

## THE HANOI GRAPH $H_4^3$

ANDREAS M. HINZ

*Department of Mathematics*  
*Ludwig-Maximilians-Universität (LMU) München*  
*80333 Munich, Germany*

**e-mail:** hinz@math.lmu.de

AND

NAZANIN MOVARRAEI

*International Research Centre (IRC)*  
*Kalasalingam University, Tamil Nadu, India*

**e-mail:** nazanin.movarraei@gmail.com

### Abstract

Metric properties of Hanoi graphs  $H_p^n$  are not as well understood as those of the closely related, but structurally simpler Sierpiński graphs  $S_p^n$ . The most outstanding open problem is to find the domination number of Hanoi graphs. Here we concentrate on the first non-trivial case of  $H_4^3$ , which contains no 1-perfect code. The metric dimension and the dominator chromatic number of  $H_4^3$  will be determined as well. This leads to various conjectures for the general case and will thus provide an orientation for future research.

**Keywords:** Hanoi graphs, Sierpiński graphs, metric dimension, domination number, dominator chromatic number.

**2010 Mathematics Subject Classification:** 05C69, 05C12, 05C15.

### 0. OUTLINE

The purpose of this note is to approach the question of domination in Hanoi graphs. Although we will only deal with a specific example, we nevertheless believe this to be worthwhile in view of the unsolved general case. Our investigation demonstrates that already this seemingly simple instance is quite non-trivial, but that our techniques might lead to a solution in other cases.

In Section 1 we introduce Hanoi graphs and some of their properties. Section 2 is devoted to a brief description of the concept of domination in graphs to set the stage for domination in Hanoi graphs as addressed in Section 3. The last section will give an outline for further research.

## 1. HANOI GRAPHS

*Hanoi graphs*  $H_p^n$  were introduced as a mathematical model for the *Tower of Hanoi* game with  $3 \leq p \in \mathbb{N}$  pegs and  $n \in \mathbb{N}_0$  discs.<sup>1</sup> They have reached a certain popularity already. Therefore and in order to keep this note as brief as possible, we will not give a formal definition nor a complete overview of results and relations to other mathematical objects, for which we refer to the comprehensive monograph [5]. Suffice it here to say that Hanoi graphs can be defined recursively on the set of vertices  $s \in [p]_0^n$ ,  $[p]_0 = \{0, \dots, p-1\}$ , consisting of  $n$ -tuples written as  $s = s_n \cdots s_1$ , where  $s$  stands for the state of the puzzle with discs  $d$  numbered from 1 to  $n$  according to increasing size, and  $s_d$  referring to the peg, labelled from 0 to  $p-1$ , where disc  $d$  is lying. Starting with  $n=0$  for a single vertex labelled by the empty word  $e$ ,  $H_p^{1+n}$  is constructed from  $p$  copies of graphs  $iH_p^n$ , where the  $i \in [p]_0$  indicates the variable concatenated to the left of each vertex in  $H_p^n$ , linked by suitable edges corresponding to legal moves of the Tower of Hanoi game. It is obvious from the definition that Hanoi graphs are connected; the canonical distance function, defined by the length of shortest paths between vertices, is denoted by  $d$ . The classical case is  $p=3$ , but since in this note we concentrate on  $p=4$ , we present the first two steps of this construction in Figure 1.

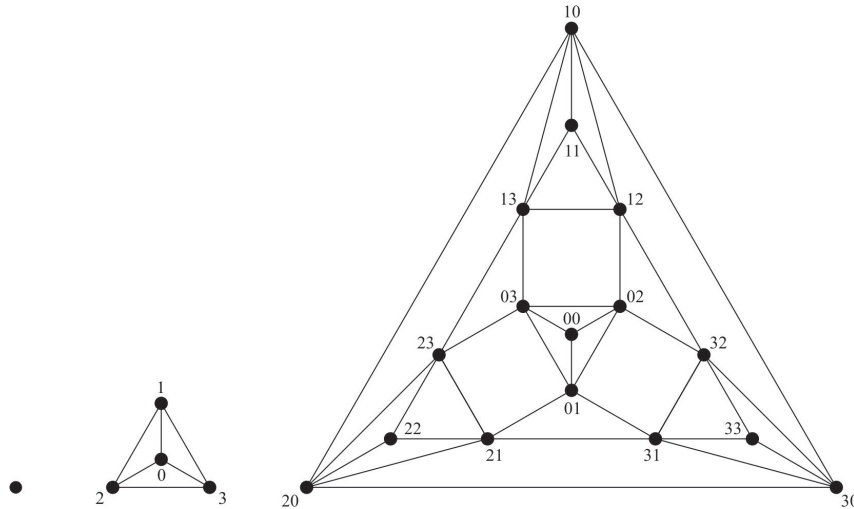
There is also an explicit definition of the edge sets  $E(H_p^n)$ , for which we refer to [5, (5.44)]. Apart from the trivial  $H_p^0 = (\{e\}, \emptyset)$ , we see that  $H_p^1 \cong K_p$ , the complete graph of order  $p$ . So (almost) everything is known for them.<sup>2</sup> This also applies to a large extent to the original Hanoi graphs  $H_3^n$  which are isomorphic to the *Sierpiński graphs*  $S_3^n$  [5, Chapter 4]. Since the general Sierpiński graphs  $S_p^n$  are obtained recursively by taking  $p$  copies of the graph  $S_p^{n-1}$  (with  $S_p^0 = (\{e\}, \emptyset)$ ) and linking each pair of these copies by exactly one edge, they can be viewed as iterated complete graphs. Many properties, metric and topological, are therefore known for them; we refer to the seminal paper [7]. But although the order of these graphs is  $|S_p^n| = p^n = |H_p^n|$ , we have for the size

$$\|S_p^n\| = \frac{p}{2}(p^n - 1) < \frac{p(p-1)}{4}(p^n - (p-2)^n) = \|H_p^n\|$$

as soon as  $p > 3$  and  $n > 1$ , whence  $H_p^n$  and  $S_p^n$  are not isomorphic anymore in these cases; cf. [5, Proposition 5.42]. (Obviously,  $S_p^0 = H_p^0$  and  $S_p^1 \cong K_p \cong H_p^1$ .)

<sup>1</sup>The case  $n=0$  is considered just to have a simple base case for induction proofs.

<sup>2</sup>The “almost” refers, e.g., to the crossing numbers.

Figure 1. The graphs  $H_4^0$ ,  $H_4^1$ , and  $H_4^2$  (see [5, Figure 5.6]).

So we cannot profit from our knowledge about Sierpiński graphs when interested in Hanoi graphs, e.g., with respect to metric properties (cf. [4]). On the other hand, some topological properties are accessible for both classes of graphs. For instance, for  $n \in \mathbb{N}$ , they are all hamiltonian [5, Exercises 4.5 and 5.10], connectivity is the same:  $\kappa(S_p^n) = p - 1 = \kappa(H_p^n)$  [5, Exercise 4.8 and Proposition 5.49], independent of  $n$  their automorphism groups are, respectively, isomorphic to the permutation group on  $p$  elements [5, Theorems 4.13 and 5.53], they share the same range of parameters for planarity [9], and basic colorings can be obtained [10]. But many questions remain open for Hanoi graphs, cf. [3].

The first case to consider is  $n = 2$ . Here virtually everything can be determined just by inspection, except, e.g., *complexity*, i.e., the number of spanning subgraphs, the number of (perfect) matchings and, for  $p \geq 5$ , the questions of crossing number and genus. For instance, perfect codes exist, leading to the domination number (see below) and even power domination and propagation radius could be approached for any  $H_p^2$  [15, Theorems 3.1 and 3.2]. For  $p = 4$  *center* and *periphery*, i.e., the sets of vertices with minimal or maximal eccentricity,<sup>3</sup> respectively, coincide with the whole vertex set. The *median*, i.e., the set of vertices with minimal average distance to the other vertices,<sup>4</sup> consists of all non-perfect vertices,<sup>5</sup>  $M(H_4^2) = Q^2 \setminus \{ii \mid i \in Q\}$ , where  $Q = \{0, 1, 2, 3\}$ , and  $\text{prox}(H_4^2) = \frac{9}{5}$ ,  $\text{rem}(H_4^2) = \frac{11}{5}$ ; the average distance<sup>6</sup> is  $\frac{19}{10}$ . For some other numerical values, see [8, Table 2].

<sup>3</sup>These values are called *radius* rad and *diameter* diam.

<sup>4</sup>This value is called *proximity* prox; the corresponding maximum is the *remoteness* rem.

<sup>5</sup>Vertices  $i^n$  in  $H_p^n$ ,  $i \in [p]_0$ , are called *perfect*.

<sup>6</sup>Average taken over all pairs  $(s, t)$  with  $s \neq t$ .

Since there is currently no chance to approach the graphs  $H_p^3$  for general  $p > 3$ , we will from now on concentrate on the graph  $H_4^3$  whose order is  $|H_4^3| = 64$ , size  $\|H_4^3\| = 168$ , minimal degree  $\delta(H_4^3) = 3$ , maximal degree  $\Delta(H_4^3) = 6$ , chromatic number  $\chi(H_4^3) = 4$ , chromatic index  $\chi'(H_4^3) = 6$ , and total chromatic number  $\chi''(H_4^3) = 7$ . Figure 2 shows a drawing of  $H_4^3$  in the plane with 72 crossings, thus showing that for the crossing number we have  $\text{cr}(H_4^3) \leq 72$ ; whether this upper bound is optimal is not known. Note that the corresponding  $S_4^3$  is the only non-planar Sierpiński graph with  $n \geq 3$  for which the crossing number has been determined:  $\text{cr}(S_4^3) = 12$ , see [12, Proposition 3.2].

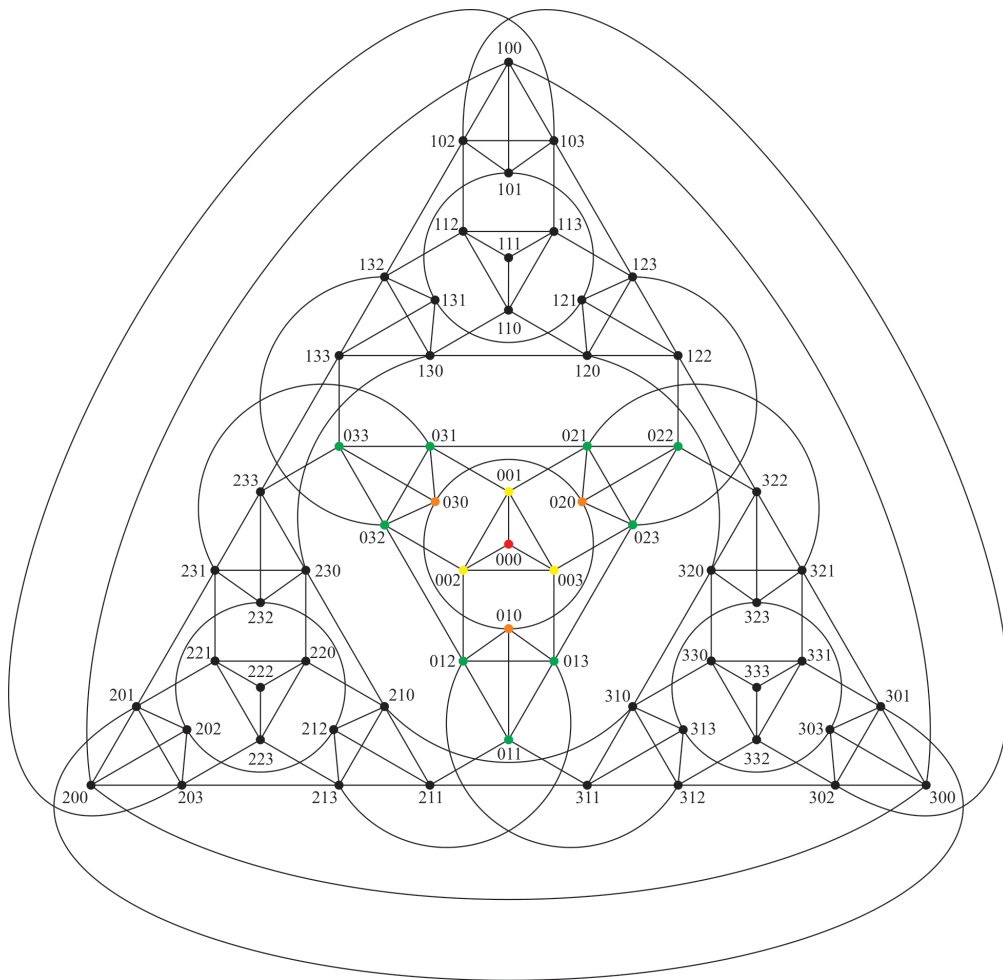


Figure 2. The graph  $H_4^3$  (cf. [5, Figure 5.13]).

Let us define  $F = \{ijk \in Q^3 \mid |\{i, j, k\}| = 3\}$ , the set of vertices corresponding to “flat” states where no two discs lie on the same peg. Note that this set induces a  $K_{3,3}$ -subdivision, showing that  $H_4^3$  is not planar. Obviously  $|F| = 24$  and we have the following facts about eccentricities.

**Theorem 1.** *If  $s \in F$ , then  $\varepsilon(s) = 4$ , otherwise  $\varepsilon(s) = 5$ , i.e.,  $\text{rad}(H_4^3) = 4 = \varepsilon(012)$ ,  $\text{diam}(H_4^3) = 5 = \varepsilon(000)$ ; in particular, the average eccentricity is  $\bar{\varepsilon}(H_4^3) = \frac{37}{8}$ . The center of  $H_4^3$  is  $C(H_4^3) = F$  and the periphery is  $P(H_4^3) = Q^3 \setminus F$ .*

**Proof.** In an easy case analysis it suffices for symmetry reasons to show that  $d(012, t) \leq 4 = d(012, 111)$  and  $d(000, t) \leq 5 = d(000, 111)$  for all  $t \in Q^3$ . Then for  $s \in \{001, 010, 011\}$  we also have  $d(s, t) \leq 5 = d(s, 111)$  because all these vertices  $s$  have a flat neighbor and the distance to state 111 can be checked easily. ■

Although these metric properties are very basic,  $H_4^3$  is the prototype for the *Peripheral phenomenon* which cannot be explained yet; see [5, p. 227]. Moreover, proximity, remoteness and the median have not been determined yet; the average distance is approximately 3.083 (see [8, Table 2]).

Another interesting graph parameter is *metric dimension*, i.e., the size of a smallest resolving set.<sup>7</sup> For Sierpiński graphs it has been determined by Klavžar and Zemljič in [13, Corollary 6] to be  $\mu(S_p^n) = p - 1$  for  $n \in \mathbb{N}$  with any  $p - 1$  out of the vertices  $i^n$ ,  $i \in [p]_0$ , forming a minimal resolving set. The same is true for  $H_p^2$ ; see [3, Section 6]. However, even the set of *all* perfect vertices does not constitute a resolving set for  $H_4^3$ ; e.g., 100 and 101 have the same distance vector with respect to  $\{iii \mid i \in Q\}$ . No resolving set  $U \subset Q^3$  can have less than 3 elements: let  $U \ni u = ijk$ , then  $ij\ell$ ,  $\ell \neq k$ , all have distance 1 to  $u$  and if  $v \in U$  not all  $d(ij\ell, v)$  can be different, because otherwise

$$d(ij\ell_1, v) < d(ij\ell_2, v) < d(ij\ell_3, v) \leq d(ij\ell_3, ij\ell_1) + d(ij\ell_1, v) = 1 + d(ij\ell_1, v),$$

a contradiction. Schlosser [14] has found the resolving set  $\{001, 121, 202, 311\}$  of size 4 and was able to exclude all triples of vertices as resolving sets with the aid of a computer program. So we have the following result.

**Theorem 2.**  $\mu(H_4^3) = 4 > 3 = \mu(S_4^3)$ .

This has been confirmed by Petr whose calculations lead to the following.

**Conjecture.** For every  $p \geq 3$  and  $n \geq 2$  we have that

$$\mu(H_p^n) = p - 1 + (p - 3)(n - 2) = \mu(S_p^n) + (p - 3)(n - 2).$$

We now turn to domination numbers.

<sup>7</sup>A subset  $\{r_1, \dots, r_m\}$ ,  $m \in \mathbb{N}_0$ , of the vertex set is called *resolving*, if the *distance vector*  $(d(v, r_1), \dots, d(v, r_m))$  determines every vertex  $v$  uniquely.

## 2. DOMINATION

The results in this section are well-known, but we present them nevertheless to keep this note as self-contained as possible.

Let  $G$  be a simple graph and define the (*closed*) *neighborhood* of a vertex  $u \in V(G)$  by

$$N[u] = \{u\} \cup \{v \in V(G) \mid \{u, v\} \in E(G)\}.$$

Moreover, the (*closed*) *neighborhood* of a subset  $U$  of  $V(G)$  is  $N[U] = \bigcup_{u \in U} N[u]$ . We say that  $v \in V(G)$  is *dominated* by  $d \in V(G)$  (or by  $D \subset V(G)$ ), if  $v \in N[d]$  (or  $v \in N[D]$ ). A  $D \subset V(G)$  is called (a  $G$ -) *dominating (set)* if  $N[D] = V(G)$ , i.e., if all  $v \in V(G)$  are dominated by  $D$ . Since obviously  $V(G)$  is dominating, it makes sense to define the *domination number* of  $G$  as

$$\gamma(G) = \min \{|D| \mid D \subset V(G), N[D] = V(G)\}.$$

An immediate upper bound is  $\gamma(G) \leq |G|$  with equality if and only if  $\Delta(G) = 0$ . ( $\gamma(G) = |G|$  means that  $V(G)$  is the only dominating set. If  $\Delta(G) = 0$ , then no vertex can be dominated by any other vertex. If  $\Delta(G) \in \mathbb{N}$ , then delete some  $v$  with  $\deg(v) \neq 0$  such that other vertices are dominated by themselves and  $v$  by its neighbor(s), whence  $V(G)$  is not minimal.) For a lower bound we have the following.

**Proposition 3.** *If  $C \subset V(G)$  is 1-error correcting, i.e., for every  $\{c, c'\} \in \binom{C}{2}$  we have that  $N[c] \cap N[c'] = \emptyset$ ,<sup>8</sup> and  $D$  is  $G$ -dominating. Then  $|C| \leq |D|$ .*

**Proof.** Let  $|C| = k$  and  $C = \{c_1, \dots, c_k\}$  and let  $D = \{d_1, \dots, d_\ell\}$  with  $|D| = \ell$ . Then for every  $i \in [k]$  there is a  $j \in [\ell]$  such that  $d_j \in N[c_i]$ . By taking, e.g., the minimal such  $j$ , we get a mapping from  $[k]$  to  $[\ell]$ , which is injective because  $N[c] \cap N[c'] = \emptyset$  if  $\{c, c'\} \in \binom{C}{2}$ . Therefore  $k \leq \ell$  by the Pigeonhole principle. ■

An immediate consequence is the following.

**Corollary 4.** *If  $C$  is a perfect code<sup>9</sup> of  $G$ , then  $\gamma(G) = |C|$ . In particular, all perfect codes of  $G$  have the same cardinality.*

**Proof.** Since  $C$  is a  $G$ -dominating set, we have  $\gamma(G) \leq |C|$ . The converse inequality follows from Proposition 3. ■

<sup>8</sup> $\binom{C}{2}$  is the set of 2-element subsets of  $C$ .

<sup>9</sup>By *perfect code* we always mean a 1-perfect code  $C \subset V(G)$  with respect to the canonical graph distance on connected components of  $G$ , i.e.,  $N[C]$  is a partition of  $V(G)$ . In other words,  $C$  is  $G$ -dominating and 1-error correcting.

Another lower bound comes from the obvious fact that

$$(1) \quad \gamma(G)(1 + \Delta(G)) \geq |G|.$$

Here equality holds if and only if all minimal dominating sets are perfect codes and all their elements have maximal degree. From (1) it follows that

$$(2) \quad \gamma(G) \geq \left\lceil \frac{|G|}{1 + \Delta(G)} \right\rceil.$$

Since the domination number of a graph is the sum of the domination numbers of its connected components, we may restrict ourselves to connected graphs for some examples. For  $\Delta(G) = 0$  we have  $G = K_0$  or  $G = K_1$  with  $\gamma(K_0) = 0$  and  $\gamma(K_1) = 1$ , respectively. For  $\Delta(G) = 1$  we get  $G = K_2$  with  $\gamma(K_2) = 1$ . In all these cases there is equality in (1) and (2). Now let  $\Delta(G) = 2$ . Then  $G$  is either a path  $P_k$  or a cycle  $C_k$ ,  $k \geq 3$ , and  $\gamma(P_k) = \lceil \frac{k}{3} \rceil = \gamma(C_k)$ . There is equality in (2), but equality in (1) only if  $k$  is a multiple of 3. Clearly, equality in (2) and (1) holds for  $G = K_p$ ,  $p \in \mathbb{N}$ , where  $|K_p| = p$ ,  $\Delta(K_p) = p - 1$ , and  $\gamma(K_p) = 1$ . Therefore, when looking at the somewhat more demanding class of Sierpiński graphs  $S_p^n$ , we may restrict ourselves to  $p, n \geq 2$ . Here  $|S_p^n| = p^n$  and  $\Delta(S_p^n) = p$ . As it is known (see [5, (4.14)]) that

$$\gamma(S_p^n) = \frac{p^n + p^{(n+1) \bmod 2}}{p + 1},$$

we have equality in (2), but never in (1). By isomorphy, the same then applies to Hanoi graphs  $H_3^n$ , but whereas  $|H_p^n| = p^n = |S_p^n|$ , we have  $\Delta(H_p^n) = \binom{p}{2} - \binom{p-n}{2}$ , if  $2 \leq n < p - 1$ , and  $\Delta(H_p^n) = \binom{p}{2}$ , if  $2 \leq p - 1 \leq n$ . Hence, for  $2 \leq n < p - 1$  we have

$$(3) \quad \frac{|H_p^n|}{1 + \Delta(H_p^n)} = \frac{2p^n}{n(2p - n - 1) + 2} < \frac{p^n}{p + 1} = \frac{|S_p^n|}{1 + \Delta(S_p^n)}.$$

For  $2 \leq p - 1 \leq n$ , we get

$$(4) \quad \frac{|H_p^n|}{1 + \Delta(H_p^n)} = \frac{2p^n}{p(p - 1) + 2} \leq \frac{p^n}{p + 1} = \frac{|S_p^n|}{1 + \Delta(S_p^n)}$$

with equality only if  $p = 3$ .

### 3. DOMINATION OF HANOI GRAPHS

Special cases for the relation between Hanoi and Sierpiński graphs are ( $p \geq 3$ ,  $n \in \mathbb{N}_0$ ):

$$\begin{aligned} \gamma(S_p^1) &= 1 &= \gamma(H_p^1), \\ \gamma(S_p^2) &= p &= \gamma(H_p^2), \\ \gamma(S_3^n) &= \frac{1}{4}(3^n + 2 + (-1)^n) = \gamma(H_3^n), \end{aligned}$$

where the second identity in the central line follows from Corollary 4 because  $H_p^2$  has a perfect code consisting of the perfect vertices (cf. [5, Exercise 5.11]) and the last identity is a consequence of  $H_3^n \cong S_3^n$ . We have strict inequalities in both, (1) and (2), the latter only if  $p > 3$ , because for  $p = 3$  and  $n > 1$  strict inequality only holds for (1), i.e.,  $\gamma(H_3^n) = \lceil \frac{3^n}{4} \rceil$ . This is sequence A122983 in the OEIS; its sequence of differences is 2\*A015518.

So we might conjecture:  $\gamma(H_p^n) = \gamma(S_p^n)$ , or, less ambitious,

$$\gamma(H_p^n) \leq \gamma(S_p^n).$$

The latter is supported by the following result.

**Proposition 5.** *If  $p \in \mathbb{N}_3$  is odd and  $n \in \mathbb{N}_0$ , then  $\gamma(H_p^n) \leq \gamma(S_p^n)$ .*

**Proof.** The domination number of a spanning subgraph cannot be smaller than the domination number of the graph itself. From [6, Theorem 3.1] (cf. [5, Theorem 5.48]) we know that  $S_p^n$  can be embedded isomorphically into  $H_p^n$  if (and only if)  $p$  is odd. ■

The smallest unknown case is  $p = 4$  and  $n = 3$  where  $\gamma(S_4^3) = 13$ . From (2) we know  $\gamma(H_4^3) \geq 10$ . The same result follows from Proposition 3 with the 1-error correcting set

$$C = \{000, 011, 022, 033, 101, 111, 202, 222, 303, 333\} \subset V(H_4^3),$$

containing 4 vertices of degree 3 and 6 vertices of degree 5 with non-overlapping neighborhoods; the set  $C$  leaves 12 vertices uncovered.

Moreover, with the  $H_4^3$ -dominating set

$$D = \{003, 011, 020, 033, 113, 121, 132, 202, 210, 221, 300, 323, 332\}$$

we see that indeed  $\gamma(H_4^3) \leq 13$ . The set  $D$  contains 11 vertices of degree 5 and 2 vertices of degree 6; it covers 16 vertices twice.

We will now present a combinatorial analysis to decide upon  $\gamma(H_4^3) \in \{10, 11, 12, 13\}$ . This will be done by distinguishing certain types of vertices, a strategy which may be employed in the general case of  $H_p^n$  as well. This is an advantage of our analytical approach because computational methods will be limited by the fact that finding the domination number for general graphs is of NP-complete complexity (see, e.g., [2, Theorem 1.7]). Of course, it would be interesting to know whether there is an efficient, i.e., polynomial, algorithm for Hanoi graphs which could perhaps be constructed on the base of our analysis. In order to appreciate how complex computations can be for metric properties of Hanoi graphs, the reader is referred to [5, Section 5.7].



So let us subdivide  $iQ^2 = \{ijk \mid jk \in Q^2\} \subset Q^3 = V(H_4^3)$ , for any fixed  $i \in Q$ , into four mutually disjoint subsets, namely

$$\begin{aligned} V_0 &= \{iii\}, \\ V_1 &= \{ijj \mid j \neq i\}, \\ V_2 &= \{iji \mid j \neq i\}, \\ V_3 &= \{ijk \mid j \neq i \neq k\}. \end{aligned}$$

In Figure 2 we have colored, for  $i = 0$ , the vertices from  $V_0$ ,  $V_1$ ,  $V_2$ , and  $V_3$  in red, yellow, orange, and green, respectively. Elements of  $V_0 \cup V_1 \cup V_2$  will be called *interior*, those from  $V_3$  *exterior*, because the latter are those with adjacent *external* vertices in other subgraphs than  $iH_4^2 \subset H_4^3$ , induced by  $iQ^2$ . Every  $H_4^3$ -dominating set  $D$  then contains

1. an element from  $V_0 \cup V_1$  (otherwise  $iii$  is not covered), where we may assume that this element is from  $V_1$ , because replacing  $iii$  by some  $ijj$ ,  $j \neq i$ , does not change the size of  $D$ , nor its dominating property;
2. (a) either an element from  $V_2$   
(b) or three elements from  $V_3$  with different  $k$   
(otherwise the elements from  $V_2$  would not be covered).

In Case 2(b) already four vertices of  $iQ^2$  lie in  $D$ .

So we have shown that  $|D \cap iQ^2| \geq 2$  for every  $i \in Q$ . Now assume that  $|D| = 12$ ; then the elements of  $D$  can be distributed among the  $iQ^2$  by the numbers

$$\begin{aligned} (5) \quad & 12 = 6 + 2 + 2 + 2, \\ (6) \quad & = 5 + 3 + 2 + 2, \\ (7) \quad & = 4 + 4 + 2 + 2, \\ (8) \quad & = 4 + 3 + 3 + 2, \\ (9) \quad & = 3 + 3 + 3 + 3. \end{aligned}$$

We will now analyse whether one of these distributions is indeed possible. Let us first assume that  $|D \cap iQ^2| = 2$ . Then  $D \cap iQ^2$  consists of one element from  $V_1$  and one from  $V_2$ . It can dominate either 11 or 12 vertices (all in  $iQ^2$ ), as can be seen from Figure 2.

Now let  $|D \cap iQ^2| = 3$ . Then  $D \cap iQ^2$  consists of one element from  $V_1$ , one from  $V_2$ , and one from  $V_3$ . A somewhat tedious case analysis, albeit facilitated by the many symmetries of the graph, shows that  $D \cap iQ^2$  can dominate at most 16 vertices altogether and that there are only three types of sets of size 3 that do cover 16 vertices. They are, with  $\{i, j, k, \ell\} = Q$ ,

$$(10) \quad \{ijj, iji, ikk\} \text{ dominates } (iQ^2 \setminus \{i\ell\ell, i\ell k\}) \cup \{jkk, \ell k k\},$$

$$(11) \quad \{ijj, iki, ijj\} \text{ dominates } (iQ^2 \setminus \{i\ell\ell, i\ell k\}) \cup \{kjj, \ell jj\},$$

$$(12) \quad \{ijj, iki, ijk\} \text{ dominates } (iQ^2 \setminus \{i\ell\ell\}) \cup \{\ell jk\}.$$

For  $|D \cap iQ^2| = 5$  we know that at least one of the elements of  $D \cap iQ^2$  is interior and that among the at most four exterior elements at most three are linked to two external vertices and then one has only one external neighbor. So in addition to the 16 elements of  $D \cap iQ^2$  itself, only up to 7 external vertices can be dominated, all in all at most 23. The same kind of argument shows that  $D \cap iQ^2$  with  $|D \cap iQ^2| = 6$  can dominate at most 24 vertices, because now there may be two elements of  $D \cap iQ^2$  with exactly one external neighbor. With what we have found so far, Cases (5) and (6) can already be excluded, because they lead to at most  $24 + 12 + 12 + 12 = 60 < 64$  and  $23 + 16 + 12 + 12 = 63 < 64$  covered vertices, respectively.

We now look at  $|D \cap iQ^2| = 4$ . Here  $D \cap iQ^2$  must contain one interior vertex and can therefore have at most three exterior ones, potentially leading to six external vertices covered, whence an upper bound for the number of covered vertices would be 22. This argument is not sufficient for Cases (7) and (8) though. It is, however, sufficient if  $|D| = 11$ , since then the decomposition is  $11 = 5 + 2 + 2 + 2 = 4 + 3 + 2 + 2 = 3 + 3 + 3 + 2$ , with upper bounds 59, 62, and 60, respectively. So we have reached the result

$$\gamma(H_4^3) \in \{12, 13\}.$$

Next we exclude Case (7). Assume that  $|D \cap iQ^2| = 2$ ,  $i \in \{2, 3\}$ . Then  $D \cap 2Q^2$  and  $D \cap 3Q^2$  just have two interior vertices each and four vertices of  $D \cap iQ^2$  are not dominated by these two vertices. A case analysis among the 9 possible pairs of interior vertices shows that only for  $D \cap iQ^2 = \{iik, iki\}$ ,  $k \in \{0, 1\}$ , the uncovered sets  $\{i\bar{k}\bar{k}, i\bar{k}j, ij\bar{k}, ijj\}$ , where  $j = 5 - i$  and  $\bar{k} = 1 - k$ , can be dominated by vertices from  $(D \cap 0Q^2) \cup (D \cap 1Q^2)$ , namely  $k\bar{k}\bar{k}$ ,  $k\bar{k}j$ ,  $kjk$ , and  $kjj$ , where the last vertex may be replaced by  $\bar{k}jj$ . This would require altogether 8 exterior vertices in  $(D \cap 0Q^2) \cup (D \cap 1Q^2)$ , but the latter set contains at most 6 of them, so the covering is not possible. (Note that this is also an alternative argument for excluding Cases (5) and (6).)

We now turn to (8). Let  $|D \cap 0Q^2| = 4$ ,  $|D \cap 1Q^2| = 2$ , and  $|D \cap 2Q^2| = 3 = |D \cap 3Q^2|$ . We then have, up to symmetry, five cases to be addressed:

$$(13) \quad D \cap 1Q^2 = \{110, 101\},$$

$$(14) \quad D \cap 1Q^2 = \{112, 121\},$$

$$(15) \quad D \cap 1Q^2 = \{110, 121\},$$

$$(16) \quad D \cap 1Q^2 = \{112, 101\},$$

$$(17) \quad D \cap 1Q^2 = \{112, 131\}.$$

Note that in Cases (13) and (14)  $D \cap 1Q^2$  dominates 12 vertices, while in the other cases only 11. This means that  $D \cap 0Q^2$  has to dominate at least 20 or 21 vertices, respectively, because otherwise only up to  $19 + 12 + 16 + 16 = 63$  or  $20 + 11 + 16 + 16 = 63$  could be dominated by  $D$ . For (13) this can only be fulfilled if  $D \cap 0Q^2 = \{00\ell, 023, 032, 011\}$  for some  $\ell$ , but then  $D \cap iQ^2 = \{iij, iki, i\bar{i}\bar{i}\}$ ,  $\{i, j\} = \{2, 3\}$ ,  $\bar{i} = 5 - i$ , and  $k \neq i$ , respectively. Consequently at least one of the vertices 200, 210, 213 is not dominated by  $D$ .

In Case (14) vertices 203 and 230 must be in  $D \cap 2Q^2$ , but this subgraph cannot contain two exterior vertices of  $D$ .

In Case (15) vertices 200 and 300 cannot be in  $D$ , because the respective subgraphs cannot contain additional exterior vertices to 203 and 302; therefore 100 is not dominated by  $D$ .

To exclude Case (16) we note that since 230 and 320 are in  $D$ , vertices 233 and 322 are not. Hence  $D \cap 0Q^2 = \{00j, 023, 022, 033\}$  for some  $j$  and 010 is not covered.

Finally, for Case (17) vertices 203 and 320 are in  $D$  and therefore 200 and 300 are not. But then 100 is not covered. This concludes Case (8).

So we are left with Case (9) which in principle could lead to a covering of  $16 + 16 + 16 + 16 = 64$  vertices. Therefore, we have to decide whether it is possible to combine choices from the alternatives (10) to (12) for each  $i \in Q$  for which the sets on the right do not overlap. Note that not all four subgraphs can have triples from  $D$  of type (12) simultaneously, because in each a vertex  $ijj$  would not be dominated by  $D \cap iQ^2$ , but the same set does not contain a vertex  $ijj$  either. So, making use of symmetries, we may just consider (10) and (11) for the values  $i = 0$ ,  $j = 1$ ,  $k = 2$ , and  $\ell = 3$ . In the first case, because  $D \cap 0Q^2 = \{001, 010, 022\}$ , we necessarily have  $D \cap 1Q^2 = \{112, 121, 132\}$ , resulting in vertex 122 being covered twice. Similarly, if  $D \cap 0Q^2 = \{001, 020, 011\}$ , we successively get  $132 \in D$ ,  $133 \notin D$ , and  $233 \in D$ , but then 133 is covered twice. This completes the proof of

**Theorem 6.**  $\gamma(H_4^3) = 13 = \gamma(S_4^3)$ .

Finally, we want to prove directly the following statement (cf. [5, p. 262]).

**Theorem 7.** *The graph  $H_4^3$  has no perfect code.*

**Proof.** 0. The graph  $H_4^2$  has exactly one perfect code  $C = \{ii \mid i \in Q\}$ ; for the proof see [5, p. 383]. In particular,  $\gamma(H_4^2) = 4$ .

1. Assume that  $C \subset Q^3$  is a perfect code for  $H_4^3$ . Then  $C \cap iQ^2$  is 1-error correcting in  $iH_4^2 \subset H_4^3$ . From Proposition 3 we get  $4 = \gamma(H_4^2) = \gamma(iH_4^2) \geq |C \cap iQ^2|$ , so no subgraph contains more than 4 codewords.

2. Suppose that a subgraph,  $0Q^2$  say, contains 4 codewords. An easy case analysis shows that the only subset of  $0Q^2$  which is 1-error correcting and has

size 4 is  $\{0ii \mid i \in Q\}$ . Therefore we have  $12j, 13j \notin C$  for all  $j$ . But then necessarily  $101 \in C$ , which in turn excludes  $10j$ ,  $j \neq 1$  from  $C$ . Moreover, only one of the vertices  $11j$  belongs to  $C$ . This results in  $C \cap 1Q^2 < 3$  and similarly in  $C \cap 2Q^2 < 3$  and  $C \cap 3Q^2 < 3$ , whence  $|C| \leq 10$  or  $|C| \leq 4 \cdot 3 = 12$ , but  $|C| = 13$  would be necessary by virtue of Corollary 4 and Theorem 6. ■

#### 4. FURTHER STUDY ON $H_p^n$

Once colorings and domination are known, one might address the question of *dominator coloring* (see, e.g., [1] or [11]), which is a proper vertex coloring such that each vertex dominates all vertices of at least one color class. The smallest number of colors required for a dominator coloring of graph  $G$  is called its *dominator chromatic number* and denoted by  $\chi_d(G)$ . This is properly defined and  $\max\{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq |G|$ , because choosing one representative from each color class of a dominator coloring leads to a dominating set and a coloring where each vertex gets its individual color is obviously a dominator coloring. A non-trivial example for equality on the left is  $G = C_4$  and equality on the right holds for complete graphs:  $\chi_d(K_p) = p$  because there is essentially only one proper vertex coloring. Another easy upper bound for the dominator chromatic number is (cf. [1, Theorem 3.3])

$$(18) \quad \chi_d(G) \leq \gamma(G) + \chi(G),$$

because assigning singleton color classes to the elements of a minimal dominating set and coloring the remaining vertices according to a minimal vertex coloring will produce a (possibly not minimal) dominator coloring.

For Hanoi graphs we have  $\chi(H_p^n) = p$  for  $n \in \mathbb{N}$  [10, Theorem 2]. Inequality (18) is *not* sharp for  $n \leq 2$ .

**Proposition 8.** *For every  $n \in \{0, 1, 2\}$  we have that*

$$\chi_d(H_p^n) = \gamma(H_p^n) + \chi(H_p^n) - 1.$$

**Proof.** For  $n < 2$  this is clear. For  $n = 2$  things get a little more complicated. A perfect vertex  $ii$ ,  $i \in [p]_0$ , can only dominate a singleton color class, because all its neighbors are mutually adjacent. Therefore each subgraph  $iH_p^1$  contains a singleton color class  $\{ij_i\}$  and the subgraph induced by the vertices  $ij$ ,  $j \neq j_i$ , which is a  $K_{p-1}$  and consequently needs another  $p - 1$  colors. So  $\chi_d(H_p^2) \geq 2p - 1 = \gamma(H_p^2) + \chi(H_p^2) - 1$ .

It can easily be checked that  $[p]_0^2 \ni ij \mapsto (i - j) \bmod p \in [p]_0$  defines a proper vertex coloring of  $H_p^2$  with color class 0 containing only the perfect vertices. Assigning individual singleton color classes for the  $p$  perfect vertices and otherwise using color classes 1 to  $p - 1$  leads to a dominator coloring with  $p + p - 1 = 2p - 1$  colors. ■

For  $n > 2$ , inequality (18) is sharp for the classical case  $p = 3$ .

**Proposition 9.** For  $p = 3 \leq n$  we get  $\chi_d(H_3^n) = \gamma(H_3^n) + \chi(H_3^n) = \lceil \frac{3^n}{4} \rceil + 3$ .

**Proof.** By virtue of (18) we only have to show that  $\chi_d(H_3^n) \geq \gamma(H_3^n) + \chi(H_3^n)$ . Let  $c$  be a (minimal) dominator coloring and  $C$  be a 1-perfect code of  $H_3^n$  (cf. [5, Section 2.3.2]). Then  $V(H_3^n)$  is a disjoint union of the neighborhoods of elements from  $C$ , each of which must contain at least one singleton color class of  $c$  (as can be seen by an easy combinatorial analysis). The graph  $H_3^n$  has  $3^{n-1}$  subgraphs isomorphic to  $K_3$ , a number which is strictly larger than  $\gamma(H_3^n) = \lceil \frac{3^n}{4} \rceil$  if and only if  $n > 2$ , so that in these cases there is a  $K_3$  whose vertices are not in the union of the singleton classes already fixed. Therefore, at least  $\chi(K_3) = 3 = \chi(H_3^n)$  more colors are present in  $c$ . ■

We are now tempted to formulate the following.

**Conjecture.** For every  $p, n \geq 3$  we have that  $\chi_d(H_p^n) = \gamma(H_p^n) + \chi(H_p^n)$ .

The Conjecture is supported by the following result.

**Theorem 10.**  $\chi_d(H_4^3) = 17 = \gamma(H_4^3) + \chi(H_4^3)$ .

**Proof.** Again we only have to show that  $\chi_d(H_4^3) \geq 17$ . Note that by Theorem 7 we do not have a perfect code. We have to start with a (minimal) dominator coloring  $c : Q^3 \rightarrow [\kappa]$  and to show that it has at least 17 colors, i.e.,  $\kappa \geq 17$ . Define  $\sigma$  such that for all  $\rho \in [\kappa]$

$$|c^{-1}(\rho)| = 1 \Leftrightarrow \rho \in [\sigma],$$

i.e.,  $c^{-1}(\rho)$  for  $\rho \in [\sigma]$  are all singleton color classes; the union of these will be denoted by  $\Sigma$ . We may assume that  $\sigma < 17$ .

For  $14 \leq \sigma \leq 16$  there exist  $K_3$ -subgraphs whose vertices have at least 3 different colors in  $Q^3 \setminus \Sigma$ , whence  $\kappa \geq 14 + 3 = 17$ .

The graph  $H_4^3$  comprises 16 subgraphs  $K_4$ , so at least one of them does not contain an element from  $\Sigma$  if  $\sigma < 16$ . In particular for  $\sigma = 13$  this means  $\kappa \geq 13 + 4 = 17$ .

For  $\sigma < 13 = \gamma(H_4^3)$  the set  $\Sigma$  will not dominate  $Q^3$ , so there are color classes with at least 2 elements. Each of these color classes can dominate at most two vertices as can be seen by looking at the graph carefully. For  $\sigma = 12$  we would have at least 14 dominating vertices, but a  $K_3$  has not received its 3 colours yet. For  $\sigma = 11$  we have 13 or 15 dominating vertices, but with a  $K_4$  uncolored so far. For  $\sigma = 10$  only at most  $4 \times 5 + 6 \times 6$  vertices would be dominated by  $\Sigma$  (each of the neighborhoods of a perfect vertex must contain an element of  $\Sigma$ ), i.e., 56. So at least 4 more color classes with at least 2 elements each must be present, and again a  $K_3$  has to receive 3 extra colors. Similarly for  $\sigma < 10$ . ■

The question of dominator coloring for general parameters  $p$  and  $n$  has not been addressed yet for Sierpiński graphs either.

### Acknowledgements

The research originated in our participation in the *International Conference on Current Trends in Graph Theory and Computation* held in 2016 at the South Asian University New Delhi (India). We are grateful to Deepa Sinha and Purnima Gupta for their kind hospitality. Thanks go to Ciril Petr (Maribor, Slovenia) for his support and to Sandi Klavžar (Ljubljana, Slovenia) for valuable discussions. The constructive suggestions of two anonymous reviewers helped to improve the readability of the paper.

A.M.H. has been supported as a member of the Institute for Mathematics, Physics, and Mechanics (IMFM Ljubljana, Slovenia) by the Slovenian Research Agency ARRS (research core funding No. J1-7110). N.M. appreciates the fellowship provided by Kalasalingam University.

### REFERENCES

- [1] R.M. Gera, *On dominator colorings in graphs*, Graph Theory Notes N.Y. **52** (2007) 25–30.
- [2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998).
- [3] A.M. Hinz, *Open problems for Hanoi and Sierpiński graphs*, Electron. Notes Discrete Math. **63** (2017) 23–31.  
doi:10.1016/j.endm.2017.10.058
- [4] A.M. Hinz and C. Holz auf der Heide, *An efficient algorithm to determine all shortest paths in Sierpiński graphs*, Discrete Appl. Math. **177** (2014) 111–120.  
doi:10.1016/j.dam.2014.05.049
- [5] A.M. Hinz, S. Klavžar and C. Petr, *The Tower of Hanoi—Myths and Maths*, Second Edition (Springer/Birkhäuser, Cham, 2018).
- [6] A.M. Hinz, S. Klavžar and S.S. Zemljič, *Sierpiński graphs as spanning subgraphs of Hanoi graphs*, Cent. Eur. J. Math. **11** (2013) 1153–1157.  
doi:10.2478/s11533-013-0227-7
- [7] A.M. Hinz, S. Klavžar and S.S. Zemljič, *A survey and classification of Sierpiński-type graphs*, Discrete Appl. Math. **217** (2017) 565–600.  
doi:10.1016/j.dam.2016.09.024
- [8] A.M. Hinz, A. Kostov, F. Kneißl, F. Sürer and A. Danek, *A mathematical model and a computer tool for the Tower of Hanoi and Tower of London puzzles*, Inform. Sci. **179** (2009) 2934–2947.  
doi:10.1016/j.ins.2009.04.010

- [9] A.M. Hinz and D. Parisse, *On the planarity of Hanoi graphs*, Expo. Math. **20** (2002) 263–268.  
doi:10.1016/S0723-0869(02)80023-8
- [10] A.M. Hinz and D. Parisse, *Coloring Hanoi and Sierpiński graphs*, Discrete Math. **312** (2012) 1521–1535.  
doi:10.1016/j.disc.2011.08.019
- [11] J.M. Jeyaseeli, N. Movarraei and S. Arumugam, *Dominator coloring of generalized Petersen graphs*, Lecture Notes in Comput. Sci. **10398** (2017) 144–151.  
doi:10.1007/978-3-319-64419-6\_19
- [12] S. Klavžar and B. Mohar, *Crossing numbers of Sierpiński-like graphs*, J. Graph Theory **50** (2005) 186–198.  
doi:10.1002/jgt.20107
- [13] S. Klavžar and S.S. Zemljic, *On distances in Sierpiński graphs: Almost-extreme vertices and metric dimension*, Appl. Anal. Discrete Math. **7** (2013) 72–82.  
doi:10.2298/AADM130109001K
- [14] M. Schlosser, *Abstände in Sierpiński- und Hanoi-Graphen und metrische Dimension* (BSc Thesis, Ludwig-Maximilians-Universität, München, 2017).
- [15] S. Varghese, A. Vijayakumar and A.M. Hinz, *Power domination in Knödel graphs and Hanoi graphs*, Discuss. Math. Graph Theory **38** (2018) 63–74.  
doi:10.7151/dmgt.1993

Received 9 March 2018

Revised 10 December 2018

Accepted 10 December 2018