*Discussiones Mathematicae*
*Graph Theory 39 (2019) 705–730*
*doi:10.7151/dmgt.2196*

# ON RADIO CONNECTION NUMBER OF GRAPHS

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**Abstract**

Given a graph $G$ and a vertex coloring $c$, $G$ is called $l$-radio connected if between any two distinct vertices $u$ and $v$ there is a path such that coloring $c$ restricted to that path is an $l$-radio coloring. The smallest number of colors needed to make $G$ $l$-radio connected is called the $l$-radio connection number of $G$. In this paper we introduce these notions and initiate the study of connectivity through radio colored paths, providing results on the 2-radio connection number, also called $L(2,1)$-connection number: lower and upper bounds, existence problems, exact values for known classes of graphs and graph operations.

**Keywords:** radio connection number, radio coloring, $L(2,1)$-connection number, $L(2,1)$-connectivity, $L(2,1)$-labeling.

**2010 Mathematics Subject Classification:** 05C15, 05C40, 05C38.

## 1. Introduction

Various types of graph colorings were introduced in the literature motivated by problems in communication networks. An important property in communication networks is connectivity, that is to have paths for communication between each pair of vertices. Many times it is not sufficient to have arbitrary paths, but paths that assure a safe communication. For example, if interference may occur in communication, it is necessary to have paths along which interferences are avoided. Also, in security problems, each link may have an associated password or firewall and a path is considered secured if the passwords along it satisfy some requests. This might mean, for example, that the labels associated to the edges or vertices of the path should be pairwise distinct. Motivated by this types of problems, rainbow colorings were introduced by Chartrand *et al.* in [4].

We remind that, given a nontrivial connected graph $G$ and $c$ an edge-coloring of $G$, a path $P$ in $G$ is a *rainbow path* if no two edges of $P$ are colored with the same color. The graph $G$ is *rainbow connected* (with respect to $c$) if for every pair $u, v$ of distinct vertices there exists in $G$ a rainbow path from $u$ to $v$. The *rainbow connection number* of $G$ is the minimum number of colors needed to make $G$ rainbow connected. Related to the rainbow connectivity, different types of constrains were imposed to the colors of the edges in a path. For example, if only adjacent edges in a path are required to have distinct colors, such a path is called *proper path*. The notion of *proper connectivity*, similar to rainbow connectivity but considering proper paths instead of rainbow paths, was introduced by Borozan *et al.* in [2] and Andrews *et al.* in [1]. Also, a more general notion was considered in [12] $(k, l)$-proper connection number. In this case, for a fixed distance $l$, it is constrained that no two edges of the same color can appear at distance less than $l$ edges on the path. Similar problems were studied for vertex-colorings (rainbow vertex-connectivity [11], proper vertex-connectivity [10]).

Yet, there are situations when, in order to have no interference, it is necessary that the difference between labels of close edges or vertices — close meaning at distance less than a fixed level $l$, to be greater than a certain limit. To model this type of requests radio colorings were introduced. First, Hale [8] considered only two levels of interference and defined a $L(2, 1)$-labeling (also called $\lambda$-labeling) of a graph. This type of labeling was later generalized to more levels of interferences $L(d_1, d_2, \ldots, d_l)$-labelings, among which the most known are radio colorings, where $d_i = l+1-i$. For a fixed level $l$, an *$l$-radio coloring* is a function $c : V(G) \to \mathbb{N}^*$ assigning positive integers (colors, labels) to vertices with the following property (called *radio condition*): $|c(u) - c(v)| \geq l + 1 - d(u, v)$, for all $u, v \in V(G)$, $u \neq v$.

The *value* of a coloring $c$, denoted by $val(c)$, is defined as the maximum value of $c$, that is the maximum label assigned by $c$ to a vertex; the *span* of $c$, denoted $span(c)$ is the difference between largest and smallest label assigned to vertices by $c$. The minimum value of a $l$-radio coloring of a graph $G$ is the *$l$-radio number* of $G$.

If $l = 1$, then a 1-radio coloring is a classic proper coloring and we have $rc_1(G) = \chi(G)$. A 2-radio coloring is an $L(2, 1)$-labeling where all colors are positive integers; there is a difference in literature between the definition of a 2-radio coloring and an $L(2, 1)$-labeling, namely in an $L(2, 1)$-labeling zero can also be used as a color. Thus, a 2-radio coloring of $G$ is an $L(2, 1)$-labeling of $G$ that uses only positive labels. This is the reason why, for a graph $G$, we have the relation:

$$rc_2(G) = 1 + \lambda(G) \text{ [5]},$$

where $\lambda(G)$ is the $L(2, 1)$-number or $\lambda$-number of $G$, usually defined as the minimum span of an $L(2, 1)$-labeling of $G$. Note that if a coloring $c$ uses color 0, then $val(c) = span(c)$, hence $\lambda(G)$ is the minimum value of an $L(2,1)$-labeling of graph $G$.

Note also that if an $L(2,1)$-labeling of $G$ has value $\lambda(G)$, its actually uses $\lambda(G)+1$ colors, hence the minimum number of colors nedeed in an $L(2,1)$-labeling of $G$ is $\lambda(G)+1$, which is less natural than assuming that colors are positive integers. When radio colorings were introduced as an extension of $L(2,1)$-labelings, only positive colors were considered. A survey on these types of colorings can be found in book [5].

Finding the $l$-radio chromatic number proved to be difficult even for simple graphs like paths and cycles [13]. But, in order to solve interference or security problems sometimes it is not necessary to color all vertices of the graph such that every pair of vertices satisfy the radio condition, but to assure that between every pair of vertices there is at least one path such that the coloring restricted to that path is a radio coloring, as in the case of proper connectivity. Motivated by this, the aim of this paper is to introduce the notion of $l$-radio connectivity for a vertex-colored graph and present results for the case when $l = 2$ regarding upper and lower bounds, exact values for some classes of graphs and graph operations, existence problems.

Let $G$ be a connected graph and $c : V(G) \to \mathbb{N}^*$ a coloring of $G$ (using positive integers). Consider $l$ a number representing the number of levels of interference. A path $P$ in $G$ is called *$l$-radio path* if coloring $c$ restricted to $V(P)$ is an $l$-radio coloring for $P$. The coloring $c$ is called *$l$-radio path coloring* if there exists an $l$-radio path between every pair of distinct vertices of $G$. A graph is *$l$-radio connected* if it admits an $l$-radio path coloring. The minimum value of a $l$-radio path coloring of $G$ is called the *$l$-radio connection number* of $G$ and is denoted $rcc^l(G)$. An $l$-radio path coloring with value equal to $rcc^l(G)$ is called an *optimal $l$-radio path coloring.* For $l = 2$, since a 2-radio coloring is similar to an $L(2,1)$-labeling or $\lambda$-labeling (except using color 0), we will use the notions of *$L(2,1)$-path coloring*, *$L(2,1)$-paths*, *$L(2,1)$-connected graph.* Denote $\lambda c(G) = rcc^2(G)$ and refer to it as *$L(2,1)$-connection number* or *$\lambda$-connection number* of $G$.

More generally, if between every pair of vertices there exist $k$ internally vertex-disjoint $L(2,1)$-paths, $G$ is called *$k$-$L(2,1)$-connected.* The minimum number of colors needed to label the vertices of $G$ to make it $k$-$L(2,1)$-connected is the *$k$-$L(2,1)$-connection number* of graph $G$ and is denoted by $\lambda c_k(G)$. We have $\lambda c_1(G) = \lambda c(G)$.

References for exact values of the $L(2,1)$-number for known classes of graphs can be found in [3].

For basic notions and notations we refer to [14]. Denote by $[n] = \{1, 2, \dots, n\}$.

Let $G$ be graph and $c$ a vertex coloring of $G$.

For a set of vertices $S \subseteq V(G)$ define $c(S) = \{c(s) \mid s \in S\}$. We will use notation $G[S]$ for the subgraph induced by $S$ in $G$.

Denote by $b(G)$ the maximum number of bridges in $G$ incident in the same vertex. If $P$ is a path in $G$ and $u, v$ are vertices of $P$, the subpath of $P$ from $u$ to $v$ will be denoted $u \overset{P}{-} v$.

Next we will prove results on $L(2,1)$-connection number of a graph.

We will mainly consider 2-(edge) connected graphs, since robust networks present interest as models for communication networks. Also, the upper bounds that will be determined for the $L(2,1)$-connection number of a 2-connected graph would be used to provide upper bounds for general connected graphs.

Next, for an integer $a$, we will denote by $a \pm 1$ the sequence with elements $a-1, a+1$.

## 2. Basic Properties

By definition, it is not difficult to see that a 2-radio coloring of a graph $G$ is also $L(2,1)$-path coloring of $G$. Indeed, for a given 2-radio coloring of $G$, any path in $G$ is actually an $L(2,1)$-path, since two vertices at distance 2 in a path of $G$ are at distance at most 2 in $G$, hence they have distinct labels. Thus the next result follows.

**Proposition 1.** *If $G$ is a connected graph, then $\lambda c(G) \leq rc_2(G) = 1 + \lambda(G)$.*

We remind the following results on the $L(2,1)$-number of a tree.

**Proposition 2** [7]**.** *For a tree $T$, we have $\lambda(T) \in \{\Delta(T)+1, \Delta(T)+2\}$.*

**Remark 3.** A linear algorithm for deciding the exact value is given in [9].

**Proposition 4.** *If $G$ is a tree, then $\lambda c(G) = rc_2(G) = 1 + \lambda(G)$.*

***Proof.*** By Proposition 1, we have $\lambda c(G) \leq rc_2(G)$. Let $c$ be an optimal $L(2,1)$-path coloring of $G$ and $u, v$ two distinct vertices of $G$. There is a unique path $P$ in $G$ between $u$ and $v$, and this path is an $L(2,1)$-path. If $d(u,v) = 2$, then $P$ has length 2 and since it is an $L(2,1)$-path we have $|c(u) - c(v)| \geq 1$. If $u$ and $v$ are adjacent, we have $P = [u, v]$ and then $|c(u) - c(v)| \geq 2$. It follows that $c$ is also a 2-radio coloring of $G$, hence the reverse inequality holds. ■

**Corollary 5.** *Let $n \geq 1$. Then*

$$\lambda c(P_n) = \begin{cases} 1, & \text{if } n = 1, \\ 3, & \text{if } n = 2, \\ 4, & \text{if } n = 3, 4, \\ 5, & \text{if } n \geq 5. \end{cases}$$

***Proof.*** By Proposition 4 we have $\lambda c(P_n) = 1 + \lambda(P_n)$ and by Proposition 3.1 from [7] the result follows. ∎

**Corollary 6.** *Let $n \geq 2$. Then for the star graph $S_n$ with $n$ terminal vertices we have $\lambda c(S_n) = n + 2$.*

***Proof.*** By Proposition 4 we have $\lambda c(S_n) = 1+\lambda(S_n)$. By Proposition 2, $\lambda(S_n) \geq \Delta(S_n)+1 = n+1$, hence $\lambda c(S_n) \geq n+2$. It is easy to see that the labeling which assigns colors $1, \ldots, n$ to the terminal vertices of $S_n$ and color $n+2$ to the center is an $L(2,1)$-path coloring (actually a 2-radio coloring), hence $\lambda c(S_n) = n+2$. ∎

**Proposition 7.** *Let $G$ be a connected graph with $n \geq 2$ vertices.*

1. $\lambda c(G) \geq \lambda c\left(P_{diam(G)+1}\right) \geq 3$.
2. *If $H$ is a spanning connected graph of $G$, then $\lambda c(G) \leq \lambda c(H)$.*
3. $\lambda c(G) \leq \Delta^* + 3$ *where* $\Delta^* = \min\{\Delta(T) \,|\, T$ *is a spanning tree of* $G\}$.
4. $\lambda c(G) \geq b(G) + 2$.

***Proof.*** 1. Let $c$ be an optimal $L(2,1)$-path coloring of $G$ and $u, v$ be two vertices such that the distance between $u$ and $v$ equals $diam(G)$. Let $P$ be an $L(2,1)$-path between $u$ and $v$. Then $P$ has at least $diam(G)+1$ vertices and contains a subpath isomorphic to $P_{diam(G)+1}$. Using Corollary 5 we have

$$\lambda c(G) \geq \lambda c(P) \geq \lambda c\left(P_{diam(G)+1}\right) \geq \lambda c(P_2) = 3.$$

2. Let $H$ be a spanning connected graph of $G$. By the definitions of a spanning connected graph and of an $L(2,1)$-path coloring it follows that any $L(2,1)$-path coloring of $H$ is also an $L(2,1)$-path coloring of $G$, hence we have $\lambda c(G) \leq \lambda c(H)$.

3. Since a spanning tree of $G$ is a spanning connected graph, by the previous item we have

$$\lambda c(G) \leq \min\left\{\lambda c(T) \,|\, T \text{ is a spanning tree of } G\right\}.$$

But, for a tree $T$, from Propositions 4 and 2 we obtain

$$\lambda c(T) = rc_2(T) = 1 + \lambda(T) \leq \Delta(T) + 3,$$

hence the stated inequality follows.

4. If $vv_1$ and $vv_2$ are adjacent bridges, then the only path between $v_1$ and $v_2$ is $[v_1, v, v_2]$. Hence $b$ bridges incident in the same vertex induce a subgraph $H$ in $G$ isomorphic to $S_b$ such that any $L(2,1)$-path coloring for $G$ induces an $L(2,1)$-path coloring for $H$. Thus we have $\lambda c(G) \geq \lambda c(S_b) = b + 2$ (by Corollary 6). ∎

**Remark 8.** Let $G$ be a graph and $c : V(G) \to \mathbb{N}^*$ be an $L(2,1)$-path coloring of $G$ with $k = val(c)$. The complementary coloring of $c$, denoted $c'$, is defined as $c'(v) = k + 1 - c(v)$, for all $v \in V(G)$. We have that $c'$ is also an $L(2,1)$-path coloring of $G$ with value $k$.

**Proposition 9.** *Let $G$ be a connected graph with $n \geq 2$ vertices.*

1. $\lambda c(G) = 3$ *if and only if* $G = P_2$.
2. $\lambda c(G) = 4$ *if and only if* $3 \leq n \leq 4$ *and* $G \neq S_3$.
3. $\lambda c(G) \geq 5$ *if and only if* $n \geq 5$ *or* $G = S_3$.

***Proof.*** Note first that if in $G$ there are two vertices with the same color, then any $L(2,1)$-path between them has at least 4 vertices. Moreover, it is easy to verify that there is no $L(2,1)$-path coloring for $P_4$ with 4 colors such that the extremities have the same color. Indeed, if the color of extremities is $a$, then both internal vertices must have colors in $[4] - \{a, a \pm 1\}$. It suffices to consider $a = 1$ or 2, since the complementary of an $L(2,1)$-path coloring is also an $L(2,1)$-path coloring. If $a = 1$ then the internal vertices must have colors $3, 4$, if $a = 2$ then both internal vertices must have color 4. In all cases the obtained coloring is not an $L(2,1)$-path coloring for $P_4$.

Since $\lambda c(P_5) = 5$ (by Corollary 5), then, in order to have $\lambda c(G) \leq 4$, there must exists an injective $L(2,1)$-path coloring of $G$ using colors $\{1, 2, \ldots, \lambda c(G)\}$, hence we must have $n \leq 4$.

Since $\lambda c(G) \geq \lambda c(P_{diam(G)+1})$ (Proposition 7), we can have $\lambda c(G) = 3$ if and only if $G = P_2$.

Otherwise, if $3 \leq n \leq 4$ and $G \neq S_3$, then $G$ has a Hamiltonian path. Since a Hamiltonian path is a spanning connected graph, by Proposition 7 point 2 we obtain $\lambda c(G) \leq \lambda c(P_n) \leq \lambda c(P_4) = 4$, hence the result follows.

In all other cases we have $\lambda c(G) \geq 5$. ■

## 3. $L(2,1)$-Connection Number of Some Classes of Graphs

Based on the idea from the proof of the last point of Proposition 9, the next result on graphs with Hamiltonian paths follows.

**Proposition 10.** *If $G$ is a graph with $n \geq 5$ vertices having a Hamiltonian path, then $\lambda c(G) = 5$.*

***Proof.*** By Proposition 9 we have $\lambda c(G) \geq 5$. Let $P$ be a Hamiltonian path in $G$. Then $P$ is a spanning connected graph of $G$ and, by Proposition 7 point 2, we have $\lambda c(G) \leq \lambda c(P) = 5$. ■

**Corollary 11.** *Let* $n \geq 3$. *Then*

$$\lambda c(C_n) = \begin{cases} 4, & \text{if } n = 3, 4, \\ 5, & \text{if } n \geq 5. \end{cases}$$

***Proof.*** The result follows from Propositions 9 and 10. ∎

**Corollary 12.** *Let* $n \geq 5$. *Then* $\lambda c(K_n) = 5$.

**Proposition 13.** *Let* $1 \leq m \leq n$. *Then*

$$\lambda c(K_{m,n}) = \begin{cases} n+2, & \text{if } m = 1, \\ 4, & \text{if } n = m = 2, \\ 5, & \text{otherwise.} \end{cases}$$

***Proof.*** For $m = 1$ the graph is $S_n$. For $n + m \leq 4$ the result follows from Proposition 9. Assume $2 \leq m \leq n$ with $m + n \geq 5$. Denote $V(K_{m,n}) = \{x_1, \ldots, x_m\} \,\dot\cup\, \{y_1, \ldots, y_n\}$. Consider the following coloring $c$.

- $c(x_1) = 1$, $c(x_i) = 2$, for $2 \leq i \leq m$;
- $c(y_1) = 4$, $c(y_i) = 5$, for $2 \leq i \leq n$.

We will prove that $c$ is an $L(2,1)$-path coloring by considering all types of pairs of vertices.

- $x_1$, $x_i$ with $2 \leq i \leq m$ — consider path $[x_1, y_1, x_i]$;
- $x_i$, $x_j$ with $2 \leq i < j \leq m$ — path $[x_i, y_1, x_1, y_2, x_j]$;
- $y_1$, $y_i$ with $2 \leq i \leq m$ — path $[y_1, x_1, y_i]$;
- $y_i$, $y_j$ with $2 \leq i < j \leq n$ — path $[y_i, x_1, y_1, x_2, y_j]$;
- $x_i$, $y_j$ with $1 \leq i \leq m$, $1 \leq j \leq n$ — path $[x_i, y_j]$. ∎

**Theorem 14.** *Let* $G$ *be a* 2*-edge connected split graph with at least* 5 *vertices. Then* $\lambda c(G) = 5$.

***Proof.*** Denote $G = (V, E)$. Since $G$ is a 2-edge connected split graph, $V$ can be partition into two subsets $C$ and $S$ such that $G[C]$ is a clique with at least three vertices and $S$ is an independent set.

By Proposition 9, we have $\lambda c(G) \geq 5$. In order to prove that equality holds it suffices to provide an $L(2,1)$-path coloring of $G$ with 5 colors. Define such a coloring $c$ as follows.

**Step 1.** Color vertices from $C$ using only colors $1, 3, 5$ such that each color is used at least once, but only one vertex has color 3.

**Step 2.** For every vertex $s \in S$ choose $f_s$ and $f'_s$ two neighbors of $s$ in $C$ and denote $F(s) = \{f_s, f'_s\}$. We say $s$ is of type 1 if $c(f_s) \neq c(f'_s)$, of type 2 if

$c(f_s) = c(f'_s) = 1$, and of type 3 if $c(f_s) = c(f'_s) = 5$ (note that since only one vertex from $C$ has color 3, we cannot have $c(f_s) = c(f'_s) = 3$). Color $s$ as follows:

$$c(s) = \begin{cases} \text{the unique element from set } \{1,3,5\} - c(F(s)), & \text{if } s \text{ is of type 1,} \\ 4, & \text{if } s \text{ is of type 2,} \\ 2, & \text{if } s \text{ is of type 3.} \end{cases}$$

Note that we have $|c(s) - c(f)| \geq 2$, for all $f \in F(s)$. Also, if $s$ is of type 1, then $\{c(s)\} \cup c(F(s)) = \{1,3,5\}$.

In order to prove that $c$ is an $L(2,1)$-path coloring it suffices to provide an $L(2,1)$-path $P$ between each pair $(x,y)$ of distinct vertices of $G$. For that we consider the following cases.

*Case* 1. $x, y \in C$. If $c(x) \neq c(y)$ then let $P = [x,y]$. Otherwise, consider $a$ and $b$ the two colors from $\{1,3,5\} - \{c(x)\}$ and $u, v$ two vertices from $C$ of color $a$, respectively $b$. Let $P = [x,u,v,y]$.

*Case* 2. $x \in S$, $y \in C$.

*Case* 2.1. $x$ is of type 2 or 3. Choose $f_x \in F(x)$. By Case 1 there exists an $L(2,1)$-path $Q$ from $f_x$ to $y$. Let $P = [x,Q]$ be the path obtained by adding $x$ at the beginning of $Q$. Since $c(x) \notin \{1,3,5\}$, $P$ is an $L(2,1)$-path.

*Case* 2.2. $x$ is of type 1. Since $\{c(x)\} \cup c(F(x)) = \{1,3,5\}$, there exists $f_x \in F(x)$ with $c(f_x) \neq c(y)$. If $c(x) \neq c(y)$ consider $P = [x, f_x, y]$. Otherwise choose $z$ in $C$ of the unique color from $\{1,3,5\} - \{c(f_x), c(y)\}$ and let $P = [x, f_x, z, y]$.

*Case* 3. $x, y \in S$.

*Case* 3.1. Both $x$ and $y$ are of type 2 or 3. Choose $f_x \in F(x)$ and $f_y \in F(y)$ with $f_x \neq f_y$. Let $Q$ be an $L(2,1)$-path in $G[C]$ from $f_x$ to $f_y$ (exists from Case 1). Let $P = [x,Q,y]$. Since $c(x), c(y) \notin \{1,3,5\}$ and $c(P) \subseteq \{1,3,5\}$, $P$ is an $L(2,1)$-path.

*Case* 3.2. $x$ is of type 2 or 3 and $y$ is of type 1. Let $f_x \in F(x)$. By Case 2 there is $Q$ an $L(2,1)$-path from $f_x$ to $y$, having only vertices of colors $\{1,3,5\}$. Consider $P = [x,Q]$.

*Case* 3.3. Both $x$ and $y$ are of type 1. We then have

$$\{c(x)\} \cup c(F(x)) = \{c(y)\} \cup c(F(y)) = \{1,3,5\}.$$

If $c(x) = c(y)$, choose $f_x \in F(x)$ and $f_y \in F(y)$ such that $c(f_x) \neq c(f_y)$. Then $|c(f_x) - c(f_y)| \geq 2$ and $P = [x, f_x, f_y, y]$ is an $L(2,1)$-path.

If $c(x) \neq c(y)$, let $f_x \in F(x)$ of color $c(y)$ and $f_y \in F(y)$ of color $c(x)$. Let $z$ be a vertex from $Q$ of color $\{1,3,5\} - \{c(x), c(y)\}$. Consider $P = [x, f_x, z, f_y, y]$.

In all cases it can be easily verified that path $P$ is an $L(2,1)$-path. Hence $c$ is an $L(2,1)$-path coloring. ∎

As a consequence of the previous theorem, $\lambda c(G)$ can be determined for a graph $G$ that has a 2-dominating clique. We remind that, for a graph $G$, a set of vertices $D$ is called a 2-*dominating set* if each vertex from $V(G) - D$ has at least two neighbors in $D$. A clique $Q$ in $G$ is a 2-*dominating clique* if $V(Q)$ is a 2-dominating set.

**Corollary 15.** *Let $G$ be a graph with at least 5 vertices. If $G$ has a 2-dominating clique, then $\lambda c(G) = 5$.*

***Proof.*** Let $Q$ be a 2-dominating clique in $G$ of maximum size and $S = V(G) - V(Q)$. By definition $|V(Q)| \geq 2$. Assume that $Q$ has only 2 elements, denoted $u$ and $v$. Then there is a vertex $s \in S$ and this vertex is adjacent to $u$ and $v$. It follows that $\{u, v, s\}$ is a 2-dominating clique, hence $Q$ is not maximum. It follows that $|V(Q)| \geq 3$.

Let $H$ be the spanning graph of $G$ obtained from $G$ by removing the edges having both ends in $S$. Then $H$ is a split graph. Moreover, it can be easily proved that $H$ has no bridge, hence is 2-edge connected. Indeed, let $e$ be an egde of $H$. If $e$ has both ends in $Q$, then, since $Q$ is a clique with at least three vertices, $e$ is contained in at least one triangle in $Q$, hence is not a bridge in $H$ [14]. Otherwise, denote the ends of $e$ with $u$ and $s$ such that $s \in S$ and $u \in Q$. Since $Q$ is 2-dominating, there exists $v \in Q$ adjacent to $s$, with $v \neq u$. Then $e$ is again contained in triangle induced by $s$, $u$ and $v$, hence it is not a bridge.

By Theorem 14, we have $\lambda c(H) = 5$. But, since $H$ is a spanning connected graph of $G$, by Proposition 7 point 2, $\lambda c(G) \leq \lambda c(H) = 5$. By Proposition 9 the reverse inequality also holds, hence $\lambda c(G) = 5$. ■

## 4. GRAPH OPERATIONS

Next we study the $L(2,1)$-connectivity for graphs obtained by some classical graph operations — Cartesian product and join of graphs.

We remind that for two graphs $G$ and $H$ the *Cartesian product* $G\Box H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G\Box H)$ defined as

$$\big\{\big((u,v),(u',v')\big) \mid \big(u = u' \text{ and } vv' \in E(H)\big) \text{ or } \big(uu' \in E(G) \text{ and } v = v'\big)\big\}.$$

For a vertex $v \in V(H)$ denote by $G_v$ the graphs induced in $G\Box H$ by vertices from $V(G) \times \{v\}$. Then $G_v$ is isomorphic to $G$.

**Theorem 16.** *Let $G$ and $H$ be two connected nontrivial graphs. Then*

$$\lambda c(G\Box H) = \begin{cases} 4, & \text{if } |V(G)| = |V(H)| = 2, \\ 5, & \text{otherwise.} \end{cases}$$

***Proof.*** If $|V(G)| = |V(H)| = 2$ then $G$ and $H$ are isomorphic to $P_2$, hence $\lambda c(G\square H) = \lambda c(C_4) = 4$. Otherwise, let $T$ be a spanning tree of $G$. We fix a root for $T$ and denote by $level(u)$ the level of a vertex $u$ of $G$ in $T$. Similar, consider $T'$ a spanning tree of $H$, fix a root for $T'$ and denote by $level'(v)$ the level of a vertex $v$ of $H$ in $T'$. The level of the root is 0.

Define a color $c$ for the vertices $(u, v)$ of $G\square H$ according to the levels of $u$ and $v$ in $T$ and $T'$, respectively as follows.

$$c((u,v)) = \begin{cases} 1, & \text{if } level'(v) \text{ and } level(u) \text{ are even,} \\ 5, & \text{if } level'(v) \text{ is even and } level(u) \text{ is odd,} \\ 2, & \text{if } level'(v) \text{ and } level(u) \text{ are odd,} \\ 4, & \text{if } level'(v) \text{ is odd and } level(u) \text{ is even.} \end{cases}$$

Informal, we color the spanning tree of each copy $G_v$ alternately with colors 1 and 5 starting from the root if $level'(v)$ is even and with colors 4 and 2 if $level'(v)$ is odd.

We prove that $c$ is an $L(2,1)$-path coloring. Let $(u, v)$ and $(u', v')$ be two vertices of $G\square H$. Let $P = [u = u_1, u_2, \ldots, u_p = u']$ be a path in $T$ from $u$ to $u'$.

We can have the following cases.

*Case* 1. $v = v'$. In this case $p \geq 2$ and the vertices are in the same copy $G_v$. Let $w$ be a vertex adjacent to $v$ in $T'$.

If $p$ is even, consider the path

$$Q_1 = \big[(u_1, v), (u_2, v), (u_2, w), (u_3, w), (u_3, v), \ldots, (u_{p-2}, v), (u_{p-2}, w), (u_{p-1}, w), (u_{p-1}, v), (u_p, v)\big];$$

otherwise consider

$$Q_2 = \big[(u_1, v), (u_2, v), (u_2, w), (u_3, w), (u_3, v), \ldots, (u_{p-1}, v), (u_{p-1}, w), (u_p, w), (u_p, v))\big].$$

*Case* 2. $v$ and $v'$ are adjacent. If $p = 1$ $(u = u')$ consider the path with an edge $[(u, v), (u, v')]$. Otherwise, from the previous case there exists an $L(2,1)$-path from $(u_1, v)$ to $(u_p, v)$ containing only vertices from $G_v$ and $G_{v'}$. If $p$ is even consider the path obtained by $Q_1$ by adding vertex $(u_p, v')$, otherwise consider the subpath from $Q_2$ between $(u_1, v)$ to $(u_p, v')$.

*Case* 3. There exists a path $P = [v = v_1, v_2, \ldots, v_q = v']$ with $q \geq 3$ in $T'$ from $v$ to $v'$. If $p = 1$ $(u = u')$ we proceed as in the first case, but changing $T'$ with $T$. Otherwise, there is an $L(2,1)$-path from $(u_1, v_1)$ to $(u_1, v_{q-1})$ containing only internal vertices that are not in $G_{v_{q-1}}$ or $G_{v_q}$. Indeed, if $q - 1$ is even we can consider the path

$$R_1 = \big[(u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3), (u_1, v_3), (u_1, v_4), \ldots, (u_2, v_{q-2}), (u_1, v_{q-2}), (u_1, v_{q-1})\big];$$

otherwise the path

$$R_2 = \big[(u_1,v_1),(u_2,v_1),(u_2,v_2),(u_1,v_2),(u_1,v_3),(u_2,v_3),\dots,(u_2,v_{q-2}),\\ (u_1,v_{q-2}),(u_1,v_{q-1})\big].$$

Then, as in Case 2, there exists an $L(2,1)$-path from $(u_1,v_{q-1})$ to $(u_p,v_q)$ containing only vertices from $G_{v_{q-1}}$ and $G_{v_q}$, and starting with vertices $(u_1,v_{q-1}),(u_2,v_{q-1})$. By joining these two paths we obtain an $L(2,1)$-path from $(u_1,v_1)$ to $(u_p,v_q)$. ■

We remind that it is difficult to determine the $L(2,1)$-number for Cartesian product of two graphs, even if they are simple graphs like paths and cycles [3].

For two graphs $G$ and $H$ the *join* $G \vee H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G) \text{ and } v \in V(H)\}$.

**Theorem 17.** *Let $G$ and $H$ be two connected nontrivial graphs. Then*

$$\lambda c(G \vee H) = \begin{cases} 4, & \textit{if } |V(G)| = |V(H)| = 2,\\ 5, & \textit{otherwise.}\end{cases}$$

***Proof.*** If $|V(G)| = |V(H)| = 2$ then $G$ and $H$ are isomorphic to $P_2$, hence $\lambda c(G \Box H) = \lambda c(K_4) = 4$. Otherwise $G \vee H$ contains a spanning subgraph isomorphic to $K_{n,m}$, where $n = |V(G)|$ and $m = |V(H)|$. By Propositions 7 and 13 we have $5 \le \lambda c(G \vee H) \le \lambda c(K_{n,m}) = 5$. ■

Note that for any other graphs operation through which we obtain a graph containing a spanning graph that is a 2-connected bipartite complete graph we have similar results as for join.

## 5. Existing Problems

Most of the bridgless graphs studied in the previous section proved to have the $L(2,1)$-connection number equal to 5. One natural question is if there are bridgeless connected graphs with $L(2,1)$-connection number greater than 5. Also, note that $\lambda(K_{m,n}) = n+m+1$, hence the difference between $\lambda(K_{m,n})$ and $\lambda c(K_{m,n})$ in this case is large. We will prove a more general result on the existence of graphs with given 2-radio connection number (or given $L(2,1)$-number) and $L(2,1)$-connection number.

First, we give an example of a 2-edge-connected graph with $L(2,1)$-connection number greater than 5. Since such graphs exist, it is useful to find upper bounds for $L(2,1)$-connection number for this type of graphs.

**Lemma 18.** *Let $G$ be the graph obtained from the bipartite complete graph $K_{4,2}$ by attaching 26 triangles to each of its vertices. Then $G$ is 2-edge-connected and $\lambda c(G) = 6$.*

***Proof.*** Denote by $V_{4,2}$ the set of vertices of the subgraph of $G$ isomorphic to $K_{4,2}$. By construction $G$ has no bridges, hence is 2-edge-connected. Assume by contradiction that $G$ has an $L(2,1)$-path coloring $c$ with $val(c) = 5$. Then $c$ restricted to $V_{4,2}$ is an $L(2,1)$-path coloring with 5 colors of $K_{4,2}$. We will prove the following claim.

**Claim.** *$K_{4,2}$ does not admit an $L(2,1)$-path coloring with 5 colors that uses only colors $\{1,3,5\}$.*

***Proof.*** Denote by $X$ and $Y$ the two sets of $K_{4,2}$ bipartition, such that $|X| = 4$. Assume there is an $L(2,1)$-path coloring of $K_{4,2}$ using only colors $1,3,5$. Since $|X| = 4$, there are two vertices $x, x' \in X$ with $c(x) = c(x')$. Then an $L(2,1)$-path from $x$ to $x'$ must have length multiple of 3. But from $x$ to $x'$ are only paths of length 2 or 4, contradiction. □

From the Claim we obtain that there is $v \in V_{4,2}$ such that $c(v) \in \{2,4\}$. Without loss of generality assume $c(v) = 2$ (otherwise consider the complement coloring). There are 26 triangles attached to $v$ and the other two vertices of each triangle have colors in set $\{1,2,3,4,5\}$. Since there are only 25 distinct pair of colors from this set, it follows that two triangles incident in $v$ have the vertices of the same color. Denote the two vertices of these triangles that are distinct from $v$ with $u_1, u_2$, respectively $w_1, w_2$ such that $c(u_i) = c(w_i) = c_i$ for $i = 1,2$. Since $c$ is an $L(2,1)$-path coloring, there is an $L(2,1)$-path from $u_1$ to $w_1$, and this path must contain $v$, which is a cut vertex. Thus, one of the paths $[u_1, v, w_2, w_1]$ or $[u_1, u_2, v, w_1]$ is an $L(2,1)$-path, hence we must have $|c_1 - c_2| \geq 2$ and also $|c_i - c(v)| \geq 2$ for $i = 1,2$, which is not possible since $c(v) = 2$ and $1 \leq c_1, c_2 \leq 5$. It follows that $\lambda c(G) > 5$. Moreover, note that if $c$ uses 6 colors instead of 5 and $c(v)$ is an arbitrary color in $\{1,\ldots,6\}$, we can always choose two colors $c_1, c_2$ with $1 \leq c_1, c_2 \leq 6$ satisfying the above inequalities and color each pair of vertices that induce a triangle together with $v$ using these colors. In this way we obtain an $L(2,1)$-path coloring with 6 colors. ■

**Proposition 19.** *For any pair of integers $a, b \geq 5$ with $a + 1 < b$ there exists a graph $G$ with $\lambda c(G) = a$ and $rc_2(G) = b$.*

***Proof.*** Consider graph $G$ with $V(G) = \{x_1,\ldots,x_p\} \cup \{y_1,\ldots,y_q\} \cup \{u,v\}$ and

$$E(G) = \{x_i u \mid i = 1,\ldots,p\} \cup \{y_i u, y_i v \mid i = 1,\ldots,q\},$$

with $p = a-2 \geq 3$ and $q = b-a \geq 2$. Graph $G$ contains a subgraph isomorphic to $S_p$ with terminal vertices being terminal in $G$ also, hence $\lambda c(G) \geq \lambda c(S_p) = p+2$ (Corollary 6). This bound is achieved by the following $L(2,1)$-path coloring: $c(u) = 1$, $c(v) = 2$, $c(x_i) = i+2$, $1 \leq i \leq p$, $c(y_1) = 4$, $c(y_i) = 5$, $2 \leq i \leq q$. Thus

$\lambda c(G) = p + 2$. Since $G$ contains a subgraph isomorphic to $S_{p+q}$, by Proposition 4 and Corollary 6 we have

$$rc_2(G) \geq rc_2(S_{p+q}) = \lambda c(S_{p+q}) = p + q + 2.$$

Consider the following coloring: $c(u) = 1$, $c(v) = 3$, $c(x_i) = i + 2$, $1 \leq i \leq p$, $c(y_i) = i + p + 2$, $1 \leq i \leq q$. It is easy to verify that $c$ is an 2-radio coloring for $G$, hence $rc_2(G) = p + q + 2$. ■

## 6. Upper Bounds

Since we proved there are 2-edge connected graphs with $\lambda c$ greater than 5, it is useful to know upper bounds for $L(2,1)$-connection number of this type of graphs. In this section we determine a constant upper bound for this type of graphs and use the provided $L(2,1)$-path coloring to obtain upper bounds for the general case. The result is based on the existence of an ear decomposition, starting with a particular cycle.

We will use some notation similar to [12]. For a path $P = [v_1, \ldots, v_p]$ from $v_1$ to $v_p$ with $p \geq 2$, we will denote by $start_2(P) = v_2$ the second vertex of $P$, by $end_2(P) = v_{p-1}$ the last but one vertex of $P$, and by $P^{-1} = [v_p, v_{p-1}, \ldots, v_1]$ the reverse of path $P$, seen as a path from $v_p$ to $v_1$.

**Theorem 20** [14]**.** *If $G$ is a 2-connected graph, then $G$ has an (open) ear decomposition. Furthermore, every cycle in $G$ is the initial cycle in some ear decomposition.*

**Lemma 21.** *Let $G$ be a 2-connected graph with $n \geq 4$ vertices and $s$ a vertex of $G$. Then $s$ is contained in a cycle of length at least 4.*

***Proof.*** Assume that $s$ is not contained in a cycle of length at least 4. We remind that since $G$ is 2-connected there are at least two internal-disjoint paths between each pair of vertices, hence any two vertices are contained in a cycle. Let $u \neq s$ be a vertex of $G$. Then there is a triangle $[s, u, v, s]$ in $G$. Let now $w$ be another vertex, distinct from $s, u, v$.

There is a path $P$ from $w$ to one of the vertices $u, v$ such that this path does not contain any other vertex from cycle $[s, u, v, s]$. Indeed, there is a path from $w$ to $u$ not containing $s$. If this path contains $v$, consider the subpath from $w$ to $v$. Assume wlog $P$ is a path from $w$ to $u$ that does not contain $s$ and $v$. There is a path $Q$ from $w$ to $s$ that does not contain $u$. Let $x$ be the last common vertex for $P$ and $Q$. Note that $x \neq u, s$, therefore $[s, x, u, s]$ must be a triangle. Then $[x, s, v, u, x]$ is a cycle of length 4 containing $s$, contradiction. ■

In the next theorem we provide a method for finding an $L(2,1)$-path coloring having the value at most 10 for a 2-connected graph with at least 4 vertices with some particular properties. This particular type of coloring would be used to color blocks of an arbitrary graphs in order to obtain upper bounds for any connected graphs.

**Theorem 22.** *Let $G$ be a 2-connected graph with $n \geq 4$ vertices and $L \geq 10$ a natural number. Let $s$ be a vertex in $V(G)$ and $cs, cs_1, cs_2 \in [L]$ such that $cs_1 \neq cs_2$ and $|cs - cs_i| \geq 2$ for $i = 1, 2$. Then there exists an $L(2,1)$-path coloring $c$ of $G$ with value at most $L$ such that $c(s) = cs$ and each vertex $v$ in $G$ has an associated multiset containing two colors, namely: for $v \neq s$, $C(v) = \{cv_1, cv_2\}$ with $|c(v) - cv_i| \geq 2$, $i = 1, 2$ and $C(s) = \{cs_1, cs_2\}$, satisfying the properties.*

1. *For every pair of vertices $x \neq y \in V$ there exists an $L(2,1)$-path $P_{xy}$ from $x$ to $y$ such that $c(start_2(P_{xy})) \in C(x)$ and $c(end_2(P_{xy})) \in C(y)$.*
2. *For every vertex $x \neq s$ there exist two $L(2,1)$-paths $P_x$ and $P'_x$ from $x$ to $s$ such that $c(start_2(P_x)), c(start_2(P'_x)) \in C(x)$, $c(end_2(P_x)) = cs_1$ and $c(end_2(P'_x)) = cs_2$.*

***Proof.*** It suffices to consider $L = 10$. Let $C_p = [v_1 = s, v_2, \ldots, v_p, v_{p+1} = v_1]$ be a cycle containing $s$ with $p \geq 4$. Such a cycle exists by Lemma 21.

By Theorem 20, $G$ has an (open) ear decomposition such that the initial cycle is $C_p$. Such a decomposition is obtained starting from cycle $C_p$ and sequentially adding a path (which is not a cycle) and has both extremities in the graph obtained at previous step and no other vertices in common with this graph.

We will prove the result by induction on the number of ears added to $C_p$.

Consider first the graph $C_p$. Color $c(v_1) = c(s) = cs$, $c(v_2) = cs_1$ and $c(v_p) = cs_2$. Then we color the other vertices of $C_p$ such that we obtain a $L(2,1)$-path coloring as follows. If $p = 4$ choose $c(v_3) \in [L] - \{c(s), cs_1, cs_2, cs_1 \pm 1, cs_2 \pm 1\}$. Otherwise color in this order $v_3, \ldots, v_{p-3}$ such that $c(v_i) \in [L] - \{c(v_{i-1}), c(v_{i-2}), c(v_{i-1}) \pm 1\}$; then color $v_{p-2}$ such that

$$c(v_{p-2}) \notin [L] - \big\{c(v_{p-4}), c(v_{p-3}), c(v_{p-3}) \pm 1, c(v_p)\big\};$$

then color $v_{p-1}$ such that

$$c(v_{p-1}) \notin [L] - \big\{c(v_{p-3}), c(v_{p-2}), c(v_{p-2}) \pm 1, cs_2, c(s), cs_2 \pm 1\big\}.$$

Let $C(v_i) = \{c(v_{i-1}), c(v_{i+1})\}$ for $2 \leq i \leq p$. Since $c$ is actually a 2-radio coloring of $C_p$, properties 1 and 2 are satisfied, both paths $[v_i, \ldots, v_p, s]$ and $[v_i, v_{i-1}, \ldots, v_2, s]$ being $L(2,1)$-paths for each $i$.

Assume now that the statement is true before ear $P$ is added. Denote by $G'$ the graph before adding ear $P$ and by $G$ the obtained graph. By induction, there exists an $L(2,1)$-path coloring $c'$ of $G'$ with value at most $L$ and sets $C'(v)$

associated to each vertex $v$ of $G'$ satisfying stated properties. Denote $P = [x = v_1, v_2, \dots, v_p = y]$ the ear added to $G'$ to obtain $G$, denoted such that $y \neq S$. We will extend the coloring $c'$ from $V(G')$ to a coloring $c$ of $G$ and define set of vertices $C(v)$ for the vertices in $V(G) - V(G')$ such that properties 1 and 2 are satisfied.

Consider $c(v) = c'(v)$ and $C(v) = C'(v)$ for $v \in V(G')$.

If $p = 2$, then $V(G) = V(G')$ and obviously $c$ is the required coloring.

Let $P_{xy}$ be an $L(2,1)$-path from $x$ to $y$ in $G'$ with $c(start_2(P_{xy})) \in C'(x) = C(x)$ and $c(end_2(P_{xy})) \in C'(y) = C(y)$.

Denote $C(x) = \{cx_1, cx_2\}$ and $C(y) = \{cy_1, cy_2\}$ such that $c(start_2(P_{xy})) = cx_1$.

If $p = 3$ then choose $c(v_2) \in [L]$, but

$$c(v_2) \notin C(y) \cup \{cx_1, c(x), c(y), c(x) \pm 1, c(y) \pm 1\}.$$

Since there are at most 9 such forbidden values for $c(v_2)$, we can choose such a value. Set $C(v_2) = \{c(x), c(y)\}$.

For connecting $v_2$ with the rest of vertices from $G'$ such that property 1 is satisfied we use the following paths:

- from $v_2$ to $x$ — path $\left[v_2, y \overset{P_{xy}^{-1}}{-} x\right]$ (note that $c(start_2(P_{xy})) \in C(y)$ and $c(v_2) \notin C(y)$),
- from $v_2$ to $y$ — path $\left[v_2, x \overset{P_{xy}}{-} y\right]$ ($c(start_2(P_{xy})) = cx_1$ and $c(v_2) \neq cx_1$),
- from $v_2$ to $z \in V(G') - \{x, y\}$ — path $\left[v_2, y \overset{P_{yz}}{-} z\right]$, where $P_{yz}$ is an $L(2,1)$-path from $y$ to $z$ in $G'$ having $c(start_2(P_{yz})) \in C(y)$, $c(end_2(P_{yz})) \in C(z)$.

For connecting $v_2$ with $s$ such that property 2 is satisfied, consider two paths $P_y^1$ and $P_y^2$ in $G'$ from $y$ to $s$ verifying property 2 and extend them by adding edge $v_2 y$ at the beginning.

Thus properties 1 and 2 are satisfied for $v_2$ and remain true also for $C(z)$ with $z \in V(G')$ since we only considered path having starts and ends in $C'(z)$.

If $p \geq 4$ we color each vertex $v_2, \dots, v_{p-2}$ in this order such that

$$c(v_i) \notin [L] - \{c(v_{i-1}), c(v_{i-2}), c(v_{i-1}) \pm 1\}$$

where by convention $c(v_0) = cx_1$. In the end color $v_{p-1}$ such that

$$c(v_{p-1}) \in [L] - \big\{c(v_{p-3}), c(v_{p-2}), c(v_{p-2}) \pm 1, c(y), c(y) \pm 1, cy_1, cy_2\big\}.$$

Consider $C(v_i) = \{c(v_{i-1}), c(v_{i+1})\}$, $2 \leq i < p-1$, $C(v_{p-1}) = \{c(v_{p-2}), c(y)\}$.

We use the following paths to connect $v_2, \dots, v_{p-1}$ between them and with vertices from $V(G')$ such that property 1 is satisfied:

- from $v_i$ to $x$ — path $\left[v_i \overset{P}{-} y \overset{P_{xy}^{-1}}{-} x\right]$,
- from $v_i$ to $y$ — path $\left[v_i \overset{P}{-} x \overset{P_{xy}}{-} y\right]$,
- from $v_i$ to $z \in V(G') - \{x, y\}$ — path $\left[v_i \overset{P}{-} y \overset{P_{yz}}{-} z\right]$,
- from $v_i$ to $v_{i+k}$ — use the subpath from $v_i$ to $v_{i+k}$ of $P$.

For connecting $v_i$ with $s$ such that property 2 is satisfied, consider again two paths $P_y^1$ and $P_y^2$ in $G'$ from $y$ to $s$ satisfying property 2 and extend them by adding path $\left[v_i \overset{P}{-} y\right]$ at the beginning. ■

**Corollary 23.** *If $G$ is a 2-connected graph, then $\lambda c(G) \leq \min\{10, \Delta(G) + 3\}$.*

***Proof.*** The result follows from Theorem 22 for $cs = 1$, $cs_1 = 3$, $cs_2 = 5$ and Proposition 7. ■

**Theorem 24.** *Let $G$ be a connected graph with $n \geq 5$ vertices. Then*

$$\max\{b(G) + 2, 5\} \leq \lambda c(G) \leq \max\{10, b(G) + 5\}.$$

***Proof.*** The lower bound follows from Proposition 7.

Let $L = \max\{10, b(G) + 5\}$. We will provide an algorithm for finding an $L(2,1)$-path coloring of $G$ using at most $L$ colors. Consider $T$ the block-cut vertex tree associated to $G$ and fix as root a cut vertex $r$. Traverse $T$ starting from $r$ level by level, exploring only vertices corresponding to cut vertices. When a cut vertex $x$ is explored, we color as described bellow the vertices of the blocks that are direct descendants of $x$. The coloring is done such that, for the set $V_c$ of vertices already colored, the following property (similar to property 1 from Theorem 22) is satisfied at each step.

*For each vertex $v \in V_c$ there exists a multiset $C(v)$ of two colors from $[L]$ such that for every $w \in V'$ with $w \neq v$ there exists an $L(2,1)$-path $P_{vw}$ from $v$ to $w$ in the $G[V_c]$ such that $c(start_2(P_{vw})) \in C(v)$ and $c(end_2(P_{vw})) \in C(w)$; moreover if $v$ is unexplored we have $|C(v)| = 2$ $(*)$.*

For $r$ consider $c(r) = 1$ and $C(r) = \{3, 3\}$. Let $x$ be the cut vertex currently explored and $C(x) = \{cx_1, cx_2\}$. If $cx_2 = cx_1$ then choose $cx_3 \in [L] - \{cx_1, c(x), c(x) \pm 1\}$. Otherwise set $cx_3 = cx_2$. Modify set $C(x) = \{cx_1, cx_3\}$. Property $(*)$ is still satisfied since $cx_3$ is either $cx_2$ or a new value added to $\{cx_1, cx_2\}$.

Color the blocks that are direct descendants of $x$ in $T$ in the following order: first blocks with 4 vertices, then blocks with 3 vertices, and last blocks with 2 vertices (corresponding to bridges) as follows, such that property $(*)$ remains true.

• Let $B$ be a block with at least 4 vertices. Apply Theorem 22 for $s = x$, $cs = c(x)$, $cs_2 = cx_1$, $cs_2 = cx_3$ in order to color the vertices of $B$. Note that the associated set for $x$ in $B$ is $\{cx_1, cx_3\}$, which is actually $C(x)$. We need to prove that for every two vertices $v \in V(B) - \{x\}$ and $y \in V_c$ there is a path from $v$ to $y$ satisfying property $(*)$. Since the property is satisfied for $G[V_c]$, there exists a path $P_{xy}$ from $x$ to $y$ in $G[V_c]$ such that $c(start_2(P_{xy})) = a \in C(x) = \{cx_1, cx_3\}$ and $c(end_2(P_{xy})) \in C(y)$. But by property 2 from Theorem 22, there exists a path $P_{vx}$ in $B$ from $v$ to $x$ with $c(end_2(P_{vx})) \in C(x) - \{a\}$ and $c(start_2(P_{vx})) \in C(v)$. Consider the path $\left[v \overset{P_{vx}}{-} x \overset{P_{xy}}{-} y\right]$ which is an $L(2,1)$-path from $v$ to $y$ satisfying property $(*)$.

• Let $B$ be a block with exactly 3 vertices $x, u_1, u_4$. Choose $cx_4 \in [L]$ such that $|cx_4 - cx_1| \geq 2$ and $|cx_4 - c(x)| \geq 2$ ($cx_4$ can be equal to $cx_3$). Set $c(u_i) = cx_i$, $C(u_i) = \{c(x), cx_{5-i}\}$ for $i = 1, 4$. As in the previous case, for a vertex $y \in V_c$ there exists an $L(2,1)$-path $P_{xy}$ from $x$ to $y$ in $G[V_c]$ such that $c(end_2(P_{xy})) \in C(y)$ and $c(start_2(P_{xy})) = cx_j \in \{cx_1, cx_3\}$. Extend this path to a path to $u_i$ for $i = 1, 4$ using one of paths $[x, u_i]$ or $[x, u_{5-i}, u_i]$ having the second vertex with color different of $cx_j$. For $i = 1, 4$ we consider as $L(2,1)$-path from $x$ to $u_i$ one of the paths $[x, u_i]$ or $[x, u_{5-i}, u_i]$, namely that having the color of the second vertex $cx_1$.

• Let $B_1, \ldots, B_k$ be the blocks with 2 vertices that are direct descendants of $x$, corresponding to bridges $xv_1, \ldots, xv_k$. Note that $k \leq b(G)$. Choose for $c(v_1), \ldots, c(v_k)$ distinct colors from $[L] - \{c(x), cx_1, cx_3, c(x) \pm 1\}$ and add them to $C(x)$. For every $i = 1, \ldots, k$ set $C(v_i) = \{c(x), c(x)\}$. Since property $(*)$ is verified before this step, there exists an $L(2,1)$-path from $x$ to any other vertex $y$ already colored that has the second vertex of color $cx_1$ or $cx_3$ and the last but one vertex of color from $C(y)$; we can extend this path by adding vertex $v_i$ at the beginning and property $(*)$ is satisfied. Note that at this step we modified $C(x)$ by adding new values, but $x$ is already explored. ∎

**Corollary 25.** *If $G$ is an 2-edge connected graph, then*

$$\lambda c(G) \leq \min\{10, \Delta(G) + 3\}.$$

## 7. On $k$-$L(2,1)$-Connection Number

In some of the colorings constructed for the graphs considered in previous section, such as $K_{n,m}$, there is more than one $L(2,1)$-path between some pairs of vertices. Since assuring $k$-connectivity is important in communication networks, in this section we will study the $k$-$L(2,1)$-connectivity of the complete bipartite graph.

First, for a given $k$, we define an $L(2,1)$-path coloring that makes graph $K_{n,n}$ $k$-$L(2,1)$-path connected, which gives an upper bounds for $\lambda c_k(K_{n,n})$. Then we will prove that these upper bounds are also lower bounds.

Remark that for a connected graph $G$ and $1 < k \leq \kappa(G)$ we have $\lambda c_{k-1}(G) \leq \lambda c_k(G) \leq rc_2(G)$. Also, if $H$ is a connected spanning subgraph of $G$, then $\lambda c_k(G) \leq \lambda c_k(H)$.

For a graph $G$ and a coloring $c$ of $G$, we will use the following notations.

For two vertices $u, v$ of $G$ denote $\kappa_c(u,v)$ the maximum number of internally-disjoint $L(2,1)$-paths from $u$ and $v$.

For a vertex $u$ denote $F_c(u) = \{v \in N_G(u) \mid |c(u) - c(v)| \leq 1\}$ the set of neighbors of $u$ that cannot be adjacent to $u$ on an $L(2,1)$-path, called the *set of forbidden neighbors* of $u$. We have $|c(F_c(u))| \leq 3$.

Denote $(X,Y)$ the bipartition of $K_{n,n}$, with $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$.

In order to provide $L(2,1)$-paths in $K_{n,n}$ the following lemma will be used to associate to each vertex from $X$ a possible neighbor in an $L(2,1)$-path.

**Lemma 26.** *Let $n \geq 8$ and $c$ be a coloring of $K_{n,n}$ such that there are no two vertices of same color in $X$ or in $Y$. Let $x \in X$ and $y \in Y$ be two arbitrary vertices such that $|F_c(x)| \leq |F_c(y)|$. Let $X' = X - F_c(y) - \{x\}$ and $Y' = Y - F_c(x) - \{y\}$. Consider $G'$ the subgraph of $K_{n,n}$ having vertex set $X' \cup Y'$ and edge set $E' = \{uv \mid u \in X', v \in Y', |c(u) - c(v)| \geq 2\}$. Then $G'$ has a matching that saturates all vertices in $X'$.*

***Proof.*** First note that $E' = \{uv \mid u \in X', v \in Y', u \notin F_c(v), v \notin F_c(u)\}$, hence two vertices are adjacent in $G'$ if they can be adjacent in an $L(2,1)$-path. Also, since for a vertex $u$, $c(F_c(u)) = \{c(u), c(u) \pm 1\}$ and vertices in the same set of bipartition have distinct colors, then $|F_c(u)| \leq 3$ and $|Y'| \geq |X'| \geq n - 4$.

In order to prove the result we will use Hall's Theorem. Let $S \subseteq X$. Then $N_{G'}(S) = Y' - \left(\bigcap_{s \in S} F_c(s)\right)$. For $|S| > 3$ we have $\bigcap_{s \in S} c(F_c(s)) = \emptyset$, hence $|N_{G'}(S)| = |Y'| \geq |X'| \geq |S|$. If $|S| \leq 3$, then $\left|\bigcap_{s \in S} c(F_c(s))\right| \leq 4 - |S|$. We obtain

$$|N_{G'}(S)| \geq |Y'| + |S| - 4 \geq |X'| + |S| - 4 \geq n - 8 + |S| \geq |S|.$$

■

**Corollary 27.** *Let $n \geq 8$ and $c$ be a coloring of $K_{n,n}$ such that there are no two vertices of same color in $X$ or in $Y$. Let $x \in X$ and $y \in Y$ be two arbitrary vertices. Then the following properties hold.*

1. *There is a matching $M$ in $K_{n,n}$ of cardinality $n - \max\{|F_c(x) \cup \{y\}|, |F_c(y) \cup \{x\}|\}$ such that for every edge $uv \in M$ with $u \in X$ and $v \in Y$, path $[x, v, u, y]$ is an $L(2,1)$-path.*
2. *$\kappa_c(x,y) = n - \max\{|F_c(x)|, |F_c(y)|\}$.*

***Proof.*** Note that $\kappa_c(x,y) \le n - \max\{|F_c(x)|, |F_c(y)|\}$ since $x$ cannot be adjacent on an $L(2,1)$-path with a vertex from $F_c(x)$ and similar for $y$.

From Lemma 26 applied for bipartition $(X,Y)$ if $|F_c(x)| \ge |F_c(y)|$ or bipartition $(Y,X)$ otherwise, there is a matching in $G$ with $N = n - \max\{|F_c(x) \cup \{y\}|, |F_c(y) \cup \{x\}|\}$ elements such that for every edge $uv$ from the matching with $u \in X$, $v \in Y$ we have $u \notin F_c(v) \cup F_c(y) \cup \{x\}$ and $v \notin F_c(u) \cup F_c(y) \cup \{x\}$. Since there are no vertices with the same color in $X$ or in $Y$, path $[x,v,u,y]$ is an $L(2,1)$-path. It follows that there are $N$ internally disjoint $L(2,1)$-paths between $x$ and $y$ of length 4. If $y \in F_c(x)$ then also $x \in F_c(y)$, hence in this case $N = n - \max\{|F_c(x)|, |F_c(y)|\}$. Otherwise $N = n - 1 - \max\{|F_c(x)|, |F_c(y)|\}$, but in this case $[x,y]$ is also an $L(2,1)$-path, hence there are $n - \max\{|F_c(x)|, |F_c(y)|\}$ internally disjoint paths as stated. ■

It is easy to see that the following remark holds.

**Remark 1.** Consider $c$ an arbitrary coloring of $K_{n,n}$ and $i,j \in [n]$.

1. For a pair $(x_i, x_j)$ with $i \ne j$, a path $[x_i, y, x_j]$ with $y \in Y$ is an $L(2,1)$-path if and only if $y \notin F_c(x_i) \cup F_c(x_j)$, hence $\kappa_c(x_i, x_j) = n - |F_c(x_i) \cup F_c(x_j)|$.
2. For a pair $(y_i, y_j)$ with $i \ne j$, we have $\kappa_c(y_i, y_j) = n - |F_c(y_i) \cup F_c(y_j)|$.
3. For a pair $(x_i, y_j)$, by Corollary 27, we have $\kappa_c(x_i, y_j) = n - \max\{|F_c(x_i)|, |F_c(y_j)|\}$.

**Proposition 28.** *Let $n \ge 3$ and $k$ such that $1 \le k \le n$. We have*

$$\lambda c_k(K_{n,n}) \le \begin{cases} 5, & \text{if } k \le \left[\frac{n}{2}\right], \\ n, & \text{if } \left[\frac{n}{2}\right] < k \le n-6, \\ k+4+\left[\frac{n+1}{3}\right], & \text{if } n-5 \le k \le n-4, \\ k+1+n, & \text{if } n-3 \le k \le n. \end{cases}$$

***Proof.*** Let $k \in [n]$. We will describe a coloring of $K_{n,n}$ for various cases of $k$ and use Remark 1 in order to prove that the coloring makes graph $K_{n,n}$ $k$-$L(2,1)$-path connected.

*Case* 1. $k \le \left[\frac{n}{2}\right]$. Consider the coloring $c$ define as follows:

$$c(v) = \begin{cases} 1, & \text{if } v \in \left\{x_1, \ldots, x_{[n/2]}\right\}, \\ 2, & \text{if } v \in \left\{x_{[n/2]+1}, \ldots, x_n\right\}, \\ 4, & \text{if } v \in \left\{y_1, \ldots, y_{[n/2]}\right\}, \\ 5, & \text{if } v \in \left\{y_{[n/2]+1}, \ldots, y_n\right\}. \end{cases}$$

In order to prove that $c$ is an $[n/2]$-connected $L(2,1)$-coloring we consider each type of pairs of vertices and provide $[n/2]$ internally disjoint $L(2,1)$-paths.

- $(x_i, x_j)$, $i \neq j \leq [n/2]$: paths $\big(x_i, y_{t+[n/2]}, x_{t+[n/2]}, y_t, x_j\big)$, $t \in [n/2]$,
- $(x_i, x_j)$, $i \neq j > [n/2]$: paths $\big(x_i, y_t, x_t, y_{t+[n/2]}, x_j\big)$, $t \in [n/2]$,
- $(x_i, x_j)$, $i \leq [n/2] < j$: paths $(x_i, y_t, x_j)$, $t \in [n/2]$,
- $(x_i, y_j)$, $i, j \leq n$: any path $[x_i, y, x, y_j]$ with $x \in X - \{x_i\}$ and $y \in Y - \{y_j\}$ is an $L(2,1)$-path. There are $n-1$ such internally disjoint paths. Moreover, $[x_i, y_j]$ is also an $L(2,1)$-path.

For all types of pairs we have at least $[n/2]$ internally disjoint paths between the pairs of vertices, hence $c$ is an $[n/2]$-connected $L(2,1)$-path coloring.

*Case* 2. $\left[\frac{n}{2}\right] < k \leq n-6$. Define the coloring $c$ as follows: $c(x_i) = c(y_i) = i$, for $i \in [n]$. Then $|F_c(u)| \leq 3$ for every $u \in V$.

Since for any two vertices $u, v$ in the same set of bipartition we have $|F_c(u) \cup F_c(v)| \leq 6$, by Remark 1 it follows that $K_{n,n}$ is $(n-6)$-$L(2,1)$-path connected with respect to $c$.

*Case* 3. $k = n-5$. Consider $c$ defined as follows, for $i \in [2n]$.

- if $i \leq 3$, define $c(x_i) = c(y_i) = i$,
- if $i > 3$ and $i = 3t+1$, define $c(x_i) = c(y_i) = 4t$,
- if $i > 3$ and $i = 3t+2$, define $c(x_i) = 4t+1$, $c(y_i) = 4t+2$,
- if $i > 3$ and $i = 3t+3$, define $c(x_i) = 4t+2$, $c(y_i) = 4t+3$.

It is easy to check that coloring $c$ has $val(c) = n - 1 + \left[\frac{n+1}{3}\right]$ by considering cases $n = 3t+1, 3t+2, 3t+3$.

The sets of forbidden neighbors for $x_i \in X$ and $y_i \in Y$ are:

- if $i = 1$: $F_c(x_1) = \{y_1, y_2\}$, $F_c(y_1) = \{x_1, x_2\}$,
- if $i = 2, 3$: $F_c(x_i) = \{y_{i-1}, y_i, y_{i+1}\}$, $F_c(y_i) = \{x_{i-1}, x_i, x_{i+1}\}$,
- if $3 < i < n$: $F_c(x_i) = \{y_{i-1}, y_i\}$, $F_c(y_i) = \{x_i, x_{i+1}\}$,
- if $i = n$: $F_c(x_i) = \{y_{i-1}, y_i\}$, $F_c(y_i) = \{x_i\}$.

For every pair of vertices $(u, v)$ we have $\max\{|F_c(u)|, |F_c(v)|\} \leq 3$. Moreover, if $u$ and $v$ are in the same class of bipartition, $|F_c(u) \cup F_c(v)| \leq 5$, hence by Remark 1 there are at least $n-5$ internally disjoint $L(2,1)$-paths from $u$ to $v$.

*Case* 4. $k = n-4$. Define $c$ as follows, for $i \in [2n]$.

- if $i = 3t+1$: $c(x_i) = c(y_i) = 4t+1$,
- if $i = 3t+2$: $c(x_i) = 4t+2$, $c(y_i) = 4t+3$,
- if $i = 3t+3$: $c(x_i) = 4t+3$, $c(y_i) = 4t+4$.

As in the previous case, it can be proved that coloring $c$ has $val(c) = n + \left[\frac{n+1}{3}\right]$ and for every pair of vertices $(u, v)$ in the same class we have $|F_c(u) \cup F_c(v)| \leq 4$, hence there are at least $n-4$ internally disjoint $L(2,1)$-paths from $u$ to $v$.

*Case* 5. $n-3 \leq k \leq n$. Define the coloring $c$ as follows: $c(x_i) = i$, $c(y_i) = i+1+k$, for $i \in [n]$.

We have $|F_c(u) \cup F_c(v)| \leq n-k$ for any pair of vertices $(u,v)$ and each vertex has at most $n-k$ forbidden neighbors. ■

**Proposition 29.** *Let $n \geq 3$ and $k$ such that $1 \leq k \leq n$. We have*

$$\lambda c_k(K_{n,n}) \geq \begin{cases} 5, & \text{if } k \leq \left[\frac{n}{2}\right], \\ n, & \text{if } \left[\frac{n}{2}\right] < k \leq n-6, \\ k+4+\left[\frac{n+1}{3}\right], & \text{if } n-5 \leq k \leq n-4, \\ k+1+n, & \text{if } n-3 \leq k \leq n. \end{cases}$$

***Proof.*** Consider an arbitrary fixed value $k$, with $1 \leq k \leq n$. Let $c$ be an optimal $k$-$L(2,1)$-path coloring of $K_{n,n}$ (with $val(c) = \lambda c_k(K_{n,n})$). We have $\lambda c_k(G) \geq \lambda c_1(G) \geq 5$.

First note that if there exist two vertices $u, v$ with $c(u) = c(v)$ in the same class of the bipartition $(X, Y)$, then $k \leq \left[\frac{n}{2}\right]$. Indeed, assume $u, v \in X$. Since $c(u) = c(v)$ it follows that every $L(2,1)$-path from $u$ to $v$ has length at least 3, hence it must contain at least two vertices from $Y$. Hence $\kappa_c(u,v) \leq [|Y|/2]$.

It results that if $k > \left[\frac{n}{2}\right]$, then $\lambda c_k(K_{n,n}) \geq n$.

It only remain to consider the cases $n-5 \leq k \leq n$.

Denote by $v_1, \ldots, v_{2n}$ the vertices of $K_{n,n}$, ordered such that

$$c(v_1) \leq \cdots \leq c(v_{2n}).$$

We have

$$val(c) = 1 + \sum_{i=1}^{2n-1} (c(v_{i+1}) - c(v_i)).$$

Denote $d_0$ the number of pairs $(v_i, v_{i+1})$ with $c(v_{i+1}) - c(v_i) = 0$ and $d_2$ the number of pairs $(v_i, v_{i+1})$ with $c(v_{i+1}) - c(v_i) \geq 2$. The following relation holds:

$$val(c) \geq 2n - d_0 + d_2.$$

In order to determine a lower bound for $val(c)$ we will determine an upper bound for $d_0 - d_2$, according to the value of $k$.

For that, denote by $S$ the sequence of colors $\{c(v_1), \ldots, c(v_{2n})\}$. If a sequence $s$ is subsequence of $S$ we will write $s \in S$. For two positive integers $a, l$ denote by $s_l(a) = \{a, a, a+1, a+1, \ldots, a+l-1, a+l-1\}$, by $s_l^+(a) = s_l(a) \cup \{a+l\}$ and by $s_l^-(a) = \{a-1\} \cup s_l(a)$. If $a$ is not fixed, will be omitted from the notation. A subsequence $s_l(a)$ of $S$ will be called a sequence of type $s_l$. If $S$ contains such a subsequence we will simply write that $s_l \in S$. A subsequence $s_l^+(a)$ or $s_l^-(a)$ will be called a subsequence of type $\overline{s}_l$. If $S$ contains such a subsequence, we will also use notation $\overline{s}_l \in S$. Note that if $S$ contains a subsequence $s_l(a)$, it also contain a subsequences $s^+_{l-1}(a)$ and $s^-_{l-1}(a+1)$. Then $d_0$ is the number subsequences of $S$ having type $s_1$.

The following claim hold.

**Claim.** (1) *If $S$ contains no $\overline{s}_1$, then $d_0 - d_2 \leq 1$ and we can have $d_0 - d_2 = 1$ only if $d_0 \geq 2$. Indeed, for every pair $(v_i, v_{i+1})$ with $c(v_{i+1}) = c(v_i)$ we have pair $(v_{i+1}, v_{i+2})$ with $c(v_{i+2}) - c(v_{i+1}) \geq 2$, with at most one exception, when $i + 1 = 2n$.*

(2) *If $S$ contains sequences of type $\overline{s}_1$, then $d_0 - d_2$ is at most the number of subsequences of type $s_1$ included in a sequence of type $\overline{s}_1$. Indeed, let $j$ be the index of the first occurrence of an $\overline{s}_1$ in $S$. For every pair $(v_i, v_{i+1})$ with $c(v_{i+1}) = c(v_i) = a$ such that $s^+_1(a), s^-_1(a) \notin S$, if $i < j$ we have pair $(v_{i+1},\ v_{i+2})$ with $c(v_{i+2}) - c(v_{i+1}) \geq 2$, otherwise pair $v_{i-1}, v_i$ has $c(v_i) - c(v_{i-1}) \geq 2$.*

(3) *If exists $l \geq 1$ such that $\overline{s}_l \in S$, then there are two vertices $x \in X$, $y \in Y$ with $|F_c(x)| \geq 1$ and $|F_c(y)| \geq 1$. Moreover, there is also a vertex $v$ having $|F_c(v)| \geq 2$.*

(4) *If exists $l \geq 2$ such that $\overline{s}_l \in S$, then there is a vertex $v$ with $|F(v)| = 3$.*

(5) *If exists $l \geq 3$ such that $\overline{s}_l \in S$, then there are two vertices $x \in X, y \in Y$ with $|F_c(x)| = |F_c(y)| = 3$. Moreover, there is another vertex $v$ such that $|F_c(v) \cup F_c(x)| \geq 4$ or $|F_c(v) \cup F_c(y)| \geq 4$.*

(6) *If exists $l \geq 4$ such that $\overline{s}_l \in S$, then there are two vertices $x \in X$, $y \in Y$ with $|F_c(x)| = |F_c(y)| = 3$. Moreover, there is another vertex $v$ such that $|F_c(v) \cup F_c(x)| \geq 5$ or $|F_c(v) \cup F_c(y)| \geq 5$.*

(7) *If exists $l \geq 5$ such that $\overline{s}_l \in S$, then there are two vertices $u, v$ in the same class of bipartition with $|F_c(u)| = |F_c(v)| = 3$ and $|F_c(u) \cup F_c(v)| = 6$.*

Next we consider all possible values for $k \geq n - 5$ and use previous Claim and Remark 1 to provide the lower bounds for $val(c)$.

*Case* 1. $k = n$. By Remark 1 we obtain that any vertex has the set of forbidden neighbors empty. From that it results that $c$ is injective and for every $x \in X$ none of the colors $c(x)$, $c(x) + 1$ and $c(x) - 1$ are in $c(Y)$. It follows that $val(c) \geq 2n + 1$.

*Case* 2. $k = n - 1$. In this case, by Remark 1, there is at most one vertex in each class of bipartition having the set of forbidden neighbors not empty, but with no more than one element. Thus, $d_0 \leq 1$. Moreover, by Claim (3), there are no $\overline{s}_1$ in $S$, hence, using Claim (1), we obtain $d_0 - d_2 \leq 0$ and $val(c) \geq 2n$.

*Case* 3. $k = n-2$. In this case $S$ contains no $\overline{s}_l$ with $l \geq 2$. If $\overline{s}_1 \in S$, it suffice to consider only the case when there is a color $a$ such that $s^+_1(a) \in S$ (the case when $s^-_1(a) \in S$ is similar, since we can consider $\overline{c}$ instead of $c$). Then there is one vertex $u$ with $|F_c(u)| \geq 2$, more precisely with $c(u) = a$ and $c(F_c(u)) = \{a, a+1\}$.

Assume $u \in X$. It results that $S$ contains no subsequence $s_1(b)$ with $b \notin \{a, a+1\}$, otherwise there will be another vertex $v \in X$ with $c(v) = b$ and $b \in c(F_c(v))$ and we will have $\kappa_c(u, v) \leq 2 + 1 = 3$.

If also $s_1(a+1) \notin S$, we have $d_0 = 1$, hence $d_0 - d_2 \leq 1$. Otherwise $s_1(a+1) \in S$ and hence $s_2(a) \in S$. But since $s_2^+(a) \notin S$ and $s_2^-(a) \notin S$ ($S$ contains no $\overline{s}_2$), we have $d_0 - d_2 \leq 2 - 1 = 1$.

If $\overline{s}_1 \notin S$, then $d_0 - d_2 \leq 1$, by Claim (1).

In all situations we obtain $d_0 - d_2 \leq 1$, thus $val(c) \geq 2n - 1$.

*Case* 4. $k = n - 3$. Then $S$ contains no $\overline{s}_l$, with $l \geq 3$.

*Case* 4.1. If $\overline{s}_2 \in S$, as in Case 2, it suffices to assume there is a color $a$ such that $s_2^+(a) \in S$. Then there is a vertex $v$ with $|F_c(v)| = 3$ and $c(F_c(v)) = \{a, a+1, a+2\}$. Using similar arguments, it follows that there is no $b \notin \{a, a+1, a+2\}$ such that $s_1(b) \in S$. If $s_1(a+2) \notin S$, we have $d_0 = 2$, hence $d_0 - d_2 \leq 2$. Otherwise, since $\overline{s}_3 \notin S$, we have $d_0 - d_2 \leq 3 - 1 = 2$.

*Case* 4.2. Assume $\overline{s}_1 \in S$ (but $\overline{s}_2 \notin S$). We will prove there are at most 2 subsequences of type $\overline{s}_1$ in $S$. If there are three such subsequences in $S$, it follows that there are three vertices $u_1, u_2, u_3$ of three distinct colors $c_1 < c_2, c_3$ such that $c(F_c(u_i)) = \{c_i, c_i + 1\}$. Two of these vertices are in the same class of bipartition. Assume $u_1, u_2 \in X$. Then $u_3 \in Y$, otherwise there are two vertices $v_1, v_2$ in $\{u_1, u_2, u_3\} \subseteq X$ with disjoint forbidden neighbor sets and then $\kappa_c(v_1, v_3) \leq n - 4$, and again contradiction. Also, if $c_1 + 1 < c_2$ then $|F_c(u_1) \cup F_c(u_2)| = 4$, contradiction. It follows that $c_2 = c_1 + 1$ and then $F_c(u_1) \cap F_c(u_2)$ contains a vertex $y$ from $Y$ of color $c_1 + 1$ that has $u_1, u_2 \in F_c(y)$. Moreover, $c_3 > c_1 + 2 = c_2 + 1$, since $S$ contains no $\overline{s}_2$. We then have two vertices $y, u_3$ in $Y$ with $|F_c(u_3) \cup F_c(y)| \geq 4$, contradiction.

In all situations we obtained contradictions, hence in this case there are at most two $\overline{s}_1$ in $S$. By Claim (2), it follows that $d_0 - d_2 \leq 2$.

*Case* 4.3. If $S$ contains no $\overline{s}_1$, then we have $d_0 - d_2 \leq 1 < 2$. We obtained in every subcase that $d_0 - d_2 \leq 2$, hence $\lambda c_{n-3}(K_{n,n}) \geq 2n - 2$.

*Case* 5. $k = n - 5$. By Claim, there are no $\overline{s}_l$ in $S$ with $l \geq 5$, hence no $s_l$ with $l \geq 6$. Also, $S$ contains no two disjoint $\overline{s}_l$ with $l \geq 3$ and there are at most two disjoint subsequences of type $\overline{s}_2$ in $S$, otherwise one class of bipartition would contain two vertices with pairwise disjoint sets of forbidden neighbors, each having size 3, which would imply $k \leq n - 6$.

Thus, there are at most 4 pairs of equal colors contained in subsequences of type $\overline{s}_l$ with $l \geq 2$. In order to maximize the difference $d_0 - d_2$, by Claim (2), the other pairs of equal labels must be included in sequences of type $\overline{s}_1$, which are of length 3 (such that we obtain no sequences of type $\overline{s}_2$). We then have

$$d_0 - d_2 \leq 4 + \left\lceil \frac{2n-8}{3} \right\rceil = \left\lceil \frac{2n+4}{3} \right\rceil,$$

and thus

$$val(c) \geq 2n - \left\lceil \frac{2n+4}{3} \right\rceil = 2n - \left\lceil \frac{3n+3-(n-1)}{3} \right\rceil = n - 1 + \left\lceil \frac{n-1}{3} \right\rceil,$$

so

$$val(c) \geq n - 1 + \left\lceil \frac{n+1}{3} \right\rceil.$$

*Case* 6. $k = n-4$. Then $S$ contains no $\overline{s}_l$, with $l \geq 4$.

*Case* 6.1. If $S$ contains no $\overline{s}_1$, then $d_0 - d_2 \leq 1$.

*Case* 6.2. If exists $a$ such that $s_3(a) \in S$, then $S$ contains no $\overline{s}_1(b)$ with $b \notin \{a, a+1, a+2\}$, hence $d_0 - d_2 \leq 4$ in this case Claim (2).

*Case* 6.3. $S$ contains a subsequence $\overline{s}_2$ (but no $s_3$). Assume without loss of generality there is $a$ such that $s_2^+(a) \in S$. Then $S$ contains no $s_2(b)$ with $b \notin \{a, a+1\}$, otherwise there are two vertices $u, v$ in the same class of bipartiton with $c(u) \in s_2^+(a)$, $c(v) \in s_2(b)$ such that $|F_c(u)| = 3$, $|F_c(v)| = 2$ and $F_c(u) \cap F_c(v) = \emptyset$, hence $|F_c(u) \cup F_c(v)| \geq 5$.

Also, there is no $b$ such that $s_1^+(b)$ and $s_1^-(b+2)$ are in $S$ (that is sequence $\{b, b, b+1, b+2, b+2\}$ is in $S$), otherwise we will again have two vertices $u, v$ in the same class of bipartiton with $|F_c(u) \cup F_c(v)| \geq 5$.

It follows that in this case $d_0 - d_2$ is maximized if each pair of equal labels not contained in $s_2^+(a)$ is included in a unique sequence of type $\overline{s}_1$. In this case we have

$$d_0 - d_2 \leq 2 + \left\lceil \frac{2n-5}{3} \right\rceil = \left\lceil \frac{2n+1}{3} \right\rceil.$$

*Case* 6.4. If $S$ contains no $\overline{s}_2$, then the maximum for $d_0 - d_2$ is obtained when pairs of equal labels are included in sequences of type $\overline{s}_1$, hence we have

$$d_0 - d_2 \leq \left\lceil \frac{2n+1}{3} \right\rceil.$$

In all cases we obtain

$$d_0 - d_2 \leq \left\lceil \frac{2n+1}{3} \right\rceil.$$

Then

$$val(c) \geq 2n - \left\lceil \frac{2n+1}{3} \right\rceil = 2n - \left\lceil \frac{3n-(n-1)}{3} \right\rceil = n + \left\lceil \frac{n-1}{3} \right\rceil = n + \left\lceil \frac{n+1}{3} \right\rceil.$$

■

## 8. Conclusions

There are many papers dedicated to variations of radio colorings, for their applications also in combinatorics, simulated annealing, genetic algorithms, neural networks [3]. In many cases, especially for 2 edge-connected networks, $L(2,1)$-connection number is much smaller than $L(2,1)$-number, hence finding an optimum $L(2,1)$-path coloring present interest not only from theoretical point of view, but also for algorithmic approaches and applications. Thus, it would be interesting to determine if there are efficient algorithms to find the $L(2,1)$-connection number of a graph or at least to decide if the $L(2,1)$-connection number is equal to 5. Note that for a fixed $l$, we can generalize the notions also to $L(d_1, d_2, \ldots, d_l)$-*path coloring* and that $k$-$L(1,1,\ldots,1)$-connection number is actually the vertex version for $(k,l)$-proper connection number [12], that has not yet been studied.

**Acknowledgements**

The author would like to thank the anonymous referees for their valuable comments which helped to improve the quality of the paper and for suggesting Corollary 15.

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Received 29 September 2017
Revised 31 July 2018
Accepted 6 December 2018