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CONVEX AND WEAKLY CONVEX DOMINATION IN PRISM GRAPHS

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Abstract

For a given graph $G = (V, E)$ and permutation $\pi : V \mapsto V$ the prism πG of G is defined as follows: $V(\pi G) = V(G) \cup V(G')$, where G' is a copy of G, and $E(\pi G) = E(G) \cup E(G') \cup M_\pi$, where $M_\pi = \{uv' : u \in V(G), v = \pi(u)\}$ and v' denotes the copy of v in G'.

We study and compare the properties of convex and weakly convex dominating sets in prism graphs. In particular, we characterize prism γ_{con}-fixers and -doublers. We also show that the differences $\gamma_{wcon}(G) - \gamma_{wcon}(\pi G)$ and $\gamma_{wcon}(\pi G) - 2\gamma_{wcon}(G)$ can be arbitrarily large, and that the convex domination number of πG cannot be bounded in terms of $\gamma_{con}(G)$.

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1. Introduction

Let $G = (V_G, E_G)$ be an undirected graph, let $\pi: V_G \mapsto V_G$ be a permutation of its vertex set and let G' be a vertex-disjoint copy of G. We denote the copy of a vertex $v \in V_G$ in G' by v'. If S is a set of vertices of G, then S' denotes the copy of S in G', i.e., the set $\{v' : v \in S\}$. By V and V' we denote the vertex sets of G and G', respectively. The *prism graph* πG is a graph with vertex set $V \cup V'$ and edge set $E_G \cup E_{G'} \cup M_\pi$, where $M_\pi = \{u\pi(u)' : u \in V_G\}$.

The *open neighborhood* of a vertex $v \in V_G$, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G. The *closed neighborhood*, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. If S is a subset of V_G, then by $N_S(v)$ and $N_S[v]$ we denote $N_G(v) \cap S$ and $N_G[v] \cap S$, respectively. The neighborhood $N_G[S]$ of a set S is the set $\bigcup_{v \in S} N_G[v]$. If A and B are disjoint sets of vertices of a graph G, then by $E(A, B)$ we denote the set of all edges of G joining a vertex from A with a vertex from B. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u, v)$, is the minimum length of a u–v path in G. The *diameter* $\mathrm{diam}(G)$ of a graph G is the maximum distance between two vertices of G.

We say that a set $A \subseteq V_G$ *dominates* $B \subseteq V_G$ if $B \subseteq N_G[A]$. We denote this by $A \succ B$. A set $A \subseteq V_G$ is called a *dominating set* of the graph G if it dominates V_G, i.e., if $N_G[A] = V_G$. The minimum cardinality of a dominating set in G is called the *domination number* of G and denoted $\gamma(G)$. A minimum dominating set of a graph is sometimes called a *γ-set*.

Prism graphs were first defined in [2] and the problem of domination in prism graphs was first studied by Burger, Mynhardt and Weakley [1].

In a connected graph G a set of vertices $A \subseteq V_G$ is said to be *convex* if for every pair of vertices $u, v \in A$ the set A contains all vertices of every shortest u–v path.

A *convex dominating set* of G is a dominating set of G which is convex. The minimum cardinality of a convex dominating set of G is called the *convex domination number* of G and denoted as $\gamma_{con}(G)$. A *γ_{con}-set* of G is a convex dominating set of cardinality $\gamma_{con}(G)$.

A set A of vertices in a connected graph G is said to be *weakly convex* if for every pair of vertices $u, v \in A$ it contains all vertices of at least one shortest u–v path.

A *weakly convex dominating set* G is a dominating set of G which is weakly convex. The minimum cardinality of a weakly convex dominating set of G is called the *weakly convex domination number* and denoted as $\gamma_{wcon}(G)$. A *γ_{wcon}-set* of G is a minimum weakly convex dominating set.

A set $A \subseteq V_G$ is said to be *connected* if it induces a connected subgraph, i.e., if for every pair of vertices $u, v \in A$ there exists a u–v path contained entirely in A.

A *connected dominating set* of G is a dominating set which is connected. The *connected domination number* of G, denoted as $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. A *γ_c-set* of G is a connected dominating set of cardinality $\gamma_c(G)$.

In this paper we compare the properties of convex and weakly convex sets in prism graphs, particularly convex and weakly convex dominating sets. We also generalize some known properties of convex domination in prism graphs to weakly convex and connected domination.

2. Connected, Convex and Weakly Convex Domination

It is clear that the notions of convex, weakly convex and connected domination are closely related. Since every convex set is weakly convex and every weakly convex set is connected, it is easy to see that $\gamma(G) \le \gamma_c(G) \le \gamma_{wcon}(G) \le \gamma_{con}(G)$. Many, but not all, properties of connected sets can be extended to both convex and weakly convex sets.

Lemma 2.1. *If $G \neq K_1$ is a connected graph and π is a permutation of V_G, then $\gamma_c(\pi G) \ge \gamma(G) + 1$.*

Proof. Obviously, if $\gamma(\pi G) \ge \gamma(G) + 1$, then $\gamma_c(\pi G) \ge \gamma(G) + 1$.

Now let $\gamma(\pi G) = \gamma(G)$ and let D be a γ-set of πG. We denote $D_1 = D \cap V$ and $D_2' = D \cap V'$. Note that neither of these sets is empty, as $|D| = \gamma(G) < |V_G|$ and a set of fewer than $|V_G|$ vertices in V or V' cannot dominate all of πG. Since the only vertices in V' dominated by D_1 are in $\pi(D_1)'$, it follows that $B = \pi(D_1) \cup D_2 \succ V$ and therefore $|B| = |D| = \gamma(G)$. Thus, $\pi(D_1) \cap D_2 = \emptyset$. Since $N_{V'}(D_1) = \pi(D_1)'$, it follows that $E(D_1, D_2') = \emptyset$ and therefore D is not connected. Thus, every connected dominating set of πG has cardinality at least $\gamma(G) + 1$. ■

Since $\gamma_c(G) \le \gamma_{wcon}(G) \le \gamma_{con}(G)$, Lemma 2.1 also applies to convex and weakly convex domination, that is, for any graph G and permutation π of V_G we have $\gamma_{con}(\pi G) \ge \gamma(G) + 1$ and $\gamma_{wcon}(\pi G) \ge \gamma(G) + 1$.

Lemma 2.2. *For any connected graph G and permutation π of V_G, if G has a dominating set A which can be partitioned into three nonempty disjoint sets A_1, A_2, A_3 such that*

(1) $A_1 \cup A_2 \succ V_G - A_3$,

(2) $A_1 \cup A_2$ *is connected,*

(3) $\pi(A_2 \cup A_3)$ *is connected,*

(4) $\pi(A_2 \cup A_3) \succ V_G - \pi(A_1)$,

then $D = A_1 \cup A_2 \cup (\pi(A_2 \cup A_3))'$ is a connected dominating set of size $|A| + |A_2|$ in πG.

Proof. Since $A_1 \cup A_2 \succ (\pi(A_1))' \cup (V - A_3)$ and $(\pi(A_2 \cup A_3))' \succ A_3 \cup (V' - (\pi(A_1))')$, it is clear that $D = A_1 \cup A_2 \cup (\pi(A_2 \cup A_3))' \succ ((\pi(A_1))' \cup (V - A_3)) \cup (A_3 \cup (V' - (\pi(A_1))')) = V \cup V'$, that is, D is a dominating set in πG. The sets $D_1 = A_1 \cup A_2$ and $D_2' = (\pi(A_2 \cup A_3))'$ are connected and $E(D_1, D_2') \neq \emptyset$, and it follows that D is connected. Hence D is a connected dominating set of πG. ■

Note that this particular result cannot be extended to convex domination as, for example, if $\pi = \mathrm{Id}$, then for any set $A = A_1 \cup A_2 \cup A_3$, where A_1, A_2 and

A_3 are disjoint and not empty, the set $A_1 \cup A_2 \cup (A_2 \cup A_3)'$ is not convex. For connected domination, however, we can use it to show exactly when the bound from Lemma 2.1 is achieved.

Proposition 2.3. *Let $G \notin \{K_1, K_2\}$ be a connected graph and let π be a permutation of V_G. Then $\gamma_c(\pi G) = \gamma(G) + 1$ if and only if there exists a γ-set A which can be partitioned into two disjoint subsets A_1, A_2, and a vertex $v \in A_1$ such that*

(1) $A_1 \succ V_G - A_2$,

(2) A_1 *is connected,*

(3) $\pi(A_2 \cup \{v\}) \succ V_G - \pi(A_1)$,

(4) $\pi(A_2 \cup \{v\})$ *is connected.*

Proof. Let $D = A_1 \cup (\pi(A_2 \cup \{v\}))'$ be a set of vertices in πG. It follows from Lemma 2.2 that D is a connected dominating set of size $\gamma(G) + 1$ in πG, so $\gamma_c(\pi G) \leq \gamma(G) + 1$. By Lemma 2.1, $\gamma_c(\pi(G)) \geq \gamma(G) + 1$. Thus, $\gamma_c(\pi G) = \gamma(G) + 1$.

Now let D be a connected dominating set of size $\gamma(G) + 1$ in πG and let $D_1 = D \cap V$ and $D_2' = D \cap V'$. Both sets D_1, D_2 are nonempty, as $\gamma_c(\pi G) < |V|$ and a set of fewer than $|V|$ vertices in V or V' cannot dominate all of πG.

Since D is a dominating set of πG, we have $D_2 \succ V - \pi(D_1)$ and $D_1 \succ V - \pi^{-1}(D_2)$, and thus $D_1 \cup \pi^{-1}(D_2) \succ V$. This implies that $\left|D_1 \cup \pi^{-1}(D_2)\right| \geq \gamma(G)$. Furthermore, since D is connected, it is necessary that $D_1 \cap \pi^{-1}(D_2) \neq \emptyset$, but $\left|D_1 \cap \pi^{-1}(D_2)\right| \leq 1$, as $\left|D_1 \cup \pi^{-1}(D_2)\right| \geq \gamma(G)$ and $|D| = \gamma(G) + 1$. Thus, there exists exactly one vertex $v \in D_1$ such that $\pi(v) \in D_2$. Finally, the fact that D is connected and every path connecting a vertex from D_1 with a vertex from D_2 contains v and $\pi(v)'$ implies that both sets D_1 and D_2 are connected.

We now define $A_1 = D_1$, $A_2 = \pi^{-1}(D_2) - \{v\}$. The set $A = A_1 \cup A_2$ is clearly a γ-set of G satisfying conditions (1)–(4). ■

Again, the above is not true for convex domination, however, in Section 4 we prove a related property of weakly convex dominating sets.

3. Generalizing Some Properties of Convex Domination

Convex domination in prism graphs was studied by Lemańska and Zuazua in [3], where they prove the following theorem.

Theorem 3.1 [3]**.** *Let G be a connected graph. If $\gamma_{con}(G) = |V_G|$ and* $\operatorname{diam}(G) \leq 2$, *then $\gamma_{con}(G) = \gamma_{con}(\pi G)$ for every permutation π of V_G.*

The proof of Theorem 3.1 relies on some properties of convex dominating sets, which they prove in the same paper. We will now compare these properties with those of weakly convex dominating sets.

Theorem 3.2 [3]**.** *For any connected graph G.*

(1) *If* $\mathrm{diam}(G) \leq 2$, *then V and V' are both convex dominating sets of πG for any permutation π.*

(2) *If* $\mathrm{diam}(G) \geq 3$, *then there exist permutations π_1 and π_2 such that V is not a convex dominating set of $\pi_1 G$ and V' is not a convex dominating set of $\pi_2 G$.*

It follows that if $\mathrm{diam}(G) \leq 2$, then $\gamma_{con}(\pi G) \leq |V_G|$ for any π. Weakly convex domination number has a similar property.

Theorem 3.3. *Every connected graph G has the following properties.*

(1) *If* $\mathrm{diam}(G) \leq 3$, *then V and V' are weakly convex dominating sets of πG for every permutation π.*

(2) *If* $\mathrm{diam}(G) > 3$, *then there exist permutations π_1 and π_2 such that V is not a weakly convex dominating set of $\pi_1 G$ and V' is not a weakly convex dominating set of $\pi_2 G$.*

Proof. Obviously, for any permutation π, V and V' are dominating sets of πG. For any pair of vertices $u, v \in V$, the shortest u–v path containing at least one vertex from V' has length at least 3. Thus, if $\mathrm{diam}(G) \leq 3$, then $d_G(u,v) = d_{\pi G}(u,v)$ and V contains at least one shortest u–v path in πG. Similarly, if $\mathrm{diam}(G) \leq 3$, then V' must contain a shortest u'–v' path in πG for every $u', v' \in V'$.

Now, let $\mathrm{diam}(G)$ be at least 4 and let $u, v \in V$ be a pair of vertices such that $d_G(u,v) \geq 4$. If π_1 is a permutation such that $\pi_1(u)$ and $\pi_1(v)$ are adjacent, then $d_{\pi G}(u,v) = 3 < d_G(u,v)$. Thus, V is not a weakly convex dominating set in $\pi_1 G$. Similarly, for $\pi_2 = \pi_1^{-1}$, the set V' is not weakly convex. ■

The first part of Theorem 3.3 implies that $\gamma_{wcon}(\pi G) \leq |V_G|$ for any graph G with diameter at most 3.

In this section as well as the next one, when D is a convex or weakly convex dominating set of πG, we will denote $D_1 = D \cap V$ and $D_2' = D \cap V'$. We will denote the equivalent of D_2' in V by D_2.

Proposition 3.4 [3]**.** *For a connected graph G and permutation π of V_G, let D be a convex dominating set of πG. Then D has the following properties.*

(1) *If* $|D| < |V_G|$, *then* $D_1 \neq \emptyset$ *and* $D_2' \neq \emptyset$.

(2) *If* $D_1 \neq \emptyset$ *and* $D_2' \neq \emptyset$, *then there exists at least one* $x \in D_1$ *such that* $\pi(x) \in D_2$.

The above can be generalized to weakly convex and connected domination.

Proposition 3.5. *For a connected graph G and permutation π of V_G, let D be a connected dominating set of πG. Then D has the following properties.*

(1) *If* $|D| < |V_G|$, *then* $D_1 \neq \emptyset$ *and* $D'_2 \neq \emptyset$.

(2) *If* $D_1 \neq \emptyset$ *and* $D'_2 \neq \emptyset$, *then there exists at least one* $x \in D_1$ *such that* $\pi(x) \in D_2$.

Proof. If $D_1 = \emptyset$, then $D = D'_2$. But every vertex in D'_2 only dominates one vertex in V. It follows that if D is a dominating set, then $D_1 = \emptyset$ implies $|D| = |V|$. The same reasoning applies if $D'_2 = \emptyset$. Thus, if $|D| < |V|$, then the subsets D_1, D'_2 are not empty.

If both sets D_1 and D_2 are nonempty, then D contains a pair of vertices $v_1 \in D_1$, $v'_2 \in D'_2$. Since the set D is connected, there exists a v_1–v'_2 path contained entirely in D. Each vertex $v \in V$ has only one neighbor in V', namely $\pi(v)'$. Hence, every path connecting $v_1 \in D_1$ with $v'_2 \in D'_2$ contains a pair of vertices $x \in V$ and $\pi(x)' \in V'$. This shows that D_1 contains a vertex x such that $\pi(x) \in D_2$. ■

Since every weakly convex dominating set is a connected dominating set, we also have the following.

Corollary 3.6. *For a connected graph* G *and permutation* π *of* V_G, *let* D *be a weakly convex dominating set of* πG. *Then* D *has the following properties.*

(1) *If* $|D| < |V_G|$, *then* $D_1 \neq \emptyset$ *and* $D'_2 \neq \emptyset$.

(2) *If* $D_1 \neq \emptyset$ *and* $D'_2 \neq \emptyset$, *then there exists at least one* $x \in D_1$ *such that* $\pi(x) \in D_2$.

Lemma 3.7 [3]**.** *Let* G *be a connected graph in which* $\operatorname{diam}(G) \leq 2$. *Let* D *be a convex dominating set of* πG. *Then the set* D *has the following properties.*

(1) *If* $\pi(D_1) \subseteq D_2$, *then* D_2 *is a convex dominating set of* G.

(2) *If* $\pi^{-1}(D_2) \subseteq D_1$, *then* D_1 *is a convex dominating set of* G.

Again, we can prove a similar property for weakly convex domination.

Lemma 3.8. *Let* G *be a connected graph in which* $\operatorname{diam}(G) \leq 2$. *Let* D *be a weakly convex dominating set of* πG. *Then the set* D *has the following properties.*

(1) *If* $\pi(D_1) \subseteq D_2$, *then* D_2 *is a weakly convex dominating set of* G.

(2) *If* $\pi^{-1}(D_2) \subseteq D_1$, *then* D_1 *is a weakly convex dominating set of* G.

Proof. If $\pi(D_1) \subseteq D_2$, then clearly D'_2 dominates V', as D_1 does not dominate any part of $V' - D'_2$. It follows that D_2 is a dominating set of G. For any two vertices $u', v' \in D'_2$ a shortest possible u'–v'-path in πG containing a vertex from V has length at least 3. If $\operatorname{diam}(G) \leq 2$, then any shortest u'–v'-path must be contained in D'_2. Thus, for every $u, v \in D_2$ the set D_2 contains a shortest u–v path. Hence, D_2 is a weakly convex dominating set of G.

Similarly, if $\pi^{-1}(D_2) \subseteq D_1$, then D_1 is a dominating set of G because D_2' does not dominate any part of $V - D_1$. D_1 is also convex, because for any $u, v \in D_1$ no shortest u–v path passes through D_2'. ■

Note that, unlike Theorem 3.3, the above does not hold for every G such that $\operatorname{diam}(G) \leq 3$. For example, if $G = P_4 = (\{1,2,3,4\}, \{12,23,34\})$ and $\pi = (12)(34)$, the set $D = \{1', 2, 3, 4'\}$ is a weakly convex dominating set of πG. The set $\{1,4\}$ contains $\pi(\{2,3\})$ and it is not a weakly convex set.

For weakly convex domination an exact analogue of Theorem 3.1 makes no sense, as there is no graph with $\operatorname{diam}(G) = 2$ and $\gamma_{wcon}(G) = |V_G|$. Since Lemańska and Zuazua's proof relies on Lemma 3.7, whose weakly convex analogue does not hold for graphs with diameter 3, as well as some properties of convex sets which weakly convex sets do not have, it seems unlikely that all graphs with diameter 3 would have such a property. Indeed, the cycle C_7 has diameter 3 and $\gamma_{wcon}(C_7) = 7$, yet a permutation π, defined in Section 5, exists such that $\gamma_{wcon}(\pi C_7) = 6$.

4. Convex and Weakly Convex Domination in IdG

We now consider the special case where the permutation is $\pi = \mathrm{Id}$. A graph G is called a *prism fixer* if $\gamma(\mathrm{Id}G) = \gamma(G)$ and a *prism doubler* if $\gamma(\mathrm{Id}G) = 2\gamma(G)$. A graph G is called a *universal fixer* if $\gamma(\pi G) = \gamma(G)$ for every permutation π of V_G and a *universal doubler* if $\gamma(\pi G) = 2\gamma(G)$ for every π. Prism fixers are characterized in [4] and universal fixers in [5] and [6]. Prism doublers and universal doublers are studied in [1].

Similarly, a graph G such that $\gamma_{con}(\mathrm{Id}G) = \gamma_{con}(G)$ is called a *prism γ_{con}-fixer* and a graph with $\gamma_{con}(\mathrm{Id}G) = 2\gamma_{con}(G)$ is called a *prism γ_{con}-doubler*. A universal γ_{con}-fixer is a graph such that for every π $\gamma_{con}(\pi G) = \gamma_{con}(G)$ and a universal γ_{con}-doubler is a graph such that $\gamma_{con}(\pi G) = 2\gamma_{con}(G)$ for every π.

We begin this section by studying some properties of convex and weakly convex sets in IdG.

Observation 4.1. *For any two vertices $u, v \in V_G$ in a connected graph G:*

(1) $d_{\mathrm{Id}G}(u, v) = d_{\mathrm{Id}G}(u', v') = d_G(u, v)$,

(2) $d_{\mathrm{Id}G}(u, v') = d_{\mathrm{Id}G}(u', v) = d_G(u, v) + 1$,

(3) *every shortest u–v' path in* IdG *has the form $u = v_0, v_1, \ldots, v_i, v_i', \ldots, v_k' = v'$ for some shortest u–v path $u = v_0, v_1, \ldots, v_i, \ldots, v_k = v$ in G,*

(4) *every path in* IdG *of the form $u = v_0, v_1, \ldots, v_i, v_i', \ldots, v_k' = v'$ for some shortest u–v path $u = v_0, v_1, \ldots, v_i, \ldots, v_k = v$ in G is a shortest u–v' path,*

(5) *every shortest u–v path in* IdG *is a shortest u–v path in G,*

(6) *every shortest u–v path in G is a shortest u–v path in* IdG.

Using the above observation we obtain the following lemma.

Lemma 4.2. *If $S \subseteq V_G$ is a convex* (*weakly convex*) *set in G, then S, S' and $S \cup S'$ are convex* (*weakly convex*) *sets in* IdG.

Proof. Let $S \subseteq V$ be a convex set in G and let u and v be any two vertices in S. It follows from Observation 4.1(5) that S contains all shortest u–v paths in IdG. The set S' is also a convex set in IdG, as P' is a shortest u'–v' path for every shortest u–v path P. Thus S and S' are convex sets in IdG.

Observation 4.1(3) implies that $S \cup S'$ contains all shortest u–v' paths in IdG. Thus $S \cup S'$ is also a convex set.

Now let $S \subseteq V$ be a weakly convex set in G and let u and v be any two vertices in S. By Observation 4.1(6) the set S contains at least one shortest u–v path in IdG. The set S' is also a weakly convex set in IdG, as P' is a shortest u'–v' path for every shortest u–v path P. Thus S and S' are weakly convex sets in IdG.

By Observation 4.1(4) the set $S \cup S'$ contains all shortest u–v' paths in IdG. Thus $S \cup S'$ is also a convex set. ■

Corollary 4.3. *$S \subset V_{\mathrm{Id}G}$ is a convex set in* IdG *if and only if $S \in \{S_1, S_1', S_1 \cup S_1'\}$ for some convex set $S_1 \in V_G$.*

Proof. If S_1 is a convex set in G then, by Lemma 4.2, S_1, S_1' and $S_1 \cup S_1'$ are convex sets in IdG.

If S is a convex set in IdG then either $S \subseteq V$, $S \subseteq V'$ or $S = S_1 \cup S_2'$, where S_1 and S_2 are convex sets in G. For any two vertices $u \in S_1, v' \in S_2'$ the set S contains all shortest u–v' paths. By Observation 4.1(4), this implies that S_1 and S_2 both contain u, v and all shortest u–v paths in G. It follows that $S_1 = S_2$ and thus $S = S_1 \cup S_1'$. ■

Weakly convex sets have an additional property.

Lemma 4.4. *A set $S \subseteq V_{\mathrm{Id}G}$, where $S_1 = S \cap V, S_2' = S \cap V'$ is weakly convex if and only if*

(1) *S_1, S_2 and $S_1 \cup S_2$ are weakly convex sets in G.*

(2) *For every $u \in S_1, v \in S_2$, a shortest u–v path in $S_1 \cup S_2$ contains a vertex from $S_1 \cap S_2$.*

Proof. Let S be a weakly convex set in IdG with $S_1 = S \cap V, S_2' = S \cap V'$. If $u, v \in S_i$ for $i \in \{1, 2\}$ then, by Observation 4.1(5), S_i contains a shortest u–v path. If $u \in S_1, v \in S_2$, then by Observation 4.1(3) S contains a shortest u–v' path of the form $u = v_0, v_1, \ldots, v_i, v_i', v_{i+1}', \ldots, v_k' = v'$ and thus $S_1 \cup S_2$

contains a shortest u–v path of the form $u = v_0, v_1, \ldots, v_i, v_{i+1}, \ldots, v_k = v$, where $v_i \in S_1 \cap S_2$.

Now let $S = S_1 \cup S_2'$ be a set satisfying conditions (1), (2). If $u, v \in S_i$ for $i \in \{1, 2\}$ then, by Observation 4.1(6), S_i contains a shortest u–v (or u'–v') path in IdG. If $u \in S_1, v \in S_2$ then $S_1 \cup S_2$ contains a shortest u–v path P containing a vertex $w \in S_1 \cap S_2$. Since S_1 and S_2 are weakly convex, they must contain a shortest u–w path P_1 and a shortest w–v path P_2, respectively. By Observation 4.1(4) $Q = P_1P_2'$ is a shortest u–v' path in IdG. Thus S is a weakly convex set in IdG. ■

This is not the case with convex sets. If a convex set in IdG contains a pair of vertices u, v' it must also contain u' and v. This leads to some differences between convex and weakly convex domination.

Theorem 4.5. *If G is any connected graph, then $\gamma_{con}(\mathrm{Id}G) = \min\{2\gamma_{con}(G), |V_G|\}$.*

Proof. Let D be a γ_{con}-set of IdG. If $D \cap V' = \emptyset$, then obviously $D = V$. Similarly, if $D \cap V = \emptyset$, then $D = V'$. In this case $|D| = |V_G|$. Otherwise, by Corollary 4.3, $D = D_1 \cup D_1'$ for some convex set $D_1 \in V_G$. Since D_1' dominates no vertices in $V - D_1$, it is clear that $D_1 \succ V$. The set D_1 is a γ_{con}-set of G. It follows that in this case $|D| = 2\,|D_1| = 2\gamma_{con}(G)$. ■

As a result, we have the following.

Corollary 4.6. *Every connected graph G has the following properties:*

(1) *G is a prism γ_{con}-fixer if and only if $\gamma_{con}(G) = |V_G|$.*

(2) *G is a prism γ_{con}-doubler if and only if $\gamma_{con}(G) \leq \frac{1}{2}\,|V_G|$.*

Proof. If $\gamma_{con}(G) = \gamma_{con}(\mathrm{Id}G) = \min\{2\gamma_{con}(G), |V|\}$, then $\gamma_{con}(G) = |V|$.

$\gamma_{con}(\mathrm{Id}G) = 2\gamma_{con}(G)$ if and only if $2\gamma_{con}(G) \leq |V|$, if and only if $\gamma_{con}(G) \leq \frac{1}{2}\,|V_G|$. ■

Since every universal γ_{con}-fixer is a prism γ_{con}-fixer, and every universal γ_{con}-doubler is a prism γ_{con}-doubler, we also have the following corollary.

Corollary 4.7. *Let G be a connected graph. Then*

(1) *If G is a universal γ_{con}-fixer, then $\gamma_{con}(G) = |V_G|$.*

(2) *If G is a universal γ_{con}-doubler, then $\gamma_{con}(G) \leq \frac{1}{2}\,|V_G|$.*

A similar property of weakly convex domination follows from Lemma 4.2.

Theorem 4.8. *If G is a connected graph, then $\gamma_{wcon}(\mathrm{Id}G) \leq \min\{|V_G|, 2\gamma_{wcon}(G)\}$.*

Proof. Obviously, V is a dominating set in IdG. By Lemma 4.2, it is also a weakly convex set in IdG. Thus, $\gamma_{wcon}(\mathrm{Id}G) \leq |V_G|$.

If S is a γ_{wcon}-set of G, then $S \cup S'$ is a dominating set in IdG, as $S \succ V$ and $S' \succ V'$. Lemma 4.2 implies that $S \cup S'$ is a (not necessarily minimal) weakly convex dominating set in IdG and thus $\gamma_{wcon}(\mathrm{Id}G) \leq 2\gamma_{wcon}(G)$. ■

However, thanks to Lemma 4.4, $\gamma_{wcon}(\mathrm{Id}G)$ is not necessarily equal to $\min\{|V_G|, 2\gamma_{wcon}(G)\}$. In fact, the following is true.

Theorem 4.9. *Let G be a connected graph. The graph* IdG *has a weakly convex dominating set $D \notin \{V, V'\}$ of cardinality $\gamma_{wcon}(G) + k$ if and only if G has a weakly convex dominating set A which can be partitioned into three nonempty sets A_1, A_2, A_3 such that $|A| + |A_2| = \gamma_{wcon}(G) + k$ and*

(1) *$A_1 \cup A_2$ and $A_2 \cup A_3$ are weakly convex,*

(2) *for every $u \in A_1 \cup A_2, v \in A_2 \cup A_3$, a shortest u–v path in A contains a vertex from A_2,*

(3) *$A_1 \cup A_2 \succ V - A_3$ and $A_3 \cup A_2 \succ V - A_1$.*

In particular, $\gamma_{wcon}(\mathrm{Id}G) = \gamma_{wcon}(G) + 1$ if and only if G has a γ_{wcon}-set $A = A_1 \cup A_2 \cup A_3$ such that conditions (1)–(3) *are fulfilled and $|A_2| = 1$.*

To prove this result we will use the following lemma.

Lemma 4.10. *Let $D = D_1 \cup D_2'$ be a weakly convex dominating set of* IdG. *Then $D_G = D_1 \cup D_2$ is a weakly convex dominating set of G.*

Proof. Since D is a dominating set of IdG, we have $D_1 \succ V - D_2$ and $D_2 \succ V - D_1$. It follows that $D_1 \cup D_2$ dominates V_G. By Lemma 4.4, $D_1 \cup D_2$ is also a weakly convex set. Thus, it is a weakly convex dominating set of G.

If $D_1 = \emptyset$ or $D_2 = \emptyset$, then $D_G = V$, which is also a convex dominating set in G. ■

***Proof of Theorem* 4.9.** Let $A = A_1 \cup A_2 \cup A_3$ be a weakly convex dominating set of a connected graph G. The set $D = A_1 \cup A_2 \cup A_2' \cup A_3'$ is a dominating set of IdG, as $A_1 \cup A_2 \succ V - A_3 \cup A_1'$ and $A_2' \cup A_3' \succ V' - A_1' \cup A_3$. By Lemma 4.4, D is a weakly convex set. Thus, IdG has a weakly convex dominating set $D \notin \{V, V'\}$ of size $|A| + |A_2|$.

Now let D be a weakly convex set of IdG. By Lemma 4.10, $A = D_1 \cup D_2$ is a weakly convex dominating set in G of cardinality $|D| - |D_1 \cap D_2|$. We define $A_1 = D_1 - D_2, A_2 = D_1 \cap D_2$ and $A_3 = D_2 - D_1$. Then $A_1 \cup A_2 \succ V - A_3$ and $A_3 \cup A_2 \succ V - A_1$ because D is a dominating set of IdG and, by Lemma 4.4, the set A satisfies conditions (1)–(2). ■

For example, graph G in Figure 1 has such a γ_{wcon}-set. As a result $\gamma_{wcon}(\mathrm{Id}G) < \min\{|V_G|\,,\, 2\gamma_{wcon}(G)\}$.

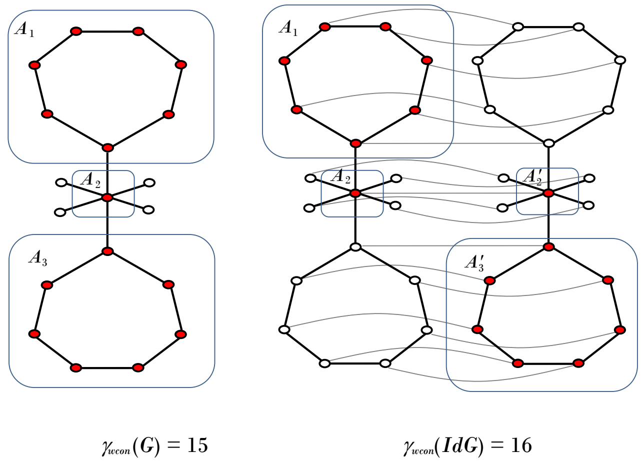

Figure 1. The set $A_1 \cup A_2 \cup A_2' \cup A_3'$ is a weakly convex dominating set of $\mathrm{Id}G$ with $\gamma_{wcon}(G) + 1$ vertices.

5. Upper and Lower Bounds

It is well known that the inequalities

$$(1) \qquad \gamma(G) \le \gamma(\pi G) \le 2\gamma(G)$$

hold for any graph G and any permutation π of its vertex set. At the conference "Colorings, Independence and Domination" in 2015 Rita Zuazua asked whether similar inequalities hold for convex and weakly convex domination, i.e.,

$$(2) \qquad \gamma_{wcon}(G) \le \gamma_{wcon}(\pi G) \le 2\gamma_{wcon}(G)$$

and

$$(3) \qquad \gamma_{con}(G) \le \gamma_{con}(\pi G) \le 2\gamma_{con}(G).$$

However, this is not true in general. The smallest counterexample is the path P_3 with $V_{P_3} = \{1, 2, 3\}, E_{P_3} = \{12, 23\}$ and the permutation $\pi = (12)$. In this case $\gamma_{con}(P_3) = \gamma_{wcon}(P_3) = 1$ while $\gamma_{con}(\pi P_3) = \gamma_{wcon}(\pi P_3) = 3$.

For a star $K_{1,k}$ with $k > 2$ and the permutation $\pi = (01)$, where 0 is the central vertex and 1 is one of the other vertices, we have $\gamma_{con}(K_{1,k}) = \gamma_{wcon}(K_{1,k}) = 1$ and $\gamma_{con}(\pi K_{1,k}) = 4$ while $\gamma_{wcon}(K_{1,k}) = 3$. Thus, the upper bounds in (2) and (3) do not hold for $K_{1,k}$.

Furthermore, for every $k \in \mathbb{N}$ there is a graph G and permutation π such that $\gamma_{wcon}(G) - \gamma_{wcon}(\pi G) \geq k$.

Let us begin with the cycle $C_7 = (\{0,1,2,3,4,5,6\}, \{01,12,23,34,45,56,60\})$ and the permutation $\pi = (13)(46)$. The weakly convex domination number of C_7 is 7, but the graph πC_7 can be dominated by a weakly convex set with only 6 vertices: $\{0, 0', 1, 1', 6, 6'\}$.

In fact, the difference can be arbitrarily large. For any $k \in \mathbb{N}$ we can construct a graph G_k as follows (see Figure 2).

1. Take k copies of C_7. Denote the i-th copy of the vertex j by (i, j).
2. Replace the vertices $(1,0), \ldots, (k,0)$ with a single vertex $(0,0)$.

The permutation π_k is defined as $\pi_k(i,j) = (i, \pi(j))$. Then $\gamma_{wcon}(G_k) = 6k+1$ and $\gamma_{wcon}(\pi_k G_k) = 4k+2$. (The set $\{(0,0), (0,0)', (1,1), \ldots, (k,1), (1,1)', \ldots, (k,1)', (1,6), \ldots, (k,6), (1,6)', \ldots, (k,6)'\}$ is a weakly convex dominating set of $\pi_k G_k$.) Hence $\gamma_{wcon}(G_k) - \gamma_{wcon}(\pi_k G_k) = 2k - 1$.

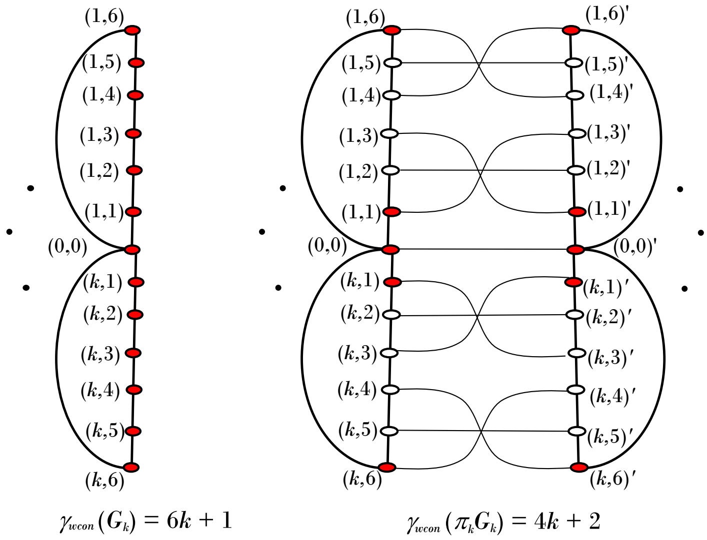

Figure 2. The graphs G_k and $\pi_k G_k$ and their γ_{con}-sets.

The second inequality in (2) can also be violated.

Let us consider the path $P_6 = (\{0,1,2,3,4,5\}, \{01,12,23,34,45\})$ and the permutation $\sigma = (14)(23)$. The weakly convex domination number of P_6 is 4, but the weakly convex domination number of σP_6 is 12.

For $k \geq 2$ we construct the graph H_k as follows.

1. Take k paths P_6. Denote the i-th copy of the vertex j as (i, j),
2. Replace the vertices $(1,0), \ldots, (k,0)$ with a single vertex $(0,0)$.

The permutation σ_k is defined as $\sigma_k(i,j) = (i, \sigma(j))$.

It is easy to see that $\gamma_{wcon}(H_k) = 4k+1$ and $\gamma_{wcon}(\pi_k H_k) = 10k+2$. Thus $\gamma_{wcon}(\sigma_k H_k) - 2\gamma_{wcon}(H_k) = 2k$. Once again, the difference can be arbitrarily large.

Thus for any $k \in \mathbb{N}$ there exist graphs G, H and permutations $\pi : V_G \mapsto V_G$, $\sigma : V_H \mapsto V_H$ such that $\gamma_{wcon}(G) - \gamma_{wcon}(\pi G) \geq k$ and $\gamma_{wcon}(\sigma H) - 2\gamma_{wcon}(H) \geq k$.

Both inequalities (3) are also violated by entire families of graphs.

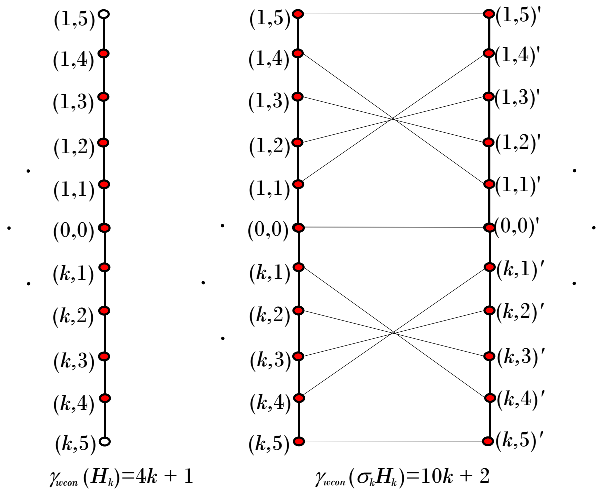

Figure 3. The graphs H_k and $\sigma_k H_k$ and their γ_{wcon}-sets.

Let $T_{k,l}$ be a tree with $V_{T_{k,l}} = \{0, 1, \ldots, k, (1,1), \ldots, (1,l), \ldots, (k,1), \ldots, (k,l)\}$ and $E_{T_{k,l}} = \{0i : 1 \leq i \leq k\} \cup \{i(i,j) : 1 \leq i \leq k, 1 \leq j \leq l\}$ for $k \geq 2$ and $l \geq 1$ (see Figure 4) and let $\pi_{k,l} = (1, \ldots, k)$.

Every convex set of $\pi_{k,l}T_{k,l}$ which dominates $S = \{(i,j) : 1 \leq i \leq k, 1 \leq j \leq l\} \cup \{(i,j)' : 1 \leq i \leq k, 1 \leq j \leq l\}$ contains $\{1, \ldots, k, 1', \ldots, k'\}$. Every convex set containing $\{1, \ldots, k, 1', \ldots, k'\}$ also contains $S \cup \{0, 0'\}$, as $i, 0, 0', i'$ and $i, (i,j), (i,j)', i'$ are all shortest $i - i'$ paths for $1 \leq i \leq k$. Thus, we have $\gamma_{con}(\pi_{k,l}T_{k,l}) = 2kl + 2k + 2$. At the same time we have $\gamma_{con}(T_{k,l}) = k+1$. Therefore, $\gamma_{con}(\pi_{k,l}T_{k,l}) - 2\gamma_{con}(T_{k,l}) = 2kl$.

The first inequality in (3) can also be violated. For $k \geq 3$ let G be a graph constructed as follows (see Figure 5).

1. Take k copies of the path P_7 with $V_{P_7^i} = \{(j,i) : 1 \leq j \leq 7\}$ and $E_{P_7^i} = \{(j,i)(j+1,i) : 2 \leq j \leq 6\}$.
2. For $j \in \{1,2,6,7\}$ replace the set $\{(j,i) : 1 \leq i \leq k\}$ with a single vertex j.
3. For $2 \leq i \leq k$ add edges $(4,1)(4,i)$.

We define the permutation π_k as $\pi_k = (26(5,1)(3,1))$.

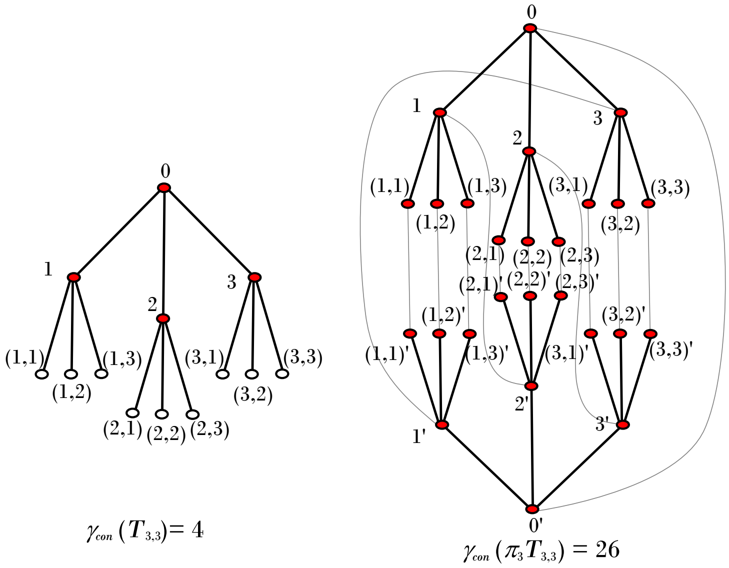

Figure 4. The graphs $T_{3,3}$ and $\pi_{3,3}T_{3,3}$ and their γ_{con}-sets.

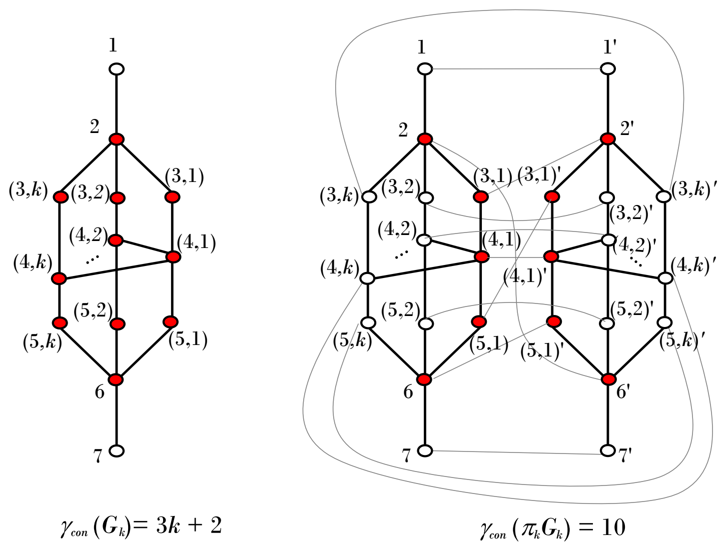

Figure 5. The graphs G_k and $\pi_k G_k$ and their γ_{con}-sets.

Every convex dominating set of G_k must contain the vertices 2 and 6, as well as all vertices of every shortest 2–6 path. Since $2, (3, i), (4, i), (5, i), 6$ for $i \in \{1, \ldots, k\}$ are all shortest 2–6 paths, $\gamma_{con}(G_k) = 3k + 2$. However, the set $\{2, 2', (3, 1), (3, 1)', (4, 1), (4, 1)', (5, 1), (5, 1)', 6, 6'\}$ is a convex dominating set of cardinality 10 in $\pi_k G_k$ for any k. Thus, the difference $\gamma_{con}(G) - \gamma_{con}(\pi G)$ can be arbitrarily large.

In fact, the above examples show a stronger property of the convex domination number.

Remark 5.1. The convex domination number of πG cannot be bounded in terms of $\gamma_{con}(G)$.

Proof. Notice that for the graphs G_k defined above $\gamma_{con}(\pi_k G_k)$ is constant, while $\gamma_{con}(G_k)$ grows with the increase of k. This shows that there is no upper bound on $\gamma_{con}(\pi G)$ depending only on $\gamma_{con}(G)$.

Similarly, if k is constant and l increases, $\gamma_{con}(\pi_{k,l} T_{k,l})$ for the tree $T_{k,l}$ increases, while $\gamma_{con}(T_{k,l})$ remains constant. Thus there is no lower bound on $\gamma_{con}(\pi G)$ depending solely on $\gamma_{con}(G)$. ∎

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