Discussiones Mathematicae
Graph Theory 41 (2021) 345–364
doi:10.7151/dmgt.2191

TOTAL 2-RAINBOW DOMINATION NUMBERS OF TREES

H. Abdollahzadeh Ahangar[1], J. Amjadi[2], M. Chellali[3],

S. Nazari-Moghaddam[2] and S.M. Sheikholeslami[2]

[1]*Department of Mathematics*
Babol Noshirvani University of Technology
Babol, I.R. Iran
e-mail: ha.ahangar@nit.ac.ir

[2]*Department of Mathematics*
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran
e-mail: j-amjadi;s.nazari;s.m.sheikholeslami@azaruniv.ac.ir

[3]*LAMDA-RO Laboratory, Department of Mathematics*
University of Blida
B.P. 270, Blida, Algeria
e-mail: m_chellali@yahoo.com

Abstract

A 2-rainbow dominating function (2RDF) of a graph $G = (V(G), E(G))$ is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for every vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. A total 2-rainbow dominating function f of a graph with no isolated vertices is a 2RDF with the additional condition that the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The total 2-rainbow domination number, $\gamma_{tr2}(G)$, is the minimum weight of a total 2-rainbow dominating function of G. In this paper, we establish some sharp upper and lower bounds on the total 2-rainbow domination number of a tree. Moreover, we show that the decision problem associated with $\gamma_{tr2}(G)$ is NP-complete for bipartite and chordal graphs.

Keywords: 2-rainbow dominating function, 2-rainbow domination number, total 2-rainbow dominating function, total 2-rainbow domination number.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E) such that G has no isolated vertices. The order of a graph G is the number of vertices in G, denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *maximum degree* of a graph G is denoted by $\Delta = \Delta(G)$. The *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. The *diameter* of G, denoted by $\operatorname{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. A *leaf* of a tree T is a vertex of degree one, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. If v is a support vertex, then $L(v)$ will denote the set of the leaves attached to v. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\operatorname{depth}(v)$, is the maximum distance from v to a vertex in $D(v)$. We denote by T_v the induced subgraph of T with vertex set $D[v]$. The *independence number* of a graph G, denoted $\alpha(G)$, is the order of a largest subset of vertices in which no two are adjacent. A *vertex cover* of G is a set of vertices S that covers all the edges, i.e., every edge is incident with a vertex of S. The *vertex cover number* $\beta(G)$ is the minimum cardinality of a vertex cover of G. It is well-known that for every graph G of order n, $\beta(G) + \alpha(G) = n$.

A *total Roman dominating function* of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the following conditions: (i) every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$, and (ii) the subgraph of G induced by the set of all vertices of positive weight has no isolated vertices. The weight of a total Roman dominating function f is the value $w(f) = \sum_{u \in V(G)} f(u)$, and the *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function of G. The concept of total Roman domination in graphs was introduced by Liu and Chang [11] and studied for example in [2].

A 2-*rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled. The weight of a 2RDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$, and the minimum weight of a 2RDF is called the 2-*rainbow domination number* of G, denoted by $\gamma_{r2}(G)$. The concept of 2-rainbow domination was introduced by Brešar *et al.* [6], and has been studied by several authors, for example [4, 5, 7, 8, 10, 12, 13].

A 2RDF f is called a *total* 2-*rainbow dominating function*, or just T2RDF, if the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertices. The *total* 2-*rainbow domination number*, $\gamma_{tr2}(G)$, is the minimum weight of a total

2-rainbow dominating function of G, and a T2RDF of G with weight $\gamma_{tr2}(G)$ is called a $\gamma_{tr2}(G)$-function. We note that if f is a T2RDF of a graph G and H is a subgraph of G, then we denote the restriction of f to H by $f|_{V(H)}$. Total 2-rainbow domination was recently introduced by Abdollahzadeh Ahangar *et al.* in [1] and has been studied in [3].

Before presenting our main results, we present some straightforward observations.

Observation 1. *If v is a strong support vertex in a graph G, then there exists a $\gamma_{tr2}(G)$-function f such that $f(v) = \{1, 2\}$.*

Observation 2. *If u_1 and u_2 are two adjacent support vertices in a graph G, then there exists a $\gamma_{tr2}(G)$-function f such that $f(u_1) = f(u_2) = \{1, 2\}$.*

Observation 3. *If v is a leaf neighbor of a support vertex of degree 2 in a graph G, then there exists a $\gamma_{tr2}(G)$-function f such that $|f(v)| = 1$.*

2. Lower Bounds

In this section, we establish some sharp lower bounds on the total 2-rainbow domination number of a tree. We begin by recalling the following result given in [1] for paths.

Proposition 4. *For $n \geq 2$, $\gamma_{tr2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.*

Our first lower bound on $\gamma_{tr2}(T)$ is in terms of the order and the number of leaves of a tree T.

Theorem 5. *Let T be a non-trivial tree of order n with $\ell(T)$ leaves. Then*

$$\gamma_{tr2}(T) \geq \left\lceil \frac{2(n + 3 - \ell(T))}{3} \right\rceil .$$

This bound is sharp for paths, stars and double stars.

Proof. We use an induction on n. It is easy to check that the statement holds for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that for every non-trivial tree T of order at most $n-1$ the result is true. Let T be a tree of order $n \geq 5$. If T is a star, then $\gamma_{tr2}(T) = 3 = \left\lceil \frac{2(n+3-(n-1))}{3} \right\rceil$. If T is a double star, then $\gamma_{tr2}(T) = 4 = \left\lceil \frac{2(n+3-(n-2))}{3} \right\rceil$. Henceforth we can assume that T has diameter at least 4.

Suppose that T has a strong support vertex u. Let $T' = T - u'$, where u' is a leaf neighbor of u. By Observation 1, there exists a $\gamma_{tr2}(T)$-function g such that $g(u) = \{1,2\}$. We may assume, without loss of generality, that $g(u') = \emptyset$. Then the function g, restricted to T' is a T2RDF. We can apply the inductive hypothesis to the tree T' and deduce that

$$\gamma_{tr2}(T) = \omega(g) \geq \gamma_{tr2}(T') \geq \left\lceil \tfrac{2((n-1)+3-(\ell(T)-1))}{3} \right\rceil = \left\lceil \tfrac{2(n+3-\ell(T))}{3} \right\rceil .$$

Therefore, from now on we suppose that T has no strong support vertex.

Let $v_1 v_2 \cdots v_k$ be a diametral path of rooted tree T with root vertex v_k. Since T has no strong support vertex, each child of v_3 is either a leaf or a support vertex of degree 2. Let f be a $\gamma_{tr2}(T)$-function, and consider the following cases.

Case 1. $\deg_T(v_3) \geq 3$. Assume first that v_3 is a support vertex. By Observation 2, we may assume that $f(v_2) = f(v_3) = \{1,2\}$. Let $T' = T - v_1$ and define $h : V(T') \to \mathcal{P}(\{1,2\})$ by $h(v_2) = \{1\}$ and $h(x) = f(x)$ for $x \in V(T') - \{v_2\}$. Clearly, h is a T2RDF of T'. Using the fact that $n' = n - 1$ and $\ell(T') = \ell(T)$, it follows from the induction hypothesis that

$$\begin{aligned} \gamma_{tr2}(T) = {} & \omega(f) = \omega(h) + 1 \geq \gamma_{tr2}(T') + 1 \\ \geq {} & \left\lceil \tfrac{2((n-1)+3-\ell(T))}{3} \right\rceil + 1 \geq \left\lceil \tfrac{2(n+3-\ell(T))+1}{3} \right\rceil , \end{aligned}$$

as desired. Hence we assume that v_3 is not a support vertex, and thus every child of v_3 is a support vertex of degree 2. Let $u_2 \neq v_2$ be a child of v_3 and u_1 the leaf neighbor of u_2. Clearly, $|f(u_1)| + |f(u_2)| \geq 2$ and $|f(v_1)| + |f(v_2)| \geq 2$. Let $T' = T - \{u_1, u_2\}$ and define $h : V(T') \to \mathcal{P}(\{1,2\})$ by $h(v_3) = \{1\} \cup f(v_3)$ and $h(x) = f(x)$ for $x \in V(T') - \{v_3\}$. Clearly, h is a T2RDF of T', $n' = n - 2$ and $\ell(T') = \ell(T) - 1$. It follows from the induction hypothesis that

$$\begin{aligned} \gamma_{tr2}(T) = {} & \omega(f) \geq \omega(h) + 1 \geq \gamma_{tr2}(T') + 1 \\ \geq {} & \left\lceil \tfrac{2((n-2)+3-(\ell(T)-1))}{3} \right\rceil + 1 \geq \left\lceil \tfrac{2(n+3-\ell(T))+1}{3} \right\rceil , \end{aligned}$$

as desired.

Case 2. $\deg_T(v_3) = 2$. As above we have $|f(v_1)| + |f(v_2)| \geq 2$. Suppose first that $|f(v_1)| + |f(v_2)| \geq 3$, and let $T' = T - v_1$. Then the function $h : V(T') \to \mathcal{P}(\{1,2\})$ defined by $h(v_3) = \{1\}$ and $h(x) = f(x)$ for $x \in V(T') - \{v_3\}$ is a T2RDF of T'. By induction on T' and using the fact that $n' = n-1$, $\ell(T') = \ell(T)$, we obtain $\gamma_{tr2}(T) \geq \left\lceil \tfrac{2(n+3-\ell(T))+1}{3} \right\rceil$, as desired. Therefore, we assume for the next that $|f(v_1)| + |f(v_2)| = 2$. Now, if $f(v_3) \neq \emptyset$, then the function f, restricted to $T - v_1$ is a T2RDF of $T - v_1$ of weight $\gamma_{tr2}(T) - 1$, and by the induction hypothesis on $T - v_1$ we obtain

$$\gamma_{tr2}(T) \geq \left\lceil \frac{2(n+3-\ell(T))+1}{3} \right\rceil .$$

Hence let $f(v_3) = \emptyset$. Let $T' = T - \{v_1, v_2, v_3\}$ and recall that T has diameter at least four. If T' has order $n' = 2$, then $T = P_5$, and by Proposition 4 the result is valid. Hence let $n' \geq 3$. Then $f|_{V(T')}$ is a T2RDF of T' of weight $\omega(f) - 2$. Using the fact that $n' = n - 3$ and $\ell(T') \leq \ell(T)$, and by applying the induction on T', we obtain

$$\begin{aligned}\gamma_{tr2}(T) = \omega(f) = \ & \omega(f|_{V(T')}) + 2 \geq \gamma_{tr2}(T') + 2 \\ & \geq \left\lceil \tfrac{2((n-3)+3-\ell(T))}{3} \right\rceil + 2 = \left\lceil \tfrac{2(n+3-\ell(T))}{3} \right\rceil .\end{aligned}$$

This completes the proof. ■

Theorem 6. *If T is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then*

$$\gamma_{tr2}(T) \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta} \right\rceil ,$$

and this bound is sharp.

Proof. The proof is by induction on n. One can easily check that the statement holds for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that the result is true for every non-trivial tree T' of order n', with $3 \leq n' < n$. Let T be a tree of order n with $\ell(T)$ leaves and $s(T)$ support vertices. If $\mathrm{diam}(T) = 2$, then T is a star, where $\gamma_{tr2}(T) = 3 = 2 + \left\lceil \frac{n-2}{n-1} \right\rceil$. If $\mathrm{diam}(T) = 3$, then T is a double star, where $4 = \gamma_{tr2}(T) \geq 2 + \left\lceil \frac{n-4}{\Delta} \right\rceil$, and clearly the result is valid since $\left\lceil \frac{n-4}{\Delta} \right\rceil \leq 2$. Henceforth we may assume that $\mathrm{diam}(T) \geq 4$.

Let $v_1 v_2 \cdots v_k$ be a diametral path of T and f be a $\gamma_{tr2}(T)$-function. Without loss of generality, we assume $\deg_T(v_2) \leq \deg_T(v_{k-1})$. Consider the following situations.

Suppose first that v_3 is a support vertex adjacent to another support vertex different from v_2, v_4 or v_3 is adjacent to a strong support vertex different from v_2, v_4. Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. Moreover, it is easy to see that $\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}\gamma_{tr2}(T) \geq \gamma_{tr2}(T') + 2 \geq \ & \gamma_t(T') + \left\lceil \tfrac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 2 \\ & \geq \gamma_t(T) + \left\lceil \tfrac{\ell(T) - s(T)}{\Delta(T)} \right\rceil .\end{aligned}$$

Next, suppose that v_3 is not a support vertex and it is adjacent to a support vertex of degree two different from v_2. Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. On the other hand, if $\deg_T(v_2) \geq 3$, then $\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$, and if $\deg_T(v_2) = 2$, then

$\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 1$. Using the induction on T' and according to each situation, the result follows.

Suppose now that v_3 is a support vertex having no neighbor as support vertex besides v_2 and (possibly) v_4. If $|f(x)| \geq 1$ for some $x \in N(v_3) - \{v_2\}$, then let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. Moreover, one can see that $\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By induction on T', we obtain as above $\gamma_{tr2}(T) \geq \gamma_t(T) + \left\lceil \frac{\ell(T)-s(T)}{\Delta(T)} \right\rceil$. Hence, we assume that $f(x) = \emptyset$ for all $x \in N(v_3) - \{v_2\}$. Thus $f(v_3) = \{1,2\}$. Since, $f(v_4) = \emptyset$, we conclude that v_4 is not a support vertex and has no child of depth 1 which is a strong support vertex. Assume that $\deg_T(v_4) \geq 3$. If v_4 has a child of depth 1 say, u_2, with u_1 as a leaf neighbor of u_2, then let $T' = T - \{u_1, u_2\}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - 1$ and $s(T') = s(T) - 1$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}\gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 2 \geq \gamma_t(T') + \left\lceil \frac{\ell(T')-s(T')}{\Delta(T')} \right\rceil + 2 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T)-s(T)}{\Delta(T)} \right\rceil.\end{aligned}$$

Therefore, we can assume that all children of v_4 have depth 2. According the diametral path and the situations already considered, we conclude that each child of v_4 is a support vertex or has degree 2. If z is a child of v_4 with degree 2 with $z_1 \in N(z) - v_4$, then let $T' = T - T_z$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(z_1)|$ and $s(T') = s(T) - 1$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. Using the induction on T', we obtain desired result. Hence, each child of v_4 is a support vertex assigned $\{1,2\}$ under f. Let $T' = T - T_{v_3}$. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - (|L(v_2)| + |L(v_3)|)$ and $s(T') = s(T) - 2$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}\gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 4 \geq \gamma_t(T') + \left\lceil \frac{\ell(T')-s(T')}{\Delta(T')} \right\rceil + 4 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T)-s(T)}{\Delta(T)} \right\rceil.\end{aligned}$$

Now, let $\deg_T(v_4) = 2$ and $T' = T - T_{v_4}$. Note T' has order $n' \geq 1$ since $\operatorname{diam}(T) \geq 4$. It is a routine matter to check that the result holds if $n' \in \{1,2\}$. Hence let $n' \geq 3$. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') \geq \ell(T) - (|L(v_2)| + |L(v_3)|)$ and $s(T') \leq s(T) - 1$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4$. Using the induction on T', the result follows.

Finally, assume that $\deg_T(v_3) = 2$. First, assume that $f|_{T'}$ is a T2RDF of $T' = T - T_{v_3}$. Recall that T has diameter at least four. If T' has order 2, then T

is obtained from a star of order at least three and a path P_2 by adding an edge joining their leaves, and clearly the result holds. So assume that T' has order at least three. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') \geq \ell(T) - |L(v_2)|$ and $s(T') \leq s(T)$. Moreover, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 3 \geq \gamma_t(T') + \left\lceil \tfrac{\ell(T')-s(T')}{\Delta(T')} \right\rceil + 3 \\ &\geq \gamma_t(T) + \left\lceil \tfrac{\ell(T')-s(T')}{\Delta(T')} \right\rceil + 1 \geq \gamma_t(T) + \left\lceil \tfrac{\ell(T)-s(T)}{\Delta(T)} \right\rceil.\end{aligned}$$

Suppose now that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$. Hence, we have the following cases.

Case 1. $f(v_4) = \emptyset$. Then v_4 is not a support vertex and has no child of depth 1 which is a strong support vertex. Seeing the previous cases, it follows that any child of v_4 other than v_3 is either a support vertex of degree two or a vertex with depth 2 and degree 2. Moreover, since every child of v_4 is assigned a non-empty set, we conclude from our assumption that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$ and that $\deg_T(v_4) \in \{2, 3\}$. We consider the following.

Subcase 1.1. $\deg_T(v_4) = 3$. Observe that T_{v_4} has exactly two support vertices, v_2 and say z. We note that z is a either at distance one or two from v_4. Let $T'' = T - T_{v_4}$. Clearly, T'' has order at least three, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(z)|)$, $s(T'') \leq s(T) - 1$ and $\gamma_t(T) \leq \gamma_t(T'') + 4$. Now, if z is at distance one from v_4, then $|L(z)| = 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Also, if z is at distance two from v_4, then $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 6$. Whatever the case, using the induction on T'', the result follows.

Subcase 1.2. $\deg_T(v_4) = 2$. Let $T'' = T - T_{v_4}$. It is easy to check the result if $n(T'') \in \{1, 2\}$. Hence let $n(T'') \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - |L(v_2)|$ and $s(T'') \leq s(T)$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on T'', the result follows.

Case 2. $|f(v_4)| \geq 1$ and thus $f(x) = \emptyset$ for each vertex $x \in N(v_4) - \{v_3\}$. Then every child of v_4 besides v_3 (if any) is leaf. To avoid the previous case when $f(v_4) = \emptyset$ we can assume that v_4 is a support vertex (else substitute the assignments of v_4 and v_5). Now if $f|_{T''}$ is a T2RDF of $T'' = T - T_{v_4}$, then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_4)|)$ and $s(T'') \leq s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$, the result follows by using the induction on T''. Hence suppose that $f|_{T''}$ is not a T2RDF of $T'' = T - T_{v_4}$ and so v_5 has no child of depth 3 other than v_4. Since $f(v_5) = \emptyset$, we conclude that v_5 is not a support vertex and has no child of depth 1 which is a strong support vertex. Consider the following situations.

Subcase 2.1. v_5 has a child of depth 1. Let u_2 be such a child of depth 1 and u_1 its the leaf neighbor. Note that $\deg_T(u_2) = 2$. Let $T'' = T - \{u_1, u_2\}$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2$, the result follows by using the induction on T''.

Subcase 2.2. All children of v_5 different to v_4 have depth 2. Since $|f(x)| \leq 1$ for $x \in N(v_5) - \{v_4\}$, we deduce that every child of v_5 other than v_4 is not a support vertex. Let $z \neq v_4$ be a child of v_5. If $\deg(z) = 2$ and $z' \in N(z) - \{v_5\}$, then let $T'' = T - T_z$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - |L(z')|$ and $s(T'') = s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on T'', the result follows. Hence suppose that $\deg_T(z) \geq 3$. If z has a child of depth 1 say, u_2, of degree two, with u_1 as the leaf neighbor of u_2, then let $T'' = T - \{u_1, u_2\}$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2$. Using the induction on T'', the result follows. Hence, all children of z are strong support vertex. Let $|C(z)| = k$ and $x_1, \ldots, x_k$ be the children of z, and let $T'' = T - T_z$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - \left(\sum_{i=1}^{k} |L(x_i)|\right)$ and $s(T'') = s(T) - k$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + k + 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2k - 1$. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{tr2}(T) &\geq \gamma_{tr2}(T'') + 2k + 1 \geq \gamma_t(T'') + \left\lceil \tfrac{\ell(T'') - s(T'')}{\Delta(T'')} \right\rceil + 2k + 1 \\ &\geq \gamma_t(T) + \left\lceil \tfrac{\ell(T'') - s(T'')}{\Delta(T'')} \right\rceil + k \geq \gamma_t(T) + \left\lceil \tfrac{\ell(T) - s(T)}{\Delta(T)} \right\rceil.\end{aligned}$$

Subcase 2.3. $\deg_T(v_5) = 2$. Let $T'' = T - T_{v_5}$. Note that T'' may have order $n'' = 0$. However, it is easy to check that the result is valid for $n'' \leq 2$. Hence, let $n'' \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_4)|)$ and $s(T'') \leq s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Using the induction on T'', the result follows. This completes the proof. ■

Obviously, $\gamma_{tr2}(G) \leq \gamma_{tR}(G)$ for every graph G without isolated vertices. In the following, we provide an upper bound on the ratio $\gamma_{tR}(G)/\gamma_{tr2}(G)$ for arbitrary graphs G. Moreover, this ratio will be slightly improved for the class of trees.

Theorem 7. *If G is a graph without isolated vertices, then $\gamma_{tR}(G) \leq \frac{3}{2}\gamma_{tr2}(G)$. This bound is sharp for the graph in Figure* 1.

Proof. Let f be a $\gamma_{tr2}(G)$-function. For every $i \in \{1, 2\}$, let X_i be the set of all vertices u for which $i \in f(u)$. Clearly, if a vertex of G is assigned $\{1, 2\}$ under f, then $X_1 \cap X_2 \neq \emptyset$. Also, it is obvious that $|X_1| + |X_2| = \gamma_{tr2}(G)$. Now assume, without loss of generality, that $|X_1| \leq |X_2|$. Then $|X_1| \leq \frac{|X_1| + |X_2|}{2} = \frac{\gamma_{tr2}(G)}{2}$, and

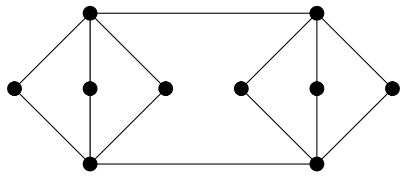

Figure 1. Graph G with $\gamma_{tR}(G) = \frac{3}{2}\gamma_{tr2}(G) = 6$.

the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x) = 0$ if $f(x) = \emptyset$, $g(x) = 1$ if $f(x) = \{2\}$, and $g(x) = 2$ if $1 \in f(x)$, is a total Roman dominating function on G, implying that

$$\gamma_{tR}(G) \leq \omega(g) = 2|X_1| + |X_2| \leq \tfrac{|X_1|+|X_2|}{2} + |X_1| + |X_2| \leq \tfrac{3}{2}\gamma_{tr2}(G). \quad \blacksquare$$

Theorem 8. *For every non-trivial tree T,*

$$\gamma_{tR}(T) \leq \frac{3}{2}\gamma_{tr2}(T) - 1,$$

and this bound is sharp for P_n such that $n \equiv 2 \pmod 3$.

Proof. The proof is by induction on n. The statement is valid for all trees of order $n \in \{2, 3, 4\}$. Let $n \geq 5$ and assume that for every tree T' of order at most $n-1$, $\gamma_{tR}(T') \leq \frac{3}{2}\gamma_{tr2}(T') - 1$. Let T be a tree of order n. Since stars and double stars T satisfy $\gamma_{tr2}(T) = 3 = \gamma_{tR}(T)$, the result holds. Therefore, we can assume that $\mathrm{diam}(T) \geq 4$.

If T has a support vertex, say u, with $|L(u)| \geq 3$, then let $T' = T - u'$, where u' is a leaf neighbor of u. Clearly $\gamma_{tR}(T) \leq \gamma_{tR}(T')$. On the other hand, by Observation 1, there exists a $\gamma_{tr2}(T)$-function g such that $g(u) = \{1, 2\}$. Also, we can assume that $g(u') = \emptyset$. It follows that $g|_{V(T')}$ is a T2RDF of T', and thus $\gamma_{tr2}(T') \leq \gamma_{tr2}(T)$. By the inductive hypothesis on T', we obtain

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') \leq 3\gamma_{tr2}(T') - 2 \leq 3\gamma_{tr2}(T) - 2.$$

Hence we assume that every support vertex in T is adjacent to at most two leaves. Let $v_1 v_2 \cdots v_k$ be a diametral path in T with root vertex v_k. We consider the following cases.

Case 1. $\deg_T(v_3) = 2$. Let $T' = T - T_{v_3}$. Then $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. It follows from the induction hypothesis that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 6 \leq 3\gamma_{tr2}(T') + 4 \leq 3\gamma_{tr2}(T) - 2.$$

Case 2. $\deg_T(v_3) \geq 3$. Consider the following subcases.

Subcase 2.1. Suppose that v_3 is a support vertex adjacent to another support vertex different from v_2 and v_4, or v_3 is adjacent to a strong support vertex different from v_2 and v_4. Let $T' = T - T_{v_2}$. It is easy to see that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. It follows from the induction hypothesis that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 4 \leq 3\gamma_{tr2}(T') + 2 \leq 3\gamma_{tr2}(T) - 4 < 3\gamma_{tr2}(T) - 2.$$

Subcase 2.2. v_3 is not a support vertex. Since $\deg_T(v_3) \geq 3$, every child of v_3 is a support vertex. Moreover, according to Subcase 2.1, all support vertices of T_{v_3}, but possibly v_2, have degree two. Let $t = \deg_T(v_3) - 1 \geq 2$. Let $T' = T - T_{v_3}$. It is easy to see that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2t + 1$. Among all $\gamma_{tr2}(T)$-functions, let g be one for which $|g(v_3)|$ is as small as possible. Clearly, for every child x of v_3 we have $|g(N[x])| \geq 2$. Now, if $g(v_3) = \emptyset$, then $g|_{V(T')}$ is a T2RDF of T' of weight $\omega(g) - 2t$, and thus $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2t$. Hence assume that $g(v_3) \neq \emptyset$. The choice of g implies that $|g(v_3)| = 1$, and thus the weight of T_{v_3} under g is $2t + 1$. The choice of g also implies that $g(v_4) = \emptyset$. In that case, the function g' defined on $V(T')$ defined by $g'(v_4) = g(v_3)$ and $g'(x) = g(x)$ for all $x \in V(T') - \{v_4\}$ is a T2RDF of T' of weight $\gamma_{tr2}(T) - 2t$, and thus $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2t$. In all cases, it follows from the induction hypothesis on T' that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 2 + 4t \leq 3\gamma_{tr2}(T') + 4t \leq 3\gamma_{tr2}(T) - 6t + 4t < 3\gamma_{tr2}(T) - 2.$$

Subcase 2.3. v_3 is a support vertex adjacent to no support vertex besides v_2 and (possibly) v_4. Let f be a $\gamma_{tr2}(T)$-function. If $|f(v_4)| \geq 1$ or there exists a vertex $x \in N_T(v_4) - \{v_3\}$ with $|f(x)| \geq 1$, then let $T' = T - T_{v_3}$. Obviously, $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. It follows from the induction hypothesis that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 8 \leq 3\gamma_{tr2}(T') + 6 \leq 3\gamma_{tr2}(T) - 9 + 6 < 3\gamma_{tr2}(T) - 2.$$

Hence we can assume that $f(x) = \emptyset$ for each $x \in N_T[v_4] - \{v_3\}$. Therefore, all children of v_4 have depth 2. According to Case 1 and the diametral path, we conclude that each child of v_4 is a support vertex. Since we assumed that $f(x) = \emptyset$ for each $x \in N_T[v_4] - \{v_3\}$, we deduce that $d_T(v_4) = 2$. In this case, let $T' = T - T_{v_4}$. Recall that T has diameter at least four. Suppose that T' has order one. Clearly, T is a tree with three support vertices v_2, v_3, v_4 and the remaining vertices are leaves. Hence $\gamma_{tR}(T) = \gamma_{tR}(T) = 6$, and thus the result holds. So suppose that T' is nontrivial. Then $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4$. By induction on T' we deduce that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 8 \leq 3\gamma_{tr2}(T') + 6 \leq 3\gamma_{tr2}(T) - 12 + 6 < 3\gamma_{tr2}(T) - 2.$$

This completes the proof. ■

3. Upper Bounds

In this section, we provide two upper bounds on the total 2-rainbow domination number of a tree. The first one we present is in terms of the order and the number of support vertices of a tree.

Theorem 9. *If T is a tree of order $n \geq 4$ with s support vertices, then*

$$\gamma_{tr2}(T) \leq \frac{2(n+s)}{3},$$

and this bound is sharp for P_n such that $n \equiv 1 \pmod 3$.

Proof. The proof is by induction on n. It is a routine matter to check that the statement holds if $n \in \{4, 5\}$. Hence, let $n \geq 6$ and assume that for every T' or order $n' < n$ with s' support vertices satisfies $\gamma_{tr2}(T') \leq \frac{2(n'+s')}{3}$. Let T be a tree of order n. If T is a star, then $\gamma_{tr2}(T) = 3 < \frac{2(n+1)}{3}$. Likewise, if T is a double star, then $\gamma_{tr2}(T) = 4 < \frac{2(n+2)}{3}$. Henceforth we can assume T has diameter at least four.

If T has a strong support vertex u adjacent to at least three leaves, then let $T' = T - u'$, where u' is a leaf neighbor of u. Clearly, any $\gamma_{tr2}(T')$-function can be extended to T2RDF of T by assigning $\emptyset$ to vertex u', and thus $\gamma_{tr2}(T) \leq \gamma_{tr2}(T')$. The result follows by using the induction on T', with $n' = n - 1$ and $s' = s$. Therefore, we will assume that every support vertex of T is adjacent to at most two leaves.

Let $v_1 v_2 \cdots v_k$ be a diametral path in T and root T in v_k. We consider the following cases.

Case 1. $\deg_T(v_2) = 3$. Thus v_2 has two leaf neighbors. We distinguish between the following situations.

Subcase 1.1. $\deg_T(v_3) \geq 3$. Suppose first that v_3 is a support vertex. Let $T' = T - T_{v_2}$. Then $n' = n - 3$ and $s' = s - 1$. Let f be a $\gamma_{tr2}(T')$-function. Since v_3 is a support vertex of T', we must have $|f(v_3)| \geq 1$. Then the function $g : V(T) \to \mathcal{P}(\{1, 2\})$ defined by $g(v_2) = \{1, 2\}$, $g(x) = \emptyset$ for $x \in L(v_2)$ and $g(x) = f(x)$ otherwise, is a T2RDF of T of weight $\gamma_{tr2}(T') + 2$. By induction on T', we have

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n'+s')}{3} + 2 = \frac{2(n-3+s-1)}{3} + 2 < \frac{2(n+s)}{3}.$$

Suppose now that v_3 is not a support vertex. Thus every child of v_3 is a support vertex with degree either 2 or 3. Let u_2 be a child of v_3 different from v_2. If $\deg_T(u_2) = 3$, then let $T' = T - T_{v_2}$. By using a similar argument to that used above, we obtain $\gamma_{tr2}(T) < \frac{2(n+s)}{3}$. Thus let $\deg_T(u_2) = 2$ with u_1 as the unique leaf of u_2. Let $T' = T - \{u_1, u_2\}$. Clearly, any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of T by assigning the set $\{1\}$ to both u_1 and u_2. Since $n' = n - 2$ and $s' = s - 1$, using the induction on T' we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

Subcase 1.2. $\deg_T(v_3) = 2$. Recall that since T has diameter at least four, $\deg_T(v_4) \geq 2$. Assume that $\deg_T(v_4) \geq 3$, and let $T' = T - T_{v_3}$. Observe that T' has order $n' \geq 3$. If $n' = 3$, then T is a tree of order 7 with 2 support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3} = 6$. Hence we assume that $n' \geq 4$. Clearly, any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of T by assigning $\{1, 2\}$ to v_2, $\{1\}$ to v_3 and $\emptyset$ to the leaves of $L(v_2)$. By induction on T' and using the fact that $n = n - 4$ and $s' = s - 1$ we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$

So, suppose for the sequel that $\deg_T(v_4) = 2$. Let $T' = T - T_{v_2}$. Note that $n' \geq 3$. If $n' = 3$, then T' has order 6 with 2 support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3} = \frac{16}{3}$. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T)$-function f such that $|f(v_3)| = 1$ and clearly such a function can be extended to a T2RDF of T by assigning $\{1, 2\}$ to v_2 and $\emptyset$ to the leaves of $L(v_2)$. Hence $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2$. By induction on T' and using the fact that $n = n - 3$ and $s' = s$, we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$

Case 2. $\deg_T(v_2) = 2$. Seeing the previous case, we may assume that every child of v_3 which is a support vertex has degree two. Consider the following subcases.

Subcase 2.1. $\deg_T(v_3) \geq 3$. Let $T' = T - \{v_1, v_2\}$. Since any $\gamma_{tr2}(T')$-function can be extended to a T2RDF of T by assigning the set $\{1\}$ to v_1 and v_2, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2$. Using the induction on T', where $n = n - 2$ and $s' = s - 1$, we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

Subcase 2.2. $\deg_T(v_3) = 2$. We consider some additional subcases.

Subcase 2.2.1. $\deg_T(v_4) \geq 3$. Let $T' = T - \{v_1, v_2, v_3\}$. Note that $n' \geq 3$. If $n' = 3$, then T is a tree of order 6 with two support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3} = \frac{16}{3}$, and thus the result is valid. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$-functions, let f be one such that $|f(v_4)|$ is as large as possible. If $|f(v_4)| \geq 1$, then define the function g on $V(T)$ as follows: $g(x) = f(x)$ for all $x \in V(T')$, $g(v_3) = \emptyset$ and $g(v_1) = g(v_2) = \{1\}$ or $\{2\}$ so that $g(N[v_3]) = \{1, 2\}$. Clearly, g is a T2RDF of T of weight $\gamma_{tr2}(T') + 2$. By induction on T' and using the fact that $n' = n - 3$ and $s' = s - 1$ we deduce that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.$$

For the sequel we can assume that $f(v_4) = \emptyset$. Clearly in that case, v_4 is not a support vertex. By the choice of the diametral path and taking into account the previous cases, we can assume that every child of v_4 with depth two and different from v_3 has degree 2. We consider the following.

(i) v_4 has a child u_2 which is a support vertex. Since $f(v_4) = \emptyset$, we conclude that $\deg_T(u_2) = 2$. Let u_1 be the leaf neighbor of u_2 and let $T'' = T - \{u_1, u_2\}$. Clearly, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$, $n'' = n - 2$ and $s'' = s - 1$. By induction on T'', it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

(ii) There is a pendant path $v_4u_3u_2u_1$ in T, where $u_3 \neq v_3$. Since $|f(v_4)| = 0$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| = 3$. Define the function g on T' by $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(v_4) = \{2\}$, and $g(x) = f(x)$ otherwise. Clearly g is a $\gamma_{tr2}(T')$-function $|g(v_4)| > |f(v_4)| = 0$, contradicting our choice of f.

Subcase 2.2.2. $\deg_T(v_4) = 2$. If $\deg_T(v_5) = 2$, then let $T' = T - \{v_1, v_2, v_3\}$. Note that T' has order $n' \geq 3$. If $n = 3$, then T is a path P_6, where $\gamma_{tr2}(P_6) = 5$ (by Proposition 4) and the result is valid. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T)$-function f such that $|f(v_4)| = 1$, and such a function can be extended to a T2RDF of T by assigning $\emptyset$ to v_3, $\{1\}$ to v_1 and $\{1, 2\} - f(v_4)$ to v_2. It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$

Assume now that $\deg_T(v_5) \geq 3$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that T' has order $n' \geq 3$. If $n' = 3$, then T is a tree of order 7 obtained from a path P_6 by adding a new vertex attached to one of the two support vertices of the path P_6. It is easy to check that $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3}$. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$-functions, let f be one such that $|f(v_5)|$ is as large as possible. If $|f(v_5)| \geq 1$, then f can be extended to a T2RDF of T by assigning $\emptyset$ to $v_4, \{1\}$ to v_1 and v_2, and either $\{1\}$ or $\{2\}$ to v_3 so that $f(N[v_4]) = \{1, 2\}$. By induction on T', it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$

For the sequel, we can assume that $f(v_5) = \emptyset$. Trivially, v_5 is not a support vertex. Also, every child of v_5 with depth one has degree two. We consider the following.

(i) v_5 has a child with depth 3. Let $u_1 \neq v_1$ be a leaf at distance four from v_5 and let $v_5u_4u_3u_2u_1$ be the unique path between u_1 and v_5. According to Cases 1 and 2 and Subcases 2.1 and 2.2, we must assume that each of u_4, u_3 and u_2 has degree two. Moreover, since $f(v_5) = \emptyset$ as assumed and according to the choice of f maximizing $|f(v_5)|$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| + |f(u_4)| = 4$. Define the function g on $V(T')$ as follows: $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(u_4) = g(v_5) = \{2\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a $\gamma_{tr2}(T')$-function with $|g(v_5)| > |f(v_5)| = 0$, a contradiction.

(ii) v_5 has a child u_2 with depth one. Let u_1 be the leaf neighbor of u_2. Let $T'' = T - \{u_1, u_2\}$. Obviously, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$. It follows by induction on T'' that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

(iii) v_5 has a child, say w, with depth two having degree at least 3. Suppose first that w has at least two children as support vertices and let z be one of them having minimum degree. Note that $\deg_T(z) \in \{2, 3\}$ since every support vertex of T has at most two leaves. Let $T'' = T - (\{z\} \cup L(z))$. Then $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$, $n'' = n - 1 - |L(z)|$ and $s'' = s - 1$. Using the induction on T' we obtain the desired result. Now, let w has exactly one child, say t, as a support neighbor. Since $\deg_T(w) \geq 3$, we deduce that w is a support vertex. Let $T'' = T - T_w$. Note that T_w has order $n_w \in \{4, 5, 6\}$. Moreover, it is clear that $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 4$. It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 4 \leq \frac{2(n'' + s'')}{3} + 4 \leq \frac{2(n - n_w + s - 2)}{3} + 4 \leq \frac{2(n + s)}{3}.$$

(iv) v_5 has a child, say w, with depth two and having degree 2. Suppose first that the child z of w is a strong support. Let $L(z) = \{z_1, z_2\}$ and let $T'' = T - \{w, z, z_1, z_2\}$. Then $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 3$, $n'' = n - 4$ and $s'' = s - 1$. It follows from the induction on T' that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 3 \leq \frac{2(n'' + s'')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$

Now, suppose that the child z of w is a support vertex of degree two. Let $\deg_T(v_5) = k \geq 3$ and H_t for $t \geq 2$ be the tree obtained from a star $K_{1,t}$ by subdividing one edge three times and each of the remaining edges exactly twice. Seeing the previous situations, clearly T_{v_5} is isomorphic to H_{k-1}. Now let $T' = T - T_{v_5}$. We note that T' has order $n' \geq 3$. If $n' = 3$, then $T = H_k$, where $n = 3k + 2$,

$s(T) = k$ and $\gamma_{tr2}(T) = 2k+2 < \frac{2(n+s)}{3}$. Hence we can assume that $n' \geq 4$. Then $\gamma_{tr2}(T) \leq \gamma_{tr2}(T^{'})+2k$, $n' = n-3k+1$ and $s(T') \leq s(T)-(k-1)+1$. It follows from the induction on T' that

$$\begin{aligned}\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2k &\leq \frac{2(n'+s')}{3} + 2k \\ &= \frac{2(n-3k+1+s-k+2)}{3} + 2k \leq \frac{2(n+s)}{3}.\end{aligned}$$

This completes the proof. ■

Next we establish an upper bound on the total 2-rainbow domination number of a tree in terms of the vertex cover number. We first give an upper bound for arbitrary graphs.

Lemma 10. *Let G be a graph of order $n \geq 2$ with no isolated vertex and V_c a minimum vertex cover of G. Then*

$$\gamma_{tr2}(G) \leq 2\beta(G) + r,$$

where r is the number of isolated vertices in the subgraph induced by V_c. This bound is sharp for the graphs in Figure 2.

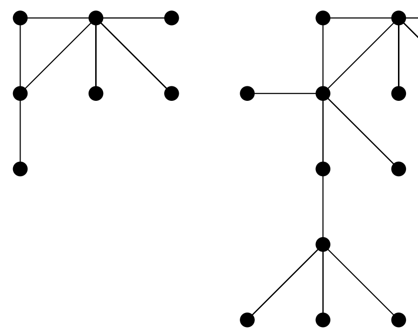

Figure 2. Two graphs G with $\gamma_{tr2}(G) = 2\beta(G) + r$.

Proof. Let V_c be a minimum vertex cover of G and I the set of isolated vertices in $G[V_c]$. Let $K = V(G) - V_c$. Since K is a maximum independent set, every vertex of V_c has a neighbor in K. Let D be a smallest subset of vertices of K that dominates all vertices of I. Obviously, $|D| \leq |I| = r$. Now define a function $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1,2\}$ if $x \in V_c$, $f(x) = \{1\}$ if $x \in D$ and $f(x) = \emptyset$ otherwise. Clearly, f is a T2RDF of G of weight $2|V_c| + |D| \leq 2|V_c| + r$. ■

The proof of the next the result is inspired by the proof of Theorem 2 in [9].

Theorem 11. *Let T be a tree of order $n \geq 3$ and let S' be the set of isolated vertices in the subgraph induced by the set of support vertices of T. Then*

$$\gamma_{tr2}(T) \leq 2\beta(T) + |S'|.$$

This bound is sharp for the graph in Figure 3.

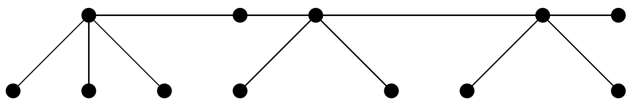

Figure 3. A tree T with $\gamma_{tr2}(T) = 2\beta(T) + |S'|$.

Proof. Let L and S denote the set of leaves and support vertices of a tree T, respectively. Let V_I be a maximum independent set that contains all leaves of T. Then $V_c = V - V_I$ is a vertex cover set of T. Note that $S \subseteq V_c$. If no support vertex of T is isolated in $T[V_c]$, then the result holds by Lemma 10. Hence, assume that u is a support vertex which is isolated in $T[V_c]$. Root T at u and let $A_1 = \{u\}$ and $A_2 = N(u)$. Clearly, $A_1 \subseteq V_c$ and $A_2 \subseteq V_I$. Assume that $A_3 = (N(A_2) - A_1) \cup B_{N(A_2)-A_1}$, where $B_{N(A_2)-A_1} = \{v \in V_c \,|\, v$ is in a component of $T[V_c]$ with a vertex of $N(A_2) - A_1\}$. Set $A_4 = N(A_3) - A_2$. Then we have $A_3 \subseteq V_c$ and $A_4 \subseteq V_I$.

We repeat this process so that at some odd number step $2k + 1$, we put

$$A_{2k+1} = (N(A_{2k}) - A_{2k-1}) \cup B_{N(A_{2k})-A_{2k-1}},$$

where $B_{N(A_{2k})-A_{2k-1}} = \{v \in V_c \,|\, v$ is in a component of $T[V_c]$ with a vertex of $N(A_{2k})-A_{2k-1}\}$ and we set $A_{2k+2} = N(A_{2k+1})-A_{2k}$. This process will terminate at some m^{th} step where m is even and A_m composed only of leaves. Note that $A_1 \cup \cdots \cup A_m$ is a partition of $V(T)$. Obviously, $V_I = A_2 \cup A_4 \cup \cdots \cup A_{m-2} \cup A_m$ and $V_c = A_1 \cup A_3 \cup \cdots \cup A_{m-3} \cup A_{m-1}$. Note that if $v \in A_i$, for $i > 1$, has a neighbor in A_{i-1}, then it has only one neighbor in A_{i-1}.

Let $D_1 = V_c$. If $T[V_c]$ has isolated vertices that are support vertices in T, then let K be a smallest subset of vertices of $V_I - L$ that dominates these isolated support vertices. Clearly, $|K| \leq |S'|$. Now we consider the isolated vertices of $T[V_c]$ that are not support vertex in T. In decreasing order, we visit each A_i with odd index i, where $3 \leq i \leq m - 1$. We start with A_{m-1} and observe that if there is an isolate of $T[V_c]$ in A_{m-1}, then it is a support vertex and some vertex of K is adjacent to it. Now for each non-support isolated vertex v of $T[V_c]$ which is in A_{m-3}, if $N(v) \cap A_{m-2}$ is dominated by $A_{m-1} \cap V_c$, then remove v from D_1 and add to D_1 its unique neighbor in A_{m-4}, otherwise we leave v in D_1. Continue this way for each odd i in decreasing order. That is, in general for A_i where i is odd, if a non-support isolated vertex v of $T[V_c]$ is in A_i and $N(u) \cap A_{i+1}$ are dominated by $A_{i+2} \cap V_c$, then remove v from D_1 and add its unique neighbor in A_{i-1} to D_1, otherwise we leave v in D_1. This process terminates after $i = 3$. Now, if some vertex of A_2 is in K, then we are done. Otherwise remove u from D_1 and add to D_1 one of its neighbors. Note that $|D_1|$ has not increased. Now let $D_2 = D_1 \cup K$. Using an argument similar to that described in the proof of Theorem 2 in [9], we see that the induced subgraph $T[D_2]$ has no isolated vertex. Define the function $f : V(T) \to \mathcal{P}(\{1,2\})$ by $f(x) = \{1,2\}$ for $x \in D_1$, $f(x) = \{1\}$ for $x \in K$ and $f(x) = \emptyset$ otherwise. Clearly, f is a T2RDF of T and thus

$$\gamma_{tr2}(T) \leq 2|V_c| + |K| \leq 2\beta(T) + |S'|.$$

This achieves that proof. ∎

4. Complexity

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as TOTAL 2-RAINBOW DOMINATION:

TOTAL 2-RAINBOW DOMINATION

Instance. Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question. *Does G have a total 2-rainbow dominating function of weight at most k?*

We show that this problem is NP-complete by reducing the well-known NP-complete problem, EXACT-3-COVER (X3C), to TOTAL 2-RAINBOW DOMINATION.

EXACT 3-COVER (X3C)

Instance. A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

Question. *Is there a subset C' of C such that every element of X appears in exactly one element of C'?*

Theorem 12. TOTAL 2-RAINBOW DOMINATION *is* NP*-complete for bipartite graphs.*

Proof. TOTAL 2-RAINBOW DOMINATION is a member of NP, since we can check in polynomial time that a function $f : V \to \{0, 1, 2\}$ has weight at most k and is a T2RDF. Now let us show how to transform any instance of X3C into an instance of TOTAL 2-RAINBOW DOMINATION so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we build a graph H_i obtained from a path $P_2 : x_i$-y_i and two stars $K_{1,3}$ with centers a_i and b_i, by adding edges y_ia_i and y_ib_i. Hence, each H_i has order 10. For each $C_j \in C$, we build a double star $S_{3,3}$ with support vertices u_j and v_j. Let c_j be a leaf of the double star $S_{3,3}$. Let $Y = \{c_1, c_2, \ldots, c_t\}$. Now to obtain a graph G, we add edges c_jx_i if $x_i \in C_j$. Clearly, G is a bipartite graph (for example, see Figure 4). Set $k = 4t + 16q$. Observe that for every T2RDF f on G, each H_i has weight at least 5 and each double star $S_{3,3}$ has weight at least 4.

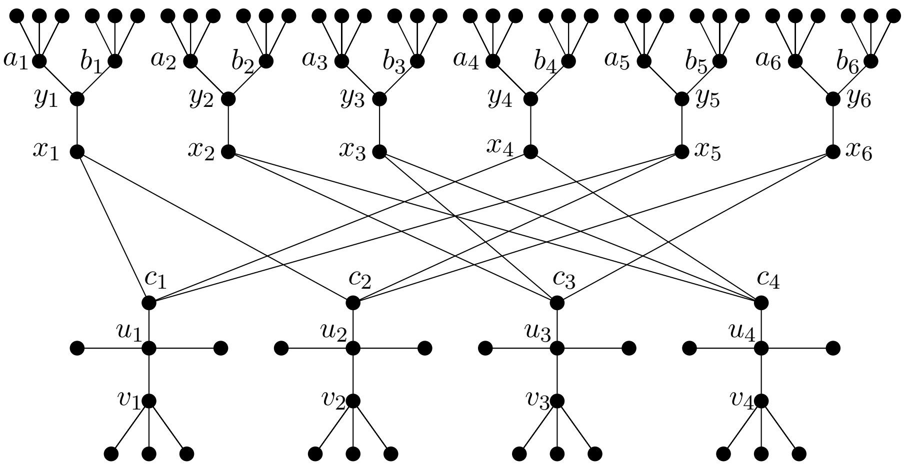

Figure 4. NP-completeness for bipartite graphs.

Suppose that the instance X, C of X3C has a solution C'. We construct a T2RDF f on G of weight k. For each i, assign the set $\{1, 2\}$ to a_i, b_i, the set $\{1\}$ to y_i and $\emptyset$ to the remaining vertices of H_i. For every j, assign $\{1, 2\}$ to u_j and v_j, and $\emptyset$ to each leaf. In addition, if for every C_j, assign to c_j the set $\{2\}$ if $C_j \in C'$ and $\emptyset$ if $C_j \notin C'$. Note that since C' exists, its cardinality is precisely q, and so the number of c_j's assigned $\{2\}$ is q, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$, Since C' is a solution for X3C, every vertex x_i in X satisfies $f(N[x_i]) = \{1, 2\}$. Hence, it is straightforward to see that f is a T2RDF with weight $f(V) = 4t + q + 15q = k$.

Conversely, suppose that G has a T2RDF with weight at most k. Among all such functions, let $g = (V_\emptyset, V_1, V_2, V_{12})$ be one such that the number of vertices of $\{y_1, y_2, \ldots, y_{3q}\}$ assigned $\{1, 2\}$ is as small as possible. As observed above, since each H_i has weight at least 5, we may assume that $g(a_i) = g(b_i) = \{1, 2\}$ and $|g(y_i)| > 0$ so that vertices a_i, b_i are not isolated in the subgraph induced by $V_1 \cup V_2 \cup V_{12}$. Hence each leaf neighbor of a_i or b_i is assigned $\emptyset$ under g. Assume that $g(y_i) = \{1, 2\}$ for some i. Observe that if $|g(x_i)| > 0$, then reassigning $\{1\}$ to y_i provides a T2RDF g' with less vertices y_i assigned $\{1, 2\}$ than under g, contradicting our choice of g. Hence $g(x_i) = \emptyset$. But then reassigning $\{1\}$ to each of y_i and x_i instead of $\{1, 2\}$ and $\emptyset$, respectively, provides a T2RDF g' with less vertices y_i assigned $\{1, 2\}$ than under g, a contradiction too. Therefore $|g(y_i)| = 1$ for every $i \in \{1, 2, \ldots, 3q\}$. On the other hand, the total weight of all double stars corresponding to elements of C is $4t$. In this case, we can assume that $g(u_j) = g(v_j) = \{1, 2\}$ and so each leaf neighbor of u_j or v_j is assigned $\emptyset$ under g. Note that each c_j can be assigned $\emptyset$ since $g(u_j) = \{1, 2\}$. Since $w(g) \leq 4t + 16q$ and the total weight assigned to vertices of $V(G) - (X \cup Y)$ is $4t + 15q$, we have to assign to vertices of $(X \cup Y)$ sets whose total cardinalities not exceeding q so that each vertex $x_i \in X$ has either $|g(x_i)| > 0$ or has two neighbors in $V_1 \cup V_2$ so that $f(N[x_i]) = \{1, 2\}$. Since $|X| = 3q$, it is clear that this is only possible if there are q vertices of $\{c_1, c_2, \ldots, c_t\}$ belonging to $V_1 \cup V_2$. Since each c_j has a exactly three neighbors in $\{x_1, x_2, \ldots, x_{3q}\}$, we deduce that $C' = \{C_j : |g(c_j)| = 1\}$ is an exact cover for C. ■

The next result is obtained by using the same proof as for Theorem 12 on the (same) graph G built for the transformation by adding all edges between the c_j's so that the resulting graph is chordal.

Theorem 13. TOTAL 2-RAINBOW DOMINATION *is* NP*-complete for chordal graphs.*

Acknowledgements

The authors are grateful to anonymous referees for their remarks and suggestions that helped improve the manuscript. H. Abdollahzadeh Ahangar was supported by the Babol Noshirvani University of Technology under research Grant Number BNUT/385001/97.

References

[1] H. Abdollahzadeh Ahangar, J. Amjadi, N. Jafari Rad and V. Samodivkin, *Total k-rainbow domination in graphs*, Comm. Combin. Optim. **3** (2018) 37–50. doi:10.22049/CCO.2018.25719.1021

[2] H. Abdollahzadeh Ahangar, M.A. Henning, V. Samodivkin and I.G. Yero, *Total Roman domination in graphs*, Appl. Anal. Discrete Math. **10** (2016) 501–517. doi:10.2298/AADM160802017A

[3] H. Abdollahzadeh Ahangar, M. Khaibari, N. Jafari Rad and S.M. Sheikholeslami, *Graphs with large total 2-rainbow domination number*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2018) 841–846. doi:10.1007/s40995-017-0465-9

[4] G.J. Chang, J. Wu and X. Zhu, *Rainbow domination on trees*, Discrete Appl. Math. **158** (2010) 8–12.
doi:10.1016/j.dam.2009.08.010

[5] M. Chellali and N. Jafari Rad, *On* 2*-Rainbow domination and Roman domination in graphs*, Australas. J. Combin. **56** (2013) 85–93.

[6] B. Brešar, M.A. Henning and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008) 201–213.

[7] B. Brešar and T.K. Sumenjak, *On the* 2*-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007) 2394–2400.
doi:10.1016/j.dam.2007.07.018

[8] N. Dehgardi, S.M. Sheikholeslami and L. Volkmann, *The rainbow domination subdivision numbers of graphs*, Mat. Vesnik **67** (2015) 102–114.

[9] E. DeLaVina, C.E. Larson, R. Pepper and B. Waller, *On total domination and support vertices of a tree*, AKCE Int. J. Graphs Comb. **7** (2010) 85–95.

[10] M. Falahat, S.M. Sheikholeslami and L. Volkmann, *New bounds on the rainbow domination subdivision number*, Filomat **28** (2014) 615–622.
doi:10.2298/FIL1403615F

[11] C.-H. Liu and G.J. Chang, *Roman domination on strongly chordal graphs*, J. Comb. Optim. **26** (2013) 608–619.
doi:10.1007/s10878-012-9482-y

[12] Y. Wu and N. Jafari Rad, *Bounds on the* 2*-Rainbow domination number of graphs*, Graphs Combin. **29** (2013) 1125–1133.
doi:10.1007/s00373-012-1158-y

[13] Y. Wu and H. Xing, *Note on* 2*-rainbow domination and Roman domination in graphs*, Appl. Math. Lett. **23** (2010) 706–709.
doi:10.1016/j.aml.2010.02.012

Received 11 April 2018
Revised 22 October 2018
Accepted 3 November 2018