universal $\gamma_{c o n}$-fixer is a graph such that for every $\pi \gamma_{c o n}(\pi G)=\gamma_{c o n}(G)$ and a universal $\gamma_{c o n}$-doubler is a graph such that $\gamma_{c o n}(\pi G)=2 \gamma_{c o n}(G)$ for every $\pi$.

We begin this section by studying some properties of convex and weakly convex sets in $\operatorname{Id} G$.

Observation 4.1. For any two vertices $u, v \in V_{G}$ in a connected graph $G$ :
(1) $d_{\mathrm{Id} G}(u, v)=d_{\mathrm{Id} G}\left(u^{\prime}, v^{\prime}\right)=d_{G}(u, v)$,
(2) $d_{\mathrm{Id} G}\left(u, v^{\prime}\right)=d_{\mathrm{Id} G}\left(u^{\prime}, v\right)=d_{G}(u, v)+1$,
(3) every shortest $u-v^{\prime}$ path in $\operatorname{Id} G$ has the form $u=v_{0}, v_{1}, \ldots, v_{i}, v_{i}^{\prime}, \ldots, v_{k}^{\prime}=v^{\prime}$ for some shortest $u-v$ path $u=v_{0}, v_{1}, \ldots, v_{i}, \ldots, v_{k}=v$ in $G$,
(4) every path in $\operatorname{Id} G$ of the form $u=v_{0}, v_{1}, \ldots, v_{i}, v_{i}^{\prime}, \ldots, v_{k}^{\prime}=v^{\prime}$ for some shortest $u-v$ path $u=v_{0}, v_{1}, \ldots, v_{i}, \ldots, v_{k}=v$ in $G$ is a shortest $u-v^{\prime}$ path,
(5) every shortest $u-v$ path in $\operatorname{Id} G$ is a shortest $u-v$ path in $G$,
(6) every shortest $u-v$ path in $G$ is a shortest $u-v$ path in $\operatorname{Id} G$.

Using the above observation we obtain the following lemma.
Lemma 4.2. If $S \subseteq V_{G}$ is a convex (weakly convex) set in $G$, then $S, S^{\prime}$ and $S \cup S^{\prime}$ are convex (weakly convex) sets in $\operatorname{Id} G$.

Proof. Let $S \subseteq V$ be a convex set in $G$ and let $u$ and $v$ be any two vertices in $S$. It follows from Observation $4.1(5)$ that $S$ contains all shortest $u-v$ paths in $\operatorname{Id} G$. The set $S^{\prime}$ is also a convex set in $\operatorname{Id} G$, as $P^{\prime}$ is a shortest $u^{\prime}-v^{\prime}$ path for every shortest $u-v$ path $P$. Thus $S$ and $S^{\prime}$ are convex sets in $\operatorname{Id} G$.

Observation 4.1(3) implies that $S \cup S^{\prime}$ contains all shortest $u-v^{\prime}$ paths in $\operatorname{Id} G$. Thus $S \cup S^{\prime}$ is also a convex set.

Now let $S \subseteq V$ be a weakly convex set in $G$ and let $u$ and $v$ be any two vertices in $S$. By Observation 4.1(6) the set $S$ contains at least one shortest $u-v$ path in $\operatorname{Id} G$. The set $S^{\prime}$ is also a weakly convex set in $\operatorname{Id} G$, as $P^{\prime}$ is a shortest $u^{\prime}-v^{\prime}$ path for every shortest $u-v$ path $P$. Thus $S$ and $S^{\prime}$ are weakly convex sets in $\operatorname{Id} G$.

By Observation 4.1(4) the set $S \cup S^{\prime}$ contains all shortest $u-v^{\prime}$ paths in $\operatorname{Id} G$. Thus $S \cup S^{\prime}$ is also a convex set.

Corollary 4.3. $S \subset V_{\mathrm{Id} G}$ is a convex set in $\operatorname{Id} G$ if and only if $S \in\left\{S_{1}, S_{1}^{\prime}, S_{1} \cup S_{1}^{\prime}\right\}$ for some convex set $S_{1} \in V_{G}$.

Proof. If $S_{1}$ is a convex set in $G$ then, by Lemma $4.2, S_{1}, S_{1}^{\prime}$ and $S_{1} \cup S_{1}^{\prime}$ are convex sets in $\operatorname{Id} G$.

If $S$ is a convex set in $\operatorname{Id} G$ then either $S \subseteq V, S \subseteq V^{\prime}$ or $S=S_{1} \cup S_{2}^{\prime}$, where $S_{1}$ and $S_{2}$ are convex sets in $G$. For any two vertices $u \in S_{1}, v^{\prime} \in S_{2}^{\prime}$ the set $S$ contains all shortest $u-v^{\prime}$ paths. By Observation 4.1(4), this implies that $S_{1}$ and $S_{2}$ both contain $u, v$ and all shortest $u-v$ paths in $G$. It follows that $S_{1}=S_{2}$ and thus $S=S_{1} \cup S_{1}^{\prime}$.

Weakly convex sets have an additional property.
Lemma 4.4. A set $S \subseteq V_{\mathrm{Id} G}$, where $S_{1}=S \cap V, S_{2}^{\prime}=S \cap V^{\prime}$ is weakly convex if and only if
(1) $S_{1}, S_{2}$ and $S_{1} \cup S_{2}$ are weakly convex sets in $G$.
(2) For every $u \in S_{1}, v \in S_{2}$, a shortest $u-v$ path in $S_{1} \cup S_{2}$ contains a vertex from $S_{1} \cap S_{2}$.

Proof. Let $S$ be a weakly convex set in Id $G$ with $S_{1}=S \cap V, S_{2}^{\prime}=S \cap V^{\prime}$. If $u, v \in S_{i}$ for $i \in\{1,2\}$ then, by Observation 4.1(5), $S_{i}$ contains a shortest $u-v$ path. If $u \in S_{1}, v \in S_{2}$, then by Observation 4.1(3) $S$ contains a shortest $u-v^{\prime}$ path of the form $u=v_{0}, v_{1}, \ldots, v_{i}, v_{i}^{\prime}, v_{i+1}^{\prime}, \ldots, v_{k}^{\prime}=v^{\prime}$ and thus $S_{1} \cup S_{2}$
contains a shortest $u-v$ path of the form $u=v_{0}, v_{1}, \ldots, v_{i}, v_{i+1}, \ldots, v_{k}=v$, where $v_{i} \in S_{1} \cap S_{2}$.

Now let $S=S_{1} \cup S_{2}^{\prime}$ be a set satisfying conditions (1), (2). If $u, v \in S_{i}$ for $i \in\{1,2\}$ then, by Observation $4.1(6), S_{i}$ contains a shortest $u-v$ (or $u^{\prime}-v^{\prime}$ ) path in Id $G$. If $u \in S_{1}, v \in S_{2}$ then $S_{1} \cup S_{2}$ contains a shortest $u-v$ path $P$ containing a vertex $w \in S_{1} \cap S_{2}$. Since $S_{1}$ and $S_{2}$ are weakly convex, they must contain a shortest $u-w$ path $P_{1}$ and a shortest $w-v$ path $P_{2}$, respectively. By Observation 4.1(4) $Q=P_{1} P_{2}^{\prime}$ is a shortest $u-v^{\prime}$ path in $\operatorname{Id} G$. Thus $S$ is a weakly convex set in $\operatorname{Id} G$.

This is not the case with convex sets. If a convex set in $\operatorname{Id} G$ contains a pair of vertices $u, v^{\prime}$ it must also contain $u^{\prime}$ and $v$. This leads to some differences between convex and weakly convex domination.

Theorem 4.5. If $G$ is any connected graph, then $\gamma_{c o n}(\operatorname{Id} G)=\min \left\{2 \gamma_{c o n}(G)\right.$, $\left.\left|V_{G}\right|\right\}$.

Proof. Let $D$ be a $\gamma_{c o n}$-set of Id $G$. If $D \cap V^{\prime}=\emptyset$, then obviously $D=V$. Similarly, if $D \cap V=\emptyset$, then $D=V^{\prime}$. In this case $|D|=\left|V_{G}\right|$. Otherwise, by Corollary 4.3, $D=D_{1} \cup D_{1}^{\prime}$ for some convex set $D_{1} \in V_{G}$. Since $D_{1}^{\prime}$ dominates no vertices in $V-D_{1}$, it is clear that $D_{1} \succ V$. The set $D_{1}$ is a $\gamma_{c o n}$-set of $G$. It follows that in this case $|D|=2\left|D_{1}\right|=2 \gamma_{\text {con }}(G)$.

As a result, we have the following.
Corollary 4.6. Every connected graph $G$ has the following properties:
(1) $G$ is a prism $\gamma_{c o n}$-fixer if and only if $\gamma_{c o n}(G)=\left|V_{G}\right|$.
(2) $G$ is a prism $\gamma_{c o n}$-doubler if and only if $\gamma_{c o n}(G) \leq \frac{1}{2}\left|V_{G}\right|$.

Proof. If $\gamma_{c o n}(G)=\gamma_{c o n}(\operatorname{Id} G)=\min \left\{2 \gamma_{c o n}(G),|V|\right\}$, then $\gamma_{c o n}(G)=|V|$.
$\gamma_{c o n}(\operatorname{Id} G)=2 \gamma_{c o n}(G)$ if and only if $2 \gamma_{c o n}(G) \leq|V|$, if and only if $\gamma_{c o n}(G) \leq$ $\frac{1}{2}\left|V_{G}\right|$.

Since every universal $\gamma_{c o n}$-fixer is a prism $\gamma_{c o n}$-fixer, and every universal $\gamma_{c o n^{-}}$ doubler is a prism $\gamma_{c o n}$-doubler, we also have the following corollary.

Corollary 4.7. Let $G$ be a connected graph. Then
(1) If $G$ is a universal $\gamma_{\text {con }}$-fixer, then $\gamma_{\text {con }}(G)=\left|V_{G}\right|$.
(2) If $G$ is a universal $\gamma_{\text {con- }}$-doubler, then $\gamma_{\text {con }}(G) \leq \frac{1}{2}\left|V_{G}\right|$.

A similar property of weakly convex domination follows from Lemma 4.2.
Theorem 4.8. If $G$ is a connected graph, then $\gamma_{w c o n}(\operatorname{Id} G) \leq \min \left\{\left|V_{G}\right|\right.$, $\left.2 \gamma_{\text {wcon }}(G)\right\}$.

Proof. Obviously, $V$ is a dominating set in IdG. By Lemma 4.2, it is also a weakly convex set in $\operatorname{Id} G$. Thus, $\gamma_{w c o n}(\operatorname{Id} G) \leq\left|V_{G}\right|$.

If $S$ is a $\gamma_{w c o n}$-set of $G$, then $S \cup S^{\prime}$ is a dominating set in $\operatorname{Id} G$, as $S \succ V$ and $S^{\prime} \succ V^{\prime}$. Lemma 4.2 implies that $S \cup S^{\prime}$ is a (not necessarily minimal) weakly convex dominating set in $\operatorname{Id} G$ and thus $\gamma_{w c o n}(\operatorname{Id} G) \leq 2 \gamma_{w c o n}(G)$.

However, thanks to Lemma 4.4, $\gamma_{w c o n}(\operatorname{Id} G)$ is not necessarily equal to $\min \left\{\left|V_{G}\right|, 2 \gamma_{w c o n}(G)\right\}$. In fact, the following is true.

Theorem 4.9. Let $G$ be a connected graph. The graph $\operatorname{Id} G$ has a weakly convex dominating set $D \notin\left\{V, V^{\prime}\right\}$ of cardinality $\gamma_{w c o n}(G)+k$ if and only if $G$ has a weakly convex dominating set $A$ which can be partitioned into three nonempty sets $A_{1}, A_{2}, A_{3}$ such that $|A|+\left|A_{2}\right|=\gamma_{w c o n}(G)+k$ and
(1) $A_{1} \cup A_{2}$ and $A_{2} \cup A_{3}$ are weakly convex,
(2) for every $u \in A_{1} \cup A_{2}, v \in A_{2} \cup A_{3}$, a shortest $u-v$ path in $A$ contains a vertex from $A_{2}$,
(3) $A_{1} \cup A_{2} \succ V-A_{3}$ and $A_{3} \cup A_{2} \succ V-A_{1}$.

In particular, $\gamma_{w c o n}(\operatorname{Id} G)=\gamma_{w c o n}(G)+1$ if and only if $G$ has a $\gamma_{w c o n-s e t ~}$ $A=A_{1} \cup A_{2} \cup A_{3}$ such that conditions (1)-(3) are fulfilled and $\left|A_{2}\right|=1$.

To prove this result we will use the following lemma.
Lemma 4.10. Let $D=D_{1} \cup D_{2}^{\prime}$ be a weakly convex dominating set of $\operatorname{Id} G$. Then $D_{G}=D_{1} \cup D_{2}$ is a weakly convex dominating set of $G$.

Proof. Since $D$ is a dominating set of $\operatorname{Id} G$, we have $D_{1} \succ V-D_{2}$ and $D_{2} \succ$ $V-D_{1}$. It follows that $D_{1} \cup D_{2}$ dominates $V_{G}$. By Lemma 4.4, $D_{1} \cup D_{2}$ is also a weakly convex set. Thus, it is a weakly convex dominating set of $G$.

If $D_{1}=\emptyset$ or $D_{2}=\emptyset$, then $D_{G}=V$, which is also a convex dominating set in $G$.

Proof of Theorem 4.9. Let $A=A_{1} \cup A_{2} \cup A_{3}$ be a weakly convex dominating set of a connected graph $G$. The set $D=A_{1} \cup A_{2} \cup A_{2}^{\prime} \cup A_{3}^{\prime}$ is a dominating set of $\operatorname{Id} G$, as $A_{1} \cup A_{2} \succ V-A_{3} \cup A_{1}^{\prime}$ and $A_{2}^{\prime} \cup A_{3}^{\prime} \succ V^{\prime}-A_{1}^{\prime} \cup A_{3}$. By Lemma 4.4, $D$ is a weakly convex set. Thus, $\operatorname{Id} G$ has a weakly convex dominating set $D \notin\left\{V, V^{\prime}\right\}$ of size $|A|+\left|A_{2}\right|$.

Now let $D$ be a weakly convex set of $\operatorname{Id} G$. By Lemma $4.10, A=D_{1} \cup D_{2}$ is a weakly convex dominating set in $G$ of cardinality $|D|-\left|D_{1} \cap D_{2}\right|$. We define $A_{1}=D_{1}-D_{2}, A_{2}=D_{1} \cap D_{2}$ and $A_{3}=D_{2}-D_{1}$. Then $A_{1} \cup A_{2} \succ V-A_{3}$ and $A_{3} \cup A_{2} \succ V-A_{1}$ because $D$ is a dominating set of $\operatorname{Id} G$ and, by Lemma 4.4, the set $A$ satisfies conditions (1)-(2).

For example, graph $G$ in Figure 1 has such a $\gamma_{w c o n}$-set. As a result $\gamma_{w c o n}(\operatorname{Id} G)$ $<\min \left\{\left|V_{G}\right|, 2 \gamma_{\text {wcon }}(G)\right\}$.


Figure 1. The set $A_{1} \cup A_{2} \cup A_{2}^{\prime} \cup A_{3}^{\prime}$ is a weakly convex dominating set of $\operatorname{Id} G$ with $\gamma_{\text {wcon }}(G)+1$ vertices.

## 5. Upper and Lower Bounds

It is well known that the inequalities

$$
\begin{equation*}
\gamma(G) \leq \gamma(\pi G) \leq 2 \gamma(G) \tag{1}
\end{equation*}
$$

hold for any graph $G$ and any permutation $\pi$ of its vertex set. At the conference "Colorings, Independence and Domination" in 2015 Rita Zuazua asked whether similar inequalities hold for convex and weakly convex domination, i.e.,

$$
\begin{equation*}
\gamma_{w c o n}(G) \leq \gamma_{w c o n}(\pi G) \leq 2 \gamma_{w c o n}(G) \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{c o n}(G) \leq \gamma_{c o n}(\pi G) \leq 2 \gamma_{c o n}(G) \tag{3}
\end{equation*}
$$

However, this is not true in general. The smallest counterexample is the path $P_{3}$ with $V_{P_{3}}=\{1,2,3\}, E_{P_{3}}=\{12,23\}$ and the permutation $\pi=(12)$. In this case $\gamma_{c o n}\left(P_{3}\right)=\gamma_{w c o n}\left(P_{3}\right)=1$ while $\gamma_{c o n}\left(\pi P_{3}\right)=\gamma_{w c o n}\left(\pi P_{3}\right)=3$.

For a star $K_{1, k}$ with $k>2$ and the permutation $\pi=(01)$, where 0 is the central vertex and 1 is one of the other vertices, we have $\gamma_{c o n}\left(K_{1, k}\right)=\gamma_{w c o n}\left(K_{1, k}\right)=1$ and $\gamma_{c o n}\left(\pi K_{1, k}\right)=4$ while $\gamma_{w c o n}\left(K_{1, k}\right)=3$. Thus, the upper bounds in (2) and (3) do not hold for $K_{1, k}$.

Furthermore, for every $k \in \mathbb{N}$ there is a graph $G$ and permutation $\pi$ such that $\gamma_{w c o n}(G)-\gamma_{w c o n}(\pi G) \geq k$.

Let us begin with the cycle $C_{7}=(\{0,1,2,3,4,5,6\},\{01,12,23,34,45,56,60\})$ and the permutation $\pi=(13)(46)$. The weakly convex domination number of $C_{7}$ is 7 , but the graph $\pi C_{7}$ can be dominated by a weakly convex set with only 6 vertices: $\left\{0,0^{\prime}, 1,1^{\prime}, 6,6^{\prime}\right\}$.

In fact, the difference can be arbitrarily large. For any $k \in \mathbb{N}$ we can construct a graph $G_{k}$ as follows (see Figure 2).

1. Take $k$ copies of $C_{7}$. Denote the $i$-th copy of the vertex $j$ by $(i, j)$.
2. Replace the vertices $(1,0), \ldots,(k, 0)$ with a single vertex $(0,0)$.

The permutation $\pi_{k}$ is defined as $\pi_{k}(i, j)=(i, \pi(j))$. Then $\gamma_{w c o n}\left(G_{k}\right)=6 k+1$ and $\gamma_{w c o n}\left(\pi_{k} G_{k}\right)=4 k+2$. (The set $\left\{(0,0),(0,0)^{\prime},(1,1), \ldots,(k, 1),(1,1)^{\prime}, \ldots\right.$, $\left.(k, 1)^{\prime},(1,6), \ldots,(k, 6),(1,6)^{\prime}, \ldots,(k, 6)^{\prime}\right\}$ is a weakly convex dominating set of $\pi_{k} G_{k}$.) Hence $\gamma_{w c o n}\left(G_{k}\right)-\gamma_{w c o n}\left(\pi_{k} G_{k}\right)=2 k-1$.


Figure 2. The graphs $G_{k}$ and $\pi_{k} G_{k}$ and their $\gamma_{c o n}$-sets.
The second inequality in (2) can also be violated.
Let us consider the path $P_{6}=(\{0,1,2,3,4,5\},\{01,12,23,34,45\})$ and the permutation $\sigma=(14)(23)$. The weakly convex domination number of $P_{6}$ is 4 , but the weakly convex domination number of $\sigma P_{6}$ is 12 .

For $k \geq 2$ we construct the graph $H_{k}$ as follows.

1. Take $k$ paths $P_{6}$. Denote the $i$-th copy of the vertex $j$ as $(i, j)$,
2. Replace the vertices $(1,0), \ldots,(k, 0)$ with a single vertex $(0,0)$.

The permutation $\sigma_{k}$ is defined as $\sigma_{k}(i, j)=(i, \sigma(j))$.

It is easy to see that $\gamma_{w c o n}\left(H_{k}\right)=4 k+1$ and $\gamma_{w c o n}\left(\pi_{k} H_{k}\right)=10 k+2$. Thus $\gamma_{w c o n}\left(\sigma_{k} H_{k}\right)-2 \gamma_{w c o n}\left(H_{k}\right)=2 k$. Once again, the difference can be arbitrarily large.

Thus for any $k \in \mathbb{N}$ there exist graphs $G, H$ and permutations $\pi: V_{G} \mapsto V_{G}$, $\sigma: V_{H} \mapsto V_{H}$ such that $\gamma_{w c o n}(G)-\gamma_{w c o n}(\pi G) \geq k$ and $\gamma_{w c o n}(\sigma H)-2 \gamma_{w c o n}(H)$ $\geq k$.

Both inequalities (3) are also violated by entire families of graphs.


Figure 3. The graphs $H_{k}$ and $\sigma_{k} H_{k}$ and their $\gamma_{w c o n}$-sets.
Let $T_{k, l}$ be a tree with $V_{T_{k, l}}=\{0,1, \ldots, k,(1,1), \ldots,(1, l), \ldots,(k, 1), \ldots$, $(k, l)\}$ and $E_{T_{k, l}}=\{0 i: 1 \leq i \leq k\} \cup\{i(i, j): 1 \leq i \leq k, 1 \leq j \leq l\}$ for $k \geq 2$ and $l \geq 1$ (see Figure 4) and let $\pi_{k, l}=(1, \ldots, k)$.

Every convex set of $\pi_{k, l} T_{k, l}$ which dominates $S=\{(i, j): 1 \leq i \leq k, 1 \leq$ $j \leq l\} \cup\left\{(i, j)^{\prime}: 1 \leq i \leq k, 1 \leq j \leq l\right\}$ contains $\left\{1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}\right\}$. Every convex set containing $\left\{1, \ldots, k, 1^{\prime}, \ldots, k^{\prime}\right\}$ also contains $S \cup\left\{0,0^{\prime}\right\}$, as $i, 0,0^{\prime}, i^{\prime}$ and $i,(i, j),(i, j)^{\prime}, i^{\prime}$ are all shortest $i-i^{\prime}$ paths for $1 \leq i \leq k$. Thus, we have $\gamma_{c o n}\left(\pi_{k, l} T_{k, l}\right)=2 k l+2 k+2$. At the same time we have $\gamma_{c o n}\left(T_{k, l}\right)=k+1$. Therefore, $\gamma_{c o n}\left(\pi_{k, l} T_{k, l}\right)-2 \gamma_{c o n}\left(T_{k, l}\right)=2 k l$.

The first inequality in (3) can also be violated. For $k \geq 3$ let $G$ be a graph constructed as follows (see Figure 5).

1. Take $k$ copies of the path $P_{7}$ with $V_{P_{7}^{i}}=\{(j, i): 1 \leq j \leq 7\}$ and $E_{P_{7}^{i}}=$ $\{(j, i)(j+1, i): 2 \leq j \leq 6\}$.
2. For $j \in\{1,2,6,7\}$ replace the set $\{(j, i): 1 \leq i \leq k\}$ with a single vertex $j$.
3. For $2 \leq i \leq k$ add edges $(4,1)(4, i)$.

We define the permutation $\pi_{k}$ as $\pi_{k}=(26(5,1)(3,1))$.

$\gamma_{\text {con }}\left(T_{3,3}\right)=4$

$\gamma_{\text {con }}\left(\pi_{3} T_{3,3}\right)=26$

Figure 4. The graphs $T_{3,3}$ and $\pi_{3,3} T_{3,3}$ and their $\gamma_{c o n}$-sets.


Figure 5. The graphs $G_{k}$ and $\pi_{k} G_{k}$ and their $\gamma_{c o n}$-sets.
Every convex dominating set of $G_{k}$ must contain the vertices 2 and 6 , as well as all vertices of every shortest $2-6$ path. Since $2,(3, i),(4, i),(5, i), 6$ for $i \in\{1, \ldots, k\}$ are all shortest $2-6$ paths, $\gamma_{c o n}\left(G_{k}\right)=3 k+2$. However, the set $\left\{2,2^{\prime},(3,1),(3,1)^{\prime},(4,1),(4,1)^{\prime},(5,1),(5,1)^{\prime}, 6,6^{\prime}\right\}$ is a convex dominating set of cardinality 10 in $\pi_{k} G_{k}$ for any $k$. Thus, the difference $\gamma_{c o n}(G)-\gamma_{c o n}(\pi G)$ can be arbitrarily large.

In fact, the above examples show a stronger property of the convex domination number.

Remark 5.1. The convex domination number of $\pi G$ cannot be bounded in terms of $\gamma_{\text {con }}(G)$.

Proof. Notice that for the graphs $G_{k}$ defined above $\gamma_{c o n}\left(\pi_{k} G_{k}\right)$ is constant, while $\gamma_{c o n}\left(G_{k}\right)$ grows with the increase of $k$. This shows that there is no upper bound on $\gamma_{c o n}(\pi G)$ depending only on $\gamma_{c o n}(G)$.

Similarly, if $k$ is constant and $l$ increases, $\gamma_{c o n}\left(\pi_{k, l} T_{k, l}\right)$ for the tree $T_{k, l}$ increases, while $\gamma_{c o n}\left(T_{k, l}\right)$ remains constant. Thus there is no lower bound on $\gamma_{c o n}(\pi G)$ depending solely on $\gamma_{c o n}(G)$.

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