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DECOMPOSITIONS OF COMPLETE BIPARTITE GRAPHS AND COMPLETE GRAPHS INTO PATHS, STARS, AND CYCLES WITH FOUR EDGES EACH

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Abstract

Let G be either a complete graph of odd order or a complete bipartite graph in which each vertex partition has an even number of vertices. In this paper, we determine the set of triples (p, q, r), with p, q, $r > 0$, for which there exists a decomposition of G into p paths, q stars, and r cycles, each of which has 4 edges.

Keywords: complete graph, complete bipartite graph, path, star, cycle, decomposition.

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1. Introduction

All graphs considered here are finite and undirected, unless otherwise noted.

Let G, H, $H_1, \ldots, H_r$ be graphs for some integer r. A *decomposition* of G is a set of edge-disjoint subgraphs of G whose union is G. An H-*decomposition* of G is a decomposition of G into copies of H. If G has an H-decomposition, we say that G is H-*decomposable*. An $\{H_1, \ldots, H_r\}$-*decomposition* of G is a decomposition of G into copies of $H_1, \ldots, H_r$ containing at least one copy of each H_i, for each $i = 1, \ldots, r$. If G has an $\{H_1, \ldots, H_r\}$-decomposition, we say that G is $\{H_1, \ldots, H_r\}$-*decomposable*. Moreover, if there is a decomposition of G containing precisely α_i elements isomorphic to H_i, then we say that G has an $\{{H_1}^{\alpha_1}, \ldots, {H_r}^{\alpha_r}\}$-*decomposition* or G is $\{{H_1}^{\alpha_1}, \ldots, {H_r}^{\alpha_r}\}$-*decomposable*. Let $\mathcal{CD}(G; H_1, \ldots, H_r)$ denote the set of all r-tuples $(\alpha_1, \ldots, \alpha_r)$ of positive integers such that G is $\{{H_1}^{\alpha_1}, \ldots, {H_r}^{\alpha_r}\}$-decomposable. Obviously, if we can find an r-tuple in $\mathcal{CD}(G; H_1, \ldots, H_r)$, then G is $\{H_1, \ldots, H_r\}$*-decomposable*.

As usual, K_n denotes the complete graph on n vertices, and $K_{m,n}$ denotes the complete bipartite graph with vertex partitions of sizes m and n. A *k-path*, denoted by P_k, is a path with k edges; a *k-star*, denoted by S_k, is the complete bipartite graph $K_{1,k}$; a *k-cycle*, denoted by C_k, is a cycle of length k.

Decompositions of graphs into isomorphic paths has attracted considerable attention (see [8, 12–14, 17–19, 28, 40, 42]). Besides, decompositions of graphs into k-stars have also attracted a fair share of interest (see [9, 25, 39, 41, 43, 44]). Moreover, decompositions of graphs into k-cycles have been a popular topic of research in graph theory (see [10, 27] for surveys of this topic).

The study of the $\{G, H\}$-decomposition was introduced by Abueida and Daven in [1]. In [2, 4], they investigated, respectively, the problem of $\{K_k, S_k\}$-decomposition of the complete graph K_n and the problem of the $\{C_4, E_2\}$-decomposition of several graph products, where E_2 is a matching of size 2. Abueida and O'Neil [3] settled the existence problem for $\{C_k, S_{k-1}\}$-decomposition of the complete multigraph λK_n for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [29,30] gave necessary and sufficient conditions for the existence of $\{G(n), H(n)\}$-decompositions of λK_n and $\lambda K_{n,n}$, where $G(n), H(n) \in \{C_n, P_{n-1}, S_{n-1}\}$.

Recently, Lee and Lin [20, 21, 23, 24] established necessary and sufficient conditions for the existence of $\{C_k, S_k\}$-decompositions of the complete bipartite graphs, the complete bipartite multigraphs, the complete bipartite graphs with a 1-factor removed, and the multicrowns, respectively. Besides, Abueida, Lian [5], and Beggas *et al.* [7] investigated the problems of $\{C_k, S_k\}$-decompositions of the complete graph K_n and λK_n respectively, giving some necessary or sufficient conditions for such decompositions to exist. In [22], Lee and Chu established necessary and sufficient conditions for the existence of $\{P_k, S_k\}$-decompositions of the balanced complete bipartite graphs. In 2016, Lin and Jou [26] established necessary and sufficient conditions for the existence of $\{P_k, C_k, S_k\}$-decompositions of the balanced complete bipartite graphs.

For the $\{G^p, H^q\}$-decompositions of a graph, Jeevadoss and Muthusamy [15, 16] determined the set of ordered pairs (p, q) of positive integers for which there exists a $\{{P_k}^p, {C_k}^q\}$-decomposition of $\lambda K_{m,n}$ when $\lambda = 1$ and $k \equiv 0 \pmod 4$; $\lambda = 2$ and $k \equiv 0 \pmod 2$; for some positive integers λ, m, n, and k. Jeevadoss and Muthusamy [15] also determined the set of ordered pairs (p, q) of positive integers for which there exists a $\{{P_k}^p, {C_k}^q\}$-decomposition of K_n when k is even and n is odd with $n > 4k$. Fu *et al.* [11] determined the set of ordered pairs (p, q) of positive integers for which there exists a $\{{C_3}^p, {S_3}^q\}$-decomposition of K_n. The author also determined the set of ordered pairs (p, q) of positive integers for which there exists a $\{{P_k}^p, {S_k}^q\}$-decompositon of K_n when $n \geq 4k$ [36]; there exists a $\{{P_k}^p, {C_k}^q\}$-decomposition of K_n when k is even, n is odd, and $n > 5k$ [33]; there exists a $\{C_k{}^p, S_k{}^q\}$-decomposition of K_n for some k and n [35]; there exists a $\{P_k{}^p, S_k{}^q\}$-decomposition of $K_{m,n}$ when $m > k$ and $n \geq 3k$ [36]. In [37], the author also investigated the $\{H^p, K^q\}$-decomposition of the complete bipartite digraphs and the complete digraphs, where H and K are, respectively, directed paths and directed cycles with k edges each.

In this paper, we determine the set of triples (p, q, r) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-decomposition of K_n and $K_{m,l}$ when n is odd, and both m and l are even.

2. Preliminaries

In this section we collect some needed terminologies and notations, and present some results which are useful for our discussions.

Let $|V(G)|$ and $e(G)$ denote, respectively, the order of a graph G and the number of edges in G; and let us call a graph *even* if all its vertex degrees are even. Let G_1 and G_2 be graphs. The *union* $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

The following theorem gives necessary conditions for the existence of a decomposition of an even graph into specified numbers of paths, cycles, and stars with same number of edges each.

Theorem 1. *Let G be an even graph and let k, p, q, and r be positive integers with $k \geq 3$. If G can be decomposed into p copies of P_k, q copies of S_k, and r copies of C_k, then $|V(G)| \geq k+1$; $k(p+q+r) = e(G)$ and $p \geq \left\lceil \frac{k}{2} \right\rceil$ when $q = 1$.*

Proof. Conditions $|V(G)| \geq k+1$ and $k(p+q+r) = e(G)$ are trivial. Assume $\mathcal{D}$ is an arbitrary decomposition of G into p copies of P_k, one copy of S_k, and r copies of C_k. Let H be the only S_k and $C^{(1)}, \ldots, C^{(r)}$ denote those r copies of C_k in $\mathcal{D}$. Then, there are $2\left\lceil \frac{k}{2} \right\rceil$ vertices with odd degree in $G - E\left(H \cup C^{(1)} \cup \cdots \cup C^{(r)}\right)$. Since $G - E\left(H \cup C^{(1)} \cup \cdots \cup C^{(r)}\right)$ has to decompose into p copies of P_k, and there are exactly two vertices with odd degree in a path, $p \geq \left\lceil \frac{k}{2} \right\rceil$. ■

Let $\mathcal{D}(G; P_k, S_k, C_k)$ denote the set of all triples (m, n, l) of non-negative integers such that a decomposition of a graph G into m copies of P_k, n copies of S_k, and l copies of C_k exists. Note that $(m, n, 0) \in \mathcal{D}(G; P_k, S_k, C_k)$ if $(m, n) \in \mathcal{CD}(G; P_k, S_k)$; $(m, 0, l) \in \mathcal{D}(G; P_k, S_k, C_k)$ if $(m, l) \in \mathcal{CD}(G; P_k, C_k)$; $(0, n, l) \in \mathcal{D}(G; P_k, S_k, C_k)$ if $(n, l) \in \mathcal{CD}(G; S_k, C_k)$; $\left(\frac{e(G)}{k}, 0, 0\right), \left(0, \frac{e(G)}{k}, 0\right), \left(0, 0, \frac{e(G)}{k}\right) \in \mathcal{D}(G; P_k, S_k, C_k)$ if G can be decomposed into $\frac{e(G)}{k}$ copies of P_k (S_k, C_k).

Let G be an even graph, and let k, p, q, and r be positive integers with $k \geq 3$, $|V(G)| \geq k+1$, and $k(p+q+r) = e(G)$. If $k = 4$, by Theorem 1, $p \geq \left\lceil \frac{k}{2} \right\rceil = 2$ if $q = 1$, and hence $\mathcal{CD}(G; P_4, S_4, C_4) \subset \Big\{(p, q, r) : p, q, r > 0,\ p + q + r = \frac{e(G)}{4},$

$(p,q) \neq (1,1)\big\}$. Note that both $\mathcal{CD}(G;P_4,S_4,C_4)$ and $\{(p,q,r) : p,q,r > 0,\ p+q+r = \frac{e(G)}{4},\ (p,q) \neq (1,1)\}$ are empty if $e(G)$ is not divisible by 4. If we can prove that $\mathcal{CD}(G;P_4,S_4,C_4) \supset \big\{(p,q,r) : p,q,r > 0, p+q+r = \frac{e(G)}{4}, (p,q) \neq (1,1)\big\}$, then $\mathcal{CD}(G;P_4,S_4,C_4) = \big\{(p,q,r) : p,q,r > 0,\ p+q+r = \frac{e(G)}{4},\ (p,q) \neq (1,1)\big\}$, and hence we determine the set of triples (p,q,r) of positive integers for which there exists a $\{{P_4}^p, {S_4}^q, {C_4}^r\}$-decomposition of G.

If $X_1, \ldots, X_n$ are n sets of triples of non-negative integers, then $X_1 + \cdots + X_n$ denotes the set $\{(p_1,q_1,r_1) + \cdots + (p_n,q_n,r_n) : (p_1,q_1,r_1) \in X_1, \ldots, (p_n,q_n,r_n) \in X_n\}$. The following two lemmas will be used for proving the main theorems.

Lemma 2. *Let n, l, and s be positive integers, and let X and Y be sets of triples of non-negative integers such that $X \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = s,\ (p,q) \neq (1,1)\}$ and $Y \supset \{(al,bl,cl) : a,b,c \geq 0,\ a+b+c = n\}$. If $l \geq 2$ and $s \geq 3l$, then $X + Y \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = s+nl,\ (p,q) \neq (1,1)\}$.*

Proof. Let (p^*,q^*,r^*) be a triple of positive integers such that $p^*+q^*+r^* = s+nl$ and $(p^*,q^*) \neq (1,1)$. Clearly, $(p^*,q^*,r^*) = (\alpha l+p', \beta l+q', \gamma l+r')$ with $1 \leq p',q',r' \leq l$ and $\alpha,\beta,\gamma \geq 0$. It is not difficult to check that $s = s'+n'l$ where $s' = p'+q'+r' \leq 3l \leq s$ and $n' \geq 0$. Let $(\alpha l+p', \beta l+q', \gamma l+r') = (\alpha' l+p', \beta' l+q', \gamma' l+r') + ((\alpha-\alpha')l, (\beta-\beta')l, (\gamma-\gamma')l)$, where $\alpha' = \min\{\alpha, n'\}$, $\beta' = \min\{\beta, n'-\alpha'\}$, and $\gamma' = n'-\alpha'-\beta'$. Clearly, $(\alpha' l+p')+(\beta' l+q')+(\gamma' l+r') = s$ and $((\alpha-\alpha')l, (\beta-\beta')l, (\gamma-\gamma')l) \in Y$.

It is left to show that $(\alpha' l+p', \beta' l+q') \neq (1,1)$. Assume for a contradiction that $\alpha' l+p' = \beta' l+q' = 1$. It follows that $p' = q' = 1$ and $\alpha' = \beta' = 0$. Therefore, either $n' = 0$ or $\alpha = \beta = 0$. If $n' = 0$, then $s = s' = 2+r' \leq 2+l \leq 2+\frac{s}{3}$, hence $s \leq 3$ which is a contradiction since $s \geq 6$. If $\alpha = \beta = 0$, then $(p^*,q^*) = (p',q') = (1,1)$ which contradicts our assumption. Hence $(\alpha' l+p', \beta' l+q') \neq (1,1)$, thus $(p^*,q^*,r^*) \in X+Y$. ∎

Lemma 3. *Let s_1 and s_2 be positive integers with $s_1, s_2 \geq 9$ and let X_1 and X_2 be sets of triples of non-negative integers such that $X_1 \supset \{(a,b,c) : a,b,c \geq 0,\ a+b+c = s_1,\ (a,b,c) \neq (1,1,c), (1,0,c), (0,1,c)$ when $c \geq 1\}$ and $X_2 \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = s_2,\ (p,q) \neq (1,1)\}$. Then $X_1 + X_2 \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = s_1+s_2,\ (p,q) \neq (1,1)\}$.*

Proof. Let (p^*,q^*,r^*) be a triple of positive integers such that $p^*+q^*+r^* = s_1+s_2$ and $(p^*,q^*) \neq (1,1)$. We consider three cases as follows.

Case 1. $p^*,q^* \geq 3$. If $r^* \geq s_2-3$, then let $(p^*,q^*,r^*) = (p^*-1, q^*-2, r^*-(s_2-3)) + (1,2,s_2-3)$. Clearly, $(p^*-1, q^*-2, r^*-(s_2-3)) \in X_1$ and $(1,2,s_2-3) \in X_2$. If $r^* \leq s_2-4$, then $p^*+q^* \geq s_1+4$. Since $p^*,q^* \geq 3$ with $p^*+q^* \geq s_1+4$, there exist positive integers p_1^*, p_2^*, q_1^* and q_2^* with $p_1^* \geq 1$, $p_2^* \geq 2$, $q_1^* \geq 2$, and $q_2^* \geq 1$ such that $p^* = p_1^*+p_2^*$, $q^* = q_1^*+q_2^*$, $p_1^*+q_1^* = s_1$, and

$p_2^* + q_2^* + r^* = s_2$. Let $(p^*, q^*, r^*) = (p_1^*, q_1^*, 0) + (p_2^*, q_2^*, r^*)$. It is easy to check that $(p_1^*, q_1^*, 0) \in X_1$ and $(p_2^*, q_2^*, r^*) \in X_2$. Hence $(p^*, q^*, r^*) \in X_1 + X_2$.

Case 2. $p^*, q^* \leq 2$. Let $(p^*, q^*, r^*) = (0, 0, s_1) + (p^*, q^*, r^* - s_1)$. In this case, $r^* \geq s_1 + s_2 - 4$ and $(p^*, q^*) \neq (1, 1)$. It implies that $(0, 0, s_1) \in X_1$ and $(p^*, q^*, r^* - s_1) \in X_2$. Hence $(p^*, q^*, r^*) \in X_1 + X_2$.

Case 3. Either $p^* \leq 2$, $q^* \geq 3$ or $p^* \geq 3$, $q^* \leq 2$. Assume $p^* \leq 2$ and $q^* \geq 3$. If $q^* \leq s_2 - 3$, then $p^* + q^* \leq s_2 - 1$, and hence $r^* \geq s_1 + 1$. Let $(p^*, q^*, r^*) = (0, 0, s_1) + (p^*, q^*, r^* - s_1)$. Clearly, $(0, 0, s_1) \in X_1$ and $(p^*, q^*, r^* - s_1) \in X_2$.

If $q^* \geq s_2 - 2$ and $r^* \geq 6$, then let $(p^*, q^*, r^*) = (0, s_1 - (r^* - 5), r^* - 5) + (p^*, s_2 - (p^* + 5), 5)$. Since $1 \leq p^* \leq 2$, $s_1 + s_2 - 2 \leq q^* + r^* \leq s_1 + s_2 - 1$. Moreover, since $q^* \geq s_2 - 2$, $r^* \leq s_1 + 1$, and hence $s_1 - (r^* - 5) \geq 4$. Besides, $s_2 - (p^* + 5) \geq 2$ since $s_2 \geq 9$ and $p^* \leq 2$. It implies that $(0, s_1 - (r^* - 5), r^* - 5) \in X_1$ and $((p^*, s_2 - (p^* + 5), 5) \in X_2$.

If $q^* \geq s_2 - 2$ and $r^* \leq 5$, then let $(p^*, q^*, r^*) = (0, s_1, 0) + (p^*, s_2 - (p^* + r^*), r^*)$. Since $s_2 \geq 9$, $p^* \leq 2$, and $r^* \leq 5$, $s_2 - (p^* + r^*) \geq 2$. Clearly, $(0, s_1, 0) \in X_1$ and $(p^*, s_2 - (p^* + r^*), r^*) \in X_2$. Hence $(p^*, q^*, r^*) \in X_1 + X_2$.

The case where $p^* \geq 3$ and $q^* \leq 2$ is similar to the case $p^* \leq 2$ and $q^* \geq 3$, therefore we omit its proof. ■

3. $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-Decomposition of $K_{m,n}$

In this section we study the $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-decomposition of $K_{m,n}$ when both m and n are even. In particular, we prove that $\mathcal{CD}(K_{m,n}; P_4, S_4, C_4) = \{(p, q, r) : p, q, r > 0;\ m + n \geq 6;\ 4(p + q + r) = mn;\ (p, q) \neq (1, 1);\ q \text{ is even when } m = 2;\ (p, q, r) \neq (1, 2, 1) \text{ when } m = n = 4\}$. We first recall three results on P_k-decomposition, S_k-decomposition, and C_k-decomposition of $K_{m,n}$ as follows.

Theorem 4 (Parker [28]). *Let k, m, and n be positive integers. There exists a P_k-decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod{k}$ and one of cases in Table 1 occurs.*

Theorem 5 (Yamamoto *et al.* [44]). *Let k, m, and n be positive integers with $m \leq n$. There exists an S_k-decomposition of $K_{m,n}$ if and only if one of the following conditions holds.*

(1) *$m \geq k$ and $mn \equiv 0 \pmod{k}$;*

(2) *$m < k \leq n$ and $n \equiv 0 \pmod{k}$.*

Theorem 6 (Sotteau [38]). *Let k, m, and n be positive integers. $K_{m,n}$ has a C_{2k}-decomposition if and only if m and n are even, $k \geq 2$, $m \geq k$, $n \geq k$, and $mn \equiv 0 \pmod{2k}$.*

Case	k	m	n	Characterization
1.	even	even	even	$k \leq 2m, k \leq 2n$, not both equalities
2.	even	even	odd	$k \leq 2m-2, k \leq 2n$
3.	even	odd	even	$k \leq 2m, k \leq 2n-2$
4.	odd	even	even	$k \leq 2m-1, k \leq 2n-1$
5.	odd	even	odd	$k \leq 2m-1, k \leq n$
6.	odd	odd	even	$k \leq m, k \leq 2n-1$
7.	odd	odd	odd	$k \leq m, k \leq n$

Table 1. Necessary and Sufficient Conditions for P_k-Decomposition of $K_{m,n}$.

Before going into more detail, we need the following lemma.

Lemma 7 ([36, Theorem 2.10])**.** *Let p and q be non-negative integers, and let k, m, and s be positive integers such that k is even and $m < k$. There exists a decomposition of $K_{sk,m}$ into p copies of P_k and q copies of S_k if and only if $k(p+q) = e(K_{sk,m})$, and there is $t \in \{0, \ldots, s\}$ such that $\lceil \frac{tk}{2} \rceil \leq p \leq tm$.*

Let $(x_1, \ldots, x_k)$ and $(x_1, \ldots, x_k, x_1)$ denote, respectively, the k-path and the k-cycle through vertices $x_1, \ldots, x_k$ in order, and let $(y; x_1, \ldots, x_k)$ denote the k-star with center y and leafs $x_1, \ldots, x_k$. An *internal* vertex of a path is a vertex of degree 2. In the following lemma, we determine the set of ordered pairs (p, q) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q\}$-decomposition of $K_{2,2n}$.

Lemma 8. *Let n, p, and q be positive integers. $(p, q) \in \mathcal{CD}(K_{2,2n}; P_4, S_4)$ if and only if $n \geq 2$; $p + q = n$ and q is even.*

Proof. Let n, p, and q be positive integers. Assume that $(p, q) \in \mathcal{CD}(K_{2,2n}; P_4, S_4)$. It is easily seen that $n \geq 2$ and $p + q = n$.

Let $\mathcal{D}$ be an arbitrary decomposition of $K_{2,2n}$ into p copies of P_4 and q copies S_4. Let (A, B) be the bipartition of $K_{2,2n}$ where $A = \{a_0, a_1\}$ and $B = \{b_0, b_1, \ldots, b_{2n-1}\}$. It is easily seen that each S_4 in $\mathcal{D}$ has to center at either a_0 or a_1, and each P_4 in $\mathcal{D}$ has to contain both a_0 and a_1 as its internal vertices. It implies that the number of copies of S_4 centered in a_0 in $\mathcal{D}$ is the same as the number of copies of S_4 centered in a_1 in $\mathcal{D}$, and hence q is even.

Conversely, assume that $n \geq 2$; $p + q = n$ and q is even. If $2n = 4s$ for some integer s, by Lemma 7, then $(p, n - p) \in \mathcal{CD}(K_{2,2n}; P_4, S_4)$ for each $p \in \{2, 4, \ldots, 2s\}$ (i.e., $q = n - p \in \{2, 4, \ldots, 2s\}$). Assume $2n = 4s + 2$ for some integer s. For each $q \in \{2, 4, \ldots, 2(s-1)\}$, the graph $K_{2,4s+2}$ is the edge-disjoint union of a copy H_1^q of $K_{2,2q}$ and a copy H_2^q of $K_{2,4s-2q+2}$. By Theorem 5, H_1^q is S_4-decomposable, and by Theorem 4, H_2^q is P_4-decomposable. If $q = 2s$, then let $K_{2,4s+2}$ decompose into $K_{2,4s-4}$ and $K_{2,6}$. As mentioned above, $K_{2,4s-4}$ can be decomposed into $2s - 2$ copies of S_4. Besides, $K_{2,6}$ can be decomposed into one copy of P_4 and two copies of S_4 as follows: $(b_0, a_1, b_5, a_0, b_4)$, $(a_0;\ b_0, b_1, b_2, b_3)$, $(a_1;\ b_1, b_2, b_3, b_4)$. ■

In the following lemma, we determine the set of triples (p, q, r) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-decomposition of $K_{2,2n}$.

Lemma 9. *Let n, p, q, and r be positive integers with $n \geq 3$. $(p, q, r) \in \mathcal{CD}(K_{2,2n}; P_4, S_4, C_4)$ if and only if $p + q + r = n$ and q is even.*

Proof. Let n, p, q, and r be positive integers with $n \geq 3$. Assume that $(p, q, r) \in \mathcal{CD}(K_{2,2n}; P_4, S_4, C_4)$. It is easily seen that $p + q + r = n$.

Let D be an arbitrary decomposition of $K_{2,2n}$ into p copies of P_4, q copies of S_4, and r copies of C_4, and let $C^{(1)}, \ldots, C^{(r)}$ denote the r copies of C_4 in D. It is easily seen that $K_{2,2n} - E\left(C^{(1)} \cup \cdots \cup C^{(r)}\right) \cong K_{2,2(n-r)}$. It implies that $K_{2,2(n-r)}$ can be decomposed into p copies of P_4 and q copies of S_4, and hence q is even by Lemma 8.

Conversely, assume that $p + q + r = n$ and q is even. Let (A, B) be the bipartition of $K_{2,2n}$ where $A = \{a_0, a_1\}$ and $B = \{b_0, b_1, \ldots, b_{2n-1}\}$, and let $C^{(i)} = \left(b_{2i-2}, a_0, b_{2i-1}, a_1, b_{2i-2}\right)$ for each $i \in \{1, \ldots, r\}$. It clear that $C^{(i)}$ is a C_4 and $K_{2,2n} - E\left(C^{(1)} \cup \cdots \cup C^{(r)}\right) \cong K_{2,2(n-r)}$. By Lemma 8, $K_{2,2(n-r)}$ is $\{P_4{}^p, S_4{}^q\}$-decomposable. ■

In the following lemma, we determine the set of triples (p, q, r) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-decomposition of $K_{4,2n}$.

Lemma 10. *Let n, p, q, and r be positive integers with $n \geq 2$. $(p, q, r) \in \mathcal{CD}(K_{4,2n}; P_4, S_4, C_4)$ if and only if $p + q + r = 2n$ and $(p, q) \neq (1, 1)$; $(p, q, r) \neq (1, 2, 1)$.*

Proof. (*Necessity*) By Theorem 1, condition $p + q + r = 2n$ and $(p, q) \neq (1, 1)$ holds.

On the contrary, suppose $(1, 2, 1) \in \mathcal{CD}(K_{4,4}; P_4, S_4, C_4)$. Let D be an arbitrary decomposition of $K_{4,4}$ into one copy of P_4, two copies of S_4, and one copy of C_4; and let $S^{(1)}, S^{(2)}$, and C denote, respectively, the two copies of S_4 and the copy of C_4 in D. It is easily seen that $K_{4,4} - E(S^{(1)} \cup S^{(2)}) \cong K_{2,4}$ and $K_{2,4} - E(C) \cong K_{2,2}$. It follows that $K_{4,4} - E(S^{(1)} \cup S^{(2)} \cup C)$ is not P_4-decomposable, a contradiction.

(*Sufficiency*) By assumption, $\mathcal{CD}(K_{4,2n}; P_4, S_4, C_4) \subset \{(p, q, r) : p, q, r > 0,\ p + q + r = 2n,\ (p, q) \neq (1, 1);\ (p, q, r) \neq (1, 2, 1)\}$, and hence $\mathcal{CD}(K_{4,4}; P_4, S_4, C_4) \subset \{(2, 1, 1)\}$. Let (A, B) be the bipartition of $K_{4,4}$ where $A = \{a_0, a_1, a_2, a_3\}$ and $B = \{b_0, b_1, b_2, b_3\}$. $K_{4,4}$ can be decomposed into two copies of P_4, one copy of S_4, and one copy of C_4 as follows: $(b_0, a_0, b_1, a_1, b_2)$, $(b_1, a_2, b_2, a_0, b_3)$, $(a_3;\ b_0, b_1, b_2, b_3)$, $(b_0, a_1, b_3, a_2, b_0)$.

Assume $n = 3$. We show that $\mathcal{CD}(K_{4,6}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0, p+q+r = 6, (p,q) \neq (1,1)\} = \{(1,2,3), (1,3,2), (1,4,1), (2,1,3), (2,2,2), (2,3,1), (3,1,2), (3,2,1), (4,1,1)\}$.

We decompose $K_{4,6}$ into one copy of $K_{4,4}$ and one copy of $K_{4,2}$. By Theorems 4, 5, and 6, $K_{4,2}$ is P_4-decomposable, S_4-decomposable, and C_4-decomposable, respectively. Since $(2,1,1) \in \mathcal{CD}(K_{4,4}; P_4, S_4, C_4)$, $\{(2,1,1) + (2,0,0), (2,1,1) + (0,2,0), (2,1,1) + (0,0,2)\} = \{(4,1,1), (2,3,1), (2,1,3)\} \subset \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$. Besides, it is easy to check that $K_{4,4}$ is $\{{P_4}^2, {C_4}^2\}$-decomposable, $\{{P_4}^3, {C_4}^1\}$-decomposable, and $\{{P_4}^3, {S_4}^1\}$-decomposable, respectively. Thus $\{(2,0,2) + (0,2,0), (3,0,1) + (0,2,0), (3,1,0) + (0,0,2)\} = \{(2,2,2), (3,2,1), (3,1,2)\} \subset \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$. We now turn our attention to the case $(1,2,3)$. The graph $K_{4,6}$ is the edge-disjoint union of two copies of $K_{2,6}$. By Lemma 8 and Theorem 6, $K_{2,6}$ is $\{{P_4}^1, {S_4}^2\}$-decomposable and C_4-decomposable, respectively. Thus $(1,2,3) \in \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$. Let (A, B) be the bipartition of $K_{4,6}$ where $A = \{a_0, a_1, a_2, a_3\}$ and $B = \{b_0, b_1, b_2, b_3, b_4, b_5\}$. We now show that $(1,4,1), (1,3,2) \in \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$ as follows: $(b_0, a_3, b_4, a_0, b_5)$, $(a_0;\ b_0, b_1, b_2, b_3)$, $(a_1; b_1, b_2, b_3, b_4)$, $(a_2; b_1, b_2, b_3, b_4)$, $(a_3; b_1, b_2, b_3, b_5)$, $(b_0, a_1, b_5, a_2, b_0)$; $(b_0, a_1, b_1, a_3, b_3)$, $(a_0; b_0, b_1, b_2, b_3)$, $(a_1;\ b_2, b_3, b_4, b_5)$, $(a_2;\ b_1, b_3, b_4, b_5)$, $(a_0, b_4, a_3, b_5, a_0)$, $(b_0, a_2, b_2, a_3, b_0)$.

Assume $n \geq 4$. We decompose $K_{4,2n}$ into one copy of $K_{4,6}$ and one copy of $K_{4,2(n-3)}$, and then we decompose $K_{4,2(n-3)}$ into $(n-3)$ copies of $K_{4,2}$. By Theorems 4, 5 and 6, $\{(2,0,0), (0,2,0), (0,0,2)\} \subset \mathcal{D}(K_{4,2}; P_4, S_4, C_4)$, and thus $\mathcal{D}(K_{4,2(n-3)}; P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a,b,c \geq 0,\ a+b+c = n-3)\}$. Moreover, since $\mathcal{CD}(K_{4,6};\ P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0,\ p+q+r = 6,\ (p,q) \neq (1,1)\}$, $\mathcal{CD}(K_{4,2n}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = 2n,\ (p,q) \neq (1,1)\}$ by Lemma 2, and hence $\mathcal{CD}(K_{4,2n}; P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0,\ p+q+r = 2n,\ (p,q) \neq (1,1)\}$. ■

In the following lemma, we determine the set of triples (p,q,r) of positive integers for which there exists a $\{{P_4}^p, {S_4}^q, {C_4}^r\}$-decomposition of $K_{6,2n}$.

Lemma 11. *Let n, p, q, and r be positive integers with $n \geq 3$. $(p,q,r) \in \mathcal{CD}(K_{6,2n}; P_4, S_4, C_4)$ if and only if $p+q+r = 3n$ and $(p,q) \neq (1,1)$.*

Proof. (*Necessity*) By Theorem 1, condition $p+q+r = 3n$ and $(p,q) \neq (1,1)$ holds.

(*Sufficiency*) Assume $n = 3$. It is easily seen that $K_{6,6}$ can be decomposed into one copy of $K_{4,6}$ and one copy of $K_{2,6}$. By Lemma 10, $\mathcal{CD}(K_{4,6}; P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0, p+q+r = 6, (p,q) \neq (1,1)\}$. By Theorem 4, 6 and Lemma 8, $\{(3,0,0), (0,0,3), ((1,2,0)\} \subset \mathcal{D}(K_{2,6}; P_4, S_4, C_4)$. Besides, $K_{2,6} - E(C_4) \cong K_{2,4}$, hence $\{(2,0,1), (0,2,1)\} \subset \mathcal{D}(K_{2,6}; P_4, S_4, C_4)$ by Theorems 4, 5. We show that $\mathcal{CD}(K_{6,6}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = 9,\ (p,q) \neq (1,1)\}$ as follows.

Suppose $q = 1$ or 2. If $p > r$, then let $(p,q,r) = (p-3,q,r) + (3,0,0)$, and if $p \le r$ then let $(p,q,r) = (p,q,r-3) + (0,0,3)$. Since $\{(p-3,q,r), (p,q,r-3)\} \subset \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$ and $\{(3,0,0), (0,0,3)\} \subset \mathcal{D}(K_{2,6}; P_4, S_4, C_4)$, $(p,q,r) \in \mathcal{CD}(K_{6,6}; P_4, S_4, C_4)$.

Suppose $q = 3$, 4, or 5. If $r \ge 4$, then let $(p,q,r) = (p,q,r-3) + (0,0,3)$; if $2 \le r \le 3$, then let $(p,q,r) = (p,q-2,r-1) + (0,2,1)$ (note that $p \ge 3$ if $q = 3$); if $r = 1$, then let $(p,q,r) = (p-1,q-2,r) + (1,2,0)$ (note that $p \ge 3$). Since $\{(p,q,r-3), (p,q-2,r-1), (p-1,q-2,r)\} \subset \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$ and $\{(0,0,3), (0,2,1), (1,2,0)\} \subset \mathcal{D}(K_{2,6}; P_4, S_4, C_4)$, $(p,q,r) \in \mathcal{CD}(K_{6,6}; P_4, S_4, C_4)$.

Suppose $q = 6$. In this case $p + r = 3$. If $r = 2$, then let $(1,6,2) = (1,4,1) + (0,2,1)$, and if $r = 1$, then let $(2,6,1) = (1,4,1) + (1,2,0)$. Since $(1,4,1) \in \mathcal{CD}(K_{4,6}; P_4, S_4, C_4)$ and $(0,2,1), (1,2,0) \in \mathcal{D}(K_{2,6}; P_4, S_4, C_4)$, $(1,6,2), (2,6,1) \in \mathcal{CD}(K_{6,6}; P_4, S_4, C_4)$.

Suppose $q = 7$. Let (A,B) be the bipartition of $K_{6,6}$ where $A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_0, b_1, b_2, b_3, b_4, b_5\}$. We show that $(1,7,1) \in \mathcal{CD}(K_{6,6}; P_4, S_4, C_4)$ below: $(b_0, a_5, b_2, a_3, b_3)$, $(a_0;\ b_0, b_1, b_2, b_3)$, $(a_1;\ b_0, b_1, b_2, b_3)$, $(a_2;\ b_0, b_1, b_2, b_3)$, $(a_3;\ b_0, b_1, b_4, b_5)$, $(a_4;\ b_0, b_2, b_4, b_5)$, $(b_4;\ a_0, a_1, a_2, a_5)$, $(b_5;\ a_0, a_1, a_2, a_5)$, $(b_1, a_4, b_3, a_5, b_1)$.

Assume $n \ge 4$. If n is even, then write $n = 2k$ for some integer k with $k \ge 2$. We decompose $K_{6,4k}$ into one copy of $K_{6,4}$ and one copy of $K_{6,4(k-1)}$, and then we decompose $K_{6,4(k-1)}$ into $3(k-1)$ copies of $K_{2,4}$. By Theorems 4, 5 and 6, $\{(2,0,0), (0,2,0), (0,0,2)\} \subset \mathcal{D}(K_{4,2}; P_4, S_4, C_4)$, and thus $\mathcal{D}(K_{6,4(k-1)}; P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a, b, c \ge 0,\ a+b+c = 3(k-1)\}$. By Lemma 10, $\mathcal{CD}(K_{6,4}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = 6,\ (p,q) \ne (1,1)\}$, and hence $\mathcal{CD}(K_{6,2n}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = 3n,\ (p,q) \ne (1,1)\}$ by Lemma 2.

If n is odd, then write $n = 2k+1$ for some integer k with $k \ge 2$, and thus $2n = 4k+2 = 4(k-1)+6$. We decompose $K_{6,4k+2}$ into one copy of $K_{6,6}$ and one copy of $K_{6,4(k-1)}$, and then we decompose $K_{6,4(k-1)}$ into $3(k-1)$ copies of $K_{2,4}$. As mentioned above, $\mathcal{D}(K_{6,4(k-1)};\ P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a, b, c \ge 0,\ a+b+c = 3(k-1)\}$. Since $\mathcal{CD}(K_{6,6};\ P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0,\ p+q+r = 9,\ (p,q) \ne (1,1)\}$, $\mathcal{CD}(K_{6,2n}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = 3n,\ (p,q) \ne (1,1)\}$, by Lemma 2. ■

In the following lemma, we determine the set of triples (p,q,r) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-decomposition of $K_{m,n}$ when both m and n are positive even integers with $n \ge m \ge 8$.

Lemma 12. *Let p, q, and r be positive integers, and let m and n be positive even integers with $n \ge m \ge 8$. $(p,q,r) \in \mathcal{CD}(K_{m,n}; P_4, S_4, C_4)$ if and only if $4(p+q+r) = mn$ and $(p,q) \ne (1,1)$.*

Proof. (*Necessity*) By Theorem 1, condition $4(p+q+r) = mn$ and $(p,q) \neq (1,1)$ holds.

(*Sufficiency*) We divided the proof into two cases as follows.

Case 1. $m \equiv 0 \pmod 4$. Write $m = 4k$ for some integer k with $k \geq 2$. We decompose $K_{4k,n}$ into one copy of $K_{4,n}$ and one copy of $K_{4(k-1),n}$, and then we decompose $K_{4(k-1),n}$ into $\frac{n}{2}(k-1)$ copies of $K_{4,2}$. By Theorems 4, 5 and 6, $\{(2,0,0),(0,2,0),(0,0,2)\} \subset \mathcal{D}(K_{4,2}; P_4, S_4, C_4)$, and thus $\mathcal{D}(K_{4(k-1),n}; P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a, b, c \geq 0,\ a+b+c = \frac{n}{2}(k-1)\}$. By Lemma 10, $\mathcal{CD}(K_{4,n}; P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0,\ p+q+r = n,\ (p,q) \neq (1,1)\}$, and hence $\mathcal{CD}(K_{m,n}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = kn,\ (p,q) \neq (1,1)\}$ by Lemma 2.

Case 2. $m \equiv 2 \pmod 4$. Write $m = 4k+2 = 4(k-1)+6$ for some integer k with $k \geq 2$. We decompose $K_{4k+2,n}$ into one copy of $K_{6,n}$ and one copy of $K_{4(k-1),n}$, and then we decompose $K_{4(k-1),n}$ into $\frac{n}{2}(k-1)$ copies of $K_{4,2}$. As mentioned above, $\mathcal{D}(K_{4(k-1),n};\ P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a, b, c \geq 0,\ a+b+c = \frac{n}{2}(k-1)\}$. By Lemma 11, $\mathcal{CD}(K_{6,n}; P_4, S_4, C_4) = \{(p,q,r) : p,q,r > 0,\ p+q+r = \frac{6n}{4},\ (p,q) \neq (1,1)\}$, and hence $\mathcal{CD}(K_{4k+2,n}; P_4, S_4, C_4) \supset \{(p,q,r) : p,q,r > 0,\ p+q+r = \frac{(4k+2)n}{4},\ (p,q) \neq (1,1)\}$ by Lemma 2. ■

Now, we are ready for the main result of this section. It is obtained by combining Theorem 1 and Lemmas 9, 10, 11, and 12.

Theorem 13. *Let m, n, p, q, and r be positive integers such that both m and n are even, and $m \leq n$. $(p,q,r) \in \mathcal{CD}(K_{m,n}; P_4, S_4, C_4)$ if and only if $m+n \geq 6$; $4(p+q+r) = mn$; $(p,q) \neq (1,1)$; q is even when $m = 2$; $(p,q,r) \neq (1,2,1)$ when $m = n = 4$.*

4. $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-Decomposition of K_n

In this section, we study the $\{P_4{}^p, S_4{}^q, C_4{}^r\}$-decomposition of K_n when n is odd. In particular, we prove that $\mathcal{CD}(K_n; P_4, S_4, C_4) = \{(p,q,r) :\ p,q,r > 0,\ 4(p+q+r) = \binom{n}{2},\ (p,q) \neq (1,1)\}$. Let us begin with three well-known results on P_k-decomposition, S_k-decomposition, and C_k-decomposition of K_n, respectively.

Theorem 14 (Tarsi [40])**.** *Let k and n be positive integers. There exists a P_k-decomposition of K_n if and only if $k+1 \leq n$ and $n(n-1) \equiv 0 \pmod{2k}$.*

Theorem 15 (Tarsi [39] and Yamamoto *et al.* [44])**.** *Let k and n be positive integers. There exists an S_k-decomposition of K_n if and only if $2k \leq n$ and $n(n-1) \equiv 0 \pmod{2k}$.*

Theorem 16 (Alspach, Gavlas [6] and Šajna [31]). *Let n and k be positive integers. K_n has a C_k-decomposition if and only if n is odd, $3 \leq k \leq n$, and $n(n-1) \equiv 0 \pmod{2k}$.*

In the following, we will introduce three known results on $\{P_4{}^p, C_4{}^r\}$-decomposition, $\{S_4{}^q, C_4{}^r\}$-decomposition, and $\{P_4{}^p, S_4{}^q\}$-decomposition of K_n, respectively.

Theorem 17 [33]. *Let p and r be positive integers, and let n be a positive odd integer. $(p, r) \in \mathcal{CD}(K_n; P_4, C_4)$ if and only if $4(p+q) = e(K_n)$ and $p \neq 1$.*

Theorem 18 [35]. *Let q and r be positive integers, and let n be a positive odd integer. $(q, r) \in \mathcal{CD}(K_n; S_4, C_4)$ if and only if $4(p+q) = e(K_n)$ and $q \neq 1$.*

Theorem 19 [36]. *Let p, q, and n be positive integers with $n \geq 16$. $(p, q) \in \mathcal{CD}(K_n; P_4, S_4)$ if and only if $4(p+q) = e(K_n)$.*

Theorem 19 determined the set of ordered pairs (p, q) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q\}$-decomposition of K_n when $n \geq 16$. In the following lemma, we determine the set of ordered pairs (p, q) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q\}$-decomposition of K_n when $n < 16$ and n is odd, thus we determine the set of ordered pairs (p, q) of positive integers for which there exists a $\{P_4{}^p, S_4{}^q\}$-decomposition of K_n when n is odd.

Theorem 20. *Let p and q be positive integers, and let n be a positive odd integer. $(p, q) \in \mathcal{CD}(K_n; P_4, S_4)$ if and only if $4(p+q) = e(K_n)$.*

Proof. (*Necessity*) Condition $4(p+q) = e(K_n)$ is trivial.

(*Sufficiency*) Observe that $4 \mid \frac{n(n-1)}{2}$ implies $8 \mid (n-1)$. It follows that $n = 8m+1$ for some positive integer m. By Theorem 19, we need only consider the case $n = 9$. Assume $V(K_9) = \{1, \ldots, 9\}$. We show that $\mathcal{CD}(K_9; P_4, S_4) \supset \{(p, q) : p, q > 0,\ p+q = 9\}$.

Assume $(p, q) = (8, 1)$. K_9 can be decomposed into 8 copies of P_4 and one copy of S_4 as follows: $(3, 1, 9, 2, 4)$, $(7, 5, 9, 6, 8)$, $(4, 3, 2, 1, 5)$, $(5, 2, 6, 1, 4)$, $(5, 4, 6, 3, 7)$, $(7, 4, 8, 3, 5)$, $(1, 7, 2, 8, 5)$, $(5, 6, 7, 8, 1)$, $(9; 3, 4, 7, 8)$.

Assume $(p, q) = (7, 2)$. It is easily seen that K_9 is the edge-disjoint union of a copy H_1^q of K_8 and a copy H_2^q of S_8. By Theorem 14, H_1^q is P_4-decomposable, and H_2^q can be decomposed into two copies of S_4. Hence the assertion follows.

Assume $(p, q) = (6, 3)$. K_9 can be decomposed into 6 copies of P_4 and 3 copies of S_4 as follows: $(5, 1, 3, 2, 6)$, $(6, 3, 4, 2, 5)$, $(3, 7, 4, 5, 6)$, $(6, 4, 8, 5, 3)$, $(6, 9, 5, 7, 1)$, $(1, 8, 2, 7, 6)$, $(1; 2, 4, 6, 9)$, $(8; 3, 6, 7, 9)$, $(9; 2, 3, 4, 7)$.

Assume $(p, q) = (5, 4)$. K_9 can be decomposed into 5 copies of P_4 and 4 copies of S_4 as follows: $(2, 4, 3, 6, 5)$, $(5, 8, 4, 6, 7)$, $(7, 5, 9, 6, 2)$, $(2, 3, 1, 5, 4)$, $(4, 7, 3, 5, 2)$, $(1; 4, 6, 7, 8)$, $(2; 1, 7, 8, 9)$, $(8; 3, 6, 7, 9)$, $(9; 1, 3, 4, 7)$.

Assume $(p,q)=(4,5)$. K_9 can be decomposed into 4 copies of P_4 and 5 copies of S_4 as follows: $(5,1,3,2,6)$, $(6,3,4,2,5)$, $(7,3,5,4,8)$, $(8,5,6,4,7)$, $(1;4,6,8,9)$, $(2;1,7,8,9)$, $(7;1,5,6,9)$, $(8;3,6,7,9)$, $(9;3,4,5,6)$.

Assume $(p,q)=(3,6)$. K_9 can be decomposed into 3 copies of P_4 and 6 copies of S_4 as follows: $(4,1,2,3,9)$, $(9,4,3,8,7)$, $(7,6,5,8,4)$, $(1;3,5,7,9)$, $(2;4,6,7,9)$, $(5;2,3,4,9)$, $(6;1,3,4,9)$, $(7;3,4,5,9)$, $(8;1,2,6,9)$.

Assume $(p,q)=(2,7)$. K_9 can be decomposed into 2 copies of P_4 and 7 copies of S_4 as follows: $(1,2,3,4,5)$, $(5,6,7,8,9)$, $(1;3,4,8,9)$, $(2;4,5,8,9)$, $(3;6,7,8,9)$, $(4;6,7,8,9)$, $(5;1,3,8,9)$, $(6;1,2,8,9)$, $(7;1,2,5,9)$.

Assume $(p,q)=(1,8)$. K_9 can be decomposed into one copy of P_4 and 8 copies of S_4 as follows: $(4,5,6,7,8)$, $(1;2,3,4,5)$, $(2;3,4,5,6)$, $(3;4,6,7,8)$, $(5;3,7,8,9)$, $(6;1,4,8,9)$, $(7;1,2,4,9)$, $(8;1,2,4,9)$. $(9;1,2,3,4)$. ∎

The following lemma gives sufficient conditions for decomposing an edge-disjoint union of cycles of length k into copies of P_k. In fact, the proof of the following lemma is essentially given in [33, Lemma 3.8]. We present it here for completeness.

Lemma 21. *Let k and n be integers such that $k\ge 3$ and $n\ge 2$. For each $i\in\{1,2,\dots,n\}$, let $C(i)$ denote the cycle of length k, $\big(x_{(i,1)},x_{(i,2)},\dots,x_{(i,k)},x_{(i,1)}\big)$. If $x_{(1,1)}=x_{(2,1)}=\cdots=x_{(n,1)}$, $x_{(i-1,2)}\notin V(C(i))$ for each $i\in\{2,3,\dots,n\}$, and $x_{(n,2)}\notin V(C(1))$, then $\bigcup_{i=1}^{n}C(i)$ can be decomposed into n paths of length k.*

Proof. By assumptions, $\bigcup_{i=1}^{n}C(i)$ can be decomposed into n paths of length k as follows: $\big(x_{(2,2)},x_{(2,3)},\dots,x_{(2,k)},x_{(2,1)},x_{(1,2)}\big)$, $\big(x_{(3,2)},x_{(3,3)},\dots,x_{(3,k)},x_{(3,1)},x_{(2,2)}\big)$, $\dots$, $\big(x_{(n,2)},x_{(n,3)},\dots,x_{(n,k)},x_{(n,1)},x_{(n-1,2)}\big)$, $\big(x_{(1,2)},x_{(1,3)},\dots,x_{(1,k)},x_{(1,1)},x_{(n,2)}\big)$. ∎

In the following lemma, we determine the set of triples (p,q,r) of positive integers for which there exists a $\{P_4{}^p,S_4{}^q,C_4{}^r\}$-decomposition of K_9.

Lemma 22. *Let p, q, and r be positive integers. $(p,q,r)\in\mathcal{CD}(K_9;P_4,S_4,C_4)$ if and only if $p+q+r=9$ and $(p,q)\ne(1,1)$.*

Proof. (*Necessity*) The assertion follows immediately from Theorem 1.

(*Sufficiency*) Let $V(K_9)=\{1,\dots,9\}$. We split the proof into 7 cases according to the value of q.

Assume $q=1$ (note that $p\ge 2$ in this case). K_9 can be decomposed into two copies of P_4, one copy of S_4, and 6 copies of C_4 as follows: $(3,1,9,2,4)$, $(7,5,9,6,8)$, $(9;3,4,7,8)$, $C(1)=(1,4,3,2,1)$, $C(2)=(1,5,2,6,1)$, $C(3)=(3,5,4,6,3)$, $C(4)=(3,7,4,8,3)$, $C(5)=(8,1,7,2,8)$, $C(6)=(8,5,6,7,8)$. Since $1\in V(C(1))\cap V(C(2))$, $5\notin V(C(1))$, and $4\notin (C(2))$, $C(1)\cup C(2)$ can be decomposed into two copies of P_4, by Lemma 21. By the same argument, $C(3)\cup C(4)$ and $C(5)\cup C(6)$ can also be decomposed into two copies of P_4. Hence $\mathcal{CD}(K_9;P_4,S_4,C_4)\supset\{(p,1,8-p):2\le p\le 7$ and p is even$\}$.

On the other hand, K_9 can be decomposed into three copies of P_4, one copy of S_4, and 5 copies of C_4 as follows: $(3,1,9,2,4)$, $(9,5,8,6,7)$, $(9,8,7,5,6)$, $(9;3,4,6,7)$, $C(1) = (1,4,3,2,1)$, $C(2) = (1,5,2,6,1)$, $C(3) = (1,7,2,8,1)$, $C(4) = (3,5,4,6,3)$, $C(5) = (3,7,4,8,3)$. By the same argument mentioned above, $C(1) \cup C(2)$ and $C(4) \cup C(5)$ can be decomposed into two copies of P_4. Thus $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 1, 8-p) : 2 \le p \le 7$ and p is odd $\}$.

Assume $q = 2$. K_9 can be decomposed into two copies of S_4, and 7 copies of C_4 as follows: $(1;3,4,8,9)$, $(2;3,4,8,9)$, $C(1) = (3,7,1,6,3)$, $C(2) = (3,5,6,4,3)$, $C(3) = (3,8,4,9,3)$, $C(4) = (2,7,5,1,2)$, $C(5) = (2,6,9,5,2)$, $C(6) = (7,6,8,9,7)$, $C(7) = (7,4,5,8,7)$, Since $3 \in V(C(1)) \cap V(C(2) \cap V(C(3))$, $8 \notin V(C(1))$, $7 \notin V(C(2))$, and $5 \notin V(C(3))$, $C(1) \cup C(2) \cup C(3)$ can be decomposed into three copies of P_4, by Lemma 21. By the same argument, $C(1) \cup C(2)$, $C(4) \cup C(5)$, and $C(6) \cup C(7)$ can also be decomposed into two copies of P_4. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 2, 7-p) : p = 2, \ldots, 6\}$. Besides, K_9 can also be decomposed into one copy of P_4, two copies of S_4, and 6 copies of C_4 as follows: $(4,1,7,3,6)$, $(1;3,6,8,9)$, $(2;3,4,8,9)$, $(3,5,6,4,3)$, $(3,8,4,9,3)$, $(2,7,5,1,2)$, $(2,6,9,5,2)$, $(7,6,8,9,7)$, $(7,4,5,8,7)$, Thus $(1,2,6) \in \mathcal{CD}(K_9; P_4, S_4, C_4)$.

Assume $q = 3$. K_9 can be decomposed into three copies of S_4, and 6 copies of C_4 as follows: $(1;2,4,6,9)$, $(8;3,6,7,9)$, $(9;2,3,4,7)$, $C(1) = (2,5,1,3,2)$, $C(2) = (2,6,3,4,2)$, $C(3) = (4,7,3,5,4)$, $C(4) = (4,8,5,6,4)$, $C(5) = (7,6,9,5,7)$, $C(6) = (7,1,8,2,7)$. By Lemma 21, both $C(1) \cup C(2)$ and $C(3) \cup C(4)$ can be decomposed into two copies of P_4. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 3, 6-p) : p = 2, 4\}$. Besides, K_9 can also be decomposed into one copy of P_4, three copies of S_4, and 5 copies of C_4 as follows: $(8,2,7,1,4)$, $(1;2,6,8,9)$, $(8;3,6,7,9)$, $(9;2,3,4,7)$, $C(1) = (3,1,5,2,3)$, $C(2) = (3,6,2,4,3)$, $C(3) = (3,7,4,5,3)$, $C(4) = (5,8,4,6,5)$, $C(5) = (5,9,6,7,5)$. By Lemma 21 again, both $C(1) \cup C(2)$ and $C(4) \cup C(5)$ can be decomposed into two copies of P_4. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 3, 6-p) : p = 1, 3, 5\}$.

Assume $q = 4$. K_9 can be decomposed into 4 copies of S_4, and 5 copies of C_4 as follows: $(1;4,6,7,8)$, $(2;1,7,8,9)$, $(8;3,6,7,9)$, $(9;1,3,4,7)$, $C(1) = (3,1,5,2,3)$, $C(2) = (3,6,2,4,3)$, $C(3) = (3,7,4,5,3)$, $C(4) = (5,8,4,6,5)$, $C(5) = (5,9,6,7,5)$. By Lemma 21, both $C(1) \cup C(2)$ and $C(4) \cup C(5)$ can be decomposed into two copies of P_4, and $C(1) \cup C(2) \cup C(3)$ can be decomposed into three copies of P_4. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 4, 5-p) : p = 2, 3, 4\}$. Besides, K_9 can also be decomposed into one copy of P_4, 4 copies of S_4, and 4 copies of C_4 as follows: $(3,2,5,1,4)$, $(1;3,6,7,8)$, $(2;1,7,8,9)$, $(8;3,6,7,9)$, $(9;1,3,4,7)$, $(2,6,3,4,2)$, $(3,7,4,5,3)$, $(4,8,5,6,4)$, $(5,9,6,7,5)$. Hence $(1,4,4) \in \mathcal{CD}(K_9; P_4, S_4, C_4)$.

Assume $q = 5$. K_9 can be decomposed into 5 copies of S_4, and 4 copies of C_4 as follows: $(1;4,6,8,9)$, $(2;1,7,8,9)$, $(7;1,5,6,9)$, $(8;3,6,7,9)$, $(9;3,4,5,6)$, $C(1) = (3,4,2,6,3)$, $C(2) = (3,1,5,2,3)$, $C(3) = (3,7,4,5,3)$, $C(4) = (4,8,5,

$6,4)$. By Lemma 21, $C(1) \cup C(2)$ can be decomposed into two copies of P_4, and $C(1) \cup C(2) \cup C(3)$ can be decomposed into three copies of P_4. Hence $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(p, 5, 4-p) : p = 2, 3\}$. Besides, K_9 can also be decomposed into one copy of P_4, 5 copies of S_4, and three copies of C_4 as follows: $(4, 6, 5, 8, 3)$, $(1; 4, 6, 8, 9)$, $(2; 1, 7, 8, 9)$, $(7; 1, 5, 6, 9)$, $(8; 4, 6, 7, 9)$, $(9; 3, 4, 5, 6)$, $(1, 5, 2, 3, 1)$, $(2, 6, 3, 4, 2)$, $(3, 7, 4, 5, 3)$. Hence $(1, 5, 3) \in \mathcal{CD}(K_9; P_4, S_4, C_4)$.

Assume $q = 6$. K_9 can be decomposed into two copies of P_4, 6 copies of S_4, and one copy of C_4 as follows: $(4, 9, 3, 8, 7)$, $(7, 6, 5, 8, 4)$, $(1; 3, 5, 7, 9)$, $(2; 4, 6, 7, 9)$, $(5; 2, 3, 4, 9)$, $(6; 1, 3, 4, 9)$, $(7; 3, 4, 5, 9)$, $(8; 1, 2, 6, 9)$, $(1, 2, 3, 4, 1)$. Besides, K_9 can also be decomposed into one copy of P_4, 6 copies of S_4, and two copies of C_4 as follows: $(1, 8, 5, 6, 7)$, $(1; 3, 5, 7, 9)$, $(2; 4, 6, 7, 9)$, $(5; 2, 3, 4, 9)$, $(6; 1, 3, 4, 9)$, $(7; 3, 4, 5, 9)$, $(8; 2, 6, 7, 9)$, $(1, 2, 3, 4, 1)$, $(3, 9, 4, 8, 3)$. Thus $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(2, 6, 1), (1, 6, 2)\}$.

Assume $q = 7$. K_9 can be decomposed into one copy of P_4, 7 copies of S_4, and one copy of C_4 as follows: $(2, 8, 3, 9, 4)$, $(1; 3, 7, 8, 9)$, $(2; 5, 6, 7, 9)$, $(4; 2, 5, 6, 7)$, $(5; 1, 3, 8, 9)$, $(6; 1, 3, 5, 9)$, $(7; 3, 5, 6, 9)$, $(8; 4, 6, 7, 9)$, $(1, 2, 3, 4, 1)$. Thus $\mathcal{CD}(K_9; P_4, S_4, C_4) \supset \{(1, 7, 1)\}$. ■

Now, we prove the main result of this section.

Theorem 23. *Let p, q, and r be positive integers, and let n be a positive odd integer. $(p, q, r) \in \mathcal{CD}(K_n; P_4, S_4, C_4)$ if and only if $4(p + q + r) = \binom{n}{2}$ and $(p, q) \neq (1, 1)$.*

Proof. (*Necessity*) The assertion follows immediately from Theorem 1.

(*Sufficiency*) Observe that $4 \mid \frac{n(n-1)}{2}$ implies $8 \mid (n - 1)$. It follows that $n = 8m + 1$ for some positive integer m. The proof is by induction on m. By Lemma 22, the assertion holds for $m = 1$. Assume $m \geq 2$. When m is even, write $m = 2k$ for some integer k. It is easily seen that K_{16k+1} can be decomposed into two copies of K_{8k+1} and a copy of $K_{8k,8k}$. By the induction hypotheses, $\mathcal{CD}(K_{8k+1}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0, p+q+r = k(8k+1), (p, q) \neq (1, 1)\}$. By Theorems 14, 15, 16, 17, 18, and 20, $\mathcal{D}(K_{8k+1}; P_4, S_4, C_4) \supset \{(a, b, c) : a, b, c \geq 0$ with at least one of a, b, c is 0, $a + b + c = k(8k + 1)$, $(a, b, c) \neq (1, 0, c), (0, 1, c)$ when $c \geq 1\}$. Therefore, $\mathcal{D}(K_{8k+1}; P_4, S_4, C_4) \supset \{(a, b, c) : a, b, c \geq 0,\ a + b + c = k(8k + 1),\ (a, b, c) \neq (1, 1, c), (1, 0, c), (0, 1, c)$ when $c \geq 1\}$. By Lemma 3, $\mathcal{CD}(K_{8k+1} \cup K_{8k+1}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0,\ p + q + r = 2k(8k + 1),\ (p, q) \neq (1, 1)\}$. Besides, $K_{8k,8k}$ can be decomposed into $8k^2$ copies of $K_{2,4}$, and by Theorems 4, 5, and 6, $\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\} \subset \mathcal{D}(K_{2,4}; P_4, S_4, C_4)$. Hence $\mathcal{D}(K_{8k,8k}; P_4, S_4, C_4) \supset \{(2a, 2b, 2c) : a, b, c \geq 0,\ a + b + c = 8k^2\}$. By Lemma 2, $\mathcal{CD}(K_{8k+1} \cup K_{8k,8k} \cup K_{8k+1}; P_4, S_4, C_4) \supset \{(p, q, r) : p, q, r > 0,\ 4(p + q + r) = \binom{16k+1}{2},\ (p, q) \neq (1, 1)\}$, that is, $\mathcal{CD}(K_{8m+1}; P_4, S_4, C_4) \supset \big\{(p, q, r) : p, q, r > 0,\ 4(p + q + r) = \binom{8m+1}{2},\ (p, q) \neq (1, 1)\big\}$.

When m is odd, write $m = 2k + 1$ for some integer k. It is easily seen that K_{16k+9} can be decomposed into one copy of K_{8k+1}, one copy of $K_{8k,8(k+1)}$, and one copy of K_{8k+9}. Besides, $K_{8k,8(k+1)}$ can be decomposed into $8k(k + 1)$ copies of $K_{2,4}$. The case where $m = 2k + 1$ is similar to the case $m = 2k$, therefore we omit its proof. ■

Remark. As mentioned on page 3, $\mathcal{D}(K_n; P_4, S_4, C_4)$ denote the set of all triples (a, b, c) of non-negative integers such that a decomposition of K_n into a copies of P_4, b copies of S_4, and c copies of C_4 exists. In fact, when n is odd, all triples in $\mathcal{D}(K_n; P_4, S_4, C_4)$ can be determined by combining Theorems 14, 15, 16, 17, 18, 20 and 23.

For the set $\mathcal{D}(K_{m,n}; P_4, S_4, C_4)$, we can also determine all triples in $\mathcal{D}(K_{m,n}; P_4, S_4, C_4)$ when both m and n are even. Let p, q, and r be positive integers, and let m and n be positive even integers with $m \leq n$. Jeevadoss and Muthusamy [15] showed that $\mathcal{CD}(K_{m,n}; P_4, C_4) = \{(p, r) : m \geq 2 \text{ and } n \geq 4;\ 4(p + r) = mn \text{ and } p \neq 1\}$. Besides, we proved that $\mathcal{CD}(K_{m,n}; S_4, C_4) = \{(q, r) : m \geq 2 \text{ and } n \geq 4;\ 4(q + r) = mn \text{ and } q \neq 1;\ q \text{ is even when } m = 2;\ r \neq 1 \text{ when } m = 4\}$ and $\mathcal{CD}(K_{m,n}; P_4, S_4) = \{(p, q) : m \geq 2 \text{ and } n \geq 4;\ 4(p + q) = mn;\ q \text{ is even when } m = 2;\ p \neq 1 \text{ when } m = 4\}$. Because the proofs are rather lengthy and the arguments are similar to the proofs of Lemmas 8, 9, 10, 11, and 12, we omit the proofs here. Thus all triples in $\mathcal{D}(K_{m,n}; P_4, S_4, C_4)$ can be determined by combining Theorems 4, 5, 6, and 13, $\mathcal{CD}(K_{m,n}; P_4, S_4)$, $\mathcal{CD}(K_{m,n}; P_4, C_4)$, and $\mathcal{CD}(K_{m,n}; S_4, C_4)$.

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