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# THE MINIMUM SIZE OF A GRAPH WITH GIVEN TREE CONNECTIVITY

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**Abstract**

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an $S$-tree is a such subgraph $T$ of $G$ that is a tree with $S \subseteq V(T)$. Two $S$-trees $T_1$ and $T_2$ are said to be internally disjoint if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$, and edge-disjoint if $E(T_1) \cap E(T_2) = \emptyset$. The generalized local connectivity $\kappa_G(S)$ (generalized local edge-connectivity $\lambda_G(S)$, respectively) is the maximum number of internally disjoint (edge-disjoint, respectively) $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the generalized $k$-connectivity (generalized $k$-edge-connectivity, respectively) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$ ($\lambda_k(G) = \min\{\lambda_G(S) \mid S \subseteq V(G), |S| = k\}$, respectively).

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Let $f(n,k,t)$ ($g(n,k,t)$, respectively) be the minimum size of a connected graph $G$ with order $n$ and $\kappa_k(G) = t$ ($\lambda_k(G) = t$, respectively), where $3 \leq k \leq n$ and $1 \leq t \leq n - \lceil \frac{k}{2} \rceil$. For general $k$ and $t$, Li and Mao obtained a lower bound for $g(n,k,t)$ which is tight for the case $k = 3$. We show that the bound also holds for $f(n,k,t)$ and is tight for the case $k = 3$. When $t$ is general, we obtain upper bounds of both $f(n,k,t)$ and $g(n,k,t)$ for $k \in \{3,4,5\}$, and all of these bounds can be attained. When $k$ is general, we get an upper bound of $g(n,k,t)$ for $t \in \{1,2,3,4\}$ and an upper bound of $f(n,k,t)$ for $t \in \{1,2,3\}$. Moreover, both bounds can be attained.

**Keywords:** generalized connectivity, tree connectivity, generalized $k$-connectivity, generalized $k$-edge-connectivity, packing.

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## 1. Introduction

We refer to [1] for graph theoretical notation and terminology not described here. For a graph $G$, let $V(G)$, $E(G)$ be the set of vertices, the set of edges of $G$, respectively. For $X \subseteq V(G)$, we use $G - X$ to denote the subgraph obtained by deleting from $G$ the vertices of $X$ together with the edges incident with them, and use $G[X]$ to denote the induced subgraph of $G$ with vertex set $X$. For $Y \subseteq E(G)$, we use $G - Y$ to denote the subgraph obtained by deleting from $G$ the edges of $Y$, and use $G[Y]$ to denote the subgraph of $G$ with vertex set $V(Y)$ and edge set $Y$. For a set $S$, we use $|S|$ to denote the number of its elements. We use $P_n$, $C_m$ and $K_\ell$ to denote a path of order $n$, a cycle of order $m$ and a complete graph of order $\ell$, respectively.

Connectivity is one of the most basic concepts in graph theory, both in a combinatorial sense and in an algorithmic sense. The classical connectivity has two equivalent definitions. The connectivity of a connected graph $G$, written $\kappa(G)$, is the minimum size of a vertex set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. This definition is called the *cut-version* definition of the connectivity. A well-known theorem of Menger provides an equivalent definition, which can be called the *path-version* definition of the connectivity. For any two distinct vertices $x$ and $y$ in $G$, the local connectivity $\kappa_G(x,y)$ is the maximum number of internally disjoint paths connecting $x$ and $y$. Then $\kappa(G) = \min\{\kappa_G(x,y) \mid x, y \in V(G), x \neq y\}$ is defined to be the connectivity of $G$. Similarly, there are cut-version and path-version definitions for the edge-connectivity of graphs.

The generalized $k$-connectivity $\kappa_k(G)$ of a graph $G$ which was introduced by Hager [9] in 1985 is a natural generalization of the path-version definition of the connectivity. For a graph $G = (V,E)$ and a set $S \subseteq V$ of at least two vertices, an *$S$-Steiner tree* or a *Steiner tree connecting $S$* (or simply, an *$S$-tree*) is a such subgraph $T$ of $G$ that is a tree with $S \subseteq V(T)$. Two $S$-trees $T_1$ and $T_2$ are said to be *internally disjoint* if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. The *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the *generalized $k$-connectivity* is defined as

$$\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}.$$

Thus, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when $S$ runs over all the $k$-subsets of $V(G)$. By definition, we clearly have $\kappa_2(G) = \kappa(G)$. By convention, for a connected graph $G$ with less than $k$ vertices, we set $\kappa_k(G) = 1$, and $\kappa_k(G) = 0$ when $G$ is disconnected. For more details about this topic, the reader can see [2, 3, 9, 11, 12, 15, 20, 28, 29].

As a natural counterpart of the generalized $k$-connectivity, Li, Mao and Sun [20] introduced the following concept of generalized edge-connectivity which is a generalization of the path-version definition of the edge-connectivity. Two $S$-trees $T_1$ and $T_2$ are said to be *edge-disjoint* if $E(T_1) \cap E(T_2) = \emptyset$. The *generalized local edge-connectivity* $\lambda_G(S)$ is the maximum number of edge-disjoint $S$-trees in $G$. For an integer $k$ with $2 \leq k \leq n$, the *generalized $k$-edge-connectivity* is defined as

$$\lambda_k(G) = \min\{\lambda_G(S) \mid S \subseteq V(G), |S| = k\}.$$

Thus, $\lambda_k(G)$ is the minimum value of $\lambda_G(S)$ when $S$ runs over all the $k$-subsets of $V(G)$. Hence, we have $\lambda_2(G) = \lambda(G)$. By definitions of $\kappa_k(G)$ and $\lambda_k(G)$, $\kappa_k(G) \leq \lambda_k(G)$ holds. There are many results on this type of generalized edge-connectivity, see [3, 16, 17, 20, 25, 26, 27, 29].

The generalized edge-connectivity is related to two important problems [14]. For a given graph $G$ and $S \subseteq V(G)$, the problem of finding a set of maximum number of edge-disjoint Steiner trees connecting $S$ in $G$ is called the *Steiner tree packing problem*. The difference between the Steiner tree packing problem and the generalized edge-connectivity is as follows: The Steiner tree packing problem studies local properties of graphs since $S$ is given beforehand, but the generalized edge-connectivity focuses on global properties of graphs since it first needs to compute the maximum number $\lambda_G(S)$ of edge-disjoint trees connecting $S$ and then $S$ runs over all $k$-subsets of $V(G)$ to get the minimum value of $\lambda_G(S)$.

The problem for $S = V(G)$ is called the *spanning tree packing problem*. Note that spanning tree packing problem is a specialization of Steiner tree packing problem (For $k = n$, each $S$-Steiner tree is a spanning tree of $G$.) For any graph $G$ of order $n$, the *spanning tree packing number* is the maximum number of edge-disjoint spanning trees contained in $G$. From the definitions of $\kappa_k(G)$ and $\lambda_k(G)$, $\kappa_n(G) = \lambda_n(G)$ is exactly the spanning tree packing number of $G$. (For $k = n$, both internally disjoint $S$-Steiner trees and edge-disjoint $S$-Steiner trees are edge-disjoint spanning trees.) For the spanning tree packing number, we refer to [23, 24]. Observe that the spanning tree packing number is a special case of both the generalized $k$-connectivity and the generalized $k$-edge-connectivity.

In addition to being natural combinatorial measures, the generalized (edge-) connectivity can be motivated by its interesting interpretation in practice as well as theoretical consideration [14]. From a theoretical perspective, both extremes of this problem relate to fundamental theorems in combinatorics. One extreme of the problem is when we have two terminals. In this case internally (edge-)disjoint trees are just internally (edge-)disjoint paths between the two terminals, and so the problem is relevant to the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case internally disjoint trees and edge-disjoint trees are just edge-disjoint spanning trees of the graph, and so the problem is relevant to the classical Nash-Williams-Tutte theorem [22, 30].

The generalized edge-connectivity and the Steiner tree packing problem have applications in VLSI circuit design, see [7, 8]. In these applications, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. Steiner trees are also used in computer communication networks [5, 6] and optical wireless communication networks [4]. Another application, which is our primary focus, arises in the Internet Domain. Imagine that a given graph $G$ represents a network. We choose arbitrary $k$ vertices as nodes. Suppose one of the nodes in $G$ is a *broadcaster*, and all other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via a tree connecting all the users and the broadcaster. Hence, in essence we need to find the maximum number Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda_G(S)$, where $S$ is the set of the $k$ nodes. Clearly, it is a Steiner tree packing problem. Furthermore, if we want to know whether for any $k$ nodes the network $G$ has above properties, then we need to compute $\lambda_k(G) = \min\{\lambda_G(S)\}$. It is also worth noting that the concept of generalized $k$-(edge-)connectivity is related to Steiner distance, see [2, 21].

Nowadays, more and more researchers are working in the topic of generalized connectivity with applications. In the literature, generalized $k$-connectivity and generalized $k$-edge-connectivity are also called *tree connectivity.* The reader is referred to a new book [19] for a detailed introduction of this field.

For $3 \leq k \leq n$ and $1 \leq t \leq n - \lceil \frac{k}{2} \rceil$, let $f(n,k,t)$ ($g(n,k,t)$, respectively) be the minimum size of a connected graph $G$ with order $n$ and $\kappa_k(G) = t$ ($\lambda_k(G) = t$, respectively). It is not easy to determine the exact values of $f(n,k,t)$ and $g(n,k,t)$ for general $k$ and $t$, so people try to obtain nice bounds for these two parameters.

For general $k$ and $t$, Li and Mao [17] obtained a lower bound of $g(n,k,t)$ which is tight when $k = 3$ (Theorem 11). We show that the same bound also holds for $f(n,k,t)$ and is tight for the case $k = 3$ (Theorem 12).

People also investigate these two parameters in the following two directions. For the first direction that $t$ is general, Li and Mao [17] investigated $g(n,3,t)$ and derived a lower bound and some precise values (Theorem 7). In this paper, we will get a similar lower bound and precise values for $f(n,3,t)$ (Theorem 13). Furthermore, we will obtain upper bounds of both $f(n,k,t)$ and $g(n,k,t)$ for $k \in \{3,4,5\}$, and all of these bounds can be attained (Theorems 14, 15 and 16).

For the second direction that $k$ is general, Li, Mao and Sun [20] obtained precise values for $f\left(n,k,n-\left\lceil \frac{k}{2} \right\rceil\right)$ and $g\left(n,k,n-\left\lceil \frac{k}{2} \right\rceil\right)$ (Theorem 8). For the case that $t = n - \left\lceil \frac{k}{2} \right\rceil - 1$ and $k$ is an even integer, Li and Mao [14] obtained the precise values of $f(n,k,t)$ and $g(n,k,t)$ (Theorem 9). In this paper, we will get an upper bound of $g(n,k,t)$ for $t \in \{1,2,3,4\}$ (Theorem 18), and an upper bound of $f(n,k,t)$ for $t \in \{1,2,3\}$ (Theorem 22). Moreover, both bounds can be attained for the case $k = n$.

## 2. Preliminaries

The following two propositions concern sharp bounds for $\kappa_k(G)$ and $\lambda_k(G)$.

**Proposition 1** [20]**.** *Let $k,n$ be two integers with $2 \le k \le n$. For a connected graph $G$ of order $n$ we have $1 \le \kappa_k(G) \le n - \left\lceil \frac{k}{2} \right\rceil$. Moreover, the upper and lower bounds are sharp.*

**Proposition 2** [20]**.** *Let $k,n$ be two integers with $2 \le k \le n$. For a connected graph $G$ of order $n$ we have $1 \le \lambda_k(G) \le n - \left\lceil \frac{k}{2} \right\rceil$. Moreover, the upper and lower bounds are sharp.*

Li and Mao [16] showed that the monotone property of $\lambda_k(G)$ is true for $2 \le k \le n$.

**Proposition 3** [16]**.** *Let $k,n$ be two integers with $2 \le k \le n-1$. For a connected graph $G$ we have $\lambda_{k+1}(G) \le \lambda_k(G)$.*

Li and Mao [17] gave a sufficient condition for $\lambda_k(G) \le \delta - 1$. Li [13] obtained a similar result on the generalized $k$-connectivity.

**Proposition 4** [17]**.** *Let $G$ be a connected graph of order $n$ with minimum degree $\delta$. If there are two adjacent vertices of degree $\delta$, then $\lambda_k(G) \le \delta - 1$ for $3 \le k \le n$. Moreover, the upper bound is sharp.*

**Proposition 5** [13]**.** *Let $G$ be a connected graph of order $n$ with minimum degree $\delta$. If there are two adjacent vertices of degree $\delta$, then $\kappa_k(G) \le \delta - 1$ for $3 \le k \le n$. Moreover, the upper bound is sharp.*

Li, Li, Mao and Sun characterized graphs with $\kappa_3(G) = n - 3$.

**Theorem 6** [10]**.** *Let $G$ be a connected graph of order $n$ ($n \geq 3$). Then $\kappa_3(G) = n-3$ if and only if $G$ is a graph obtained from the complete graph $K_n$ by deleting an edge set $M$ such that $K_n[M] = P_4$ or $K_n[M] = P_3 \cup rP_2$ ($r = 1, 2$) or $K_n[M] = C_3 \cup rP_2$ ($r = 1, 2$) or $K_n[M] = sP_2$ $\left(2 \leq s \leq \lfloor \frac{n}{2} \rfloor\right)$.*

For the case that $k = 3$ and $t$ is general, Li and Mao [17] investigated $g(n, 3, t)$ and derived a lower bound and some precise values.

**Theorem 7** [17]**.** *Let $n$ be an integer with $n \geq 3$. Then*

(i) $g(n, 3, n-2) = \binom{n}{2} - 1$;

(ii) $g(n, 3, n-3) = \binom{n}{2} - \lfloor \frac{n+3}{2} \rfloor$;

(iii) $g(n, 3, 1) = n - 1$;

(iv) $g(n, 3, t) \geq \left\lceil \frac{t(t+1)}{2t+1} n \right\rceil$ *for $n \geq 11$ and $2 \leq t \leq n-4$. Moreover, the bound is sharp.*

The complete bipartite graph $G = K_{t,t+1}$ is a sharp example for the bound of Theorem 7(iv).

For the case that $t = n - \lceil \frac{k}{2} \rceil$ and $k$ is a general integer, Li, Mao and Sun [20] obtained the following result.

**Theorem 8** [20]**.**

$$f\left(n, k, n - \left\lceil \frac{k}{2} \right\rceil\right) = g\left(n, k, n - \left\lceil \frac{k}{2} \right\rceil\right) = \begin{cases} \binom{n}{2}, & \text{for } k \text{ even;} \\ \binom{n}{2} - \frac{k-1}{2}, & \text{for } k \text{ odd.} \end{cases}$$

For the case that $t = n - \lceil \frac{k}{2} \rceil - 1$ and $k$ is an even integer, Li and Mao [18] obtained the following result.

**Theorem 9** [18]**.** *Let $k$ be an even integer. We have*

$$f\left(n, k, n - \left\lceil \frac{k}{2} \right\rceil - 1\right) = \binom{n}{2} - k + 1$$

*and*

$$g\left(n, k, n - \left\lceil \frac{k}{2} \right\rceil - 1\right) = \binom{n}{2} - \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, k - 1\right\}.$$

## 3. LOWER BOUNDS

The following result concerns a relationship between $f(n, k, t)$ and $g(n, k, t)$.

**Lemma 10.**

$$f(n, k, t) \geq g(n, k, t).$$

***Proof.*** We need the following claim.

**Claim.** *For a connected graph $G$, there exists a connected spanning subgraph $H$ of $G$ such that $\lambda_k(H) = \lambda_k(G) - 1$.*

***Proof.*** Without loss of generality, assume that $G$ is a minimal graph with $\lambda_k(G) = t$, that is, $\lambda_k(G) = t$ and $\lambda_k(G - e) < t$ for any $e \in E(G)$. We now show that $\lambda_k(G - e) = t - 1$. Indeed, for any set $S \subseteq V(G)$ with $|S| = k$, there is a set of $t$ edge-disjoint $S$-trees in $G$, namely $\mathcal{T} = \{T_i \mid 1 \leq i \leq t\}$. For any $e \in E(G)$, we know that $e$ belongs to at most one element of $\mathcal{T}$, say $T_t$. Then $\mathcal{T}' = \{T_i \mid 1 \leq i \leq t - 1\}$ is a set of $t - 1$ edge-disjoint $S$-trees in $G - e$, so $\lambda_{G-e}(S) \geq t - 1$ and $\lambda_k(G - e) \geq t - 1$. Hence, $\lambda_k(G - e) = t - 1$ and the claim holds. □

Let $G$ be any connected graph with $e(G) = f(n, k, t)$ and $\kappa_k(G) = t$. Clearly, we have $\lambda_k(G) \geq \kappa_k(G) = t$. By the above claim, there exists a connected spanning subgraph $H$ of $G$ such that $\lambda_k(H) = t$. Hence, $f(n, k, t) = e(G) \geq e(H) \geq g(n, k, t)$. ■

Note that in the proof of (iv) of Theorem 7, Li and Mao [17] actually proved that $e(G) \geq \left\lceil \frac{t(t+1)}{2t+1} n \right\rceil$ for a graph $G$ of order $n$ with $\lambda_k(G) = t$, where $1 \leq t \leq n - \left\lceil \frac{k}{2} \right\rceil$. Hence, the following result holds.

**Theorem 11** [17]**.** *For $3 \leq k \leq n, 1 \leq t \leq n - \left\lceil \frac{k}{2} \right\rceil$, we have*

$$g(n, k, t) \geq \left\lceil \frac{t(t+1)}{2t+1} n \right\rceil.$$

By Theorem 7, we know that the lower bound in Theorem 11 is tight for the case $k = 3$. By Lemma 10 and Theorem 11, the following result holds.

**Theorem 12.** *For $3 \leq k \leq n, 1 \leq t \leq n - \left\lceil \frac{k}{2} \right\rceil$, we have*

$$f(n, k, t) \geq \left\lceil \frac{t(t+1)}{2t+1} n \right\rceil.$$

In fact, the lower bound in Theorem 12 is tight for the case $k = 3$. We just consider the graph $G_0 \cong K_{t,t+1}$. As shown in the proof of Theorem 14 in the next section, we have that $\kappa_3(G_0) = t$ and $|e(G_0)| = t(t + 1) = f(n, 3, t)$ in this case.

Recall that in Theorem 7, Li and Mao [17] derived a lower bound and some precise values for $g(n, 3, t)$. We can obtain similar results for $f(n, 3, t)$.

**Theorem 13.** *Let $n$ be an integer with $n \geq 3$. Then*

(i) $f(n, 3, n - 2) = \binom{n}{2} - 1$;

(ii) $f(n,3,n-3) = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$;

(iii) $f(n,3,1) = n-1$;

(iv) $f(n,3,t) \geq \left\lceil \frac{t(t+1)}{2t+1} n \right\rceil$ *for* $2 \leq t \leq n-4$. *Moreover, the bound is sharp.*

***Proof.*** The assertions (i) and (iv) are directly derived by Theorems 8 and 12, respectively. The assertion (iii) is from the fact that $\kappa_3(T) = 1$ for a tree $T$. By Theorem 6, $\kappa_3(G) = n-3$ if and only if $G$ is a graph obtained from the complete graph $K_n$ by deleting an edge set $M$ such that $K_n[M] = P_4$ or $K_n[M] = P_3 \cup rP_2$ $(r = 1,2)$ or $K_n[M] = C_3 \cup rP_2$ $(r = 1,2)$ or $K_n[M] = sP_2$ $\left(2 \leq s \leq \lfloor \frac{n}{2} \rfloor\right)$. It is not hard to show that $f(n,3,n-3) = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor$ and thus the assertion (ii) holds. ■

## 4. For Small $k$

In this section, we will study $f(n,k,t)$ and $g(n,k,t)$ when $t$ is general and $k \in \{3,4,5\}$. For the case $k = 3$, we have $1 \leq t \leq n-2$ by Propositions 1 and 2, and the following result holds.

**Theorem 14.** *Let $n$ be an integer with $n \geq 3$. The following assertions hold.*

(i) *For* $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$, *we have*

$$f(n,3,t) \leq t(n-t)$$

*and*

$$g(n,3,t) \leq t(n-t).$$

*Moreover, both bounds can be attained for the case that* $t = \lfloor \frac{n-1}{2} \rfloor$ *and $n$ is odd, that is,* $f\left(n,3,\lfloor \frac{n-1}{2} \rfloor\right) = g\left(n,3,\lfloor \frac{n-1}{2} \rfloor\right) = \frac{n^2-1}{4}$ *when $n$ is odd.*

(ii) *For* $\lfloor \frac{n-1}{2} \rfloor + 1 \leq t \leq n-2$, *we have*

$$f(n,3,t) \leq \binom{t}{2} + t(n-t)$$

*and*

$$g(n,3,t) \leq \binom{t}{2} + t(n-t).$$

*Moreover, both bounds can be attained for the case that $t = n-2$, that is,* $f(n,3,n-2) = g(n,3,n-2) = \binom{n}{2} - 1$.

***Proof.*** We first prove (i). For $1 \leq t \leq \lfloor \frac{n-1}{2} \rfloor$, let $G \cong K_{t,n-t}$ be a complete bipartite graph with two parts $A$ and $B$, where $A = \{u_i \mid 1 \leq i \leq t\}$ and $B = \{v_j \mid 1 \leq j \leq n-t\}$. Clearly, $|A| < |B|$. Choose any $S \subseteq V(G)$ with

$|S| = 3$. In the following we will show that $\kappa_G(S) \geq t$, that is, there are at least $t$ internally disjoint $S$-trees in $G$.

*Case* 1. $|S \cap A| = 3$. Without loss of generality, let $S = \{u_i \mid 1 \leq i \leq 3\}$. Let $T_j$ be a tree with $V(T_j) = \{v_j\} \cup S$ and $E(T_j) = \{v_j u_i \mid 1 \leq i \leq 3\}$, where $1 \leq j \leq n-t$. Clearly, $\{T_j \mid 1 \leq j \leq n-t\}$ is a set of $n-t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq n-t > t$.

*Case* 2. $|S \cap A| = 2$. Without loss of generality, let $S = \{u_1, u_2, v_1\}$. Let $T_1$ be the path $v_1, u_1, v_3, u_2$, $T_2$ be the path $v_1, u_2, v_2, u_1$ and $T_i$ be a tree with $V(T_i) = \{u_1, u_2, u_i, v_1, v_{i+1}\}$ and $E(T_i) = \{u_1 v_{i+1}, u_2 v_{i+1}, u_i v_{i+1}, u_i v_1\}$, where $3 \leq i \leq t < n-t$. Clearly, $\{T_i \mid 1 \leq i \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

*Case* 3. $|S \cap A| = 1$. Without loss of generality, let $S = \{u_1, v_1, v_2\}$. Let $T_1$ be the path $v_1, u_1, v_2$ and $T_i$ be a tree with $V(T_i) = \{u_1, u_i, v_1, v_2, v_{i+1}\}$ and $E(T_i) = \{u_1 v_{i+1}, u_i v_1, u_i v_2, u_i v_{i+1}\}$, where $2 \leq i \leq t$. Clearly, $\{T_i \mid 1 \leq i \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

*Case* 4. $|S \cap A| = 0$. Without loss of generality, let $S = \{v_j \mid 1 \leq j \leq 3\}$. Let $T_i$ be a tree with $V(T_i) = \{u_i\} \cup S$ and $E(T_i) = \{u_i v_j \mid 1 \leq j \leq 3\}$, where $1 \leq i \leq t$. Clearly, $\{T_i \mid 1 \leq j \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

Hence, for any $S \subseteq V(G)$ with $|S| = 3$, we have $\kappa_G(S) \geq t$, and so $\lambda_G(S) \geq \kappa_G(S) \geq t$. We have that $\lambda_3(G) \geq t$ and $\kappa_3(G) \geq t$. Since $\delta(G) = t$, we have $\lambda_3(G) \leq t$ and $\kappa_3(G) \leq t$, and so $\kappa_3(G) = \lambda_3(G) = t$. As $|e(G)| = t(n-t)$, the bounds $f(n,3,t) \leq t(n-t)$ and $g(n,3,t) \leq t(n-t)$ hold. Consider the graph $G_0 \cong K_{t,t+1}$, we have $\kappa_3(G_0) = \lambda_3(G_0) = t$ and $|e(G_0)| = t(t+1) = f(n,3,t) = g(n,3,t)$ in this case.

We next prove (ii). In this case, we have $n-t \leq t$. Let $H$ be a connected graph with $V(H) = A \cup B$ and $E(H) = \{u_{i_1} u_{i_2} \mid 1 \leq i_1 \neq i_2 \leq t\} \cup \{u_i v_j \mid 1 \leq i \leq t, 1 \leq j \leq n-t\}$, where $A = \{u_i \mid 1 \leq i \leq t\}$ and $B = \{v_j \mid 1 \leq j \leq n-t\}$. With a similar argument to that of (i), we can prove that $\kappa_3(H) = \lambda_3(H) = t$. As $|e(H)| = \binom{t}{2} + t(n-t)$, the bounds $f(n,3,t) \leq \binom{t}{2} + t(n-t)$ and $g(n,3,t) \leq \binom{t}{2} + t(n-t)$ hold. For the case that $t = n-2$, by Theorem 8, $f(n,3,n-2) = g(n,3,n-2) = \binom{n}{2} - 1 = \binom{t}{2} + t(n-t)$, so both bounds can be attained in this case. ■

We now introduce the following graph class $\mathcal{G}(n,k,t)$ with $3 \leq k \leq n$ and $1 \leq t \leq n - \lceil \frac{k}{2} \rceil$: for a graph $G \in \mathcal{G}(n,k,t)$, $V(G) = A \cup B$ and $E(G) = \{u_{i_1} u_{i_2} \mid 1 \leq i_1 \neq i_2 \leq t\} \cup \{u_i v_j \mid 1 \leq i \leq t, 1 \leq j \leq n-t\} \cup \left\{v_{2j-1} v_{2j} \mid 1 \leq j \leq \left\lfloor \frac{n-t}{2} \right\rfloor\right\}$, where $A = \{u_i \mid 1 \leq i \leq t\}$ and $B = \{v_j \mid 1 \leq j \leq n-t\}$. Clearly, $|e(G)| = \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor$.

For the case $k = 4$, we have $1 \leq t \leq n-2$ by Propositions 1 and 2, then the following result holds.

**Theorem 15.** *For $1 \leq t \leq n-2$, we have*

$$f(n,4,t) \leq \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor$$

*and*

$$g(n,4,t) \leq \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor.$$

*Moreover, both bounds can be attained for the case that $t = n-2$, that is, $f(n,4,n-2) = g(n,4,n-2) = \binom{n}{2}$.*

***Proof.*** Let $G \in \mathcal{G}(n,4,t)$, where $1 \leq t \leq n-2$. Choose any $S \subseteq V(G)$ with $|S| = 4$. We will show that $\kappa_G(S) \geq t$, that is, there are at least $t$ internally disjoint $S$-trees in $G$. We only consider the case that $|S \cap A| = 3$ since the discussions for other cases (that is, $|S \cap A| \in \{0,1,2,4\}$) are similar.

If $n-t = 2$, then $B = \{v_1, v_2\}$. Without loss of generality, let $S = \{u_1, u_2, u_3, v_1\}$. Let $T_1$ be the path $v_1, u_1, u_3, u_2$ and $T_2$ be the path $u_1, u_2, v_1, u_3$. Let $T_3$ be a tree with $V(T_3) = S \cup \{v_2\}$ and $E(T_3) = \{v_1v_2, u_1v_2, u_2v_2, u_3v_2\}$, and $T_i$ be a tree with $V(T_i) = S \cup \{u_i\}$ and $E(T_i) = \{v_1u_i, u_1u_i, u_2u_i, u_3u_i\}$ for $4 \leq i \leq t$. Clearly, $\{T_i \mid 1 \leq i \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

Otherwise, we have $n-t \geq 3$. If the element of $S \cap B$ is not isolated in $G[B]$, then we are done with a similar argument to that of the case $n-t = 2$. Otherwise, the element of $S \cap B$ is isolated in $G[B]$ and so $S = \{u_1, u_2, u_3, v_{n-t}\}$. Let $T_1$ be a tree with $V(T_1) = S$ and $E(T_1) = \{u_1u_3, u_2u_3, u_3v_{n-t}\}$, $T_2$ be a tree with $V(T_2) = S \cup \{v_2\}$ and $E(T_2) = \{u_1u_2, u_1v_2, u_1v_{n-t}, u_3v_2\}$, $T_3$ be a tree with $V(T_3) = S \cup \{v_1\}$ and $E(T_3) = \{u_1v_1, u_2v_1, u_3v_1, u_2v_{n-t}\}$. Let $T_i$ be a tree with $V(T_i) = S \cup \{u_i\}$ and $E(T_i) = \{v_{n-t}u_i, u_1u_i, u_2u_i, u_3u_i\}$ for $4 \leq i \leq t$. Clearly, $\{T_i \mid 1 \leq i \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

Hence, for any $S \subseteq V(G)$ with $|S| = 4$, we have $\kappa_G(S) \geq t$, and so $\lambda_G(S) \geq \kappa_G(S) \geq t$. Therefore, $\lambda_4(G) \geq t$ and $\kappa_4(G) \geq t$. Since $\delta(G) = t$, or $\delta(G) = t+1$ and there are two adjacent vertices of degree $t+1$, we have $\lambda_4(G) \leq t$ and $\kappa_4(G) \leq t$ by Propositions 4 and 5. So $\kappa_4(G) = \lambda_4(G) = t$. As $|e(G)| = \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor$, both bounds hold. For the case that $t = n-2$, we have $f(n,4,n-2) = g(n,4,n-2) = \binom{n}{2} = \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor$. Hence, both bounds can be attained in this case. ■

For the case $k = 5$, we have $1 \leq t \leq n-3$ by Propositions 1 and 2, then the following result holds.

**Theorem 16.** *For $1 \leq t \leq n-3$, we have*

$$f(n,5,t) \leq \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor$$

*and*

$$g(n,5,t) \leq \binom{t}{2} + t(n-t) + \left\lfloor \frac{n-t}{2} \right\rfloor.$$

*Moreover, both bounds can be attained for the case that $t = n-3$, that is, $f(n,5,n-3) = g(n,5,n-3) = \binom{n}{2} - 2$.*

***Proof.*** Let $G \in \mathcal{G}(n,5,t)$, where $1 \leq t \leq n-3$. Choose any $S \subseteq V(G)$ with $|S| = 5$. We will show that $\kappa_G(S) \geq t$, that is, there are at least $t$ internally disjoint $S$-trees in $G$. We only consider the case that $|S \cap A| = 4$ since the discussions for other cases (that is, $|S \cap A| \in \{0,1,2,3,5\}$) are similar.

Now we have $|S \cap B| = 1$. We first consider the case that the element of $S \cap B$ is not isolated in $G[B]$. Without loss of generality, let $S = \{u_1, u_2, u_3, u_4, v_1\}$. Let $T_1$ be the path $v_1, u_1, u_2, u_3, u_4$, $T_2$ be a tree with $V(T_2) = S \cup \{v_2\}$ and $E(T_2) = \{u_1v_2, u_2v_2, u_3v_2, u_4v_2, v_1v_2\}$, $T_3$ be a tree with $V(T_3) = S \cup \{v_3\}$ and $E(T_3) = \{u_1v_3, u_2v_3, u_3v_3, u_4v_3, v_1u_4\}$, and $T_4$ be the path $u_1, u_3, v_1, u_2, u_4$. For $5 \leq i \leq t$, let $T_i$ be a tree with $V(T_i) = S \cup \{u_i\}$ and $E(T_i) = \{v_1u_i, u_1u_i, u_2u_i, u_3u_i, u_4u_i\}$. Clearly, $\{T_i \mid 1 \leq i \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

Otherwise, the element of $S \cap B$ is isolated in $G[B]$, so $n-t \geq 3$ is odd and now $S = \{u_1, u_2, u_3, u_4, v_{n-t}\}$. Let $T_1$ be the path $u_1, u_4, u_3, u_2, v_{n-t}$, $T_2$ be a tree with $V(T_2) = S \cup \{v_2\}$ and $E(T_2) = \{u_1v_2, u_2v_2, u_3v_2, u_4v_2, u_3v_{n-t}\}$, $T_3$ be a tree with $V(T_3) = S \cup \{v_1\}$ and $E(T_3) = \{u_1v_1, u_2v_1, u_3v_1, u_4v_1, u_4v_{n-t}\}$, and $T_4$ be a tree $V(T_4) = S$ and $E(T_4) = \{u_1u_2, u_2u_4, u_3u_1, u_1v_{n-t}\}$. For $5 \leq i \leq t$, let $T_i$ be a tree with $V(T_i) = S \cup \{u_i\}$ and $E(T_i) = \{v_{n-t}u_i, u_1u_i, u_2u_i, u_3u_i, u_4u_i\}$. Clearly, $\{T_i \mid 1 \leq i \leq t\}$ is a set of $t$ internally disjoint $S$-trees and so $\kappa_G(S) \geq t$.

Hence, for any $S \subseteq V(G)$ with $|S| = 5$, we have $\kappa_G(S) \geq t$, and so $\lambda_G(S) \geq \kappa_G(S) \geq t$. Therefore, $\lambda_5(G) \geq t$ and $\kappa_5(G) \geq t$. Since $\delta(G) = t$, or $\delta(G) = t+1$ and there are two adjacent vertices of degree $t+1$, we have $\lambda_5(G) \leq t$ and $\kappa_5(G) \leq t$ by Propositions 4 and 5. So $\kappa_5(G) = \lambda_5(G) = t$. As $|e(G)| = \binom{t}{2} + t(n-t) + \lfloor \frac{n-t}{2} \rfloor$, both bounds hold. For the case that $t = n-3$, we have $f(n,5,n-3) = g(n,5,n-3) = \binom{n}{2} - 2 = \binom{t}{2} + t(n-t) + \lfloor \frac{n-t}{2} \rfloor$. Hence, both bounds can be attained in this case. ■

## 5. For Small $t$

In this section, we will first study $g(n,k,t)$ when $k$ is general and $t \in \{1,2,3,4\}$, and then investigate $f(n,k,t)$ when $k$ is general and $t \in \{1,2,3\}$.

A *wheel graph* $W_n$ of order $n$ is a graph that contains a cycle of order $n-1$, and every vertex in the cycle is connected to one other vertex, which is known as the *center*. By Lemma 10 and definitions of $f(n,k,t)$ and $g(n,k,t)$, we have the following observation.

**Observation 17.**

$$f(n,n,t) \geq g(n,n,t) \geq t(n-1).$$

**Theorem 18.** *For $n \geq 10$, $3 \leq k \leq n$ and $1 \leq t \leq 4$, we have*

$$g(n,k,t) \leq t(n-1).$$

*Moreover, the bound can be attained for the case $k=n$, that is, $g(n,n,t)=t(n-1)$ for $1 \leq t \leq 4$.*

***Proof.*** For $t=1$, the result clearly holds by the fact that $\lambda_k(T)=1$, where $T$ is a tree.

We now consider the case $t=2$. Let $G \cong W_n$, where $V(G)=\{u\}\cup\{u_i \mid 1 \leq i \leq n-1\}$ and $u$ is the center. Let $T_1$ be the path $u, u_2, u_3, \ldots, u_{n-1}, u_1$ and $T_2$ be a tree with $V(T_2)=V(G)$ and $E(T_2)=\{u_1u_2\}\cup\{uu_i \mid 1 \leq i \leq n-1, i \neq 2\}$. Clearly, $T_1$ and $T_2$ are two edge-disjoint spanning trees of $G$, so $\lambda_n(G) \geq 2$. Since there are two adjacent vertices with minimum degree 3, we have $\lambda_k(G) \leq 2$ for $3 \leq k \leq n$ by Proposition 4. By Proposition 3, we have $\lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_3(G) \leq 2$, and so $\lambda_k(G)=2$ for $3 \leq k \leq n$. As $e(G)=2(n-1)$, the bound holds for $t=2$. By Observation 17, we have $g(n,n,2)=2(n-1)$.

We next consider the case $t=3$. Let $H$ be a connected graph with $V(H)=\{u_1,u_2\}\cup\{v_j \mid 1 \leq j \leq n-2\}$ and $E(H)=\{u_1u_2, v_1v_3, v_1v_4\}\cup\{u_iv_j \mid 1 \leq i \leq 2, 1 \leq j \leq n-2\}\cup\{v_jv_{j+1} \mid 1 \leq j \leq n-2\}$, where $v_{n-1}=v_1$. Let $T_1$ be the path $u_1, v_1, v_2, \ldots, v_{n-2}, u_2$, $T_2$ be a tree with $V(T_2)=V(H)$ and $E(T_2)=\{u_1u_2, v_1v_4, v_1v_{n-2}\}\cup\{u_2v_j \mid 2 \leq j \leq n-3\}$, and $T_3$ be a tree with $V(T_3)=V(H)$ and $E(T_3)=\{u_2v_1, v_1v_3\}\cup\{u_1v_j \mid 2 \leq j \leq n-2\}$.

Clearly, $\{T_1,T_2,T_3\}$ is a set of three edge-disjoint spanning trees of $H$, so $\lambda_n(H) \geq 3$. Since there are two adjacent vertices with minimum degree 4, we have $\lambda_k(H) \leq 3$ for $3 \leq k \leq n$ by Proposition 4. By Proposition 3, we have $\lambda_n(H) \leq \lambda_{n-1}(H) \leq \cdots \leq \lambda_3(H) \leq 3$, and so $\lambda_k(H)=3$ for $3 \leq k \leq n$. As $e(H)=3(n-1)$, the bound holds for $t=3$. By Observation 17, we have $g(n,n,3)=3(n-1)$.

Finally we consider the case $k=4$. Let $W$ be a connected graph $V(W)=\{u_1,u_2,u_3\}\cup\{v_j \mid 1 \leq j \leq n-3\}$ and $E(W)=\{u_1u_2, u_1u_3, u_2u_3, v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_3v_5\}\cup\{u_iv_j \mid 1 \leq i \leq 3, 1 \leq j \leq n-3\}\cup\{v_jv_{j+1} \mid 1 \leq j \leq n-3\}$, where $v_{n-2}=v_1$. Let $T_1$ be a tree with $V(T_1)=V(W)$ and $E(T_1)=\{u_1v_1, u_2v_2, u_3v_{n-3}\}\cup\{v_jv_{j+1} \mid 1 \leq j \leq n-4\}$, $T_2$ be a tree with $V(T_2)=V(W)$ and $E(T_2)=\{u_1u_3, v_1v_4, v_1v_{n-3}, u_2v_5\}\cup\{u_3v_j \mid 2 \leq j \leq n-4\}$, $T_3$ be a tree with

$V(T_3) = V(W)$ and $E(T_3) = \{u_2v_3, v_1v_3, v_1u_3\} \cup \{u_1v_j \mid 2 \leq j \leq n-3\}$, and $T_4$ be a tree with $V(T_4) = V(W)$ and $E(T_4) = \{u_1u_2, u_2u_3, v_2v_4, v_3v_5, v_1v_5\} \cup \{u_2v_j \mid 1 \leq j \leq n-3, j \neq 2, 3, 5\}$.

Clearly, $\{T_i \mid 1 \leq i \leq 4\}$ is a set of four edge-disjoint spanning trees of $W$, so $\lambda_n(W) \geq 4$. Since there are two adjacent vertices with minimum degree 5, we have $\lambda_k(W) \leq 4$ for $3 \leq k \leq n$ by Proposition 4. By Proposition 3, we have $\lambda_n(W) \leq \lambda_{n-1}(W) \leq \cdots \leq \lambda_3(W) \leq 4$, and so $\lambda_k(W) = 4$ for $3 \leq k \leq n$. As $e(W) = 4(n-1)$, the bound holds for $t = 4$. By Observation 17, we have $g(n, n, 4) = 4(n-1)$. ∎

Note that in the proof of Theorem 18, we actually find three graphs $G, H, W$ such that $\kappa_n(G) = \lambda_n(G) = 2$, $\kappa_n(H) = \lambda_n(H) = 3$ and $\kappa_n(W) = \lambda_n(W) = 4$. We also know $\kappa_k(T) = 1$, where $T$ is a tree. So $f(n, n, t) \leq t(n-1)$. By Observation 17, the following result holds.

**Proposition 19.** *For $n \geq 10$ and $1 \leq t \leq 4$, we have $f(n, n, t) = t(n-1)$ for $1 \leq t \leq 4$.*

The following lemma concerns an upper bound for $f(n, k, 2)$.

**Lemma 20.** *For $n \geq 10$ and $3 \leq k \leq n$, we have*

$$f(n, k, 2) \leq 2(n-1).$$

*Moreover, the bound can be attained for the case $k = n$, that is, $f(n, n, 2) = 2(n-1)$.*

***Proof.*** Let $G \cong W_n$, where $V(G) = \{u\} \cup \{u_i \mid 1 \leq i \leq n-1\}$, $u$ is the center and $C$ is the cycle $u_1, u_2, \ldots, u_{n-1}, u_1$. We will show that $\kappa_k(G) = 2$ for $3 \leq k \leq n$. As $\kappa_n(G) = 2$, it suffices to prove the equality for $3 \leq k \leq n-1$. Let $S \subseteq V(G)$ with $|S| = k$, where $3 \leq k \leq n-1$. If $u \notin S$, let $T_1$ be the path connecting $S$ in the cycle $C$, and $T_2$ be a tree with $V(T_2) = S \cup \{u\}, E(T_2) = \{uv \mid v \in S\}$. Otherwise, $u \in S$ and there exists a vertex, say $u_1 \notin S$. Let $T_1$ be a tree with $V(T_1) = S$ and $E(T_1) = \{uv \mid v \in S \setminus \{u\}\}$, and $T_2$ be a tree with $V(T_2) = V(G), E(T_2) = E(G) \setminus (\{uu_i \mid 2 \leq i \leq n-1\} \cup \{e\})$ where $e \in E(C)$. Clearly, in both cases, $T_1$ and $T_2$ are two internally disjoint $S$-trees, so $\kappa_G(S) \geq 2$, then $\kappa_k(G) \geq 2$. Recall the fact that $\kappa_k(G) \leq \lambda_k(G)$ and $\lambda_k(G) = 2$ in the proof of Theorem 18, we have $\kappa_k(G) \leq 2$. Hence, $\kappa_k(G) = 2$. As $e(G) = 2(n-1)$, the bound holds for $t = 2$. By Proposition 19, the bound can be attained in the case $k = n$. ∎

The following lemma is about an upper bound for $f(n, k, 3)$.

**Lemma 21.** *For $n \geq 10$ and $3 \leq k \leq n$, we have*

$$f(n,k,3) \leq 3(n-1).$$

*Moreover, the bound can be attained for the case $k = n$, that is, $f(n,n,3) = 3(n-1)$.*

***Proof.*** It suffices to show that $f(n,k,3) \leq 3(n-1)$ holds for $3 \leq k \leq n-1$ by Proposition 19. We use the graph $H$ from the proof of Theorem 18. Recall that $V(H) = \{u_1, u_2\} \cup \{v_j \mid 1 \leq j \leq n-2\}$ and $E(H) = \{u_1u_2, v_1v_3, v_1v_4\} \cup \{u_iv_j \mid 1 \leq i \leq 2, 1 \leq j \leq n-2\} \cup \{v_jv_{j+1} \mid 1 \leq j \leq n-2\}$, where $v_{n-1} = v_1$. Let $A = \{u_1, u_2\}$ and $B = \{v_j \mid 1 \leq j \leq n-2\}$. As shown before that $\kappa_n(H) = 3$, it suffices to prove that $\kappa_k(H) = 3$ for $3 \leq k \leq n-1$. Let $S \subseteq V(H)$ with $|S| = k$, where $3 \leq k \leq n-1$, so $V(H) \setminus S \neq \emptyset$.

*Case* 1. At least one element, say $u_1$, of $\{u_1, u_2\}$ does not belong to $S$. Let $T_1$ be a tree with $V(T_1) = \{u_1\} \cup S$ and $V(T_1) = \{u_1v \mid v \in S\}$.

If $u_2 \notin S$, then let $T_2$ be a tree with $V(T_2) = \{u_2\} \cup S$ and $E(T_2) = \{u_2v \mid v \in S\}$, $T_3$ be a path connecting $S$ in the induced subgraph $H[B]$ of $H$. Clearly, $\{T_1, T_2, T_3\}$ is a set of three internally disjoint $S$-trees, so $\kappa_H(S) \geq 3$. Otherwise, we have $u_2 \in S$ and consider the following two subcases.

*Subcase* 1.1. $B \subseteq S$. Let $T_2$ be a tree with $V(T_2) = S$ and $E(T_2) = \{u_2v_j \mid 1 \leq j \leq n-3\} \cup \{v_{n-3}v_{n-2}\}$, $T_3$ be a tree with $V(T_3) = S$ and $E(T_3) = \{v_jv_{j+1} \mid 1 \leq j \leq n-4\} \cup \{v_1v_{n-2}, v_{n-2}u_2\}$. Clearly, $\{T_1, T_2, T_3\}$ is a set of three internally disjoint $S$-trees, so $\kappa_H(S) \geq 3$.

*Subcase* 1.2. $|B \setminus S| \neq \emptyset$, say $v_1 \notin S$. Let $T_2$ be a tree with $V(T_2) = S$ and $E(T_2) = \{u_2v \mid v \in S \setminus \{u_2\}\}$, $T_3$ be the path $u_2, v_1, v_2, \ldots, v_{n-2}$. Clearly, $\{T_1, T_2, T_3\}$ is a set of three internally disjoint $S$-trees, so $\kappa_H(S) \geq 3$.

*Case* 2. $\{u_1, u_2\} \subseteq S$. We only consider the case that $|S| = n-1$ since the argument for the case $3 \leq |S| \leq n-2$ is similar. Without loss of generality, we assume that $v_1 \notin S$. Let $T_1$ be the path with $E(T_1) = \{u_1v_2, u_2v_{n-2}\} \cup \{v_jv_{j+1} \mid 2 \leq j \leq n-3\}$, $T_2$ be a tree with $V(T_2) = V(H)$ and $E(T_2) = \{u_1v_1, v_1u_2, v_1v_2, v_1v_{n-2}\} \cup \{u_1v_j \mid 3 \leq j \leq n-3\}$, $T_3$ be a tree with $V(T_3) = S$ and $E(T_3) = \{u_1u_2, u_1v_{n-2}\} \cup \{u_2v_j \mid 2 \leq j \leq n-3\}$. Clearly, $\{T_1, T_2, T_3\}$ is a set of three internally disjoint $S$-trees, so $\kappa_H(S) \geq 3$.

Now we have that $\kappa_H(S) \geq 3$ for any $S \subseteq V(H)$ with $3 \leq k \leq n-1$, so $\kappa_k(H) \geq 3$. Recall the fact that $\kappa_k(H) \leq \lambda_k(H)$ and $\lambda_k(H) = 3$ in the proof of Theorem 18, we have $\kappa_k(H) \leq 3$. Hence, $\kappa_k(H) = 3$ for $3 \leq k \leq n$. As $e(H) = 3(n-1)$, the bound holds. By Proposition 19, the bound can be attained in the case $k = n$. ■

By Lemmas 20, 21, and the fact that $\lambda_k(T) = 1$ where $T$ is a tree, we have the following result.

**Theorem 22.** *For $n \geq 10$, $3 \leq k \leq n$ and $1 \leq t \leq 3$, we have*

$$f(n,k,t) \leq t(n-1).$$

*Moreover, the bound can be attained for the case $k = n$, that is, $f(n,n,t) = t(n-1)$ for $1 \leq t \leq 3$.*

## 6. Concluding Remarks

In this paper, we study two functions $f(n,k,t)$ and $g(n,k,t)$, where $3 \leq k \leq n$ and $1 \leq t \leq n - \left\lceil \frac{k}{2} \right\rceil$. For general $k$ and $t$, we get a lower bound for $f(n,k,t)$ which is tight for the case that $k = 3$. For the upper bounds, we investigate these two parameters in two directions. For the first direction that $t$ is general, we obtain upper bounds of both $f(n,k,t)$ and $g(n,k,t)$ for $k \in \{3,4,5\}$, and all of these bounds can be attained. For the second direction that $k$ is general, we get an upper bound of $g(n,k,t)$ for $t \in \{1,2,3,4\}$, and an upper bound of $f(n,k,t)$ for $t \in \{1,2,3\}$. Moreover, both bounds can be attained.

Recall that the generalized $k$-edge-connectivity $\lambda_k(G)$ satisfies the monotone property of $\lambda_k(G)$ for $2 \leq k \leq n$. And this property indeed plays an important role in our argument. However, the monotone property of the generalized $k$-connectivity $\kappa_k(G)$ does not hold [13]. So the research on $f(n,k,t)$ is much harder than that of $g(n,k,t)$. Hence, when $k$ is general, we may try other approach to study $f(n,k,t)$ for the case $t = 4$. One may also try to compute the exact values of $f(n,k,t)$ and $g(n,k,t)$ for some special pairs of $k$ and $t$, for example, $f(n,3,2)$ and $g(n,3,2)$.

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