

## ***H*-KERNELS IN UNIONS OF *H*-COLORED QUASI-TRANSITIVE DIGRAPHS**

JOSÉ MANUEL CAMPERO-ALONZO<sup>1</sup>

AND

ROCÍO SÁNCHEZ-LÓPEZ

*Facultad de Ciencias*  
*Universidad Nacional Autónoma de México*  
*Círculo Exterior s/n, Coyoacán*  
*Ciudad Universitaria, 04510, Ciudad de México, CDMX*

**e-mail:** (Campero-Alonzo) cloud\_1993@hotmail.com  
(Sánchez-López) usagitsukinomx@yahoo.com.mx

### **Abstract**

Let  $H$  be a digraph (possibly with loops) and  $D$  a digraph without loops whose arcs are colored with the vertices of  $H$  ( $D$  is said to be an  $H$ -colored digraph). For an arc  $(x, y)$  of  $D$ , its color is denoted by  $c(x, y)$ . A directed path  $W = (v_0, \dots, v_n)$  in an  $H$ -colored digraph  $D$  will be called  $H$ -path if and only if  $(c(v_0, v_1), \dots, c(v_{n-1}, v_n))$  is a directed walk in  $H$ . In  $W$ , we will say that there is an obstruction on  $v_i$  if  $(c(v_{i-1}, v_i), c(v_i, v_{i+1})) \notin A(H)$  (if  $v_0 = v_n$  we will take indices modulo  $n$ ). A subset  $N$  of  $V(D)$  is said to be an  $H$ -kernel in  $D$  if for every pair of different vertices in  $N$  there is no  $H$ -path between them, and for every vertex  $u$  in  $V(D) \setminus N$  there exists an  $H$ -path in  $D$  from  $u$  to  $N$ . Let  $D$  be an arc-colored digraph. The color-class digraph of  $D$ ,  $\mathcal{C}_C(D)$ , is the digraph such that  $V(\mathcal{C}_C(D)) = \{c(a) : a \in A(D)\}$  and  $(i, j) \in A(\mathcal{C}_C(D))$  if and only if there exist two arcs, namely  $(u, v)$  and  $(v, w)$  in  $D$ , such that  $c(u, v) = i$  and  $c(v, w) = j$ . The main result establishes that if  $D = D_1 \cup D_2$  is an  $H$ -colored digraph which is a union of asymmetric quasi-transitive digraphs and  $\{V_1, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(D))$  with a property  $P^*$  such that

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,
2. either  $D[\{a \in A(D) : c(a) \in V_i\}]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ ,
3.  $D_i$  has no infinite outward path for every  $i$  in  $\{1, 2\}$ ,

---

<sup>1</sup>Corresponding author.

4. every cycle of length three in  $D$  has at most two obstructions,  
then  $D$  has an  $H$ -kernel.

Some results with respect to the existence of kernels by monochromatic paths in finite digraphs will be deduced from the main result.

**Keywords:** quasi-transitive digraph, kernel by monochromatic paths, alternating kernel,  $H$ -kernel, obstruction.

**2010 Mathematics Subject Classification:** 05C20, 05C38, 05C69.

## 1. INTRODUCTION

Let  $H$  be a digraph possibly with loops and  $D$  a digraph without loops. An  $H$ -arc coloring of  $D$  is a function  $c : A(D) \rightarrow V(H)$ .  $D$  is  $H$ -colored if  $D$  has an  $H$ -arc coloring. A path  $W = (v_0, \dots, v_n)$  in  $D$  is said to be an  $H$ -path if and only if  $(c(v_0, v_1), \dots, c(v_{n-1}, v_n))$  is a walk in  $H$ . We are going to consider that an arc is an  $H$ -path, that is to say, a singleton vertex is a walk in  $H$ . A subset  $S$  of  $V(D)$  is  $H$ -absorbent if for every  $x$  in  $V(D) \setminus S$  there is an  $H$ -path from  $x$  to some point of  $S$ . A subset  $I$  of  $V(D)$  is  $H$ -independent if there is no  $H$ -path between any two distinct vertices of  $I$ . A subset  $N$  of  $V(D)$  is an  $H$ -kernel if  $N$  is both  $H$ -absorbent and  $H$ -independent. The concept of  $H$ -kernel has its origins in the works carried out by Sands, Sauer and Woodrow [15], Linek and Sands [13] and Arpin and Linek [1]. In [15] Sands, Sauer and Woodrow proved that if the arcs of a finite tournament  $T$  are colored with two colors, then there is always a vertex  $v$  in  $T$  such that for every  $w$  in  $V(T) \setminus \{v\}$  there exists a monochromatic path from  $w$  to  $v$ . In [13] Linek and Sands gave an extension of the result of Sands, Sauer and Woodrow, in which the arcs of a tournament  $T$  are colored with the elements of a partially ordered set  $P$  and in their paper they give the first notion of  $H$ -path. In [1] Arpin and Linek work with  $H$ -colored digraphs and in their paper they introduce the concept of  $H$ -walk where an  $H$ -walk is a walk  $(v_0, \dots, v_n)$  in  $D$  such that  $(c(v_0, v_1), \dots, c(v_{n-1}, v_n))$  is a walk in  $H$ . In [1] Arpin and Linek introduce the concept of  $H$ -independent set by walks as a subset of vertices  $I$  of  $D$  such that there is no  $H$ -walks between any two different vertices of  $I$ . They also define an  $H$ -sink as a subset of vertices  $S$  of  $D$  such that for any  $u$  in  $V(D) \setminus S$  there is  $v$  in  $S$  such that there exists an  $H$ -walk from  $u$  to  $v$ . Galeana-Sánchez and Delgado-Escalante were inspired by the work of Arpin and Linek and in [6] they introduced the concept of  $H$ -kernels. A subset of vertices  $N$  of  $D$  is called  $H$ -kernel by walks if  $N$  is both an  $H$ -independent set by walks and  $N$  is an  $H$ -sink. Notice that the concept of  $H$ -kernel and the concept of  $H$ -kernel by walks are different because of that the existence of an  $H$ -walk between two vertices does not guarantee the existence of an  $H$ -path between those vertices and the concatenation of two  $H$ -paths is not always an  $H$ -walk, see Figure 1.

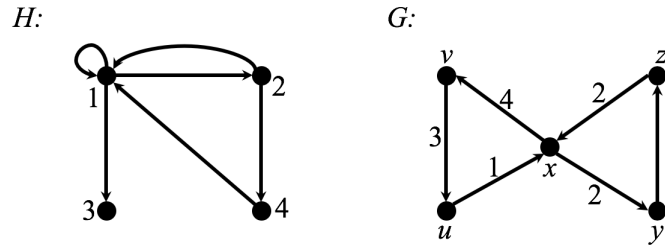


Figure 1.  $(u, x, y, z, x, v)$  is a  $uv$ - $H$ -walk in  $G$ . The only one  $uv$ -path in  $G$  is  $(u, x, v)$  but this path is not a  $uv$ - $H$ -path in  $G$ .  $\{v\}$  is an  $H$ -kernel by walks in  $G$ . Every  $H$ -independent set in  $G$  consists only of one element but none of these is an  $H$ -kernel.

Notice that it follows from the definition of  $H$ -kernel that when  $A(H) = \emptyset$ , an  $H$ -kernel is a kernel (a subset  $N$  of vertices of  $D$  such that (1) for every  $u$  and  $v$  in  $N$  it holds that  $\{(u, v), (v, u)\} \cap A(D) = \emptyset$  and (2) for every  $u$  in  $V(D) \setminus N$  there exists  $v$  in  $N$  such that  $(u, v) \in A(D)$ ); when  $A(H) = \{(u, u) : u \in V(H)\}$ , an  $H$ -kernel is a kernel by monochromatic paths ( $mp$ -kernel) (a subset  $N$  of vertices of  $D$  such that (1) for every  $u$  and  $v$  in  $N$  there exists no monochromatic directed paths between  $u$  and  $v$  and (2) for every  $u$  in  $V(D) \setminus N$  there exists  $v$  in  $N$  such that there exists a monochromatic directed path from  $u$  to  $v$ ) and when  $H$  has no loops, an  $H$ -kernel is an alternating kernel (a subset  $N$  of vertices of  $D$  such that (1) for every  $u$  and  $v$  in  $N$  it holds that there exists no directed path between  $u$  and  $v$  in which consecutive arcs have different colors and (2) for every  $u$  in  $V(D) \setminus N$  there exists  $v$  in  $N$  such that there is a directed path from  $u$  to  $v$  in which consecutive arcs have different colors). In each of these special cases for  $H$ , sufficient conditions have been established in order to guarantee the existence of  $H$ -kernels, see for example [3, 5, 7, 9, 15]. Thus we can see that the concept of  $H$ -kernels is a generalization of the concepts of kernels,  $mp$ -kernels and alternating kernels.

Due to the difficulty of finding kernels,  $mp$ -kernels and alternating kernels in arc-colored digraphs, sufficient conditions for the existence of each of these  $H$ -kernels in arc-colored digraphs have been obtained mainly by study special classes of digraphs. A digraph  $D$  is *quasi-transitive* whenever  $\{(u, v), (v, w)\} \subseteq A(D)$  implies either  $(u, w) \in A(D)$  or  $(w, u) \in A(D)$ . Quasi-transitive digraphs are of interest because these are a generalization of tournaments (due to Ghouilá-Houri [12]) and those digraphs are a special case of digraphs in which the existence of kernels,  $mp$ -kernels and alternating kernels has been studied.

In [10] Galeana-Sánchez and Rojas-Monroy proved that if  $D = D_1 \cup D_2$  (possibly  $A(D_1) \cap A(D_2) \neq \emptyset$ ) where  $D_i$  is a quasi-transitive digraph which contains no asymmetric infinite outward path (in  $D_i$ ) for  $i$  in  $\{1, 2\}$ , and that every directed cycle of length 3 contained in  $D$  has at least two symmetric arcs, then  $D$  has a kernel.

A *chromatic class* of  $D$  is the set of arcs of a same color. We say that a chromatic class  $\mathcal{C}$  is quasi-transitive if  $D[\mathcal{C}]$  is a quasi-transitive digraph. Let  $D = D_1 \cup D_2$  be a digraph. We will say that  $D$  is a union of asymmetric quasi-transitive digraphs if (1)  $D_i$  is a quasi-transitive digraph for every  $i$  in  $\{1, 2\}$ , (2)  $D_i$  is asymmetric for every  $i$  in  $\{1, 2\}$  and (3)  $A(D_1) \cap A(D_2) = \emptyset$ .

In [11] Galeana-Sánchez *et al.* worked with a finite  $m$ -colored multidigraph (a digraph with parallel arcs)  $D = D_1 \cup D_2$  which is a union of asymmetric quasi-transitive digraphs, and they proved that if  $D$  satisfies that

1. every chromatic class induces a quasi-transitive digraph,
2. every chromatic class is contained in  $D_i$  for some  $i$  in  $\{1, 2\}$  and
3.  $D$  contains neither 3-colored directed triangles nor 3-colored transitive subtournaments of order 3,

then  $D$  has an  $mp$ -kernel.

In [7], recently, Delgado-Escalante *et al.* proved the following.

**Theorem 1.** *If  $D$  is a finite  $m$ -colored quasi-transitive digraph such that every directed cycle of length 3 contained in  $D$  is 3-colored, then  $D$  has an alternating kernel.*

Basically the spirit of the conditions that guarantee the existence of kernels or  $mp$ -kernels in [10] and [11], respectively, arises from structural properties of 2-colored digraphs which were studied in [15] by Sands *et al.*

On the other hand, in [8] Galeana-Sánchez defined the *color-class digraph*  $\mathcal{C}_C(D)$  of  $D$  as the digraph whose vertices are the colors represented in the arcs of  $D$  and  $(i, j) \in A(\mathcal{C}_C(D))$  if and only if there exist two arcs, namely  $(u, v)$  and  $(v, w)$  in  $D$ , such that  $(u, v)$  has color  $i$  and  $(v, w)$  has color  $j$  (notice that  $\mathcal{C}_C(D)$  can have loops by definition). Because of that in an  $H$ -colored digraph  $D$ , it holds that  $V(\mathcal{C}_C(D)) \subseteq V(H)$ , we can establish structural properties on  $\mathcal{C}_C(D)$ , with respect to  $H$ , in order to guarantee the existence of  $H$ -kernels.

Let  $H$  be a digraph,  $D$  an  $H$ -colored digraph and  $(v_0, v_1, \dots, v_n)$  a walk in  $D$ . We will say that there is an *obstruction* on  $v_i$  if  $(c(v_{i-1}, v_i), c(v_i, v_{i+1})) \notin A(H)$  (if  $v_0 = v_n$  we will take indices modulo  $n$ ).

In this paper we continue with the study of the existence of  $H$ -kernels in unions of quasi-transitive digraphs and for this we will need the following definitions.

**Definition.** Let  $H$  be a digraph,  $D$  an  $H$ -colored digraph and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$ . We will say that  $\{V_1, \dots, V_k\}$  has the property  $P^*$  if the following conditions are satisfied.

1.  $\mathcal{C}_C(D)[V_i]$  is a subdigraph of  $H$  for every  $i$  in  $\{1, \dots, k\}$ .
2. If  $(u, v) \in A(\mathcal{C}_C(D))$ , for some  $u$  in  $V_i$  and for some  $v$  in  $V_j$  with  $i \neq j$ , then  $(u, v) \notin A(H)$ .

**Definition.** Let  $H$  be a digraph,  $D$  an  $H$ -colored digraph and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$ .  $V_i$  is said to be a quasi-transitive  $V_i$ -class if  $D[\{a \in A(D) : c(a) \in V_i\}]$  is a quasi-transitive digraph for every  $i$  in  $\{1, \dots, k\}$ .

The main result establishes that if  $H$  is a digraph,  $D = D_1 \cup D_2$  is an  $H$ -colored digraph which is a union of asymmetric quasi-transitive digraphs and  $\{V_1, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$  such that

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,
2. either  $D[\{a \in A(D) : c(a) \in V_i\}]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ ,
3.  $D_i$  has no infinite outward path for every  $i$  in  $\{1, 2\}$ ,
4. every directed cycle of length three in  $D_i$  has at most two obstructions,

then  $D$  has an  $H$ -kernel.

With the main result of this paper we show that the main result in [11] can be reduced for digraphs as follows.

Let  $D = D_1 \cup D_2$  be a finite  $m$ -colored digraph which is a union of asymmetric quasi-transitive digraphs such that

1. every chromatic class is quasi-transitive,
2. if  $\mathcal{C}$  is a chromatic class, then  $\mathcal{C} \subseteq A(D_j)$  for some  $j$  in  $\{1, 2\}$  and
3.  $D$  does not contain 3-colored directed cycles of length three.

Then  $D$  has an  $mp$ -kernel.

In terms of  $H$ -kernels Theorem 1 says that if  $H$  is a complete digraph without loops and  $D$  is a finite  $H$ -colored quasi-transitive digraph such that every directed cycle of length 3 contained in  $D$  has no obstructions, then  $D$  has an  $H$ -kernel. However, the above is not true if  $H$  is not complete; consider the directed cycle of length 3,  $C_3$ , whose arcs are colored with three different vertices of  $H$ , with  $A(H) = \emptyset$ , it is clear that  $C_3$  has no  $H$ -kernel. In this paper we will deduce from the main result the following.

Let  $H$  be a digraph (possibly with loops),  $D$  an  $H$ -colored asymmetric quasi-transitive digraph and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Suppose that

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,
2.  $D$  has no infinite outward path,
3. every cycle of length three in  $D$  has at most two obstructions.

Then  $D$  has an  $H$ -kernel.

We will need the following result.

**Corollary 2** ([2], p. 53). *If a quasi-transitive digraph  $D$  has an  $xy$ -path but  $(x, y) \notin A(D)$ , then either  $(y, x) \in A(D)$  or there exists vertices  $u$  and  $v$  in  $V(D) \setminus \{x, y\}$  such that  $(x, u, v, y)$  and  $(y, u, v, x)$  are paths in  $D$ .*

## 2. TERMINOLOGY AND NOTATION

For general concepts we refer the reader to [2] and [4]. An arc of the form  $(x, x)$  is a *loop*. An arc  $(u, v)$  in  $A(D)$  is *asymmetric* if  $(v, u) \notin A(D)$ . We will say that a digraph  $D$  is *asymmetric* if every arc of  $D$  is asymmetric. We will say that two digraphs  $D_1$  and  $D_2$  are *equal*, denoted by  $D_1 = D_2$ , if  $A(D_1) = A(D_2)$  and  $V(D_1) = V(D_2)$ . A *directed walk* is a sequence  $W = (v_0, v_1, \dots, v_n)$  such that  $(v_i, v_{i+1}) \in A(D)$  for each  $i$  in  $\{0, \dots, n-1\}$ . The number  $n$  is the *length* of the walk. We will say that the directed walk  $(v_0, v_1, \dots, v_n)$  is *closed* if  $v_0 = v_n$ . If  $v_i \neq v_j$  for all  $i$  and  $j$  such that  $\{i, j\} \subseteq \{0, \dots, n\}$  and  $i \neq j$ , it is called a *directed path*. A *directed cycle* is a directed walk  $(v_1, v_2, \dots, v_n, v_1)$  such that  $v_i \neq v_j$  for all  $i$  and  $j$  such that  $\{i, j\} \subseteq \{1, \dots, n\}$  and  $i \neq j$ , this will be denoted by  $C_n$ . If  $D$  is an infinite digraph, an infinite outward path is an infinite sequence  $(v_1, v_2, \dots)$  of distinct vertices of  $D$  such that  $(v_i, v_{i+1}) \in A(D)$  for each  $i \in \mathbb{N}$ . In this paper we are going to write walk, path, cycle, instead of directed walk, directed path, directed cycle, respectively. The union of walks will be denoted with  $\cup$ . Let  $W = (v_0, v_1, \dots, v_n)$  be a walk and  $\{v_i, v_j\} \subseteq V(W)$ , with  $i < j$ . Then the  $v_i v_j$ -walk  $(v_i, v_{i+1}, \dots, v_{j-1}, v_j)$  contained in  $W$  will be denoted by  $(v_i, W, v_j)$ . For a subset  $S$  of  $V(D)$  the subdigraph of  $D$  induced by  $S$ , denoted by  $D[S]$ , has  $V(D[S]) = S$  and  $A(D[S]) = \{(u, v) \in A(D) : \{u, v\} \subseteq S\}$ . A subset  $S$  of  $V(D)$  is said to be *independent* if the only arcs in  $D[S]$  are loops. For a subset  $B$  of  $A(D)$  the subdigraph of  $D$  induced by  $B$ , denoted by  $D[B]$ , has  $A(D[B]) = B$  and  $V(D[B]) = \{v \in V(D) : (u, v) \in B \text{ or } (v, u) \in B \text{ for some } u \in V(D)\}$ . A pair of digraphs  $D$  and  $G$  are *isomorphic* if there exists a bijection  $f : V(D) \rightarrow V(G)$  such that  $(x, y) \in A(D)$  if and only if  $(f(x), f(y)) \in A(G)$  ( $f$  will be called *isomorphism*). We will say that a digraph  $D$  is *complete* if for every pair of different vertices  $u$  and  $v$  in  $V(D)$  it holds that  $\{(u, v), (v, u)\} \subseteq A(D)$ .

A digraph  $D$  is said to be *m-colored* if the arcs of  $D$  are colored with  $m$  colors. A path is called *monochromatic* if all of its arcs are colored alike.

## 3. PREVIOUS RESULTS

For the rest of the work  $H$  is a digraph possibly with loops and  $D$  is a, possibly infinite, digraph without loops.

We need to introduce some notation in order to present our proofs more compactly.

Let  $H$  be a digraph and  $D$  an  $H$ -colored digraph. Consider  $\{u, v\}$  and  $S$  two subsets of  $V(D)$ . We will write  $u \xrightarrow[H]{D} v$  if there exists a  $uv$ - $H$ -path in  $D$ ;  $u \xrightarrow[H]{D} S$  if there exists a  $uS$ - $H$ -path in  $D$ ;  $u \not\xrightarrow[H]{D} v$  is the denial of  $u \xrightarrow[H]{D} v$ ;  $u \not\xrightarrow[H]{D} S$  is

the denial of  $u \xrightarrow[H]{H} S$ .

We will start with some results which will be useful.

From now on, the set  $\{a \in A(D) : c(a) \in V_i\}$  will be denoted by  $B_i$  for every  $i$  in  $\{1, \dots, k\}$ .

**Lemma 3.** *Let  $H$  be a digraph and  $D$  an  $H$ -colored digraph. Suppose that  $\{V_1, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Then the following properties are satisfied.*

1. *Let  $i$  be an index in  $\{1, \dots, k\}$ . Every finite path in  $D[B_i]$  is an  $H$ -path in  $D[B_i]$ .*
2. *If  $P$  is a finite  $H$ -path in  $D$ , then there exists  $i$  in  $\{1, \dots, k\}$  such that  $P$  is contained in  $D[B_i]$ .*

**Proof.** Let  $P = (u_0, \dots, u_m)$  be a path in  $D[B_i]$ . We will prove that  $P$  is an  $H$ -path in  $D$ . It follows from the definition of color-class digraphs that  $P' = (c(u_0, u_1), \dots, c(u_{m-1}, u_m))$  is a walk in  $\mathcal{C}_C(D)$ . Since  $c(u_j, u_{j+1}) \in V_i$  for every  $j$  in  $\{0, \dots, m-1\}$ , then  $P'$  is a walk in  $\mathcal{C}_C(D)[V_i]$ , which implies that  $P'$  is a walk in  $H$  (because  $\mathcal{C}_C(D)[V_i]$  is a subdigraph of  $H$ ). Therefore  $P$  is an  $H$ -path in  $D[B_i]$ .

On the other hand, let  $P = (v_0, \dots, v_n)$  be an  $H$ -path in  $D$ . Then, it follows from the definition of  $H$ -paths and the definition of color-class digraphs that  $(c(v_{j-1}, v_j), c(v_j, v_{j+1})) \in A(H) \cap A(\mathcal{C}_C(D))$  for every  $j$  in  $\{1, \dots, n-1\}$ . Therefore, we get from 2 in definition of  $P^*$  that there exists  $i$  in  $\{1, \dots, k\}$  such that  $c(v_j, v_{j+1}) \in V_i$  for every  $j$  in  $\{0, \dots, n-1\}$ . Thus  $P$  is contained in  $D[B_i]$ . ■

**Lemma 4.** *Let  $H$  be a digraph,  $D$  an  $H$ -colored digraph and  $\{w, z\} \subseteq V(D)$ . Suppose that  $\{V_1, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(D))$  such that  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ . If there exists a  $wz$ -path in  $D[B_j]$  and there exists no  $zw$ -path in  $D[B_j]$  for some  $j$  in  $\{1, \dots, k\}$ , then  $(w, z) \in A(D[B_j])$ .*

**Proof.** It follows from Corollary 2. ■

We can obtain an extension of Lemma 4 as follows.

**Lemma 5.** *Let  $H$  be a digraph and  $D = D_1 \cup D_2$  an  $H$ -colored digraph. Suppose that  $\{V_1, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$  such that*

1.  *$V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,*
2. *either  $D[B_i]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ .*

*Let  $r$  be an index in  $\{1, 2\}$ . If  $x \xrightarrow[H]{D_r} z$  and  $z \not\xrightarrow[H]{D_r} x$ , then  $(x, z) \in A(D_r)$ .*

**Proof.** Let  $P$  be an  $xz$ - $H$ -path in  $D_r$ . It follows from Lemma 3 that there exists  $i$  in  $\{1, \dots, k\}$  such that  $P$  is contained in  $D[B_i]$ . The hypothesis implies that  $D[B_i]$  is a subdigraph of  $D_r$  (because  $P$  is in  $D_r$ ). On the other hand, since  $z \xrightarrow[D_r]{H} x$ , there exists no  $zx$ - $H$ -path in  $D[B_i]$ , which implies that there exists no  $zx$ -path in  $D[B_i]$  (by 1 in Lemma 3). Therefore, we get from Lemma 4 that  $(x, z) \in A(D[B_i])$ . So,  $(x, z) \in A(D_r)$ . ■

The following result will be useful in what follows.

**Lemma 6.** *Let  $H$  be a digraph,  $D = D_1 \cup D_2$  an  $H$ -colored digraph and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . If either  $D[B_i]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ , then there exists a partition of  $V(\mathcal{C}_C(D_r))$  with the property  $P^*$  for every  $r$  in  $\{1, 2\}$ .*

**Proof.** Suppose that  $\{V_1, \dots, V_t\}$  is such that  $D[B_i]$  is a subdigraph of  $D_1$  for every  $i$  in  $\{1, \dots, t\}$  and  $\{V_{t+1}, \dots, V_k\}$  is such that  $D[B_j]$  is a subdigraph of  $D_2$  for every  $j$  in  $\{t+1, \dots, k\}$ . Then, considering that  $D_1$  and  $D_2$  are also  $H$ -colored digraphs, it follows that  $\{V_1, \dots, V_t\}$  is a partition of  $V(\mathcal{C}_C(D_1))$  with the property  $P^*$  and  $\{V_{t+1}, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(D_2))$  with the property  $P^*$  (this follows from the fact that  $\mathcal{C}_C(D_r)$  is a subdigraph of  $\mathcal{C}_C(D)$  for every  $r$  in  $\{1, 2\}$  and the fact that either  $V_i \subseteq V(\mathcal{C}_C(D_1))$  or  $V_i \subseteq V(\mathcal{C}_C(D_2))$  for every  $i$  in  $\{1, \dots, k\}$ ). ■

**Proposition 7.** *Let  $H$  be a digraph,  $D = D_1 \cup D_2$  an  $H$ -colored digraph which is a union of asymmetric quasi-transitive digraphs,  $r$  an index in  $\{1, 2\}$ ,  $\{x, y\} \subseteq V(D)$  and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Suppose that*

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,
2. either  $D[B_i]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ ,
3. every cycle of length three in  $D_i$  has at most two obstructions for every  $i$  in  $\{1, 2\}$ ,
4.  $x \xrightarrow[D_r]{H} y$  and  $y \not\xrightarrow[D_r]{H} x$ .

*If  $z$  is a vertex in  $V(D)$  such that  $y \xrightarrow[D_r]{H} z$ , then  $(x, z) \in A(D_r)$ ; if  $z \xrightarrow[D_r]{H} x$ , then  $(z, y) \in A(D_r)$ .*

**Proof.** Notice that it follows from Lemma 5 and hypothesis 4 of this proposition that  $(x, y) \in A(D_r)$ .

If  $y \xrightarrow[D_r]{H} z$ , then let  $(y = w_0, \dots, w_m = z)$  be a  $yz$ - $H$ -path in  $D_r$ . We will prove that  $(x, z) \in A(D_r)$  by induction on  $m$ .



If  $m = 0$ , it is clear that  $(x, z) \in A(D_r)$  (because in this case  $y = w_0 = z$ ).

Suppose that if  $(y = u_0, \dots, u_{m-1})$  is a  $yu_{m-1}$ - $H$ -path in  $D_r$  with length  $m - 1$ , then  $(x, u_{m-1}) \in A(D_r)$ .

Let  $P = (y = \alpha_0, \dots, \alpha_m)$  be a  $y\alpha_m$ - $H$ -path in  $D_r$  with length  $m$ . We will prove that  $(x, \alpha_m) \in A(D_r)$ . Since  $(y, P, \alpha_{m-1})$  is a  $y\alpha_{m-1}$ - $H$ -path in  $D_r$  with length  $m - 1$ , it follows from the induction hypothesis that  $(x, \alpha_{m-1}) \in A(D_r)$ . Since  $\{(x, \alpha_{m-1}), (\alpha_{m-1}, \alpha_m)\} \subseteq A(D_r)$  and  $D_r$  is a quasi-transitive digraph, it follows that  $\{(x, \alpha_m), (\alpha_m, x)\} \cap A(D_r) \neq \emptyset$ . If  $(\alpha_m, x) \in A(D_r)$ , then  $\gamma = (x, \alpha_{m-1}, \alpha_m, x)$  is a cycle of length three in  $D_r$  which has at most two obstructions by hypothesis 3 of this proposition. If there is no obstruction on  $\alpha_{m-1}$ , we have that  $P' = (x, \alpha_{m-1}, \alpha_m)$  is an  $H$ -path in  $D_r$ , then we get by Lemma 6 and by Lemma 3 that there exists  $i$  in  $\{1, \dots, k\}$  such that  $D[B_i]$  is a subdigraph of  $D_r$  and  $P'$  is contained in  $D[B_i]$ , respectively. Since  $D[B_i]$  is a quasi-transitive digraph,  $P'$  is a path with length two in  $D[B_i]$ ,  $(\alpha_m, x) \in A(D_r)$  and  $D_r$  is an asymmetric digraph, we get that  $(\alpha_m, x) \in A(D[B_i])$ . This implies that  $\gamma$  is contained in  $D[B_i]$ . In the same way, we can conclude that  $\gamma$  is contained in  $D[B_j]$  for some  $j$  in  $\{1, \dots, k\}$  if either there is no obstruction on  $x$  or there is no obstruction on  $\alpha_m$ . Therefore, in particular, we get from Lemma 3 that  $(\alpha_{m-1}, \alpha_m, x)$  is an  $H$ -path in  $D_r$ , that is  $(c(\alpha_{m-1}, \alpha_m), c(\alpha_m, x)) \in A(H)$ . Thus  $P \cup (\alpha_m, x)$  is an  $yx$ - $H$ -path in  $D_r$ , a contradiction with hypothesis 4 of this proposition. Therefore,  $(\alpha_m, x) \notin A(D_r)$ , which implies that  $(x, \alpha_m) \in A(D_r)$ .

If  $z \xrightarrow[H]{D_r} x$ , then we can consider the converse of  $D$  and the converse of  $H$  (where the converse of a digraph  $G$  is the digraph  $\overleftarrow{G}$  which one obtains from  $G$  by reversing all arcs). It is clear that the digraph  $\overleftarrow{D} = \overleftarrow{D_1} \cup \overleftarrow{D_2}$  is an  $\overleftarrow{H}$ -colored digraph which is a union of asymmetric quasi-transitive digraphs and  $\{V_1, \dots, V_k\}$  is a partition of  $V(\mathcal{C}_C(\overleftarrow{D}))$  with the property  $P^*$ , with respect to  $\overleftarrow{H}$ . In addition, the hypothesis 1, 2 and 3 fulfill in this digraph  $\overleftarrow{H}$ -colored, in the context of the new related digraphs associated, and hypothesis 4 says that  $y \xrightarrow[\overleftarrow{D_r}]{\overleftarrow{H}} x$  and  $x \xrightarrow[\overleftarrow{D_r}]{\overleftarrow{H}} y$ . Since in  $\overleftarrow{D}$  we have that  $x \xrightarrow[\overleftarrow{D_r}]{\overleftarrow{H}} z$ , then we conclude from the previous case that  $(y, z) \in A(\overleftarrow{D_r})$ . Therefore,  $(z, y) \in A(D_r)$ . ■

Notice that, since an arc  $(w, t)$  in  $D_r$  defines a  $wt$ - $H$ -path in  $D_r$ , we also can conclude from Proposition 7 that  $x \xrightarrow[H]{D_r} z$  if  $(x, z) \in A(D_r)$  or  $z \xrightarrow[H]{D_r} y$  if  $(z, y) \in A(D_r)$ .

**Proposition 8.** *Let  $H$  be a digraph,  $D = D_1 \cup D_2$  an  $H$ -colored digraph which is a union of asymmetric quasi-transitive digraphs and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Suppose that*

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,

2. either  $D[B_i]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ ,
3. every cycle of length three in  $D_i$  has at most two obstructions.

Then there exists no cycle  $\gamma = (u_0, u_1, \dots, u_n, u_0)$  in  $D_r$ , with  $r$  in  $\{1, 2\}$ , such that  $u_{i+1} \xrightarrow[H]{D_r} u_i$  for every  $i$  in  $\{0, \dots, n\}$  (indices modulo  $n+1$ ).

**Proof.** Proceeding by contradiction, suppose that there exists a cycle  $\gamma = (u_0, u_1, \dots, u_n, u_0)$  in  $D_r$ , for some  $r$  in  $\{1, 2\}$ , of minimum length such that  $u_{i+1} \xrightarrow[H]{D_r} u_i$  for every  $i$  in  $\{0, \dots, n\}$  (indices modulo  $n+1$ ). Notice that there exists  $j_0$  in  $\{0, \dots, n\}$  such that there is an obstruction on  $u_{j_0}$  in  $\gamma$ , otherwise  $u_{i+1} \xrightarrow[H]{D_r} u_i$  for every  $i$  in  $\{0, \dots, n\}$  (indices modulo  $n+1$ ), which is a contradiction. Suppose without loss of generality that there is an obstruction on  $u_1$ , that is  $(c(u_0, u_1), c(u_1, u_2)) \notin A(H)$ . Since  $u_0 \xrightarrow[H]{D_r} u_1$  (because  $(u_0, u_1) \in A(D_r)$ ),  $u_1 \xrightarrow[H]{D_r} u_0$  and  $u_1 \xrightarrow[H]{D_r} u_2$  (because  $(u_1, u_2) \in A(D_r)$ ), we get from Proposition 7 that  $(u_0, u_2) \in A(D_r)$ . Because of that  $\gamma' = (u_0, u_2) \cup (u_2, \gamma, u_0)$  is a cycle with length less than the length of  $\gamma$ , it follows from the choice of  $\gamma$  that  $u_2 \xrightarrow[H]{D_r} u_0$ . Therefore, since  $u_1 \xrightarrow[H]{D_r} u_2$  (because  $(u_1, u_2) \in A(D_r)$ ),  $u_2 \xrightarrow[H]{D_r} u_1$  and  $u_2 \xrightarrow[H]{D_r} u_0$ , we get from Proposition 7 that  $(u_1, u_0) \in A(D_r)$ , which is a contradiction.

Therefore, there exists no cycle  $\gamma = (u_0, u_1, \dots, u_n, u_0)$  in  $D_r$ , with  $r$  in  $\{1, 2\}$ , such that  $u_{i+1} \xrightarrow[H]{D_r} u_i$  for every  $i$  in  $\{0, \dots, n\}$  (indices modulo  $n+1$ ). ■

**Definition.** Let  $H$  be a digraph,  $D$  an  $H$ -colored digraph and  $G$  a subdigraph of  $D$ . We will say that a subset  $S$  of  $V(D)$  is an  $H$ -semikernel modulo  $G$  in  $D$  if

1.  $S$  is an  $H$ -independent set in  $D$ ,
2. if some vertex  $x$  in  $V(D) \setminus S$  is such that  $u \xrightarrow[H]{D[A(D) \setminus A(G)]} x$  for some vertex  $u$  in  $S$ , then there exists  $s$  in  $S$  such that  $x \xrightarrow[H]{D_r} s$ .

**Proposition 9.** Let  $H$  be a digraph,  $D = D_1 \cup D_2$  an  $H$ -colored digraph which is a union of asymmetric quasi-transitive digraphs and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Suppose that

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,
2. either  $D[B_i]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ ,
3.  $D_i$  has no infinite outward path for every  $i$  in  $\{1, 2\}$ ,
4. every cycle of length three in  $D_i$  has at most two obstructions.

Then there exists  $x$  in  $V(D)$  such that  $\{x\}$  is an  $H$ -semikernel modulo  $D_r$  in  $D$ , with  $r$  in  $\{1, 2\}$ .

**Proof.** Suppose without loss of generality that  $r = 1$ . Proceeding by contradiction, suppose that for every  $w$  in  $V(D)$  there exists  $v_w$  in  $V(D) \setminus \{w\}$  such that  $w \xrightarrow{D_2} v_w$  and  $v_w \xrightarrow{D} w$ . Therefore, for every  $n$  in  $\mathbb{N}$  given  $w_n$  in  $V(D)$  there exists  $w_{n+1}$  in  $V(D) \setminus \{w_n\}$  such that  $w_n \xrightarrow{D_2} w_{n+1}$  and  $w_{n+1} \xrightarrow{D} w_n$ . So, it follows from Lemma 5 that  $(w_n, w_{n+1}) \in A(D_2)$  for every  $n$  in  $\mathbb{N}$ . If  $w_i \neq w_j$  for every  $i$  different from  $j$ , then  $(w_n)_{n \in \mathbb{N}}$  is an infinite outward path in  $D_2$  which is not possible. Therefore, there exist  $w_i$  and  $w_j$ , with  $i < j$ , such that  $w_i = w_j$ , which implies that  $(w_i, w_{i+1}, \dots, w_j = w_i)$  is a closed walk in  $D_2$  which contains a cycle  $\gamma = (w_{i_0}, w_{i_1}, \dots, w_{i_t}, w_{i_0})$  such that  $w_{i_{s+1}} \xrightarrow{D_2} w_{i_s}$  for every  $s$  in  $\{0, \dots, t\}$  (indices modulo  $t+1$ ), a contradiction with Proposition 8. Therefore, there exists  $x$  in  $V(D)$  such that  $\{x\}$  is an  $H$ -semikernel modulo  $D_1$  in  $D$ . ■

#### 4. MAIN RESULT

**Theorem 10.** Let  $H$  be a digraph,  $D = D_1 \cup D_2$  an  $H$ -colored digraph which is a union of asymmetric quasi-transitive digraphs and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Suppose that

1.  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,
2. either  $D[B_i]$  is a subdigraph of  $D_1$  or it is a subdigraph of  $D_2$  for every  $i$  in  $\{1, \dots, k\}$ ,
3.  $D_i$  has no infinite outward path for every  $i$  in  $\{1, 2\}$ ,
4. every cycle of length three in  $D_i$  has at most two obstructions for every  $i$  in  $\{1, 2\}$ .

Then  $D$  has an  $H$ -kernel.

**Proof.**  $\mathcal{S} = \{S \subseteq V(D) : S \text{ is } H\text{-independent in } D\}$  and  $\mathcal{L} = \{S \in \mathcal{S} : S \text{ is an } H\text{-semikernel modulo } D_1 \text{ in } D\}$ .

Since  $\{w\}$  is an  $H$ -independent set for every  $w$  in  $V(D)$ , it follows that  $\mathcal{S} \neq \emptyset$ ; by Proposition 9 we get that  $\mathcal{L} \neq \emptyset$ .

For sets  $S, T$  in  $\mathcal{L}$ , put  $S \leq T$  if for all  $s$  in  $S$  there exists  $t$  in  $T$  such that either  $s = t$ , or  $s \xrightarrow{D_1} t$  and  $t \xrightarrow{D_1} s$ .

**Claim 1.**  $(\mathcal{L}, \leq)$  is a poset.

**Proof.** Consider  $\{S, T, R\}$  a subset of  $\mathcal{L}$ .

(1.1)  $\leq$  is reflexive.

Clearly  $S \leq S$  for every  $S$  in  $\mathcal{L}$ .

(1.2)  $\leq$  is antisymmetric.

Suppose that  $S \leq T$  and  $T \leq S$ . We will prove that  $S = T$ . Let  $t$  be a vertex in  $T$  and suppose that  $t \notin S$ . Then since  $T \leq S$ , we have that there exists  $s$  in  $S$  such that  $t \xrightarrow{H}_{D_1} s$  and  $s \xrightarrow{H}_{D_1} t$ . Because of that  $s \notin T$  ( $T$  is  $H$ -independent) and  $S \leq T$ , it follows that there exists  $t'$  in  $T \setminus \{t\}$  such that  $s \xrightarrow{H}_{D_1} t'$  and  $t' \xrightarrow{H}_{D_1} s$ .

Thus, we get from Proposition 7 that  $t \xrightarrow{H}_{D_1} t'$  which contradicts that  $T$  is an  $H$ -independent set in  $D$ . Therefore,  $T \subseteq S$  and in the same way we can deduce that  $S \subseteq T$ .

(1.3)  $\leq$  is transitive.

Suppose that  $S \leq T$  and  $T \leq R$ . We will prove that  $S \leq R$ , that is, for all  $s$  in  $S$  there exists  $r$  in  $R$  such that either  $s = r$  or  $\left[s \xrightarrow{H}_{D_1} r \text{ and } r \xrightarrow{H}_{D_1} s\right]$ . Let  $s$  be a vertex in  $S$ . Then  $S \leq T$  implies that there exists  $t$  in  $T$  such that either  $s = t$  or  $\left[s \xrightarrow{H}_{D_1} t \text{ and } t \xrightarrow{H}_{D_1} s\right]$ ; because of  $T \leq R$  we get that for  $t$  in  $T$  there exists  $r$  in  $R$  such that  $t = r$  or  $\left[t \xrightarrow{H}_{D_1} r \text{ and } r \xrightarrow{H}_{D_1} t\right]$ . If  $s = t$ , then we have that  $s = r$  or  $\left[s \xrightarrow{H}_{D_1} r \text{ and } r \xrightarrow{H}_{D_1} s\right]$ . If  $s \neq t$  and  $t = r$ , then  $s \xrightarrow{H}_{D_1} r$  and  $r \xrightarrow{H}_{D_1} s$ . If  $s \neq t$  and  $t \neq r$ , then  $s \xrightarrow{H}_{D_1} t$ ,  $t \xrightarrow{H}_{D_1} s$ ,  $t \xrightarrow{H}_{D_1} r$  and Proposition 7 implies that  $s \xrightarrow{H}_{D_1} r$ . Because of  $t \xrightarrow{H}_{D_1} s$  and  $s \xrightarrow{H}_{D_1} t$  it follows from Proposition 7 that  $r \xrightarrow{H}_{D_1} s$  (because  $r \xrightarrow{H}_{D_1} t$ ).  $\square$

**Claim 2.**  $(\mathcal{L}, \leq)$  has maximal elements.

**Proof.** (2.1) Any chain in  $\mathcal{L}$  has an upper bound in  $\mathcal{L}$ .

Let  $\mathcal{C}$  be a chain in  $(\mathcal{L}, \leq)$ , consider the following sets.

For  $S$  in  $\mathcal{C}$ , let  $N_S$  be the set defined as  $\{T \text{ in } \mathcal{C} : S \leq T\}$ . Notice that  $N_S \neq \emptyset$  because  $S \in N_S$ .

$\mathcal{S}^\infty = \{s \in \bigcup_{A \in \mathcal{C}} A : \text{there exists } S \text{ in } \mathcal{C} \text{ such that } s \in T \text{ for every } T \text{ in } N_S\}$ .

(2.2)  $\mathcal{S}^\infty \neq \emptyset$ .

Proceeding by contradiction, suppose that  $\mathcal{S}^\infty = \emptyset$ . Let  $S_0$  be in  $\mathcal{C}$  and  $s_0$  in  $S_0$ . Since  $s_0 \notin \mathcal{S}^\infty$ , there exists  $S_1$  in  $N_{S_0}$  such that  $s_0 \notin S_1$ . Because of  $S_0 \leq S_1$  we get that there exists  $s_1$  in  $S_1$  such that  $s_0 \xrightarrow{H}_{D_1} s_1$  and  $s_1 \xrightarrow{H}_{D_1} s_0$ . Since  $s_1 \notin \mathcal{S}^\infty$ , there exists  $S_2$  in  $N_{S_1}$  such that  $s_1 \notin S_2$ . Thus,  $S_1 \leq S_2$  implies that there exists

$s_2$  in  $S_2$  such that  $s_1 \xrightarrow{D_1} s_2$  and  $s_2 \xrightarrow{D_1} s_1$ . Therefore, for every  $n$  in  $\mathbb{N}$  given  $S_n$  in  $\mathcal{C}$  and  $s_n$  in  $S_n$  there exist  $S_{n+1}$  in  $N_{S_n}$  and  $s_{n+1}$  in  $S_{n+1}$  such that  $s_n \notin S_{n+1}$ ,  $s_n \xrightarrow{D_1} s_{n+1}$  and  $s_{n+1} \xrightarrow{D_1} s_n$ . Then for every  $n$  in  $\mathbb{N}$  it follows from Lemma 5 that  $(s_n, s_{n+1}) \in A(D_1)$ . If  $s_i \neq s_j$  for every  $i$  different from  $j$ , then  $(s_n)_{n \in \mathbb{N}}$  is an infinite outward path in  $D_1$  which is not possible. Therefore, there exist  $s_i$  and  $s_j$ , with  $i < j$ , such that  $s_i = s_j$ , which implies that  $(s_i, s_{i+1}, \dots, s_j = s_i)$  is a closed walk in  $D_1$  which contains a cycle  $\gamma = (s_{i_0}, s_{i_1}, \dots, s_{i_t}, s_{i_0})$  such that  $s_{i_{s+1}} \xrightarrow{D_2} s_{i_s}$  for every  $s$  in  $\{0, \dots, t\}$  (indices modulo  $t+1$ ), a contradiction with Proposition 8. Therefore,  $\mathcal{S}^\infty \neq \emptyset$ .

**(2.3)**  $\mathcal{S}^\infty$  is an  $H$ -independent set in  $D$ .

Proceeding by contradiction, suppose that there exists a subset  $\{u, v\}$  of  $\mathcal{S}^\infty$ ,  $u \neq v$ , such that  $u \xrightarrow{D} v$ . Since  $\{u, v\} \subseteq \mathcal{S}^\infty$ , there exists a subset  $\{S_0, T_0\}$  of  $\mathcal{C}$  such that  $u \in S$  for every  $S$  in  $N_{S_0}$  and  $v \in T$  for every  $T$  in  $N_{T_0}$ . Since  $\mathcal{C}$  is a chain, we can suppose without loss of generality that  $S_0 \leq T_0$ . Thus, because of  $T_0 \in N_{S_0}$  we get that  $u \in T_0$ , which contradicts that  $T_0$  is an  $H$ -independent set in  $D$  (because  $v \in T_0$ ). Therefore,  $\mathcal{S}^\infty$  is an  $H$ -independent set in  $D$ .

**(2.4)**  $\mathcal{S}^\infty \in \mathcal{L}$ .

Suppose that there exist  $u$  in  $V(D) \setminus \mathcal{S}^\infty$  and  $s$  in  $\mathcal{S}^\infty$  such that  $s \xrightarrow{D_2} u$ . We will prove that there exists  $w$  in  $\mathcal{S}^\infty$  such that  $u \xrightarrow{D} w$ . Proceeding by contradiction, suppose that  $u \not\xrightarrow{D} \mathcal{S}^\infty$ .

Consider  $S_1$  in  $\mathcal{C}$  such that  $s \in S_1$ . Since  $S_1 \in \mathcal{L}$ , there exists  $s_1$  in  $S_1$  such that  $u \xrightarrow{D} s_1$ . Because of  $u \not\xrightarrow{D} \mathcal{S}^\infty$  we get that  $s_1 \notin \mathcal{S}^\infty$ ; it follows from the fact  $s \xrightarrow{D_2} u$ , the fact that  $S_1$  is an  $H$ -independent set, and by Proposition 7 that  $u \not\xrightarrow{D_2} s_1$ , which implies that  $u \xrightarrow{D_1} s_1$ . Since  $s_1 \notin \mathcal{S}^\infty$ , there exists  $S_2$  in  $N_{S_1}$  such that  $s_1 \notin S_2$ . Thus,  $S_1 \leq S_2$  implies that there exists  $s_2$  in  $S_2$  such that  $s_1 \xrightarrow{D_1} s_2$  and  $s_2 \xrightarrow{D_1} s_1$ . Then, we get from Proposition 7 that  $u \xrightarrow{D_1} s_2$  (because  $u \xrightarrow{D_1} s_1$ ), which implies that  $s_2 \notin \mathcal{S}^\infty$  (because  $u \not\xrightarrow{D} \mathcal{S}^\infty$ ). Hence, since  $s_2 \notin \mathcal{S}^\infty$ , we get that there exists  $S_3$  in  $N_{S_2}$  such that  $s_2 \notin S_3$ ; the fact  $S_2 \leq S_3$  implies that there exists  $s_3$  in  $S_3$  such that  $s_2 \xrightarrow{D_1} s_3$  and  $s_3 \xrightarrow{D_1} s_2$ .

Then, from Proposition 7 and the fact  $u \xrightarrow{D_1} s_2$ , we get that  $u \xrightarrow{D_1} s_3$ , which implies that  $s_3 \notin \mathcal{S}^\infty$  (because  $u \not\xrightarrow{D} \mathcal{S}^\infty$ ). With this procedure we have that for every  $n$  in  $\mathbb{N}$  given  $S_n$  in  $\mathcal{C}$  and  $s_n$  in  $V(D) \setminus \mathcal{S}^\infty$  such that  $s_n \in S_n$  there exist

$S_{n+1}$  in  $N_{S_n}$ ,  $s_{n+1}$  in  $S_{n+1}$  such that  $s_{n+1} \notin \mathcal{S}^\infty$ ,  $s_n \xrightarrow{H}_{D_1} s_{n+1}$ ,  $s_{n+1} \not\xrightarrow{H}_{D_1} s_n$  and  $u \xrightarrow{H}_{D_1} s_{n+1}$ . Therefore, we get from Lemma 5 that  $(s_n, s_{n+1}) \in A(D_1)$  and since  $(s_n)_{n \in \mathbb{N}}$  cannot be an infinite outward path in  $D_1$ , there exist  $s_i$  and  $s_j$ , with  $i < j$ , such that  $s_i = s_j$ , which implies that  $(s_i, s_{i+1}, \dots, s_j = s_i)$  is a closed walk in  $D_1$  which contains a cycle  $\gamma = (s_{i_0}, s_{i_1}, \dots, s_{i_t}, s_{i_0})$  such that  $s_{i_{s+1}} \not\xrightarrow{H}_{D_2} s_{i_s}$  for every  $s$  in  $\{0, \dots, t\}$  (indices modulo  $t+1$ ), a contradiction with Proposition 8. Therefore, there exists  $w$  in  $\mathcal{S}^\infty$  such that  $u \xrightarrow{H}_D w$ .

**(2.5)**  $S \leq \mathcal{S}^\infty$  for every  $S$  in  $\mathcal{C}$ .

Let  $S$  be in  $\mathcal{C}$  and  $u$  in  $S$ . We will prove that there exists  $w$  in  $\mathcal{S}^\infty$  such that  $u = w$  or  $\left[ u \xrightarrow{H}_{D_1} w \text{ and } w \not\xrightarrow{H}_{D_1} u \right]$ . Suppose that  $u \notin \mathcal{S}^\infty$ . Then there exists  $S_1$  in  $N_S$  such that  $u \notin S_1$ ;  $S \leq S_1$  implies that there exists  $s_1$  in  $S_1$  such that  $u \xrightarrow{H}_{D_1} s_1$  and  $s_1 \not\xrightarrow{H}_{D_1} u$ . If  $s_1 \in \mathcal{S}^\infty$ , then we are done; otherwise since  $s_1 \notin \mathcal{S}^\infty$ , there exists  $S_2$  in  $N_{S_1}$  such that  $s_1 \notin S_2$ . Thus,  $S_1 \leq S_2$  implies that there exists  $s_2$  in  $S_2$  such that  $s_1 \xrightarrow{H}_{D_1} s_2$  and  $s_2 \not\xrightarrow{H}_{D_1} s_1$ . Then, we get from Proposition 7 that  $u \xrightarrow{H}_{D_1} s_2$  (because  $u \xrightarrow{H}_{D_1} s_1$ ). Therefore, proceeding in the same way as in (2.4) and considering that both  $D_1$  has no infinite outward paths and  $D_1$  has no cycle as the cycle in Proposition 8, we conclude that there exists a sequence of vertices  $s_1, s_2, \dots, s_n$ , for some  $n$  in  $\mathbb{N}$ , such that  $s_n$  in  $\mathcal{S}^\infty$ ,  $u \xrightarrow{H}_{D_1} s_n$ ; for every  $i$  in  $\{1, \dots, n-1\}$   $s_i \xrightarrow{H}_{D_1} s_{i+1}$ ,  $s_{i+1} \not\xrightarrow{H}_{D_1} s_i$ ,  $s_i \notin \mathcal{S}^\infty$  and  $u \xrightarrow{H}_{D_1} s_i$ . It remains to prove that  $s_n \not\xrightarrow{H}_{D_1} u$ . Proceeding by contradiction, suppose that  $s_n \xrightarrow{H}_{D_1} u$ . Then in this case considering that  $s_i \xrightarrow{H}_{D_1} s_{i+1}$  and  $s_{i+1} \not\xrightarrow{H}_{D_1} s_i$  for every  $i$  in  $\{1, \dots, n-1\}$ , we can apply  $n-1$  times Proposition 7 and conclude that  $s_j \xrightarrow{H}_{D_1} u$  for every  $j$  in  $\{1, \dots, n-1\}$ , in particular  $s_1 \xrightarrow{H}_{D_1} u$ , which is not possible. Therefore,  $s_n \not\xrightarrow{H}_{D_1} u$ .

Therefore, we have proved that any chain in  $\mathcal{L}$  has an upper bound in  $\mathcal{L}$ , and so, by Zorn's Lemma, it follows that  $(\mathcal{L}, \leq)$  contains a maximal element.  $\square$

Let  $N$  be a maximal element of  $(\mathcal{L}, \leq)$ .

**Claim 3.**  $N$  is an  $H$ -kernel of  $D$ .

**Proof.** Since  $N$  is an  $H$ -independent set in  $D$ , it remains to prove that  $N$  is an  $H$ -absorbent set in  $D$ . Proceeding by contradiction, suppose that  $N$  is not an  $H$ -absorbent set in  $D$ . Then the set  $X = \left\{ x \in V(D) \setminus N : x \xrightarrow{H}_D N \right\}$  is not empty.

**(3.1)** There exists  $x_0$  in  $X$  such that if  $x_0 \xrightarrow{D_2} y$ , for some  $y$  in  $X$ , then  $y \xrightarrow{D} x_0$ . The proof of (3.1) is similar to the proof given in Proposition 9.

Consider the sets  $T = \left\{ v \in N : v \xrightarrow{D_1} x_0 \right\}$ ,  $B = N \setminus T$  and  $K = B \cup \{x_0\}$ .

**(3.2)**  $K$  is  $H$ -independent in  $D$ .

Since  $B$  is  $H$ -independent in  $D$  and  $x_0 \xrightarrow{D} B$ , it remains to prove that  $B \xrightarrow{D} x_0$ . It follows from the definition of  $B$  that  $B \xrightarrow{D_1} x_0$ . On the other hand, since  $N$  is an  $H$ -semikernel modulo  $D_1$  in  $D$  (because  $N \in \mathcal{L}$ ), we get that  $B \xrightarrow{D_2} x_0$  (because  $x_0 \xrightarrow{D} N$ ). Therefore,  $B \xrightarrow{D} x_0$  (recall 2 in Lemma 3 and 2 of this theorem).

**(3.3)**  $K \notin \mathcal{L}$ .

Proceeding by contradiction, suppose that  $K \in \mathcal{L}$ . We will see that  $N \leq K$ . Let  $u$  be in  $N$  and suppose that  $u \notin K$ . We will prove that there exists  $t$  in  $K$  such that  $u \xrightarrow{D_1} t$  and  $t \xrightarrow{D_1} u$ . Since  $u \in T$  (because  $N = T \cup B$ ), we get that  $u \xrightarrow{D_1} x_0$ , and because of  $x_0 \xrightarrow{D} N$ , we have that  $x_0 \xrightarrow{D_1} u$ . Therefore,  $x_0$  is the vertex desired. Hence  $N \leq K$ , which is not possible because  $K \neq N$  and  $N$  is maximal.

Since  $K \notin \mathcal{L}$  and  $K$  is  $H$ -independent in  $D$ , it follows from the definition of  $\mathcal{L}$  that  $K$  is not an  $H$ -semikernel modulo  $D_1$  in  $D$ , that is, there exist  $v$  in  $K$  and  $w$  in  $V(D) \setminus K$  such that  $v \xrightarrow{D_2} w$  and  $w \xrightarrow{D} K$ . Notice that  $(v, w) \in A(D_2)$  (by Lemma 5).

**(3.4)**  $v = x_0$ .

Proceeding by contradiction, suppose that  $v \neq x_0$ . The fact  $v \in B$  implies that  $w \notin N$ . Since  $N$  is an  $H$ -semikernel modulo  $D_1$  in  $D$  and  $w \xrightarrow{D} K$ , it follows that there exists  $t$  in  $T$  such that  $w \xrightarrow{D} t$ . The fact  $t \in T$  implies that  $t \xrightarrow{D_1} x_0$  and since  $x_0 \xrightarrow{D_1} t$ , we get from Proposition 7 that  $w \xrightarrow{D_1} t$  (because  $w \xrightarrow{D} x_0$ ), which implies that  $w \xrightarrow{D_2} t$ . Therefore,  $v \xrightarrow{D_2} w$ ,  $w \xrightarrow{D_2} v$  and  $w \xrightarrow{D_2} t$  implies that  $v \xrightarrow{D_2} t$  (by Proposition 7), which contradicts that  $N$  is an  $H$ -independent set in  $D$ .

Since  $v = x_0$ , it follows from the choice of  $x_0$  that  $w \notin X$  (because  $w \xrightarrow{D} x_0$ ). Notice that  $w \notin N$  by definition of  $X$  and because  $x_0 \in X$ . Since  $w \in V(D) \setminus (N \cup X)$ , we get from the definition of  $X$  that there exists  $t$  in  $T$  such that  $w \xrightarrow{D} t$ .

The fact  $t$  in  $T$  implies that  $t \xrightarrow{D_1} x_0$ , and since  $x_0 \xrightarrow{D_1} t$ , we get from Proposition 7 that  $w \xrightarrow{D_1} t$  (because  $w \xrightarrow{D} x_0$ ), which implies that  $w \xrightarrow{D_2} t$ . Therefore,  $v \xrightarrow{D_2} w$ ,  $w \xrightarrow{D_2} v$  and  $w \xrightarrow{D_2} t$  implies that  $v \xrightarrow{D_2} t$  (by Proposition 7), which contradicts that  $x_0 \xrightarrow{D} N$ .

Therefore,  $N$  is  $H$ -absorbent in  $D$ . Thus,  $N$  is an  $H$ -kernel of  $D$ . □

■

## 5. SOME CONSEQUENCES OF THEOREM 10

**Corollary 11.** *Let  $D = D_1 \cup D_2$  be a finite  $m$ -colored which is a union of asymmetric quasi-transitive digraphs such that*

1. *every chromatic class is quasi-transitive,*
2. *if  $\mathcal{C}$  is a chromatic class, then  $\mathcal{C} \subseteq A(D_j)$  for some  $j$  in  $\{1, 2\}$ , and*
3.  *$D$  has no 3-colored  $C_3$ .*

*Then  $D$  has an mp-kernel.*

**Corollary 12.** *Let  $H$  be a digraph,  $D$  an  $H$ -colored asymmetric quasi-transitive digraph and  $\{V_1, \dots, V_k\}$  a partition of  $V(\mathcal{C}_C(D))$  with the property  $P^*$ . Suppose that*

1.  *$V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, k\}$ ,*
2.  *$D$  has no infinite outward path,*
3. *every cycle of length three in  $D$  has at most two obstructions.*

*Then  $D$  has an  $H$ -kernel.*

**Proof.** Let  $D^*$  be an asymmetric quasi-transitive digraph such that  $D^*$  and  $D$  are isomorphic, with  $V(D) \cap V(D^*) = \emptyset$ , and let  $H^*$  be a digraph such that  $H^*$  and  $H$  are isomorphic, with  $V(H) \cap V(H^*) = \emptyset$ . Consider  $f : V(D) \rightarrow V(D^*)$  and  $g : V(H) \rightarrow V(H^*)$  two isomorphisms. Suppose that  $D^*$  is an  $H^*$ -colored digraph such that  $(u, v)$  has color  $i$  in  $D$  if and only if  $(f(u), f(v))$  has color  $g(i)$  in  $D^*$ . Therefore, it follows that  $D^*$  holds the same hypotheses as  $D$ .

Let  $D' = D \cup D^*$ . Notice that  $D'$  is an  $H'$ -colored digraph (with  $V(H') = V(H) \cup V(H^*)$  and  $A(H') = A(H) \cup A(H^*)$ ) which is a union asymmetric of quasi-transitive digraphs. If  $V_i^* = \{g(j) : j \in V_i\}$  for every  $i$  in  $\{1, \dots, k\}$ , then  $\{V_1^*, \dots, V_k^*\}$  is a partition of  $V(\mathcal{C}_C(D^*))$  which has the property  $P^*$  and so  $\{V_1, \dots, V_k, V_1^*, \dots, V_k^*\}$  is a partition of  $V(\mathcal{C}_C(D'))$  which has the property  $P^*$ . Now consider that the hypotheses of Theorem 10 fulfill on  $D'$  by the definition of  $D^*$ , the definition of  $H^*$ , the  $H^*$ -coloring of  $D^*$  and the hypotheses on  $D$ .



Therefore, it follows from Theorem 10 that  $D'$  has an  $H'$ -kernel, say  $N$ . Therefore, it follows from the definition of  $D'$  that  $N \cap V(D)$  is an  $H$ -kernel of  $D$ . ■

**Corollary 13.** *Let  $D$  be a finite  $m$ -colored asymmetric quasi-transitive digraph such that*

1. *every chromatic class is quasi-transitive,*
2.  *$D$  has no 3-colored  $C_3$ .*

*Then  $D$  has an  $mp$ -kernel.*

**Proof.** Notice that in this case the arcs of  $D$  are colored with the vertices of  $H$ , where  $V(H) = \{1, \dots, m\}$  and  $A(H) = \{(u, u) : u \in V(H)\}$ . Since (a)  $\{V_1 = \{1\}, \dots, V_m = \{m\}\}$  is a partition of  $V(\mathcal{C}_C(D))$  which has the property  $P^*$ , (b)  $V_i$  is a quasi-transitive  $V_i$ -class for every  $i$  in  $\{1, \dots, m\}$  (because  $D[B_i]$  is a chromatic class of  $D$ ), and (c) every cycle with length three in  $D$  has at most two obstructions (by hypothesis in 2), it follows from Corollary 12 that  $D$  has an  $H$ -kernel which is an  $mp$ -kernel. ■

**Corollary 14.** *Let  $T$  be a finite  $m$ -colored tournament such that*

1. *every chromatic class is quasi-transitive,*
2.  *$T$  has no 3-colored  $C_3$ .*

*Then  $T$  has an  $mp$ -kernel.*

### Acknowledgements

We thank the referees for their suggestions which improved the rewriting of this paper.

### REFERENCES

- [1] P. Arpin and V. Linek, *Reachability problems in edge-colored digraphs*, Discrete Math. **307** (2007) 2276–2289.  
doi:10.1016/j.disc.2006.09.042
- [2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications* (Springer, London, 2009).  
doi:10.1007/978-1-84800-998-1
- [3] Y. Bai, S. Fujita and S. Zhang, *Kernels by properly colored paths in arc-colored digraphs*, Discrete Math. **341** (2018) 1523–1533.  
doi:10.1016/j.disc.2018.02.014
- [4] C. Berge, *Graphs* (North-Holland, Amsterdam, 1989).
- [5] C. Berge and P. Duchet, *Recent problems and results about kernels in directed graphs*, Discrete Math. **86** (1990) 27–31.  
doi:10.1016/0012-365X(90)90346-J

- [6] P. Delgado-Escalante and H. Galeana-Sánchez, *Restricted domination in arc-colored digraphs*, AKCE Int. J. Graphs Comb. **11** (2014) 95–104.
- [7] P. Delgado-Escalante, H. Galeana-Sánchez and E. O'Reilly-Regueiro, *Alternating kernels*, Discrete Appl. Math. **236** (2018) 153–164.  
doi:10.1016/j.dam.2017.10.013
- [8] H. Galeana-Sánchez, *Kernels by monochromatic paths and the color-class digraph*, Discuss. Math. Graph Theory **31** (2011) 273–281.  
doi:10.7151/dmgt.1544
- [9] H. Galeana-Sánchez, *On monochromatic paths and monochromatic cycles in edge coloured tournaments*, Discrete Math. **156** (1996) 103–112.  
doi:10.1016/0012-365X(95)00036-V
- [10] H. Galeana-Sánchez and R. Rojas-Monroy, *Kernels in quasi-transitive digraphs*, Discrete Math. **306** (2006) 1969–1974.  
doi:10.1016/j.disc.2006.02.015
- [11] H. Galeana-Sánchez, B. Llano and J.J. Montellano-Ballesteros, *Kernels by monochromatic paths in  $m$ -colored unions of quasi-transitive digraphs*, Discrete Appl. Math. **158** (2010) 461–466.  
doi:10.1016/j.dam.2009.11.005
- [12] A. Ghoulá-Houri, *Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre*, C. R. Math. Acad. Sci. Paris **254** (1962) 1370–1371.
- [13] V. Linek and B. Sands, *A note on paths in edge-colored tournaments*, Ars Combin. **44** (1996) 225–228.
- [14] K.B. Reid, *Monotone reachability in arc-colored tournaments*, Congr. Numer. **146** (2000) 131–141.
- [15] B. Sands, N. Sauer and R. Woodrow, *On monochromatic paths in edge-coloured digraphs*, J. Combin. Theory Ser. B **33** (1982) 271–275.  
doi:10.1016/0095-8956(82)90047-8

Received 14 March 2018

Revised 6 November 2018

Accepted 7 November 2018