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$\boldsymbol{H}$-KERNELS IN UNIONS OF $\boldsymbol{H}$-COLORED QUASI-TRANSITIVE DIGRAPHS

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Abstract

Let H be a digraph (possibly with loops) and D a digraph without loops whose arcs are colored with the vertices of H (D is said to be an H-colored digraph). For an arc (x, y) of D, its color is denoted by $c(x, y)$. A directed path $W = (v_0, \ldots, v_n)$ in an H-colored digraph D will be called H-path if and only if $(c(v_0, v_1), \ldots, c(v_{n-1}, v_n))$ is a directed walk in H. In W, we will say that there is an obstruction on v_i if $(c(v_{i-1}, v_i), c(v_i, v_{i+1})) \notin A(H)$ (if $v_0 = v_n$ we will take indices modulo n). A subset N of $V(D)$ is said to be an H-kernel in D if for every pair of different vertices in N there is no H-path between them, and for every vertex u in $V(D) \setminus N$ there exists an H-path in D from u to N. Let D be an arc-colored digraph. The color-class digraph of D, $\mathscr{C}_C(D)$, is the digraph such that $V(\mathscr{C}_C(D)) = \{c(a) : a \in A(D)\}$ and $(i, j) \in A(\mathscr{C}_C(D))$ if and only if there exist two arcs, namely (u, v) and (v, w) in D, such that $c(u, v) = i$ and $c(v, w) = j$. The main result establishes that if $D = D_1 \cup D_2$ is an H-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(D))$ with a property P^* such that

1. V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,
2. either $D[\{a \in A(D) : c(a) \in V_i\}]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$,
3. D_i has no infinite outward path for every i in $\{1, 2\}$,

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4. every cycle of length three in D has at most two obstructions,

then D has an H-kernel.

Some results with respect to the existence of kernels by monochromatic paths in finite digraphs will be deduced from the main result.

Keywords: quasi-transitive digraph, kernel by monochromatic paths, alternating kernel, H-kernel, obstruction.

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1. Introduction

Let H be a digraph possibly with loops and D a digraph without loops. An *H-arc coloring* of D is a function $c : A(D) \to V(H)$. D is *H-colored* if D has an H-arc coloring. A path $W = (v_0, \ldots, v_n)$ in D is said to be an *H-path* if and only if $(c(v_0, v_1), \ldots, c(v_{n-1}, v_n))$ is a walk in H. We are going to consider that an arc is an H-path, that is to say, a singleton vertex is a walk in H. A subset S of V(D) is *H-absorbent* if for every x in $V(D) \setminus S$ there is an H-path from x to some point of S. A subset I of $V(D)$ is *H-independent* if there is no H-path between any two distinct vertices of I. A subset N of $V(D)$ is an *H-kernel* if N is both H-absorbent and H-independent. The concept of H-kernel has its origins in the works carried out by Sands, Sauer and Woodrow [15], Linek and Sands [13] and Arpin and Linek [1]. In [15] Sands, Sauer and Woodrow proved that if the arcs of a finite tournament T are colored with two colors, then there is always a vertex v in T such that for every w in $V(T) \setminus \{v\}$ there exists a monochromatic path from w to v. In [13] Linek and Sands gave an extension of the result of Sands, Sauer and Woodrow, in which the arcs of a tournament T are colored with the elements of a partially ordered set P and in their paper they give the first notion of H-path. In [1] Arpin and Linek work with H-colored digraphs and in their paper they introduce the concept of H-walk where an H-walk is a walk $(v_0, \ldots, v_n)$ in D such that $(c(v_0, v_1), \ldots, c(v_{n-1}, v_n))$ is a walk in H. In [1] Arpin and Linek introduce the concept of H-independent set by walks as a subset of vertices I of D such that there is no H-walks between any two different vertices of I. They also define an H-sink as a subset of vertices S of D such that for any u in $V(D) \setminus S$ there is v in S such that there exists an H-walk from u to v. Galeana-Sánchez and Delgado-Escalante were inspired by the work of Arpin and Linek and in [6] they introduced the concept of H-kernels. A subset of vertices N of D is called *H-kernel by walks* if N is both an H-independent set by walks and N is an H-sink. Notice that the concept of H-kernel and the concept of H-kernel by walks are different because of that the existence of an H-walk between two vertices does not guarantee the existence of an H-path between those vertices and the concatenation of two H-paths is not always an H-walk, see Figure 1.

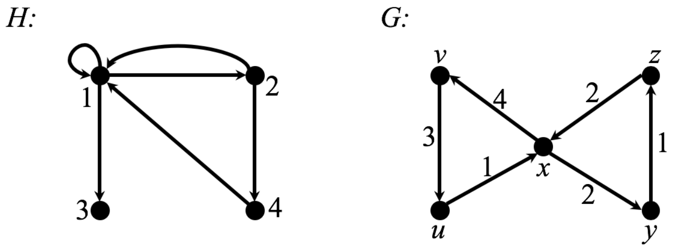

Figure 1. (u, x, y, z, x, v) is a uv-H-walk in G. The only one uv-path in G is (u, x, v) but this path is not a uv-H-path in G. $\{v\}$ is an H-kernel by walks in G. Every H-independent set in G consists only of one element but none of these is an H-kernel.

Notice that it follows from the definition of H-kernel that when $A(H) = \emptyset$, an H-kernel is a kernel (a subset N of vertices of D such that (1) for every u and v in N it holds that $\{(u, v), (v, u)\} \cap A(D) = \emptyset$ and (2) for every u in $V(D) \setminus N$ there exists v in N such that $(u, v) \in A(D)$); when $A(H) = \{(u, u) : u \in V(H)\}$, an H-kernel is a kernel by monochromatic paths (mp-kernel) (a subset N of vertices of D such that (1) for every u and v in N there exists no monochromatic directed paths between u and v and (2) for every u in $V(D) \setminus N$ there exists v in N such that there exists a monochromatic directed path from u to v) and when H has no loops, an H-kernel is an alternating kernel (a subset N of vertices of D such that (1) for every u and v in N it holds that there exists no directed path between u and v in which consecutive arcs have different colors and (2) for every u in $V(D) \setminus N$ there exists v in N such that there is a directed path from u to v in which consecutive arcs have different colors). In each of these special cases for H, sufficient conditions have been established in order to guarantee the existence of H-kernels, see for example [3, 5, 7, 9, 15]. Thus we can see that the concept of H-kernels is a generalization of the concepts of kernels, mp-kernels and alternating kernels.

Due to the difficulty of finding kernels, mp-kernels and alternating kernels in arc-colored digraphs, sufficient conditions for the existence of each of these H-kernels in arc-colored digraphs have been obtained mainly by study special classes of digraphs. A digraph D is *quasi-transitive* whenever $\{(u, v), (v, w)\} \subseteq A(D)$ implies either $(u, w) \in A(D)$ or $(w, u) \in A(D)$. Quasi-transitive digraphs are of interest because these are a generalization of tournaments (due to Ghouilá-Houri [12]) and those digraphs are a special case of digraphs in which the existence of kernels, mp-kernels and alternating kernels has been studied.

In [10] Galeana-Sánchez and Rojas-Monroy proved that if $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \emptyset$) where D_i is a quasi-transitive digraph which contains no asymmetric infinite outward path (in D_i) for i in $\{1, 2\}$, and that every directed cycle of length 3 contained in D has at least two symmetric arcs, then D has a kernel.

A *chromatic class* of D is the set of arcs of a same color. We say that a chromatic class $\mathcal{C}$ is quasi-transitive if $D[\mathcal{C}]$ is a quasi-transitive digraph. Let $D = D_1 \cup D_2$ be a digraph. We will say that D is a union of asymmetric quasi-transitive digraphs if (1) D_i is a quasi-transitive digraph for every i in $\{1, 2\}$, (2) D_i is asymmetric for every i in $\{1, 2\}$ and (3) $A(D_1) \cap A(D_2) = \emptyset$.

In [11] Galeana-Sánchez *et al.* worked with a finite m-colored multidigraph (a digraph with parallel arcs) $D = D_1 \cup D_2$ which is a union of asymmetric quasi-transitive digraphs, and they proved that if D satisfies that

1. every chromatic class induces a quasi-transitive digraph,
2. every chromatic class is contained in D_i for some i in $\{1, 2\}$ and
3. D contains neither 3-colored directed triangles nor 3-colored transitive subtournaments of order 3,

then D has an mp-kernel.

In [7], recently, Delgado-Escalante *et al.* proved the following.

Theorem 1. *If D is a finite m-colored quasi-transitive digraph such that every directed cycle of length 3 contained in D is 3-colored, then D has an alternating kernel.*

Basically the spirit of the conditions that guarantee the existence of kernels or mp-kernels in [10] and [11], respectively, arises from structural properties of 2-colored digraphs which were studied in [15] by Sands *et al.*

On the other hand, in [8] Galeana-Sánchez defined the *color-class digraph* $\mathscr{C}_C(D)$ of D as the digraph whose vertices are the colors represented in the arcs of D and $(i, j) \in A(\mathscr{C}_C(D))$ if and only if there exist two arcs, namely (u, v) and (v, w) in D, such that (u, v) has color i and (v, w) has color j (notice that $\mathscr{C}_C(D)$ can have loops by definition). Because of that in an H-colored digraph D, it holds that $V(\mathscr{C}_C(D)) \subseteq V(H)$, we can establish structural properties on $\mathscr{C}_C(D)$, with respect to H, in order to guarantee the existence of H-kernels.

Let H be a digraph, D an H-colored digraph and $(v_0, v_1, \ldots, v_n)$ a walk in D. We will say that there is an *obstruction* on v_i if $(c(v_{i-1}, v_i), c(v_i, v_{i+1})) \notin A(H)$ (if $v_0 = v_n$ we will take indices modulo n).

In this paper we continue with the study of the existence of H-kernels in unions of quasi-transitive digraphs and for this we will need the following definitions.

Definition. Let H be a digraph, D an H-colored digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$. We will say that $\{V_1, \ldots, V_k\}$ has the property P^* if the following conditions are satisfied.

1. $\mathscr{C}_C(D)[V_i]$ is a subdigraph of H for every i in $\{1, \ldots, k\}$.
2. If $(u, v) \in A(\mathscr{C}_C(D))$, for some u in V_i and for some v in V_j with $i \neq j$, then $(u, v) \notin A(H)$.

Definition. Let H be a digraph, D an H-colored digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$. V_i is said to be a quasi-transitive V_i-class if $D[\{a \in A(D) : c(a) \in V_i\}]$ is a quasi-transitive digraph for every i in $\{1, \ldots, k\}$.

The main result establishes that if H is a digraph, $D = D_1 \cup D_2$ is an H-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(D))$ with the property P^* such that

1. V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,
2. either $D[\{a \in A(D) : c(a) \in V_i\}]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$,
3. D_i has no infinite outward path for every i in $\{1, 2\}$,
4. every directed cycle of length three in D_i has at most two obstructions,

then D has an H-kernel.

With the main result of this paper we show that the main result in [11] can be reduced for digraphs as follows.

Let $D = D_1 \cup D_2$ be a finite m-colored digraph which is a union of asymmetric quasi-transitive digraphs such that

1. every chromatic class is quasi-transitive,
2. if $\mathcal{C}$ is a chromatic class, then $\mathcal{C} \subseteq A(D_j)$ for some j in $\{1, 2\}$ and
3. D does not contain 3-colored directed cycles of length three.

Then D has an mp-kernel.

In terms of H-kernels Theorem 1 says that if H is a complete digraph without loops and D is a finite H-colored quasi-transitive digraph such that every directed cycle of length 3 contained in D has no obstructions, then D has an H-kernel. However, the above is not true if H is not complete; consider the directed cycle of length 3, C_3, whose arcs are colored with three different vertices of H, with $A(H) = \emptyset$, it is clear that C_3 has no H-kernel. In this paper we will deduce from the main result the following.

Let H be a digraph (possibly with loops), D an H-colored asymmetric quasi-transitive digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Suppose that

1. V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,
2. D has no infinite outward path,
3. every cycle of length three in D has at most two obstructions.

Then D has an H-kernel.

We will need the following result.

Corollary 2 ([2], p. 53)**.** *If a quasi-transitive digraph D has an xy-path but $(x, y) \notin A(D)$, then either $(y, x) \in A(D)$ or there exists vertices u and v in $V(D) \setminus \{x, y\}$ such that (x, u, v, y) and (y, u, v, x) are paths in D.*

2. Terminology and Notation

For general concepts we refer the reader to [2] and [4]. An arc of the form (x, x) is a *loop*. An arc (u, v) in $A(D)$ is *asymmetric* if $(v, u) \notin A(D)$. We will say that a digraph D is *asymmetric* if every arc of D is asymmetric. We will say that two digraphs D_1 and D_2 are *equal*, denoted by $D_1 = D_2$, if $A(D_1) = A(D_2)$ and $V(D_1) = V(D_2)$. A *directed walk* is a sequence $W = (v_0, v_1, \ldots, v_n)$ such that $(v_i, v_{i+1}) \in A(D)$ for each i in $\{0, \ldots, n-1\}$. The number n is the *length* of the walk. We will say that the directed walk $(v_0, v_1, \ldots, v_n)$ is *closed* if $v_0 = v_n$. If $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{0, \ldots, n\}$ and $i \neq j$, it is called a *directed path*. A *directed cycle* is a directed walk $(v_1, v_2, \ldots, v_n, v_1)$ such that $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$, this will be denoted by C_n. If D is an infinite digraph, an infinite outward path is an infinite sequence $(v_1, v_2, \ldots)$ of distinct vertices of D such that $(v_i, v_{i+1}) \in A(D)$ for each $i \in \mathbb{N}$. In this paper we are going to write walk, path, cycle, instead of directed walk, directed path, directed cycle, respectively. The union of walks will be denoted with $\cup$. Let $W = (v_0, v_1, \ldots, v_n)$ be a walk and $\{v_i, v_j\} \subseteq V(W)$, with $i<j$. Then the v_iv_j-walk $(v_i, v_{i+1}, \ldots, v_{j-1}, v_j)$ contained in W will be denoted by (v_i, W, v_j). For a subset S of $V(D)$ the subdigraph of D induced by S, denoted by $D[S]$, has $V(D[S]) = S$ and $A(D[S]) = \{(u, v) \in A(D) : \{u, v\} \subseteq S\}$. A subset S of $V(D)$ is said to be *independent* if the only arcs in $D[S]$ are loops. For a subset B of $A(D)$ the subdigraph of D induced by B, denoted by $D[B]$, has $A(D[B]) = B$ and $V(D[B]) = \{v \in V(D) : (u, v) \in B \text{ or } (v, u) \in B \text{ for some } u \in V(D)\}$. A pair of digraphs D and G are *isomorphic* if there exists a bijection $f : V(D) \to V(G)$ such that $(x, y) \in A(D)$ if and only if $(f(x), f(y)) \in A(G)$ (f will be called *isomorphism*). We will say that a digraph D is *complete* if for every pair of different vertices u and v in $V(D)$ it holds that $\{(u, v), (v, u)\} \subseteq A(D)$.

A digraph D is said to be *m-colored* if the arcs of D are colored with m colors. A path is called *monochromatic* if all of its arcs are colored alike.

3. Previous Results

For the rest of the work H is a digraph possibly with loops and D is a, possibly infinite, digraph without loops.

We need to introduce some notation in order to present our proofs more compactly.

Let H be a digraph and D an H-colored digraph. Consider $\{u, v\}$ and S two subsets of $V(D)$. We will write $u \xrightarrow[D]{H} v$ if there exists a uv-H-path in D; $u \xrightarrow[D]{H} S$ if there exists a uS-H-path in D; $u \not\xrightarrow[D]{H} v$ is the denial of $u \xrightarrow[D]{H} v$; $u \not\xrightarrow[D]{H} S$ is the denial of $u \xrightarrow[D]{H} S$.

We will start with some results which will be useful.

From now on, the set $\{a \in A(D) : c(a) \in V_i\}$ will be denoted by B_i for every i in $\{1, \ldots, k\}$.

Lemma 3. *Let H be a digraph and D an H-colored digraph. Suppose that $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Then the following properties are satisfied.*

1. *Let i be an index in $\{1, \ldots, k\}$. Every finite path in $D[B_i]$ is an H-path in $D[B_i]$.*
2. *If P is a finite H-path in D, then there exists i in $\{1, \ldots, k\}$ such that P is contained in $D[B_i]$.*

Proof. Let $P = (u_0, \ldots, u_m)$ be a path in $D[B_i]$. We will prove that P is an H-path in D. It follows from the definition of color-class digraphs that $P' = (c(u_0, u_1), \ldots, c(u_{m-1}, u_m))$ is a walk in $\mathscr{C}_C(D)$. Since $c(u_j, u_{j+1}) \in V_i$ for every j in $\{0, \ldots, m-1\}$, then P' is a walk in $\mathscr{C}_C(D)[V_i]$, which implies that P' is a walk in H (because $\mathscr{C}_C(D)[V_i]$ is a subdigraph of H). Therefore P is an H-path in $D[B_i]$.

On the other hand, let $P = (v_0, \ldots, v_n)$ be an H-path in D. Then, it follows from the definition of H-paths and the definition of color-class digraphs that $(c(v_{j-1}, v_j), c(v_j, v_{j+1})) \in A(H) \cap A(\mathscr{C}_C(D))$ for every j in $\{1, \ldots, n-1\}$. Therefore, we get from 2 in definition of P^* that there exists i in $\{1, \ldots, k\}$ such that $c(v_j, v_{j+1}) \in V_i$ for every j in $\{0, \ldots, n-1\}$. Thus P is contained in $D[B_i]$. ■

Lemma 4. *Let H be a digraph, D an H-colored digraph and $\{w, z\} \subseteq V(D)$. Suppose that $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(D))$ such that V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$. If there exists a wz-path in $D[B_j]$ and there exists no zw-path in $D[B_j]$ for some j in $\{1, \ldots, k\}$, then $(w, z) \in A(D[B_j])$.*

Proof. It follows from Corollary 2. ■

We can obtain an extension of Lemma 4 as follows.

Lemma 5. *Let H be a digraph and $D = D_1 \cup D_2$ an H-colored digraph. Suppose that $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(D))$ with the property P^* such that*

1. *V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,*
2. *either $D[B_i]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$.*

Let r be an index in $\{1, 2\}$. If $x \xrightarrow[D_r]{H} z$ and $z \not\xrightarrow[D_r]{H} x$, then $(x, z) \in A(D_r)$.

Proof. Let P be an xz-H-path in D_r. It follows from Lemma 3 that there exists i in $\{1, \ldots, k\}$ such that P is contained in $D[B_i]$. The hypothesis implies that $D[B_i]$ is a subdigraph of D_r (because P is in D_r). On the other hand, since $z \underset{D_r}{\overset{H}{\not\longrightarrow}} x$, there exists no zx-H-path in $D[B_i]$, which implies that there exists no zx-path in $D[B_i]$ (by 1 in Lemma 3). Therefore, we get from Lemma 4 that $(x, z) \in A(D[B_i])$. So, $(x, z) \in A(D_r)$. ■

The following result will be useful in what follows.

Lemma 6. *Let H be a digraph, $D = D_1 \cup D_2$ an H-colored digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. If either $D[B_i]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$, then there exists a partition of $V(\mathscr{C}_C(D_r))$ with the property P^* for every r in $\{1, 2\}$.*

Proof. Suppose that $\{V_1, \ldots, V_t\}$ is such that $D[B_i]$ is a subdigraph of D_1 for every i in $\{1, \ldots, t\}$ and $\{V_{t+1}, \ldots, V_k\}$ is such that $D[B_j]$ is a subdigraph of D_2 for every j in $\{t + 1, \ldots, k\}$. Then, considering that D_1 and D_2 are also H-colored digraphs, it follows that $\{V_1, \ldots, V_t\}$ is a partition of $V(\mathscr{C}_C(D_1))$ with the property P^* and $\{V_{t+1}, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(D_2))$ with the property P^* (this follows from the fact that $\mathscr{C}_C(D_r)$ is a subdigraph of $\mathscr{C}_C(D)$ for every r in $\{1, 2\}$ and the fact that either $V_i \subseteq V(\mathscr{C}_C(D_1))$ or $V_i \subseteq V(\mathscr{C}_C(D_2))$ for every i in $\{1, \ldots, k\}$). ■

Proposition 7. *Let H be a digraph, $D = D_1 \cup D_2$ an H-colored digraph which is a union of asymmetric quasi-transitive digraphs, r an index in $\{1, 2\}$, $\{x, y\} \subseteq V(D)$ and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Suppose that*

1. *V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,*
2. *either $D[B_i]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$,*
3. *every cycle of length three in D_i has at most two obstructions for every i in $\{1, 2\}$,*
4. *$x \underset{D_r}{\overset{H}{\longrightarrow}} y$ and $y \underset{D_r}{\overset{H}{\not\longrightarrow}} x$.*

If z is a vertex in $V(D)$ such that $y \underset{D_r}{\overset{H}{\longrightarrow}} z$, then $(x, z) \in A(D_r)$; if $z \underset{D_r}{\overset{H}{\longrightarrow}} x$, then $(z, y) \in A(D_r)$.

Proof. Notice that it follows from Lemma 5 and hypothesis 4 of this proposition that $(x, y) \in A(D_r)$.

If $y \underset{D_r}{\overset{H}{\longrightarrow}} z$, then let $(y = w_0, \ldots, w_m = z)$ be a yz-H-path in D_r. We will prove that $(x, z) \in A(D_r)$ by induction on m.

If $m = 0$, it is clear that $(x, z) \in A(D_r)$ (because in this case $y = w_0 = z$).

Suppose that if $(y = u_0, \ldots, u_{m-1})$ is a yu_{m-1}-H-path in D_r with length $m-1$, then $(x, u_{m-1}) \in A(D_r)$.

Let $P = (y = \alpha_0, \ldots, \alpha_m)$ be a $y\alpha_m$-H-path in D_r with length m. We will prove that $(x, \alpha_m) \in A(D_r)$. Since (y, P, α_{m-1}) is a $y\alpha_{m-1}$-H-path in D_r with length $m-1$, it follows from the induction hypothesis that $(x, \alpha_{m-1}) \in A(D_r)$. Since $\{(x, \alpha_{m-1}), (\alpha_{m-1}, \alpha_m)\} \subseteq A(D_r)$ and D_r is a quasi-transitive digraph, it follows that $\{(x, \alpha_m), (\alpha_m, x)\} \cap A(D_r) \neq \emptyset$. If $(\alpha_m, x) \in A(D_r)$, then $\gamma = (x, \alpha_{m-1}, \alpha_m, x)$ is a cycle of length three in D_r which has at most two obstructions by hypothesis 3 of this proposition. If there is no obstruction on α_{m-1}, we have that $P' = (x, \alpha_{m-1}, \alpha_m)$ is an H-path in D_r, then we get by Lemma 6 and by Lemma 3 that there exists i in $\{1, \ldots, k\}$ such that $D[B_i]$ is a subdigraph of D_r and P' is contained in $D[B_i]$, respectively. Since $D[B_i]$ is a quasi-transitive digraph, P' is a path with length two in $D[B_i], (\alpha_m, x) \in A(D_r)$ and D_r is an asymmetric digraph, we get that $(\alpha_m, x) \in A(D[B_i])$. This implies that γ is contained in $D[B_i]$. In the same way, we can conclude that γ is contained in $D[B_j]$ for some j in $\{1, \ldots, k\}$ if either there is no obstruction on x or there is no obstruction on α_m. Therefore, in particular, we get from Lemma 3 that $(\alpha_{m-1}, \alpha_m, x)$ is an H-path in D_r, that is $(c(\alpha_{m-1}, \alpha_m), c(\alpha_m, x)) \in A(H)$. Thus $P \cup (\alpha_m, x)$ is an yx-H-path in D_r, a contradiction with hypothesis 4 of this proposition. Therefore, $(\alpha_m, x) \notin A(D_r)$, which implies that $(x, \alpha_m) \in A(D_r)$.

If $z \xrightarrow[D_r]{H} x$, then we can consider the converse of D and the converse of H (where the converse of a digraph G is the digraph $\overleftarrow{G}$ which one obtains from G by reversing all arcs). It is clear that the digraph $\overleftarrow{D} = \overleftarrow{D_1} \cup \overleftarrow{D_2}$ is an $\overleftarrow{H}$-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ is a partition of $V(\mathscr{C}_C(\overleftarrow{D}))$ with the property P^*, with respect to $\overleftarrow{H}$. In addition, the hypothesis 1, 2 and 3 fulfill in this digraph $\overleftarrow{H}$-colored, in the context of the new related digraphs associated, and hypothesis 4 says that $y \xrightarrow[\overleftarrow{D_r}]{\overleftarrow{H}} x$ and $x \not\xrightarrow[\overleftarrow{D_r}]{\overleftarrow{H}} y$. Since in $\overleftarrow{D}$ we have that $x \xrightarrow[\overleftarrow{D_r}]{\overleftarrow{H}} z$, then we conclude from the previous case that $(y, z) \in A(\overleftarrow{D_r})$. Therefore, $(z, y) \in A(D_r)$. ■

Notice that, since an arc (w, t) in D_r defines a wt-H-path in D_r, we also can conclude from Proposition 7 that $x \xrightarrow[D_r]{H} z$ if $(x,\ z) \in \mathrm{A}(D_r)$ or $z \xrightarrow[D_r]{H} y$ if $(z, y) \in A(D_r)$.

Proposition 8. *Let H be a digraph, $D = D_1 \cup D_2$ an H-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Suppose that*

1. *V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,*

2. *either $D[B_i]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$,*
3. *every cycle of length three in D_i has at most two obstructions.*

Then there exists no cycle $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ in D_r, with r in $\{1, 2\}$, such that $u_{i+1} \not\xrightarrow[D_r]{H} u_i$ for every i in $\{0, \ldots, n\}$ (indices modulo $n + 1$).

Proof. Proceeding by contradiction, suppose that there exists a cycle $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ in D_r, for some r in $\{1, 2\}$, of minimum length such that $u_{i+1} \not\xrightarrow[D_r]{H} u_i$ for every i in $\{0, \ldots, n\}$ (indices modulo $n + 1$). Notice that there exists j_0 in $\{0, \ldots, n\}$ such that there is an obstruction on u_{j_0} in γ, otherwise $u_{i+1} \xrightarrow[D_r]{H} u_i$ for every i in $\{0, \ldots, n\}$ (indices modulo $n + 1$), which is a contradiction. Suppose without loss of generality that there is an obstruction on u_1, that is $(c(u_0, u_1), c(u_1, u_2)) \notin A(H)$. Since $u_0 \xrightarrow[D_r]{H} u_1$ (because $(u_0, u_1) \in A(D_r)$), $u_1 \not\xrightarrow[D_r]{H} u_0$ and $u_1 \xrightarrow[D_r]{H} u_2$ (because $(u_1, u_2) \in A(D_r)$), we get from Proposition 7 that $(u_0, u_2) \in A(D_r)$. Because of that $\gamma' = (u_0, u_2) \cup (u_2, \gamma, u_0)$ is a cycle with length less than the length of γ, it follows from the choice of γ that $u_2 \xrightarrow[D_r]{H} u_0$. Therefore, since $u_1 \xrightarrow[D_r]{H} u_2$ (because $(u_1, u_2) \in A(D_r)$), $u_2 \not\xrightarrow[D_r]{H} u_1$ and $u_2 \xrightarrow[D_r]{H} u_0$, we get from Proposition 7 that $(u_1, u_0) \in A(D_r)$, which is a contradiction.

Therefore, there exists no cycle $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ in D_r, with r in $\{1, 2\}$, such that $u_{i+1} \not\xrightarrow[D_r]{H} u_i$ for every i in $\{0, \ldots, n\}$ (indices modulo $n + 1$). ∎

Definition. Let H be a digraph, D an H-colored digraph and G a subdigraph of D. We will say that a subset S of $V(D)$ is an H-semikernel modulo G in D if

1. S is an H-independent set in D,
2. if some vertex x in $V(D) \setminus S$ is such that $u \xrightarrow[D[A(D) \setminus A(G)]]{H} x$ for some vertex u in S, then there exists s in S such that $x \xrightarrow[D_r]{H} s$.

Proposition 9. *Let H be a digraph, $D = D_1 \cup D_2$ an H-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Suppose that*

1. *V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,*
2. *either $D[B_i]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1, \ldots, k\}$,*
3. *D_i has no infinite outward path for every i in $\{1, 2\}$,*
4. *every cycle of length three in D_i has at most two obstructions.*

Then there exists x in $V(D)$ such that $\{x\}$ is an H-semikernel modulo D_r in D, with r in $\{1,2\}$.

Proof. Suppose without loss of generality that $r=1$. Proceeding by contradiction, suppose that for every w in $V(D)$ there exists v_w in $V(D)\setminus\{w\}$ such that $w\xrightarrow[D_2]{H} v_w$ and $v_w\overset{H}{\underset{D}{\not\longrightarrow}}w$. Therefore, for every n in $\mathbb{N}$ given w_n in $V(D)$ there exists w_{n+1} in $V(D)\setminus\{w_n\}$ such that $w_n\xrightarrow[D_2]{H} w_{n+1}$ and $w_{n+1}\overset{H}{\underset{D}{\not\longrightarrow}}w_n$. So, it follows from Lemma 5 that $(w_n,w_{n+1})\in A(D_2)$ for every n in $\mathbb{N}$. If $w_i\neq w_j$ for every i different from j, then $(w_n)_{n\in\mathbb{N}}$ is an infinite outward path in D_2 which is not possible. Therefore, there exist w_i and w_j, with $i<j$, such that $w_i=w_j$, which implies that $(w_i,w_{i+1},\ldots,w_j=w_i)$ is a closed walk in D_2 which contains a cycle $\gamma=(w_{i_0},w_{i_1},\ldots,w_{i_t},w_{i_0})$ such that $w_{i_{s+1}}\overset{H}{\underset{D_2}{\not\longrightarrow}}w_{i_s}$ for every s in $\{0,\ldots,t\}$ (indices modulo $t+1$), a contradiction with Proposition 8. Therefore, there exists x in $V(D)$ such that $\{x\}$ is an H-semikernel modulo D_1 in D. ■

4. Main Result

Theorem 10. *Let H be a digraph, $D=D_1\cup D_2$ an H-colored digraph which is a union of asymmetric quasi-transitive digraphs and $\{V_1,\ldots,V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Suppose that*

1. *V_i is a quasi-transitive V_i-class for every i in $\{1,\ldots,k\}$,*
2. *either $D[B_i]$ is a subdigraph of D_1 or it is a sudigraph of D_2 for every i in $\{1,\ldots,k\}$,*
3. *D_i has no infinite outward path for every i in $\{1,2\}$,*
4. *every cycle of length three in D_i has at most two obstructions for every i in $\{1,2\}$.*

Then D has an H-kernel.

Proof. $\mathscr{I}=\{S\subseteq V(D) : S$ is H-independent in $D\}$ and $\mathscr{L}=\{S\in\mathscr{I} : S$ is an H-semikernel modulo D_1 in $D\}$.

Since $\{w\}$ is an H-independent set for every w in $V(D)$, it follows that $\mathscr{I}\neq\emptyset$; by Proposition 9 we get that $\mathscr{L}\neq\emptyset$.

For sets S, T in $\mathscr{L}$, put $S\leq T$ if for all s in S there exists t in T such that either $s=t$, or $s\xrightarrow[D_1]{H}t$ and $t\overset{H}{\underset{D_1}{\not\longrightarrow}}s$.

Claim 1. *$(\mathscr{L},\leq)$ is a poset.*

Proof. Consider $\{S,T,R\}$ a subset of $\mathscr{L}$.

(1.1) $\leq$ is reflexive.
Clearly $S \leq S$ for every S in $\mathscr{L}$.

(1.2) $\leq$ is antisymmetric.
Suppose that $S \leq T$ and $T \leq S$. We will prove that $S = T$. Let t be a vertex in T and suppose that $t \notin S$. Then since $T \leq S$, we have that there exists s in S such that $t \xrightarrow[D_1]{H} s$ and $s \not\xrightarrow[D_1]{H} t$. Because of that $s \notin T$ (T is H-independent) and $S \leq T$, it follows that there exists t' in $T \setminus \{t\}$ such that $s \xrightarrow[D_1]{H} t'$ and $t' \not\xrightarrow[D_1]{H} s$. Thus, we get from Proposition 7 that $t \xrightarrow[D_1]{H} t'$ which contradicts that T is an H-independent set in D. Therefore, $T \subseteq S$ and in the same way we can deduce that $S \subseteq T$.

(1.3) $\leq$ is transitive.
Suppose that $S \leq T$ and $T \leq R$. We will prove that $S \leq R$, that is, for all s in S there exists r in R such that either $s = r$ or $\left[s \xrightarrow[D_1]{H} r \text{ and } r \not\xrightarrow[D_1]{H} s\right]$. Let s be a vertex in S. Then $S \leq T$ implies that there exists t in T such that either $s = t$ or $\left[s \xrightarrow[D_1]{H} t \text{ and } t \not\xrightarrow[D_1]{H} s\right]$; because of $T \leq R$ we get that for t in T there exists r in R such that $t = r$ or $\left[t \xrightarrow[D_1]{H} r \text{ and } r \not\xrightarrow[D_1]{H} t\right]$. If $s = t$, then we have that $s = r$ or $\left[s \xrightarrow[D_1]{H} r \text{ and } r \not\xrightarrow[D_1]{H} s\right]$. If $s \neq t$ and $t = r$, then $s \xrightarrow[D_1]{H} r$ and $r \not\xrightarrow[D_1]{H} s$. If $s \neq t$ and $t \neq r$, then $s \xrightarrow[D_1]{H} t$, $t \not\xrightarrow[D_1]{H} s$, $t \xrightarrow[D_1]{H} r$ and Proposition 7 implies that $s \xrightarrow[D_1]{H} r$. Because of $t \not\xrightarrow[D_1]{H} s$ and $s \xrightarrow[D_1]{H} t$ it follows from Proposition 7 that $r \not\xrightarrow[D_1]{H} s$ $\left(\text{because } r \not\xrightarrow[D_1]{H} t\right)$. $\square$

Claim 2. $(\mathscr{L}, \leq)$ *has maximal elements.*

Proof. **(2.1)** Any chain in $\mathscr{L}$ has an upper bound in $\mathscr{L}$.
Let $\mathscr{C}$ be a chain in $(\mathscr{L}, \leq)$, consider the following sets.

For S in $\mathscr{C}$, let N_S be the set defined as $\{T \text{ in } \mathscr{C} : S \leq T\}$. Notice that $N_S \neq \emptyset$ because $S \in N_S$.
$\mathscr{S}^\infty = \left\{s \in \bigcup_{A \in \mathscr{C}} A : \text{there exists } S \text{ in } \mathscr{C} \text{ such that } s \in T \text{ for every } T \text{ in } N_S\right\}$.

(2.2) $\mathscr{S}^\infty \neq \emptyset$.
Proceeding by contradiction, suppose that $\mathscr{S}^\infty = \emptyset$. Let S_0 be in $\mathscr{C}$ and s_0 in S_0. Since $s_0 \notin \mathscr{S}^\infty$, there exists S_1 in N_{S_0} such that $s_0 \notin S_1$. Because of $S_0 \leq S_1$ we get that there exists s_1 in S_1 such that $s_0 \xrightarrow[D_1]{H} s_1$ and $s_1 \not\xrightarrow[D_1]{H} s_0$. Since $s_1 \notin \mathscr{S}^\infty$, there exists S_2 in N_{S_1} such that $s_1 \notin S_2$. Thus, $S_1 \leq S_2$ implies that there exists

s_2 in S_2 such that $s_1 \xrightarrow[D_1]{H} s_2$ and $s_2 \not\xrightarrow[D_1]{H} s_1$. Therefore, for every n in $\mathbb{N}$ given S_n in $\mathscr{C}$ and s_n in S_n there exist S_{n+1} in N_{S_n} and s_{n+1} in S_{n+1} such that $s_n \notin S_{n+1}$, $s_n \xrightarrow[D_1]{H} s_{n+1}$ and $s_{n+1} \not\xrightarrow[D_1]{H} s_n$. Then for every n in $\mathbb{N}$ it follows from Lemma 5 that $(s_n, s_{n+1}) \in A(D_1)$. If $s_i \neq s_j$ for every i different from j, then $(s_n)_{n \in \mathbb{N}}$ is an infinite outward path in D_1 which is not possible. Therefore, there exist s_i and s_j, with $i<j$, such that $s_i = s_j$, which implies that $(s_i, s_{i+1}, \ldots, s_j = s_i)$ is a closed walk in D_1 which contains a cycle $\gamma = (s_{i_0}, s_{i_1}, \ldots, s_{i_t}, s_{i_0})$ such that $s_{i_{s+1}} \not\xrightarrow[D_2]{H} s_{i_s}$ for every s in $\{0, \ldots, t\}$ (indices modulo $t+1$), a contradiction with Proposition 8. Therefore, $\mathscr{S}^\infty \neq \emptyset$.

(2.3) $\mathscr{S}^\infty$ is an H-independent set in D.
Proceeding by contradiction, suppose that there exists a subset $\{u, v\}$ of $\mathscr{S}^\infty$, $u \neq v$, such that $u \xrightarrow[D]{H} v$. Since $\{u, v\} \subseteq \mathscr{S}^\infty$, there exists a subset $\{S_0, T_0\}$ of $\mathscr{C}$ such that $u \in S$ for every S in N_{S_0} and $v \in T$ for every T in N_{T_0}. Since $\mathscr{C}$ is a chain, we can suppose without loss of generality that $S_0 \leq T_0$. Thus, because of $T_0 \in N_{S_0}$ we get that $u \in T_0$, which contradicts that T_0 is an H-independent set in D (because $v \in T_0$). Therefore, $\mathscr{S}^\infty$ is an H-independent set in D.

(2.4) $\mathscr{S}^\infty \in \mathscr{L}$.
Suppose that there exist u in $V(D) \setminus \mathscr{S}^\infty$ and s in $\mathscr{S}^\infty$ such that $s \xrightarrow[D_2]{H} u$. We will prove that there exists w in $\mathscr{S}^\infty$ such that $u \xrightarrow[D]{H} w$. Proceeding by contradiction, suppose that $u \not\xrightarrow[D]{H} \mathscr{S}^\infty$.

Consider S_1 in $\mathscr{C}$ such that $s \in S_1$. Since $S_1 \in \mathscr{L}$, there exists s_1 in S_1 such that $u \xrightarrow[D]{H} s_1$. Because of $u \not\xrightarrow[D]{H} \mathscr{S}^\infty$ we get that $s_1 \notin \mathscr{S}^\infty$; it follows from the fact $s \xrightarrow[D_2]{H} u$, the fact that S_1 is an H-independent set, and by Proposition 7 that $u \not\xrightarrow[D_2]{H} s_1$, which implies that $u \xrightarrow[D_1]{H} s_1$. Since $s_1 \notin \mathscr{S}^\infty$, there exists S_2 in N_{S_1} such that $s_1 \notin S_2$. Thus, $S_1 \leq S_2$ implies that there exists s_2 in S_2 such that $s_1 \xrightarrow[D_1]{H} s_2$ and $s_2 \not\xrightarrow[D_1]{H} s_1$. Then, we get from Proposition 7 that $u \xrightarrow[D_1]{H} s_2$ $\left(\text{because } u \xrightarrow[D_1]{H} s_1\right)$, which implies that $s_2 \notin \mathscr{S}^\infty$ $\left(\text{because } u \not\xrightarrow[D]{H} \mathscr{S}^\infty\right)$. Hence, since $s_2 \notin \mathscr{S}^\infty$, we get that there exists S_3 in N_{S_2} such that $s_2 \notin S_3$; the fact $S_2 \leq S_3$ implies that there exists s_3 in S_3 such that $s_2 \xrightarrow[D_1]{H} s_3$ and $s_3 \not\xrightarrow[D_1]{H} s_2$. Then, from Proposition 7 and the fact $u \xrightarrow[D_1]{H} s_2$, we get that $u \xrightarrow[D_1]{H} s_3$, which implies that $s_3 \notin \mathscr{S}^\infty$ $\left(\text{because } u \not\xrightarrow[D]{H} \mathscr{S}^\infty\right)$. With this procedure we have that for every n in $\mathbb{N}$ given S_n in $\mathscr{C}$ and s_n in $V(D) \setminus \mathscr{S}^\infty$ such that $s_n \in S_n$ there exist

S_{n+1} in N_{S_n}, s_{n+1} in S_{n+1} such that $s_{n+1} \notin \mathscr{S}^\infty$, $s_n \xrightarrow[D_1]{H} s_{n+1}$, $s_{n+1} \not\xrightarrow[D_1]{H} s_n$ and $u \xrightarrow[D_1]{H} s_{n+1}$. Therefore, we get from Lemma 5 that $(s_n, s_{n+1}) \in A(D_1)$ and since $(s_n)_{n\in\mathbb{N}}$ cannot be an infinite outward path in D_1, there exist s_i and s_j, with $i<j$, such that $s_i = s_j$, which implies that $(s_i, s_{i+1}, \ldots, s_j = s_i)$ is a closed walk in D_1 which contains a cycle $\gamma = (s_{i_0}, s_{i_1}, \ldots, s_{i_t}, s_{i_0})$ such that $s_{i_{s+1}} \not\xrightarrow[D_2]{H} s_{i_s}$ for every s in $\{0, \ldots, t\}$ (indices modulo $t+1$), a contradiction with Proposition 8. Therefore, there exists w in $\mathscr{S}^\infty$ such that $u \xrightarrow[D]{H} w$.

(2.5) $S \leq \mathscr{S}^\infty$ for every S in $\mathscr{C}$.
Let S be in $\mathscr{C}$ and u in S. We will prove that there exists w in $\mathscr{S}^\infty$ such that $u = w$ or $\left[u \xrightarrow[D_1]{H} w \text{ and } w \not\xrightarrow[D_1]{H} u\right]$. Suppose that $u \notin \mathscr{S}^\infty$. Then there exists S_1 in N_S such that $u \notin S_1$; $S \leq S_1$ implies that there exists s_1 in S_1 such that $u \xrightarrow[D_1]{H} s_1$ and $s_1 \not\xrightarrow[D_1]{H} u$. If $s_1 \in \mathscr{S}^\infty$, then we are done; otherwise since $s_1 \notin \mathscr{S}^\infty$, there exists S_2 in N_{S_1} such that $s_1 \notin S_2$. Thus, $S_1 \leq S_2$ implies that there exists s_2 in S_2 such that $s_1 \xrightarrow[D_1]{H} s_2$ and $s_2 \not\xrightarrow[D_1]{H} s_1$. Then, we get from Proposition 7 that $u \xrightarrow[D_1]{H} s_2$ $\left(\text{because } u \xrightarrow[D_1]{H} s_1\right)$. Therefore, proceeding in the same way as in (2.4) and considering that both D_1 has no infinite outward paths and D_1 has no cycle as the cycle in Proposition 8, we conclude that there exists a sequence of vertices $s_1, s_2, \ldots, s_n$, for some n in $\mathbb{N}$, such that s_n in $\mathscr{S}^\infty$, $u \xrightarrow[D_1]{H} s_n$; for every i in $\{1, \ldots, n-1\}$ $s_i \xrightarrow[D_1]{H} s_{i+1}$, $s_{i+1} \not\xrightarrow[D_1]{H} s_i$, $s_i \notin \mathscr{S}^\infty$ and $u \xrightarrow[D_1]{H} s_i$. It remains to prove that $s_n \not\xrightarrow[D_1]{H} u$. Proceeding by contradiction, suppose that $s_n \xrightarrow[D_1]{H} u$. Then in this case considering that $s_i \xrightarrow[D_1]{H} s_{i+1}$ and $s_{i+1} \not\xrightarrow[D_1]{H} s_i$ for every i in $\{1, \ldots, n-1\}$, we can apply $n-1$ times Proposition 7 and conclude that $s_j \xrightarrow[D_1]{H} u$ for every j in $\{1, \ldots, n-1\}$, in particular $s_1 \xrightarrow[D_1]{H} u$, which is not possible. Therefore, $s_n \not\xrightarrow[D_1]{H} u$.

Therefore, we have proved that any chain in $\mathscr{L}$ has an upper bound in $\mathscr{L}$, and so, by Zorn's Lemma, it follows that $(\mathscr{L}, \leq)$ contains a maximal element. □

Let N be a maximal element of $(\mathscr{L}, \leq)$.

Claim 3. *N is an H-kernel of D.*

Proof. Since N is an H-independent set in D, it remains to prove that N is an H-absorbent set in D. Proceeding by contradiction, suppose that N is not an H-absorbent set in D. Then the set $X = \left\{x \in V(D) \setminus N : x \not\xrightarrow[D]{H} N\right\}$ is not empty.

(3.1) There exists x_0 in X such that if $x_0 \xrightarrow[D_2]{H} y$, for some y in X, then $y \xrightarrow[D]{H} x_0$. The proof of (3.1) is similar to the proof given in Proposition 9.

Consider the sets $T = \left\{ v \in N : v \xrightarrow[D_1]{H} x_0 \right\}$, $B = N \setminus T$ and $K = B \cup \{x_0\}$.

(3.2) K is H-independent in D.
Since B is H-independent in D and $x_0 \underset{D}{\overset{H}{\not\longrightarrow}} B$, it remains to prove that $B \underset{D}{\overset{H}{\not\longrightarrow}} x_0$. It follows from the definition of B that $B \underset{D_1}{\overset{H}{\not\longrightarrow}} x_0$. On the other hand, since N is an H-semikernel modulo D_1 in D (because $N \in \mathscr{L}$), we get that $B \underset{D_2}{\overset{H}{\not\longrightarrow}} x_0$ $\left(\text{because } x_0 \underset{D}{\overset{H}{\not\longrightarrow}} N\right)$. Therefore, $B \underset{D}{\overset{H}{\not\longrightarrow}} x_0$ (recall 2 in Lemma 3 and 2 of this theorem).

(3.3) $K \notin \mathscr{L}$.
Proceeding by contradiction, suppose that $K \in \mathscr{L}$. We will see that $N \leq K$. Let u be in N and suppose that $u \notin K$. We will prove that there exists t in K such that $u \xrightarrow[D_1]{H} t$ and $t \underset{D_1}{\overset{H}{\not\longrightarrow}} u$. Since $u \in T$ (because $N = T \cup B$), we get that $u \xrightarrow[D_1]{H} x_0$, and because of $x_0 \underset{D}{\overset{H}{\not\longrightarrow}} N$, we have that $x_0 \underset{D_1}{\overset{H}{\not\longrightarrow}} u$. Therefore, x_0 is the vertex desired. Hence $N \leq K$, which is not possible because $K \neq N$ and N is maximal.

Since $K \notin \mathscr{L}$ and K is H-independent in D, it follows from the definition of $\mathscr{L}$ that K is not an H-semikernel modulo D_1 in D, that is, there exist v in K and w in $V(D) \setminus K$ such that $v \xrightarrow[D_2]{H} w$ and $w \underset{D}{\overset{H}{\not\longrightarrow}} K$. Notice that $(v, w) \in A(D_2)$ (by Lemma 5).

(3.4) $v = x_0$.
Proceeding by contradiction, suppose that $v \neq x_0$. The fact $v \in B$ implies that $w \notin N$. Since N is an H-semikernel modulo D_1 in D and $w \underset{D}{\overset{H}{\not\longrightarrow}} K$, it follows that there exists t in T such that $w \xrightarrow[D]{H} t$. The fact $t \in T$ implies that $t \xrightarrow[D_1]{H} x_0$ and since $x_0 \underset{D_1}{\overset{H}{\not\longrightarrow}} t$, we get from Proposition 7 that $w \underset{D_1}{\overset{H}{\not\longrightarrow}} t$ $\left(\text{because } w \underset{D}{\overset{H}{\not\longrightarrow}} x_0\right)$, which implies that $w \xrightarrow[D_2]{H} t$. Therefore, $v \xrightarrow[D_2]{H} w$, $w \underset{D_2}{\overset{H}{\not\longrightarrow}} v$ and $w \xrightarrow[D_2]{H} t$ implies that $v \xrightarrow[D_2]{H} t$ (by Proposition 7), which contradicts that N is an H-independent set in D.

Since $v = x_0$, it follows from the choice of x_0 that $w \notin X$ $\left(\text{because } w \underset{D}{\overset{H}{\not\longrightarrow}} x_0\right)$. Notice that $w \notin N$ by definition of X and because $x_0 \in X$. Since $w \in V(D) \setminus (N \cup X)$, we get from the definition of X that there exists t in T such that $w \xrightarrow[D]{H} t$.

The fact t in T implies that $t \not\xrightarrow[D_1]{H} x_0$, and since $x_0 \not\xrightarrow[D_1]{H} t$, we get from Proposition 7 that $w \not\xrightarrow[D_1]{H} t$ $\left(\text{because } w \not\xrightarrow[D]{H} x_0\right)$, which implies that $w \xrightarrow[D_2]{H} t$. Therefore, $v \xrightarrow[D_2]{H} w$, $w \not\xrightarrow[D_2]{H} v$ and $w \xrightarrow[D_2]{H} t$ implies that $v \xrightarrow[D_2]{H} t$ (by Proposition 7), which contradicts that $x_0 \not\xrightarrow[D]{H} N$.

Therefore, N is H-absorbent in D. Thus, N is an H-kernel of D. □

■

5. Some Consequences of Theorem 10

Corollary 11. *Let $D = D_1 \cup D_2$ be a finite m-colored which is a union of asymmetric quasi-transitive digraphs such that*

1. *every chromatic class is quasi-transitive,*
2. *if $\mathcal{C}$ is a chromatic class, then $\mathcal{C} \subseteq A(D_j)$ for some j in $\{1, 2\}$, and*
3. *D has no 3-colored C_3.*

Then D has an mp-kernel.

Corollary 12. *Let H be a digraph, D an H-colored asymmetric quasi-transitive digraph and $\{V_1, \ldots, V_k\}$ a partition of $V(\mathscr{C}_C(D))$ with the property P^*. Suppose that*

1. *V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, k\}$,*
2. *D has no infinite outward path,*
3. *every cycle of length three in D has at most two obstructions.*

Then D has an H-kernel.

Proof. Let D^* be an asymmetric quasi-transitive digraph such that D^* and D are isomorphic, with $V(D) \cap V(D^*) = \emptyset$, and let H^* be a digraph such that H^* and H are isomorphic, with $V(H) \cap V(H^*) = \emptyset$. Consider $f : V(D) \to V(D^*)$ and $g : V(H) \to V(H^*)$ two isomorphisms. Suppose that D^* is an H^*-colored digraph such that (u, v) has color i in D if and only if $(f(u), f(v))$ has color $g(i)$ in D^*. Therefore, it follows that D^* holds the same hypotheses as D.

Let $D' = D \cup D^*$. Notice that D' is an H'-colored digraph (with $V(H') = V(H) \cup V(H^*)$ and $A(H') = A(H) \cup A(H^*)$) which is a union asymmetric of quasi-transitive digraphs. If $V_i^* = \left\{g(j) : j \in V_i\right\}$ for every i in $\{1, \ldots, k\}$, then $\left\{V_1^*, \ldots, V_k^*\right\}$ is a partition of $V(\mathscr{C}_C(D^*))$ which has the property P^* and so $\left\{V_1, \ldots, V_k, V_1^*, \ldots, V_k^*\right\}$ is a partition of $V(\mathscr{C}_C(D'))$ which has the property P^*. Now consider that the hypotheses of Theorem 10 fulfill on D' by the definition of D^*, the definition of H^*, the H^*-coloring of D^* and the hypotheses on D.

Therefore, it follows from Theorem 10 that D' has an H'-kernel, say N. Therefore, it follows from the definition of D' that $N \cap V(D)$ is an H-kernel of D. ■

Corollary 13. *Let D be a finite m-colored asymmetric quasi-transitive digraph such that*

1. *every chromatic class is quasi-transitive,*
2. *D has no 3-colored C_3.*

Then D has an mp-kernel.

Proof. Notice that in this case the arcs of D are colored with the vertices of H, where $V(H) = \{1, \ldots, m\}$ and $A(H) = \{(u, u) : u \in V(H)\}$. Since (a) $\{V_1 = \{1\}, \ldots, V_m = \{m\}\}$ is a partition of $V(\mathscr{C}_C(D))$ which has the property P^*, (b) V_i is a quasi-transitive V_i-class for every i in $\{1, \ldots, m\}$ (because $D[B_i]$ is a chromatic class of D), and (c) every cycle with length three in D has at most two obstructions (by hypothesis in 2), it follows from Corollary 12 that D has an H-kernel which is an mp-kernel. ■

Corollary 14. *Let T be a finite m-colored tournament such that*

1. *every chromatic class is quasi-transitive,*
2. *T has no 3-colored C_3.*

Then T has an mp-kernel.

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