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A NOTE ON THE CROSSING NUMBERS OF 5-REGULAR GRAPHS [1]

Zhangdong Ouyang

Department of Mathematics
Hunan First Normal University
Changsha 410205, P.R. China

e-mail: oymath@163.com

Abstract

The crossing number $\mathrm{cr}(G)$ of a graph G is the smallest number of edge crossings in any drawing of G. In this paper, we prove that there exists a unique 5-regular graph G on 10 vertices with $\mathrm{cr}(G) = 2$. This answers a question by Chia and Gan in the negative. In addition, we also give a new proof of Chia and Gan's result which states that if G is a non-planar 5-regular graph on 12 vertices, then $\mathrm{cr}(G) \geq 2$.

Keywords: crossing number, 5-regular graph, drawing.

2010 Mathematics Subject Classification: 05C10, 05C62.

1. Introduction

All graphs considered here are simple, finite and undirected. A *drawing* of a graph $G = (V, E)$ is a mapping D that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc connecting $D(u)$ and $D(v)$. We often make no distinction between a graph-theoretical object (such as a vertex, or an edge) and its drawing. For simplicity, we impose the following conditions on a drawing: (a) no edge passes through any vertex other than its ends, (b) no three edges have an interior point in common, and (c) if two edges share an interior point p, then they cross at p. We denote by $\mathrm{cr}_D(G)$ or (when the graph is unambiguous) $\mathrm{cr}(D)$ the number of crossings in the drawing D of a graph G. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the smallest number of crossings in

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any drawing of G and the corresponding drawing is called an *optimal* drawing. Obviously, an optimal drawing is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Throughout this paper, all considered drawings are good unless otherwise specified. A graph G is said to be *planar* if $\mathrm{cr}(G) = 0$. A drawing D is called a *plane drawing* if $\mathrm{cr}(D) = 0$. A planar graph is called *maximal planar* if adding an edge between any two non-adjacent vertices results in a non-planar graph. For more about crossing number of a graph, see [5] and the references therein.

Let $\mathcal{G}(r, n)$ denote the set of all r-regular connected graphs on n vertices and let $\mathcal{G}(r, n, c)$ denote the set of all r-regular connected graphs on n vertices having crossing number c. Clearly, $\mathcal{G}(r, n) = \bigcup_{c \geq 0} \mathcal{G}(r, n, c)$. Chia and Gan [2, 3] attempted to classify 5-regular graphs according to their crossing numbers. In particular, they showed that $\mathrm{cr}(G) \geq 2$ for any $G \in \mathcal{G}(5, 10)$. Because they did not come across any 5-regular graphs on 10 vertices with crossing number 2, they put forward the following question.

Question 1 [2]**.** *Is it true that* $\mathcal{G}(5, 10, 2)$ *is an empty set*?

In this note, we prove that there exists a unique 5-regular graph on 10 vertices with crossing number 2 in Section 3. Thus this answers Question 1 in the negative. In addition, Chia and Gan [2] mainly proved that $\mathrm{cr}(G) \geq 2$ for any non-planar graph $G \in \mathcal{G}(5, 12)$. In Section 4, we give a simple proof of this result.

2. Preliminaries

Let $G = (V, E)$ be a graph and X be a subset of V or of E. We denote by $[X]$ the subgraph of G induced by X. The neighborhood of a vertex v in a graph G, denoted by $N_G(v)$ or $N(v)$ when the graph is unambiguous, is the set of all the vertices adjacent to v in G. Let $N[v] = \{v\} \cup N(v)$ denote the closed neighborhood of v. We denote by $d(G)$ the diameter of a graph G which is the maximum distance between two vertices in G.

A drawing D of a graph G imposes a circular permutation of the edges incident with $v \in V(G)$, which can be extended to its neighborhood $N(v)$. By $\pi_D(v)$ we denote the circular permutation of $N(v)$ in D. Similarly, we denote by $\pi_D(p)$ the circular permutation of the four vertices associated with p in D, where p is a crossing point in D. For a subgraph H of G, we use $D(H)$ to denote the subdrawing of D induced by H. For edge-disjoint subgraphs H_1 and H_2 of G, we denote by $\mathrm{cr}_D(H_1, H_2)$ the number of crossings in D between every pair of edges where one edge is in H_1 and the other in H_2 in D.

Given a graph G and its a vertex-induced subgraph H, *contracting subgraph* H in the graph G, denoted as G/H, is the operation which removes all edges of H

while simultaneously identifying all vertices of H as a single vertex and replacing all parallel edges by a single edge. Intuitively, it seems true that $\mathrm{cr}(G) \geq \mathrm{cr}(G/H)$. Actually, there are many examples violating the intuition, e.g., Example 2. Hence, when we consider the contracting operation, we need to combine the special drawing of G.

Example 2. For $i = 1, 2$, let G_i be the graph shown in Figure 1. Clearly, $G_2 = G_1/v_1v_2$. The drawing of G_1 shown in Figure 1 indicates that $\mathrm{cr}(G_1) \leq 1$. However, we know that $\mathrm{cr}(G_2) = 2$ from Lemma 5 in [4]. Thus, $\mathrm{cr}(G_2) \geq \mathrm{cr}(G_1)$.

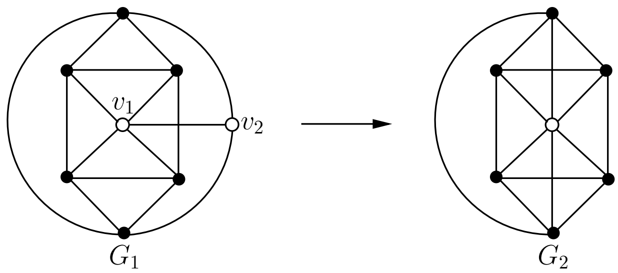


Figure 1. Contracting the edge v_1v_2.

The *responsibility*, $r_D(v)$, of a vertex v in a drawing D is defined as the total number of crossings on all edges incident with v. The *responsibility*, $r_D(e)$, of an edge e in a drawing D is defined as the total number of crossings on e. Because each crossing is in the responsibility of four vertices and of two edges respectively, it follows that

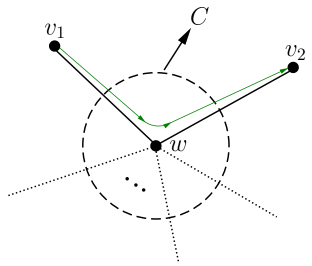
$$\sum r_D(v) = 2\sum r_D(e) = 4\mathrm{cr}(D).$$

A vertex v (respectively an edge e) is called to be *clean* in the drawing D if $r_D(v) = 0$ (respectively $r_D(e) = 0$), *unclean* otherwise. A graph H is called to be *clean* in the drawing D if all edges of H are clean in D, *unclean* otherwise.

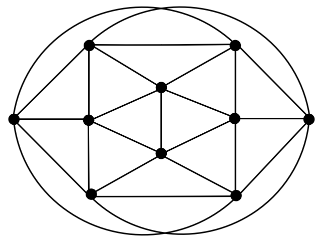
We now introduce a technique, called adding arc operation, which will be used throughout this paper.

Definition. Let D be a drawing of a graph G and $w \in V(G)$. Assume that v_1, v_2 are neighbors in $\pi_D(w)$ but $v_1v_2 \notin E(G)$. By adding an arc joining v_1 to v_2 around w in D, we mean drawing a new edge from v_1 to v_2 in D by the following way: first depart from vertex v_1 near the edge v_1w, then bypass vertex w in $N(D(w), \varepsilon)$, and finally connect to v_2 near to the edge v_2w (see Figure 2(I), where the circuit C denotes the boundary of $N(D(w), \varepsilon)$). Notice that the vertex w may be considered to be a crossing point. It is not hard to see that the arc is not crossed in the resulting drawing if v_1w and v_2w both are clean in D.

The following result, which can be easily obtained by Euler's formula, is usually used in the proofs of our results.



(I) Adding an arc around w



(II) The graph F and its drawing

Figure 2. The operation of adding arc and the graph F.

Proposition 3. *For any graph $G = (V, E)$ with $|V| \geq 3$, we have*

$$\mathrm{cr}(G) \geq |E| - 3|V| + 6.$$

Let $f_k(\Delta)$ be the maximum number of vertices in a planar graph with diameter k and maximum degree Δ. Let $g_k(\Delta)$ be the maximum number of vertices in a maximal planar graph with diameter k and maximum degree Δ. The following two results are useful to achieve our partial results.

Lemma 4 [6]**.** $g_2(5) = 9$, $g_2(6) = 11$, $g_2(7) = 12$, $f_2(4) = 9$, $f_2(5) = 10$, $f_2(6) = 11$, $f_2(7) = 12$.

For two disjoint subsets $V_1, V_2 \subseteq V(G)$, let $E(V_1, V_2)$ denote the set of edges in G whose ends are in V_1 and V_2, respectively.

Lemma 5. *Let $G = (V, E)$ be an r-regular graph with $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Then $|E([V_1])| - |E([V_2])| = r(|V_1| - |V_2|)/2$.*

Proof. By Hand-Shaking Lemma, we have $2|E([V_i])| = r|V_i| - |E(V_1, V_2)|$ for $i = 1, 2$. Then we can obtain the desired result by eliminating $|E(V_1, V_2)|$. ■

3. $\mathcal{G}(5, 10)$

Lemma 6. *Let $G = (V, E) \in \mathcal{G}(5, 10)$. Then the following properties hold.*

(i) *If $xy \notin E$, then $|N(x) \cap N(y)| \geq 2$.*

(ii) *$d(G) = 2$.*

(iii) *If $xy \in E$ and $|N(x) \cap N(y)| \geq 1$, then $d(G - xy) = 2$.*

(iv) *Assume that $e_i = x_i y_i \in E$ and $|N(x_i) \cap N(y_i)| \geq 1$ for $i = 1, 2$. If e_1 and e_2 are non-adjacent in G and there is no 4-cycle containing both e_1 and e_2, then $d(G - \{e_1, e_2\}) = 2$.*

Proof. (i) Suppose to contrary that $|N(x) \cap N(y)| \leq 1$. Then $|N[x] \cap N[y]| = |N(x) \cap N(y)| \leq 1$. Thus

$$|V| \geq |N[x] \cup N[y]| = |N[x]| + |N[y]| - |N[x] \cap N[y]| \geq 11.$$

This is absurd because $|V| = 10$.

(ii) The claim directly holds by (i).

(iii) Let $G' = G - xy$ and $u, v \in V(G')$. Assume that $uv \notin E$. Then, by (i) we have $|N(u) \cap N(v)| = |N_{G'}(u) \cap N_{G'}(v)| \geq 2$. Assume that $uv \in E$. Then $|N(u) \cap N(v)| = |N_{G'}(u) \cap N_{G'}(v)| \geq 1$ if $uv = xy$, and $uv \in E(G')$ otherwise. Thus the conclusion holds.

(iv) Let $G'' = G - \{e_1, e_2\}$. Let $u, v \in V(G'')$ and $uv \notin E(G'')$. Assume first that $uv \notin E$. As there is no 4-cycle containing both e_1 and e_2, it follows from (i) that $|N_{G''}(u) \cap N_{G''}(v)| \geq |N(u) \cap N(v)| - 1 \geq 1$. Assume now that $uv \in E$. Then $uv = e_1$ or e_2. Thus $|N(u) \cap N(v)| \geq 1$. Since e_1 and e_2 are non-adjacent in G, $|N_{G''}(u) \cap N_{G''}(v)| = |N(u) \cap N(v)| \geq 1$. Thus the claim follows. ■

A previous proof of Lemma 7 can be found in [3]. Here we shall give a shorter proof.

Lemma 7 [3]**.** *For any $G \in \mathcal{G}(5, 10)$, we have* $\mathrm{cr}(G) \geq 2$.

Proof. As $|V(G)| = 10$ and $|E(G)| = 25$ for any $G \in \mathcal{G}(5, 10)$, it follows from Proposition 3 that $\mathrm{cr}(G) \geq 1$. Hence, we only need to prove that $\mathrm{cr}(G) \neq 1$ for any $G \in \mathcal{G}(5, 10)$. Suppose now to the contrary that there exists $G \in \mathcal{G}(5, 10)$ with $\mathrm{cr}(G) = 1$, and let D be an optimal drawing of G. Assume that $e_1 = v_1v_2, e_2 = u_1u_2 \in E(G)$ cross at p in D.

We first prove that $[\{v_1, v_2, u_1, u_2\}] \cong K_4$. Suppose not. Then one can obtain a drawing D' from D by adding an arc around p in D such that $\mathrm{cr}(D') = \mathrm{cr}(D) = 1$. But D' has 10 vertices and 26 edges, implying that $\mathrm{cr}(D') \geq 2$ by Proposition 3, a contradiction. Hence, by Lemma 6(ii) and (iii) we have $d(G - e_1) = 2$.

On the other hand, clearly, $G - e_1$ is a planar graph with 10 vertices and 24 edges. Moreover, $G - e_1$ is a maximal planar graph with maximum degree $\Delta(G - e_1) = 5$. Thus, Lemma 4 implies that $d(G - e_1) \neq 2$, a contradiction. ■

Theorem 8. *Let $G \in \mathcal{G}(5, 10)$. Then* $\mathrm{cr}(G) \geq 2$ *and the equality holds if and only if G is the graph F shown in Figure* 2(II).

Proof. By Lemma 7 and the drawing of F shown in Figure 2(II), we have $\mathrm{cr}(F) = 2$. Let $G \in \mathcal{G}(5, 10, 2)$ and D be an optimal drawing of G. By Lemma 7, it is sufficient to show that $G \cong F$.

Claim 9. *If G contains $K_{2,3}$ as a subgraph, then $K_{2,3}$ is unclean in D.*

Proof. Suppose to the contrary that $K_{2,3}$ is clean in D. Let $V_1 = V(K_{2,3})$ and $V_2 = V(G) \setminus V_1$. Assume first that $[V_2]$ is connected. As $K_{2,3}$ is clean in D, all vertices of V_2 must lie in the same region of $D(K_{2,3})$. Hence, we can obtain a plane drawing of $K_{1,2,3}$ from D by contracting $[V_2]$ into a vertex. But it is impossible because $\mathrm{cr}(K_{1,2,3}) = 1$. Thus we now may assume that $[V_2]$ is not connected. Let $H_1, H_2, \ldots, H_k$ be the connected components of $[V_2]$ with $\nu_i \le \nu_{i+1}$ for $i = 1, 2, \ldots, k-1$, where $\nu_i = |V(H_i)|$ and $k \ge 2$. Clearly, $\sum_{i=1}^{k} \nu_i = 5$. Two cases now are considered, depending on whether $\nu_1 = 1$ or not.

Suppose $\nu_1 = 1$. Clearly, $[V_1 \cup V(H_1)]$ contains $K_{1,2,3}$ as a subgraph. As $\mathrm{cr}(K_{1,2,3}) = 1$ and no two edges incident with the same vertex cross in D, it is impossible that $K_{2,3}$ is clean in D.

Suppose $\nu_1 \ge 2$. It follows from Lemma 5 that $|E([V_1])| = |E([V_2])| \ge 6$. Thus,

$$6 \le |E([V_2])| = \sum_{i=1}^{k} |E(H_i)| \le \sum_{i=1}^{k} \frac{1}{2}\nu_i(\nu_i - 1) \le 4,$$

which is absurd.

Thereby, the claim follows. □

Claim 10. *If G contains K_4 as a subgraph, then K_4 is unclean in D.*

Proof. Suppose to the contrary that K_4 is clean in D. Let $V_1 = V(K_4)$ and $V_2 = V(G) \setminus V_1$. Assume that $[V_2]$ is connected. As K_4 is clean in D, all vertices of V_2 lie in the same region of $D(K_4)$. Consequently, one can obtain a plane drawing of K_5 from D by contracting $[V_2]$ into a vertex. But it is absurd because $\mathrm{cr}(K_5) = 1$. So $[V_2]$ is not connected. Let $H_1, H_2, \ldots, H_k$ be the connected components of $[V_2]$ with $\nu_i \le \nu_{i+1}$ for $i = 1, 2, \ldots, k-1$, where $\nu_i = |V(H_i)|$ and $k \ge 2$. Clearly, $\sum_{i=1}^{k} \nu_i = 6$. As $|E([V_1])| = 6$, it follows from Lemma 5 that $|E([V_2])| = 11$. Thus,

$$11 = |E([V_2])| = \sum_{i=1}^{k} |E(H_i)| \le \sum_{i=1}^{k} \frac{1}{2}\nu_i(\nu_i - 1) \le 10,$$

which is impossible. Therefore, the claim follows. □

Claim 11. *Every edge of G is crossed at most once in D.*

Proof. Suppose that $e = xy \in E(G)$ is exactly crossed twice in D. Then it is not hard to find that D must contain one of the two subdrawings shown in Figure 3 (ignore the dotted lines). Two cases now arise.

Case 1. D contains the subdrawing shown in Figure 3(I). We first conclude that $|N(x) \cap N(y)| \ge 1$. Because otherwise one can obtain a drawing D' from

D by adding an arc around p_1 or p_2, see the dotted lines in Figure 3(I). Clearly, $\mathrm{cr}(D'-e) = 0$. But $D'-e$ has 10 vertices and 25 edges, implying that $\mathrm{cr}(D'-e) \geq 1$ by Proposition 3, a contradiction. Hence, we now have $d(G-e) = 2$ by Lemma 6(iii). It is not hard to see that $G-e$ is a maximal planar graph with 10 vertices and maximum degree $\Delta(G-e) = 5$. This contradicts the fact that $g_2(5) = 9$ by Lemma 4.

Case 2. D contains the subdrawing shown in Figure 3(II). We construct a plane drawing D' from D by the following way: first treat p_1 and p_2 as two true vertices, then add two arcs joining p_1 to v_1 and v_2 around p_2, see the dotted lines in Figure 3(II). Clearly, the resulting drawing D' is planar. However, there are 12 vertices and 31 edges in D'. This implies that $\mathrm{cr}(D') \geq 1$ by Proposition 3, a contradiction. □

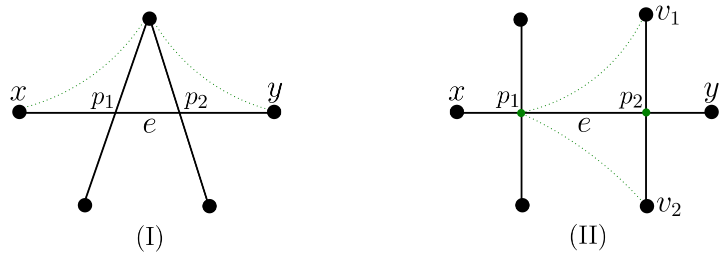


Figure 3. Two possible subdrawings of D.

Claim 12. *Adding two edges in D between any two pairs of non-adjacent vertices results in a drawing D' with $\mathrm{cr}(D') \geq 3$.*

Proof. As D' has 10 vertices and 27 edges, it follows from Proposition 3 that $\mathrm{cr}(D') \geq 3$. □

Claim 13. *Let x, y be two clean vertices in D and $xy \notin E(G)$. Then $|N(x) \cap N(y)| = 2$.*

Proof. As x, y both are clean in D, it follows from Claim 9 that $|N(x) \cap N(y)| \leq 2$. The reverse inequality holds by Lemma 6(i). □

Claim 14. *If there are exactly two clean vertices in D, then they are non-adjacent in G.*

Proof. Let x_1 and x_2 be the two clean vertices in D and let $X = \{x_1, x_2\}$. Suppose to the contrary that $x_1x_2 \in E(G)$. Assume that $y_1y_3 \in E(G)$ and $y_2y_4 \in E(G)$ (respectively, $z_1z_3 \in E(G)$ and $z_2z_4 \in E(G)$) cross at p_1 (respectively, p_2) in D. Let $Y = \{y_i : 1 \leq i \leq 4\}$ and $Z = \{z_i : 1 \leq i \leq 4\}$. Note that there are exactly two clean vertices in D. Thus $|Y \cup Z| = |V(G)| - 2 = 8$. This implies that $Y \cap Z = \emptyset$.

We now claim that either both $[Y]$ and $[Z]$ are isomorphic to K_4 or they are isomorphic to K_4 and $K_4 - e$ respectively. Otherwise we can add at least two arcs around p_1 and p_2 in D without producing new crossings, see the dotted lines in Figure 4(I), contradicting Claim 12. Without loss of generality, assume that $[Y] \cong K_4$.

Next we shall show that $|N(x_1) \cap Y| \leq 2$. Suppose not. Let $V_1 = \{x_1\} \cup Y$ and $V_2 = \{x_2\} \cup Z$. By Lemma 5, we have $|E([V_1])| = |E([V_2])| \geq 9$, implying that $[V_1]$ and $[V_2]$ both are connected. Note that $\mathrm{cr}_D([V_1], [V_2]) = 0$. Thus all vertices of V_2 must lie in the same region of $D([V_1])$. This enforces that the edges of $[V_1]$ are crossed by the edges in $E(V_1, V_2)$, a contradiction.

Similarly, we can deduce that $|N(x_2) \cap Y| \leq 2$ and $|N(x_i) \cap Z| \leq 2$ for $i = 1, 2$. Moreover, as $|N(x_i) \cap (Y \cup Z)| = 4$, we conclude that $|N(x_i) \cap Y| = |N(x_i) \cap Z| = 2$ for $i = 1, 2$.

Let $H_1 = [X \cup Y]$. As $|E(H_1)| = 11$, we have $|E([Z])| = 6$ by Lemma 5, implying that $[Z] \cong K_4$. It is not hard to verify that there are only three possible drawings for $D(H_1)$, see Figure 4(II)(III)(IV). We first may exclude D_1 because there is a clean K_4, contradicting Claim 10. For D_2, as $[Z]$ is connected and $\mathrm{cr}_D(H_1, [Z]) = 0$, all vertices of Z must lie in the same region of D_2. But the edges in $E(H_1)$ will be crossed by the edges in $E(X \cup Y, Z)$, a contradiction. Thus, D_3 is the only possible drawing for $D(H_1)$. Let $H_2 = [X \cup Z]$. With the symmetry, we can also deduce that D_3 is the only possible drawing for $D(H_2)$. Thus $D(H_1 \cup H_2)$ must be D_4 shown in Figure 4(V). However, in this case it is a routine exercise to show that there exists a pair of edges in G but not in $H_1 \cup H_2$ such that they cross each other in D, a contradiction. □

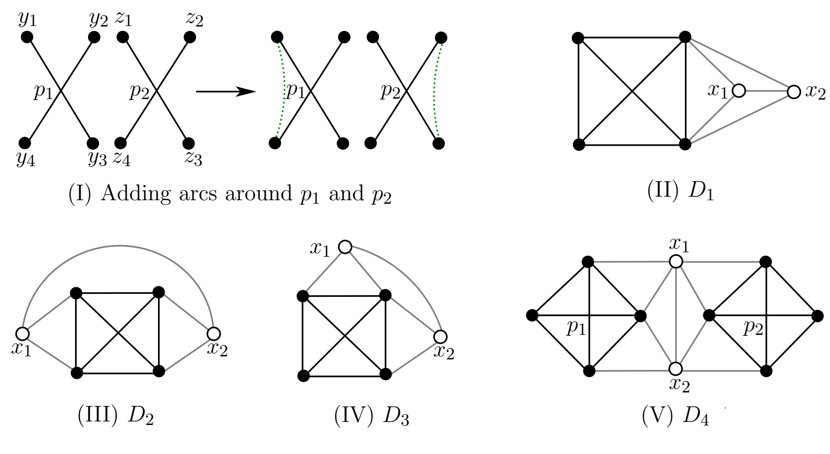


Figure 4. Adding arcs and the possible subdrawings of D.

Claim 15. *If there are exactly three clean vertices in D, then two of these vertices are not adjacent in G.*

Proof. Let x_1, x_2 and x_3 be the three clean vertices in D. Let $X = \{x_1, x_2, x_3\}$ and $Y = V(G) \setminus X$. Assume that $a_1a_3 \in E(G)$ and $a_2a_4 \in E(G)$ (respectively, $b_1b_3 \in E(G)$ and $b_2b_4 \in E(G)$) cross at p_1 (respectively, p_2) in D. Let $A = \{a_i : 1 \leq i \leq 4\}$ and $B = \{b_i : 1 \leq i \leq 4\}$. Note that $Y = A \cup B$ and $|Y| = 7$. Thus $|A \cup B| = |A| + |B| - |A \cap B| = 7$. This implies that $|A \cap B| = 1$, further implying that $D([Y])$ contains a subdrawing shown in Figure 5(I). Moreover, we claim that $D([Y])$ contains a subdrawing shown in Figure 5(II). Because otherwise one can add at least two arcs around two crossing points p_1 and p_2 in D without introducing additional crossings, contradicting Claim 12.

Now we proceed by contradiction. Suppose that x_1, x_2 and x_3 are adjacent each other in G. We now distinguish two cases.

Case 1. $wv \notin E(G)$. As $deg_G(u) = 5$, $uv \notin E(G)$. This implies that there is no 4-cycle containing both e_1 and e_2. Thus we have $d(G - \{e_1, e_2\}) = 2$ by Lemma 6(iv). Let D' be the drawing obtained from D by adding an arc joining u to v around p_2 while simultaneously deleting e_1 and e_2. Clearly, D' is a plane drawing with 10 vertices and 24 edges. This means that D' is a maximal planar graph with maximum degree 5. But this is impossible because $g_2(5) = 9$ by Lemma 4.

Case 2. $wv \in E(G)$. Clearly, $D([Y])$ contains a subdrawing shown in Figure 5(III). Note that $[X]$ is connected and clean in D, thus x_1, x_2, x_3 must be placed in the same region of $D([Y])$. However, it is not hard to verify that no matter in which region of $D([Y])$ x_1, x_2, x_3 are placed, the edges in $[Y]$ will be crossed by the edges in $E(X, Y)$, again a contradiction. $\square$

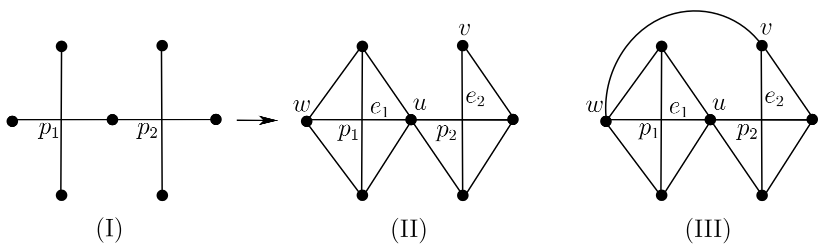


Figure 5. The possible subdrawings of $D([Y])$.

Claim 16. *There exist two clean and non-adjacent vertices in D.*

Proof. By the definition of responsibility, we have

$$\sum_{v \in V(G)} r_D(v) = 4\mathrm{cr}_D(G) = 8,$$

which implies that there are at least two clean vertices in D. By Claims 14 and 15, we only need to consider the case that there are at least four clean vertices in D. Suppose that all clean vertices in D are adjacent each other in G. Then G contains K_4 as a subgraph which is clean in D, contradicting Claim 10. $\square$

Now we continue to the proof of the theorem. By Claim 16, let us assume that $u_0, v_0 \in V(G)$ are two clean vertices in D and $u_0v_0 \notin E(G)$. Then, $|N(u_0) \cap N(v_0)| = 2$ by Claim 13. Thus, D contains a subdrawing shown in Figure 6(I).

We first claim that any two vertices in $N(u_0)$ (respectively, $N(v_0)$) are not adjacent in G unless they are neighbors in $\pi_D(u_0)$ (respectively, $\pi_D(v_0)$). Because otherwise the edge joining them is crossed at least twice in D, contradicting Claim 11. Thus, v_1 must be adjacent to u_0, v_0, u_1, u_4 and v_2; and u_1 must be adjacent to u_0, v_0, v_1, v_4 and u_2, see Figure 6(II).

We now claim that $u_4v_4 \notin E(G)$. Otherwise, observe that v_2 and v_3 both must be adjacent to at least one of u_2 and u_3. This enforces that u_4v_4 is crossed at least twice, contradicting Claim 11. Similarly, we may claim that $u_2v_2 \notin E(G)$. Thus, we can conclude the following statement, cf. Figure 6(III).

(i) v_2 is adjacent to v_0, v_1, v_3, u_4 and u_3;
(ii) u_2 is adjacent to u_0, u_1, u_3, v_4 and v_3;
(iii) v_4 is adjacent to v_0, u_1, u_2, v_3 and u_3;
(iv) u_4 is adjacent to u_0, v_1, v_2, u_3 and v_3.

Therefore, $G \cong F$ and the proof is done. ■

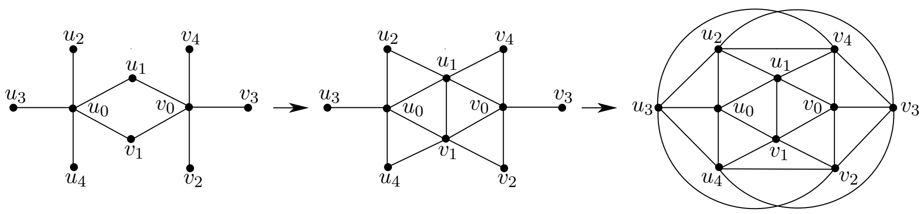


Figure 6. The possible subdrawings of D.

4. $\mathcal{G}(5, 12)$

Chia and Gan [2] mainly showed that if G is a non-planar 5-regular graph on 12 vertices, then $\mathrm{cr}(G) \geq 2$. In the rest of the paper, we give a simple proof for this result.

Theorem 17. $\mathrm{cr}(G) \geq 2$ *for any* $G \in \mathcal{G}(5,12)$ *except for the planar graph icosahedron.*

Proof. For any $G \in \mathcal{G}(5,12)$, it is well-known that $\mathrm{cr}(G) = 0$ if and only if G is the icosahedron (see [1]). Thus, it suffices to prove that $\mathrm{cr}(G) \neq 1$ for any $G \in \mathcal{G}(5,12)$. We proceed by contradiction. Suppose that there exists a graph $G \in \mathcal{G}(5,12)$ with $\mathrm{cr}(G) = 1$, and let D be an optimal drawing of G. Assume that $e_1 = x_1x_3$ and $e_2 = x_2x_4$ cross at p in D, see Figure 7(I). Let $X = \{x_i : 1 \leq i \leq 4\}$ and $Y = V(G) \setminus X = \{y_i : 1 \leq i \leq 8\}$.

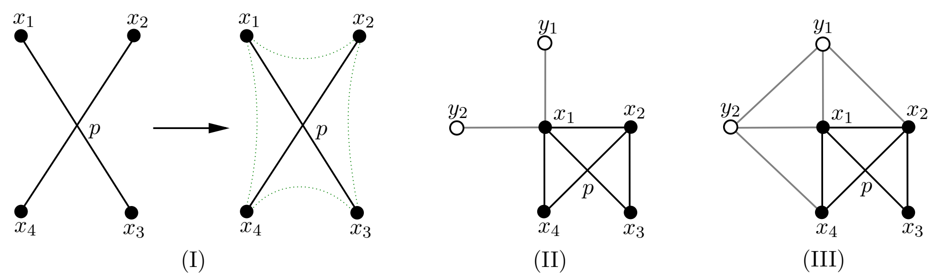


Figure 7. Some possible subdrawings of D.

Claim 18. *If* G *contains* $K_{2,3}$ *as a subgraph, then* $K_{2,3}$ *is unclean in* D.

Proof. Suppose to the contrary that $K_{2,3}$ is clean in D. Let $V_1 = V(K_{2,3})$ and $V_2 = V(G) \setminus V_1$. Assume first that $[V_2]$ is connected. As $K_{2,3}$ is clean, all vertices of V_2 lie in the same region of $D(K_{2,3})$. This means that we can obtain a plane drawing of $K_{1,2,3}$ from D by contracting $[V_2]$ into a vertex. But it is impossible because $\mathrm{cr}(K_{1,2,3}) = 1$. It remains to consider that $[V_2]$ is not connected. Let $H_1, H_2, \ldots, H_k$ be the connected components of $[V_2]$ with $\nu_i \leq \nu_{i+1}$ for $i = 1, 2, \ldots, k-1$, where $\nu_i = |V(H_i)|$ and $k \geq 2$. Two cases now arise, depending on whether $\nu_1 = 1$ or not.

Suppose $\nu_1 = 1$. Then $[V_1 \cup V(H_1)]$ contains $K_{1,2,3}$ as a subgraph. As $\mathrm{cr}(K_{1,2,3}) = 1$ and no two edges incident with the same vertex cross in D, it is impossible that $K_{2,3}$ is clean in D.

Suppose $\nu_1 \geq 2$. As $|E([V_1])| \geq 6$, we have $|E([V_2])| \geq 11$ by Lemma 5. Thus,

$$11 \leq |E([V_2])| = \sum_{i=1}^{k} |E(H_i)| \leq \sum_{i=1}^{k} \frac{1}{2}\nu_i(\nu_i - 1) \leq 11,$$

which implies that $k = 2$, $\nu_1 = 2$, $\nu_2 = 5$ and $|E(H_2)| = 10$, further implying that $H_2 \cong K_5$. So the unique crossing in D is from two edges of H_2. Note that $[V_1 \cup V(H_1)]$ is connected, thus all vertices of $V_1 \cup V(H_1)$ must lie in the same region of $D(H_2)$. Hence, we can obtain a drawing D' of K_6 from D by contracting

$[V_1 \cup V(H_1)]$ into a vertex such that $\mathrm{cr}(D') = 1$. But it is impossible because $\mathrm{cr}(K_6) = 3$. □

Claim 19. *Adding two edges in D between any two pairs of non-adjacent vertices results in a drawing D' with* $\mathrm{cr}(D') \geq 2$.

Proof. As D' has 12 vertices and 32 edges, it follows from Proposition 3 that $\mathrm{cr}(D') \geq 2$. □

Claim 20. $[X] \ncong K_4$.

Proof. Suppose to the contrary that $[X] \cong K_4$. We first assert that there are at most two vertices in Y which are not adjacent to any vertex in X. Otherwise, without loss of generality, assume that $y_i x_j \notin E(G)$, where $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. Clearly, $\bigcup_{i=1}^{3} N(y_i) \subseteq Y$ and $|N(y_i)| = 5$ for $i = 1, 2, 3$. By Claim 18, we know that $|N(y_i) \cap N(y_j)| \leq 2$ for $1 \leq i < j \leq 3$. Thus,

$$\begin{aligned} |Y| &\geq \left| \bigcup_{1 \leq i \leq 3} N(y_i) \right| = \sum_{1 \leq i \leq 3} |N(y_i)| - \sum_{1 \leq i < j \leq 3} |N(y_i) \cap N(y_j)| + \left| \bigcap_{1 \leq i \leq 3} N(y_i) \right| \\ &\geq 15 - 6 = 9, \end{aligned}$$

which is absurd because $|Y| = 8$. Thus, we have $|E(X, Y)| \geq 6$.

Let D' be a drawing obtained from D by contracting $[X]$ into a vertex. As $[X]$ is connected and $[Y]$ is clean in D, it is easy to see that $\mathrm{cr}(D') = 0$. On the other hand, it follows from Lemma 5 that $|E([Y])| = |E([X])| + 10 = 16$. So D' has $8 + 1 = 9$ vertices and at least $16 + 6 = 22$ edges. Moreover, by Proposition 3, we have

$$\mathrm{cr}(D') \geq 22 - 3 \times 9 + 6 = 1,$$

which yields a contradiction. □

Claim 21. $[X] \ncong K_4 - e$.

Proof. Suppose to the contrary that $[X] \cong K_4 - e$. Without loss of generality, assume that $x_3 x_4 \notin E(G)$ and $x_1 y_i \in E(G)$ for $i = 1, 2$, see Figure 7(II). Let $L = [N[x_1]]$ and $R = G - L$. We claim that any neighbors in $\pi_D(x_1)$ except for x_3 and x_4 are adjacent in G. Because otherwise we can obtain a drawing D^* from D by adding two arcs around x_1 and p such that $\mathrm{cr}(D^*) = 1$, contradicting Claim 19. Thus $D(L)$ contains a subdrawing shown in Figure 7(III).

Let s be the number of vertices in R which are adjacent to at least one vertex of L in G. Consider a drawing D' obtained from D by contracting L into a vertex. Observe that L is connected and R is clean in D, thus $\mathrm{cr}(D') = 0$. On the other hand, Lemma 5 implies that $|E(R)| = |E(L)| \geq 10$. So D' has $6+1=7$ vertices and at least $10+s$ edges. Thus, it follows from Proposition 3 that

$$\mathrm{cr}(D') \geq 10 + s - 3 \times 7 + 6 = s - 5.$$

This means that $s \leq 5$, implying that there exists at least one vertex of R, say y_3, which is not adjacent to any vertex of L in G. Thus $N[y_3] = V(R)$. Clearly, all vertices of R must lie in the same region of $D(L)$. We claim that $x_3y_i \notin E(G)$ for $i = 1, 2$. Because otherwise x_3y_i is crossed by the edges in $E(V(L), V(R))$. Thus $deg_L(x_3) = 2$, implying that x_3 is adjacent to exactly three vertices of R which are also in $N(y_3)$. This means that $[N[x_3] \cup N[y_3]]$ contains $K_{2,3}$ as a subgraph which is clean in D, contradicting Claim 18. □

Now we continue to the proof of the theorem. Assume that there are k pairs of neighbors in $\pi_D(p)$ which are non-adjacent in G. Then, one can obtain a drawing D' from D by adding k arcs around p such that $\mathrm{cr}(D') = 1$, see the dotted lines in Figure 7 (I). Observe that D' has 12 vertices and $30+k$ edges, thus it follows from Proposition 3 that

$$1 = \mathrm{cr}(D') \geq 30 + k - 12 \times 3 + 6 = k,$$

which implies that $k = 0$ or 1. Thus $[X] \cong K_4$ or $K_4 - e$, contradicting Claims 20 and 21. Hence, the proof is complete. ■

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