

TOTAL 2-RAINBOW DOMINATION NUMBERS OF TREES

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Abstract

A 2-rainbow dominating function (2RDF) of a graph $G = (V(G), E(G))$ is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for every vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v . A total 2-rainbow dominating function f of a graph with no isolated vertices is a 2RDF with the additional condition that the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The total 2-rainbow domination number, $\gamma_{tr2}(G)$, is the minimum weight of a total 2-rainbow dominating function of G . In this paper, we establish some sharp upper and lower bounds on the total 2-rainbow domination number of a tree. Moreover, we show that the decision problem associated with $\gamma_{tr2}(G)$ is NP-complete for bipartite and chordal graphs.

Keywords: 2-rainbow dominating function, 2-rainbow domination number, total 2-rainbow dominating function, total 2-rainbow domination number.

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1. INTRODUCTION

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E) such that G has no isolated vertices. The order of a graph G is the number of vertices in G , denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *maximum degree* of a graph G is denoted by $\Delta = \Delta(G)$. The *open neighborhood* of a set $S \subseteq V$ is the set $N_G(S) = N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N[S] = N(S) \cup S$. The *diameter* of G , denoted by $\text{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G . A *leaf* of a tree T is a vertex of degree one, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. If v is a support vertex, then $L(v)$ will denote the set of the leaves attached to v . For a vertex v in a rooted tree T , let $C(v)$ denote the set of children of v , $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v , $\text{depth}(v)$, is the maximum distance from v to a vertex in $D(v)$. We denote by T_v the induced subgraph of T with vertex set $D[v]$. The *independence number* of a graph G , denoted $\alpha(G)$, is the order of a largest subset of vertices in which no two are adjacent. A *vertex cover* of G is a set of vertices S that covers all the edges, i.e., every edge is incident with a vertex of S . The *vertex cover number* $\beta(G)$ is the minimum cardinality of a vertex cover of G . It is well-known that for every graph G of order n , $\beta(G) + \alpha(G) = n$.

A *total Roman dominating function* of a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the following conditions: (i) every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$, and (ii) the subgraph of G induced by the set of all vertices of positive weight has no isolated vertices. The weight of a total Roman dominating function f is the value $w(f) = \sum_{u \in V(G)} f(u)$, and the *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a total Roman dominating function of G . The concept of total Roman domination in graphs was introduced by Liu and Chang [11] and studied for example in [2].

A *2-rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled. The weight of a 2RDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$, and the minimum weight of a 2RDF is called the *2-rainbow domination number* of G , denoted by $\gamma_{r2}(G)$. The concept of 2-rainbow domination was introduced by Brešar *et al.* [6], and has been studied by several authors, for example [4, 5, 7, 8, 10, 12, 13].

A 2RDF f is called a *total 2-rainbow dominating function*, or just T2RDF, if the subgraph of G induced by $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertices. The *total 2-rainbow domination number*, $\gamma_{tr2}(G)$, is the minimum weight of a total

2-rainbow dominating function of G , and a T2RDF of G with weight $\gamma_{tr2}(G)$ is called a $\gamma_{tr2}(G)$ -function. We note that if f is a T2RDF of a graph G and H is a subgraph of G , then we denote the restriction of f to H by $f|_{V(H)}$. Total 2-rainbow domination was recently introduced by Abdollahzadeh Ahangar *et al.* in [1] and has been studied in [3].

Before presenting our main results, we present some straightforward observations.

Observation 1. *If v is a strong support vertex in a graph G , then there exists a $\gamma_{tr2}(G)$ -function f such that $f(v) = \{1, 2\}$.*

Observation 2. *If u_1 and u_2 are two adjacent support vertices in a graph G , then there exists a $\gamma_{tr2}(G)$ -function f such that $f(u_1) = f(u_2) = \{1, 2\}$.*

Observation 3. *If v is a leaf neighbor of a support vertex of degree 2 in a graph G , then there exists a $\gamma_{tr2}(G)$ -function f such that $|f(v)| = 1$.*

2. LOWER BOUNDS

In this section, we establish some sharp lower bounds on the total 2-rainbow domination number of a tree. We begin by recalling the following result given in [1] for paths.

Proposition 4. *For $n \geq 2$, $\gamma_{tr2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.*

Our first lower bound on $\gamma_{tr2}(T)$ is in terms of the order and the number of leaves of a tree T .

Theorem 5. *Let T be a non-trivial tree of order n with $\ell(T)$ leaves. Then*

$$\gamma_{tr2}(T) \geq \left\lceil \frac{2(n + 3 - \ell(T))}{3} \right\rceil.$$

This bound is sharp for paths, stars and double stars.

Proof. We use an induction on n . It is easy to check that the statement holds for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that for every non-trivial tree T of order at most $n - 1$ the result is true. Let T be a tree of order $n \geq 5$. If T is a star, then $\gamma_{tr2}(T) = 3 = \left\lceil \frac{2(n+3-(n-1))}{3} \right\rceil$. If T is a double star, then $\gamma_{tr2}(T) = 4 = \left\lceil \frac{2(n+3-(n-2))}{3} \right\rceil$. Henceforth we can assume that T has diameter at least 4.

Suppose that T has a strong support vertex u . Let $T' = T - u'$, where u' is a leaf neighbor of u . By Observation 1, there exists a $\gamma_{tr2}(T)$ -function g such

that $g(u) = \{1, 2\}$. We may assume, without loss of generality, that $g(u') = \emptyset$. Then the function g , restricted to T' is a T2RDF. We can apply the inductive hypothesis to the tree T' and deduce that

$$\gamma_{tr2}(T) = \omega(g) \geq \gamma_{tr2}(T') \geq \left\lceil \frac{2((n-1)+3-(\ell(T)-1))}{3} \right\rceil = \left\lceil \frac{2(n+3-\ell(T))}{3} \right\rceil.$$

Therefore, from now on we suppose that T has no strong support vertex.

Let $v_1 v_2 \cdots v_k$ be a diametral path of rooted tree T with root vertex v_k . Since T has no strong support vertex, each child of v_3 is either a leaf or a support vertex of degree 2. Let f be a $\gamma_{tr2}(T)$ -function, and consider the following cases.

Case 1. $\deg_T(v_3) \geq 3$. Assume first that v_3 is a support vertex. By Observation 2, we may assume that $f(v_2) = f(v_3) = \{1, 2\}$. Let $T' = T - v_1$ and define $h : V(T') \rightarrow \mathcal{P}(\{1, 2\})$ by $h(v_2) = \{1\}$ and $h(x) = f(x)$ for $x \in V(T') - \{v_2\}$. Clearly, h is a T2RDF of T' . Using the fact that $n' = n - 1$ and $\ell(T') = \ell(T)$, it follows from the induction hypothesis that

$$\begin{aligned} \gamma_{tr2}(T) &= \omega(f) = \omega(h) + 1 \geq \gamma_{tr2}(T') + 1 \\ &\geq \left\lceil \frac{2((n-1)+3-\ell(T))}{3} \right\rceil + 1 \geq \left\lceil \frac{2(n+3-\ell(T))+1}{3} \right\rceil, \end{aligned}$$

as desired. Hence we assume that v_3 is not a support vertex, and thus every child of v_3 is a support vertex of degree 2. Let $u_2 \neq v_2$ be a child of v_3 and u_1 the leaf neighbor of u_2 . Clearly, $|f(u_1)| + |f(u_2)| \geq 2$ and $|f(v_1)| + |f(v_2)| \geq 2$. Let $T' = T - \{u_1, u_2\}$ and define $h : V(T') \rightarrow \mathcal{P}(\{1, 2\})$ by $h(v_3) = \{1\} \cup f(v_3)$ and $h(x) = f(x)$ for $x \in V(T') - \{v_3\}$. Clearly, h is a T2RDF of T' , $n' = n - 2$ and $\ell(T') = \ell(T) - 1$. It follows from the induction hypothesis that

$$\begin{aligned} \gamma_{tr2}(T) &= \omega(f) \geq \omega(h) + 1 \geq \gamma_{tr2}(T') + 1 \\ &\geq \left\lceil \frac{2((n-2)+3-(\ell(T)-1))}{3} \right\rceil + 1 \geq \left\lceil \frac{2(n+3-\ell(T))+1}{3} \right\rceil, \end{aligned}$$

as desired.

Case 2. $\deg_T(v_3) = 2$. As above we have $|f(v_1)| + |f(v_2)| \geq 2$. Suppose first that $|f(v_1)| + |f(v_2)| \geq 3$, and let $T' = T - v_1$. Then the function $h : V(T') \rightarrow \mathcal{P}(\{1, 2\})$ defined by $h(v_3) = \{1\}$ and $h(x) = f(x)$ for $x \in V(T') - \{v_3\}$ is a T2RDF of T' . By induction on T' and using the fact that $n' = n - 1$, $\ell(T') = \ell(T)$, we obtain $\gamma_{tr2}(T) \geq \left\lceil \frac{2(n+3-\ell(T))+1}{3} \right\rceil$, as desired. Therefore, we assume for the next that $|f(v_1)| + |f(v_2)| = 2$. Now, if $f(v_3) \neq \emptyset$, then the function f , restricted to $T - v_1$ is a T2RDF of $T - v_1$ of weight $\gamma_{tr2}(T) - 1$, and by the induction hypothesis on $T - v_1$ we obtain

$$\gamma_{tr2}(T) \geq \left\lceil \frac{2(n+3-\ell(T))+1}{3} \right\rceil.$$

Hence let $f(v_3) = \emptyset$. Let $T' = T - \{v_1, v_2, v_3\}$ and recall that T has diameter at least four. If T' has order $n' = 2$, then $T = P_5$, and by Proposition 4 the result is valid. Hence let $n' \geq 3$. Then $f|_{V(T')}$ is a T2RDF of T' of weight $\omega(f) - 2$. Using the fact that $n' = n - 3$ and $\ell(T') \leq \ell(T)$, and by applying the induction on T' , we obtain

$$\begin{aligned}\gamma_{tr2}(T) &= \omega(f) = \omega(f|_{V(T')}) + 2 \geq \gamma_{tr2}(T') + 2 \\ &\geq \left\lceil \frac{2((n-3)+3-\ell(T))}{3} \right\rceil + 2 = \left\lceil \frac{2(n+3-\ell(T))}{3} \right\rceil.\end{aligned}$$

This completes the proof. ■

Theorem 6. *If T is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then*

$$\gamma_{tr2}(T) \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta} \right\rceil,$$

and this bound is sharp.

Proof. The proof is by induction on n . One can easily check that the statement holds for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that the result is true for every non-trivial tree T' of order n' , with $3 \leq n' < n$. Let T be a tree of order n with $\ell(T)$ leaves and $s(T)$ support vertices. If $\text{diam}(T) = 2$, then T is a star, where $\gamma_{tr2}(T) = 3 = 2 + \left\lceil \frac{n-2}{n-1} \right\rceil$. If $\text{diam}(T) = 3$, then T is a double star, where $4 = \gamma_{tr2}(T) \geq 2 + \left\lceil \frac{n-4}{\Delta} \right\rceil$, and clearly the result is valid since $\left\lceil \frac{n-4}{\Delta} \right\rceil \leq 2$. Henceforth we may assume that $\text{diam}(T) \geq 4$.

Let $v_1 v_2 \cdots v_k$ be a diametral path of T and f be a $\gamma_{tr2}(T)$ -function. Without loss of generality, we assume $\deg_T(v_2) \leq \deg_T(v_{k-1})$. Consider the following situations.

Suppose first that v_3 is a support vertex adjacent to another support vertex different from v_2, v_4 or v_3 is adjacent to a strong support vertex different from v_2, v_4 . Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. Moreover, it is easy to see that $\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By the induction hypothesis on T' we obtain that

$$\begin{aligned}\gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 2 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 2 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil.\end{aligned}$$

Next, suppose that v_3 is not a support vertex and it is adjacent to a support vertex of degree two different from v_2 . Let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. On the other hand, if $\deg_T(v_2) \geq 3$, then $\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$, and if $\deg_T(v_2) = 2$, then

$\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 1$. Using the induction on T' and according to each situation, the result follows.

Suppose now that v_3 is a support vertex having no neighbor as support vertex besides v_2 and (possibly) v_4 . If $|f(x)| \geq 1$ for some $x \in N(v_3) - \{v_2\}$, then let $T' = T - T_{v_2}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(v_2)|$ and $s(T') = s(T) - 1$. Moreover, one can see that $\gamma_t(T) \leq \gamma_t(T') + 1$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By induction on T' , we obtain as above $\gamma_{tr2}(T) \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil$. Hence, we assume that $f(x) = \emptyset$ for all $x \in N(v_3) - \{v_2\}$. Thus $f(v_3) = \{1, 2\}$. Since, $f(v_4) = \emptyset$, we conclude that v_4 is not a support vertex and has no child of depth 1 which is a strong support vertex. Assume that $\deg_T(v_4) \geq 3$. If v_4 has a child of depth 1 say, u_2 , with u_1 as a leaf neighbor of u_2 , then let $T' = T - \{u_1, u_2\}$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - 1$ and $s(T') = s(T) - 1$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. By the induction hypothesis on T' we obtain that

$$\begin{aligned} \gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 2 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 2 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil. \end{aligned}$$

Therefore, we can assume that all children of v_4 have depth 2. According to the diametral path and the situations already considered, we conclude that each child of v_4 is a support vertex or has degree 2. If z is a child of v_4 with degree 2 with $z_1 \in N(z) - v_4$, then let $T' = T - T_z$. Clearly, $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - |L(z_1)|$ and $s(T') = s(T) - 1$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. Using the induction on T' , we obtain desired result. Hence, each child of v_4 is a support vertex assigned $\{1, 2\}$ under f . Let $T' = T - T_{v_3}$. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') = \ell(T) - (|L(v_2)| + |L(v_3)|)$ and $s(T') = s(T) - 2$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4$. By the induction hypothesis on T' we obtain that

$$\begin{aligned} \gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 4 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 4 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil. \end{aligned}$$

Now, let $\deg_T(v_4) = 2$ and $T' = T - T_{v_4}$. Note T' has order $n' \geq 1$ since $\text{diam}(T) \geq 4$. It is a routine matter to check that the result holds if $n' \in \{1, 2\}$. Hence let $n' \geq 3$. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') \geq \ell(T) - (|L(v_2)| + |L(v_3)|)$ and $s(T') \leq s(T) - 1$. On the other hand, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4$. Using the induction on T' , the result follows.

Finally, assume that $\deg_T(v_3) = 2$. First, assume that $f|_{T'}$ is a T2RDF of $T' = T - T_{v_3}$. Recall that T has diameter at least four. If T' has order 2, then T

is obtained from a star of order at least three and a path P_2 by adding an edge joining their leaves, and clearly the result holds. So assume that T' has order at least three. Then $\Delta(T) \geq \Delta(T')$, $\ell(T') \geq \ell(T) - |L(v_2)|$ and $s(T') \leq s(T)$. Moreover, $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. It follows from the induction hypothesis that

$$\begin{aligned} \gamma_{tr2}(T) &\geq \gamma_{tr2}(T') + 3 \geq \gamma_t(T') + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 3 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T') - s(T')}{\Delta(T')} \right\rceil + 1 \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil. \end{aligned}$$

Suppose now that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$. Hence, we have the following cases.

Case 1. $f(v_4) = \emptyset$. Then v_4 is not a support vertex and has no child of depth 1 which is a strong support vertex. Seeing the previous cases, it follows that any child of v_4 other than v_3 is either a support vertex of degree two or a vertex with depth 2 and degree 2. Moreover, since every child of v_4 is assigned a non-empty set, we conclude from our assumption that $f|_{T'}$ is not a T2RDF of $T' = T - T_{v_3}$ and that $\deg_T(v_4) \in \{2, 3\}$. We consider the following.

Subcase 1.1. $\deg_T(v_4) = 3$. Observe that T_{v_4} has exactly two support vertices, v_2 and say z . We note that z is either at distance one or two from v_4 . Let $T'' = T - T_{v_4}$. Clearly, T'' has order at least three, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(z)|)$, $s(T'') \leq s(T) - 1$ and $\gamma_t(T) \leq \gamma_t(T'') + 4$. Now, if z is at distance one from v_4 , then $|L(z)| = 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Also, if z is at distance two from v_4 , then $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 6$. Whatever the case, using the induction on T'' , the result follows.

Subcase 1.2. $\deg_T(v_4) = 2$. Let $T'' = T - T_{v_4}$. It is easy to check the result if $n(T'') \in \{1, 2\}$. Hence let $n(T'') \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - |L(v_2)|$ and $s(T'') \leq s(T)$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on T'' , the result follows.

Case 2. $|f(v_4)| \geq 1$ and thus $f(x) = \emptyset$ for each vertex $x \in N(v_4) - \{v_3\}$. Then every child of v_4 besides v_3 (if any) is leaf. To avoid the previous case when $f(v_4) = \emptyset$ we can assume that v_4 is a support vertex (else substitute the assignments of v_4 and v_5). Now if $f|_{T''}$ is a T2RDF of $T'' = T - T_{v_4}$, then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_4)|)$ and $s(T'') \leq s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$, the result follows by using the induction on T'' . Hence suppose that $f|_{T''}$ is not a T2RDF of $T'' = T - T_{v_4}$ and so v_5 has no child of depth 3 other than v_4 . Since $f(v_5) = \emptyset$, we conclude that v_5 is not a support vertex and has no child of depth 1 which is a strong support vertex. Consider the following situations.

Subcase 2.1. v_5 has a child of depth 1. Let u_2 be such a child of depth 1 and u_1 its the leaf neighbor. Note that $\deg_T(u_2) = 2$. Let $T'' = T - \{u_1, u_2\}$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Since $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2$, the result follows by using the induction on T'' .

Subcase 2.2. All children of v_5 different to v_4 have depth 2. Since $|f(x)| \leq 1$ for $x \in N(v_5) - \{v_4\}$, we deduce that every child of v_5 other than v_4 is not a support vertex. Let $z \neq v_4$ be a child of v_5 . If $\deg(z) = 2$ and $z' \in N(z) - \{v_5\}$, then let $T'' = T - T_z$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - |L(z')|$ and $s(T'') = s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 3$. Using the induction on T'' , the result follows. Hence suppose that $\deg_T(z) \geq 3$. If z has a child of depth 1 say, u_2 , of degree two, with u_1 as the leaf neighbor of u_2 , then let $T'' = T - \{u_1, u_2\}$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - 1$ and $s(T'') = s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 2$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2$. Using the induction on T'' , the result follows. Hence, all children of z are strong support vertex. Let $|C(z)| = k$ and x_1, \dots, x_k be the children of z , and let $T'' = T - T_z$. Clearly, $\Delta(T) \geq \Delta(T'')$, $\ell(T'') = \ell(T) - \left(\sum_{i=1}^k |L(x_i)|\right)$ and $s(T'') = s(T) - k$. On the other hand, $\gamma_t(T) \leq \gamma_t(T'') + k + 1$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 2k - 1$. It follows from the induction hypothesis that

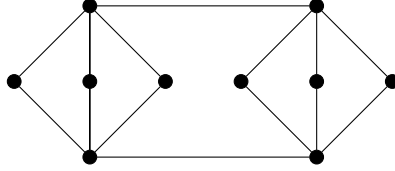
$$\begin{aligned} \gamma_{tr2}(T) &\geq \gamma_{tr2}(T'') + 2k + 1 \geq \gamma_t(T'') + \left\lceil \frac{\ell(T'') - s(T'')}{\Delta(T'')} \right\rceil + 2k + 1 \\ &\geq \gamma_t(T) + \left\lceil \frac{\ell(T'') - s(T'')}{\Delta(T'')} \right\rceil + k \geq \gamma_t(T) + \left\lceil \frac{\ell(T) - s(T)}{\Delta(T)} \right\rceil. \end{aligned}$$

Subcase 2.3. $\deg_T(v_5) = 2$. Let $T'' = T - T_{v_5}$. Note that T'' may have order $n'' = 0$. However, it is easy to check that the result is valid for $n'' \leq 2$. Hence, let $n'' \geq 3$. Then $\Delta(T) \geq \Delta(T'')$, $\ell(T'') \geq \ell(T) - (|L(v_2)| + |L(v_4)|)$ and $s(T'') \leq s(T) - 1$. Also, $\gamma_t(T) \leq \gamma_t(T'') + 3$ and $\gamma_{tr2}(T'') \leq \gamma_{tr2}(T) - 5$. Using the induction on T'' , the result follows. This completes the proof. ■

Obviously, $\gamma_{tr2}(G) \leq \gamma_{tR}(G)$ for every graph G without isolated vertices. In the following, we provide an upper bound on the ratio $\gamma_{tR}(G)/\gamma_{tr2}(G)$ for arbitrary graphs G . Moreover, this ratio will be slightly improved for the class of trees.

Theorem 7. *If G is a graph without isolated vertices, then $\gamma_{tR}(G) \leq \frac{3}{2}\gamma_{tr2}(G)$. This bound is sharp for the graph in Figure 1.*

Proof. Let f be a $\gamma_{tr2}(G)$ -function. For every $i \in \{1, 2\}$, let X_i be the set of all vertices u for which $i \in f(u)$. Clearly, if a vertex of G is assigned $\{1, 2\}$ under f , then $X_1 \cap X_2 \neq \emptyset$. Also, it is obvious that $|X_1| + |X_2| = \gamma_{tr2}(G)$. Now assume, without loss of generality, that $|X_1| \leq |X_2|$. Then $|X_1| \leq \frac{|X_1| + |X_2|}{2} = \frac{\gamma_{tr2}(G)}{2}$, and

Figure 1. Graph G with $\gamma_{tR}(G) = \frac{3}{2}\gamma_{tr2}(G) = 6$.

the function $g : V(G) \rightarrow \{0, 1, 2\}$ defined by $g(x) = 0$ if $f(x) = \emptyset$, $g(x) = 1$ if $f(x) = \{2\}$, and $g(x) = 2$ if $1 \in f(x)$, is a total Roman dominating function on G , implying that

$$\gamma_{tR}(G) \leq \omega(g) = 2|X_1| + |X_2| \leq \frac{|X_1| + |X_2|}{2} + |X_1| + |X_2| \leq \frac{3}{2}\gamma_{tr2}(G). \quad \blacksquare$$

Theorem 8. For every non-trivial tree T ,

$$\gamma_{tR}(T) \leq \frac{3}{2}\gamma_{tr2}(T) - 1,$$

and this bound is sharp for P_n such that $n \equiv 2 \pmod{3}$.

Proof. The proof is by induction on n . The statement is valid for all trees of order $n \in \{2, 3, 4\}$. Let $n \geq 5$ and assume that for every tree T' of order at most $n - 1$, $\gamma_{tR}(T') \leq \frac{3}{2}\gamma_{tr2}(T') - 1$. Let T be a tree of order n . Since stars and double stars T satisfy $\gamma_{tr2}(T) = 3 = \gamma_{tR}(T)$, the result holds. Therefore, we can assume that $\text{diam}(T) \geq 4$.

If T has a support vertex, say u , with $|L(u)| \geq 3$, then let $T' = T - u'$, where u' is a leaf neighbor of u . Clearly $\gamma_{tR}(T) \leq \gamma_{tR}(T')$. On the other hand, by Observation 1, there exists a $\gamma_{tr2}(T)$ -function g such that $g(u) = \{1, 2\}$. Also, we can assume that $g(u') = \emptyset$. It follows that $g|_{V(T')}$ is a T2RDF of T' , and thus $\gamma_{tr2}(T') \leq \gamma_{tr2}(T)$. By the inductive hypothesis on T' , we obtain

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') \leq 3\gamma_{tr2}(T') - 2 \leq 3\gamma_{tr2}(T) - 2.$$

Hence we assume that every support vertex in T is adjacent to at most two leaves. Let $v_1v_2 \cdots v_k$ be a diametral path in T with root vertex v_k . We consider the following cases.

Case 1. $\deg_T(v_3) = 2$. Let $T' = T - T_{v_3}$. Then $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. It follows from the induction hypothesis that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 6 \leq 3\gamma_{tr2}(T') + 4 \leq 3\gamma_{tr2}(T) - 2.$$

Case 2. $\deg_T(v_3) \geq 3$. Consider the following subcases.

Subcase 2.1. Suppose that v_3 is a support vertex adjacent to another support vertex different from v_2 and v_4 , or v_3 is adjacent to a strong support vertex different from v_2 and v_4 . Let $T' = T - T_{v_2}$. It is easy to see that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2$. It follows from the induction hypothesis that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 4 \leq 3\gamma_{tr2}(T') + 2 \leq 3\gamma_{tr2}(T) - 4 < 3\gamma_{tr2}(T) - 2.$$

Subcase 2.2. v_3 is not a support vertex. Since $\deg_T(v_3) \geq 3$, every child of v_3 is a support vertex. Moreover, according to Subcase 2.1, all support vertices of T_{v_3} , but possibly v_2 , have degree two. Let $t = \deg_T(v_3) - 1 \geq 2$. Let $T' = T - T_{v_3}$. It is easy to see that $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2t + 1$. Among all $\gamma_{tr2}(T)$ -functions, let g be one for which $|g(v_3)|$ is as small as possible. Clearly, for every child x of v_3 we have $|g(N[x])| \geq 2$. Now, if $g(v_3) = \emptyset$, then $g|_{V(T')}$ is a T2RDF of T' of weight $\omega(g) - 2t$, and thus $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2t$. Hence assume that $g(v_3) \neq \emptyset$. The choice of g implies that $|g(v_3)| = 1$, and thus the weight of T_{v_3} under g is $2t + 1$. The choice of g also implies that $g(v_4) = \emptyset$. In that case, the function g' defined on $V(T')$ defined by $g'(v_4) = g(v_3)$ and $g'(x) = g(x)$ for all $x \in V(T') - \{v_4\}$ is a T2RDF of T' of weight $\gamma_{tr2}(T) - 2t$, and thus $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 2t$. In all cases, it follows from the induction hypothesis on T' that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 2 + 4t \leq 3\gamma_{tr2}(T') + 4t \leq 3\gamma_{tr2}(T) - 6t + 4t < 3\gamma_{tr2}(T) - 2.$$

Subcase 2.3. v_3 is a support vertex adjacent to no support vertex besides v_2 and (possibly) v_4 . Let f be a $\gamma_{tr2}(T)$ -function. If $|f(v_4)| \geq 1$ or there exists a vertex $x \in N_T(v_4) - \{v_3\}$ with $|f(x)| \geq 1$, then let $T' = T - T_{v_3}$. Obviously, $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 3$. It follows from the induction hypothesis that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 8 \leq 3\gamma_{tr2}(T') + 6 \leq 3\gamma_{tr2}(T) - 9 + 6 < 3\gamma_{tr2}(T) - 2.$$

Hence we can assume that $f(x) = \emptyset$ for each $x \in N_T[v_4] - \{v_3\}$. Therefore, all children of v_4 have depth 2. According to Case 1 and the diametral path, we conclude that each child of v_4 is a support vertex. Since we assumed that $f(x) = \emptyset$ for each $x \in N_T[v_4] - \{v_3\}$, we deduce that $d_T(v_4) = 2$. In this case, let $T' = T - T_{v_4}$. Recall that T has diameter at least four. Suppose that T' has order one. Clearly, T is a tree with three support vertices v_2, v_3, v_4 and the remaining vertices are leaves. Hence $\gamma_{tR}(T) = \gamma_{tr2}(T) = 6$, and thus the result holds. So suppose that T' is nontrivial. Then $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 4$ and $\gamma_{tr2}(T') \leq \gamma_{tr2}(T) - 4$. By induction on T' we deduce that

$$2\gamma_{tR}(T) \leq 2\gamma_{tR}(T') + 8 \leq 3\gamma_{tr2}(T') + 6 \leq 3\gamma_{tr2}(T) - 12 + 6 < 3\gamma_{tr2}(T) - 2.$$

This completes the proof. ■

3. UPPER BOUNDS

In this section, we provide two upper bounds on the total 2-rainbow domination number of a tree. The first one we present is in terms of the order and the number of support vertices of a tree.

Theorem 9. *If T is a tree of order $n \geq 4$ with s support vertices, then*

$$\gamma_{tr2}(T) \leq \frac{2(n+s)}{3},$$

and this bound is sharp for P_n such that $n \equiv 1 \pmod{3}$.

Proof. The proof is by induction on n . It is a routine matter to check that the statement holds if $n \in \{4, 5\}$. Hence, let $n \geq 6$ and assume that for every T' or order $n' < n$ with s' support vertices satisfies $\gamma_{tr2}(T') \leq \frac{2(n'+s')}{3}$. Let T be a tree of order n . If T is a star, then $\gamma_{tr2}(T) = 3 < \frac{2(n+1)}{3}$. Likewise, if T is a double star, then $\gamma_{tr2}(T) = 4 < \frac{2(n+2)}{3}$. Henceforth we can assume T has diameter at least four.

If T has a strong support vertex u adjacent to at least three leaves, then let $T' = T - u'$, where u' is a leaf neighbor of u . Clearly, any $\gamma_{tr2}(T')$ -function can be extended to T2RDF of T by assigning \emptyset to vertex u' , and thus $\gamma_{tr2}(T) \leq \gamma_{tr2}(T')$. The result follows by using the induction on T' , with $n' = n - 1$ and $s' = s$. Therefore, we will assume that every support vertex of T is adjacent to at most two leaves.

Let $v_1 v_2 \cdots v_k$ be a diametral path in T and root T in v_k . We consider the following cases.

Case 1. $\deg_T(v_2) = 3$. Thus v_2 has two leaf neighbors. We distinguish between the following situations.

Subcase 1.1. $\deg_T(v_3) \geq 3$. Suppose first that v_3 is a support vertex. Let $T' = T - T_{v_2}$. Then $n' = n - 3$ and $s' = s - 1$. Let f be a $\gamma_{tr2}(T')$ -function. Since v_3 is a support vertex of T' , we must have $|f(v_3)| \geq 1$. Then the function $g : V(T) \rightarrow \mathcal{P}(\{1, 2\})$ defined by $g(v_2) = \{1, 2\}$, $g(x) = \emptyset$ for $x \in L(v_2)$ and $g(x) = f(x)$ otherwise, is a T2RDF of T of weight $\gamma_{tr2}(T') + 2$. By induction on T' , we have

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.$$

Suppose now that v_3 is not a support vertex. Thus every child of v_3 is a support vertex with degree either 2 or 3. Let u_2 be a child of v_3 different from v_2 . If $\deg_T(u_2) = 3$, then let $T' = T - T_{v_2}$. By using a similar argument to that used above, we obtain $\gamma_{tr2}(T) < \frac{2(n+s)}{3}$. Thus let $\deg_T(u_2) = 2$ with u_1 as the unique

leaf of u_2 . Let $T' = T - \{u_1, u_2\}$. Clearly, any $\gamma_{tr2}(T')$ -function can be extended to a T2RDF of T by assigning the set $\{1\}$ to both u_1 and u_2 . Since $n' = n - 2$ and $s' = s - 1$, using the induction on T' we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

Subcase 1.2. $\deg_T(v_3) = 2$. Recall that since T has diameter at least four, $\deg_T(v_4) \geq 2$. Assume that $\deg_T(v_4) \geq 3$, and let $T' = T - T_{v_3}$. Observe that T' has order $n' \geq 3$. If $n' = 3$, then T is a tree of order 7 with 2 support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3} = 6$. Hence we assume that $n' \geq 4$. Clearly, any $\gamma_{tr2}(T')$ -function can be extended to a T2RDF of T by assigning $\{1, 2\}$ to v_2 , $\{1\}$ to v_3 and \emptyset to the leaves of $L(v_2)$. By induction on T' and using the fact that $n = n - 4$ and $s' = s - 1$ we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$

So, suppose for the sequel that $\deg_T(v_4) = 2$. Let $T' = T - T_{v_2}$. Note that $n' \geq 3$. If $n' = 3$, then T' has order 6 with 2 support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3} = \frac{16}{3}$. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T)$ -function f such that $|f(v_3)| = 1$ and clearly such a function can be extended to a T2RDF of T by assigning $\{1, 2\}$ to v_2 and \emptyset to the leaves of $L(v_2)$. Hence $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2$. By induction on T' and using the fact that $n = n - 3$ and $s' = s$, we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$

Case 2. $\deg_T(v_2) = 2$. Seeing the previous case, we may assume that every child of v_3 which is a support vertex has degree two. Consider the following subcases.

Subcase 2.1. $\deg_T(v_3) \geq 3$. Let $T' = T - \{v_1, v_2\}$. Since any $\gamma_{tr2}(T')$ -function can be extended to a T2RDF of T by assigning the set $\{1\}$ to v_1 and v_2 , $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2$. Using the induction on T' , where $n = n - 2$ and $s' = s - 1$, we obtain

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

Subcase 2.2. $\deg_T(v_3) = 2$. We consider some additional subcases.

Subcase 2.2.1. $\deg_T(v_4) \geq 3$. Let $T' = T - \{v_1, v_2, v_3\}$. Note that $n' \geq 3$. If $n' = 3$, then T is a tree of order 6 with two support vertices, where $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3} = \frac{16}{3}$, and thus the result is valid. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$ -functions, let f be one such that $|f(v_4)|$ is as large as possible. If $|f(v_4)| \geq 1$,

then define the function g on $V(T)$ as follows: $g(x) = f(x)$ for all $x \in V(T')$, $g(v_3) = \emptyset$ and $g(v_1) = g(v_2) = \{1\}$ or $\{2\}$ so that $g(N[v_3]) = \{1, 2\}$. Clearly, g is a T2RDF of T of weight $\gamma_{tr2}(T') + 2$. By induction on T' and using the fact that $n' = n - 3$ and $s' = s - 1$ we deduce that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s - 1)}{3} + 2 < \frac{2(n + s)}{3}.$$

For the sequel we can assume that $f(v_4) = \emptyset$. Clearly in that case, v_4 is not a support vertex. By the choice of the diametral path and taking into account the previous cases, we can assume that every child of v_4 with depth two and different from v_3 has degree 2. We consider the following.

(i) v_4 has a child u_2 which is a support vertex. Since $f(v_4) = \emptyset$, we conclude that $\deg_T(u_2) = 2$. Let u_1 be the leaf neighbor of u_2 and let $T'' = T - \{u_1, u_2\}$. Clearly, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$, $n'' = n - 2$ and $s'' = s - 1$. By induction on T'' , it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

(ii) There is a pendant path $v_4 u_3 u_2 u_1$ in T , where $u_3 \neq v_3$. Since $|f(v_4)| = 0$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| = 3$. Define the function g on T' by $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(v_4) = \{2\}$, and $g(x) = f(x)$ otherwise. Clearly g is a $\gamma_{tr2}(T')$ -function $|g(v_4)| > |f(v_4)| = 0$, contradicting our choice of f .

Subcase 2.2.2. $\deg_T(v_4) = 2$. If $\deg_T(v_5) = 2$, then let $T' = T - \{v_1, v_2, v_3\}$. Note that T' has order $n' \geq 3$. If $n = 3$, then T is a path P_6 , where $\gamma_{tr2}(P_6) = 5$ (by Proposition 4) and the result is valid. Hence let $n' \geq 4$. By Observation 3, there exists a $\gamma_{tr2}(T')$ -function f such that $|f(v_4)| = 1$, and such a function can be extended to a T2RDF of T by assigning \emptyset to v_3 , $\{1\}$ to v_1 and $\{1, 2\} - f(v_4)$ to v_2 . It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2 \leq \frac{2(n' + s')}{3} + 2 = \frac{2(n - 3 + s)}{3} + 2 = \frac{2(n + s)}{3}.$$

Assume now that $\deg_T(v_5) \geq 3$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that T' has order $n' \geq 3$. If $n' = 3$, then T is a tree of order 7 obtained from a path P_6 by adding a new vertex attached to one of the two support vertices of the path P_6 . It is easy to check that $\gamma_{tr2}(T) = 5 < \frac{2(n+s)}{3}$. Hence let $n' \geq 4$. Among all $\gamma_{tr2}(T')$ -functions, let f be one such that $|f(v_5)|$ is as large as possible. If $|f(v_5)| \geq 1$, then f can be extended to a T2RDF of T by assigning \emptyset to v_4 , $\{1\}$ to v_1 and v_2 , and either $\{1\}$ or $\{2\}$ to v_3 so that $f(N[v_4]) = \{1, 2\}$. By induction on T' , it follows that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 3 \leq \frac{2(n' + s')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$

For the sequel, we can assume that $f(v_5) = \emptyset$. Trivially, v_5 is not a support vertex. Also, every child of v_5 with depth one has degree two. We consider the following.

(i) v_5 has a child with depth 3. Let $u_1 \neq v_1$ be a leaf at distance four from v_5 and let $v_5 u_4 u_3 u_2 u_1$ be the unique path between u_1 and v_5 . According to Cases 1 and 2 and Subcases 2.1 and 2.2, we must assume that each of u_4, u_3 and u_2 has degree two. Moreover, since $f(v_5) = \emptyset$ as assumed and according to the choice of f maximizing $|f(v_5)|$, we conclude that $|f(u_1)| + |f(u_2)| + |f(u_3)| + |f(u_4)| = 4$. Define the function g on $V(T')$ as follows: $g(u_1) = g(u_2) = \{1\}$, $g(u_3) = \emptyset$, $g(u_4) = g(v_5) = \{2\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a $\gamma_{tr2}(T')$ -function with $|g(v_5)| > |f(v_5)| = 0$, a contradiction.

(ii) v_5 has a child u_2 with depth one. Let u_1 be the leaf neighbor of u_2 . Let $T'' = T - \{u_1, u_2\}$. Obviously, $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$. It follows by induction on T'' that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2 \leq \frac{2(n'' + s'')}{3} + 2 = \frac{2(n - 2 + s - 1)}{3} + 2 = \frac{2(n + s)}{3}.$$

(iii) v_5 has a child, say w , with depth two having degree at least 3. Suppose first that w has at least two children as support vertices and let z be one of them having minimum degree. Note that $\deg_T(z) \in \{2, 3\}$ since every support vertex of T has at most two leaves. Let $T'' = T - (\{z\} \cup L(z))$. Then $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 2$, $n'' = n - 1 - |L(z)|$ and $s'' = s - 1$. Using the induction on T' we obtain the desired result. Now, let w has exactly one child, say t , as a support neighbor. Since $\deg_T(w) \geq 3$, we deduce that w is a support vertex. Let $T'' = T - T_w$. Note that T_w has order $n_w \in \{4, 5, 6\}$. Moreover, it is clear that $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 4$. It follows from the induction hypothesis that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 4 \leq \frac{2(n'' + s'')}{3} + 4 \leq \frac{2(n - n_w + s - 2)}{3} + 4 \leq \frac{2(n + s)}{3}.$$

(iv) v_5 has a child, say w , with depth two and having degree 2. Suppose first that the child z of w is a strong support. Let $L(z) = \{z_1, z_2\}$ and let $T'' = T - \{w, z, z_1, z_2\}$. Then $\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 3$, $n'' = n - 4$ and $s'' = s - 1$. It follows from the induction on T' that

$$\gamma_{tr2}(T) \leq \gamma_{tr2}(T'') + 3 \leq \frac{2(n'' + s'')}{3} + 3 = \frac{2(n - 4 + s - 1)}{3} + 3 < \frac{2(n + s)}{3}.$$

Now, suppose that the child z of w is a support vertex of degree two. Let $\deg_T(v_5) = k \geq 3$ and H_t for $t \geq 2$ be the tree obtained from a star $K_{1,t}$ by subdividing one edge three times and each of the remaining edges exactly twice. Seeing the previous situations, clearly T_{v_5} is isomorphic to H_{k-1} . Now let $T' = T - T_{v_5}$. We note that T' has order $n' \geq 3$. If $n' = 3$, then $T = H_k$, where $n = 3k + 2$,

$s(T) = k$ and $\gamma_{tr2}(T) = 2k + 2 < \frac{2(n+s)}{3}$. Hence we can assume that $n' \geq 4$. Then $\gamma_{tr2}(T) \leq \gamma_{tr2}(T') + 2k$, $n' = n - 3k + 1$ and $s(T') \leq s(T) - (k - 1) + 1$. It follows from the induction on T' that

$$\begin{aligned}\gamma_{tr2}(T) &\leq \gamma_{tr2}(T') + 2k \leq \frac{2(n' + s')}{3} + 2k \\ &= \frac{2(n - 3k + 1 + s - k + 2)}{3} + 2k \leq \frac{2(n + s)}{3}.\end{aligned}$$

This completes the proof. \blacksquare

Next we establish an upper bound on the total 2-rainbow domination number of a tree in terms of the vertex cover number. We first give an upper bound for arbitrary graphs.

Lemma 10. *Let G be a graph of order $n \geq 2$ with no isolated vertex and V_c a minimum vertex cover of G . Then*

$$\gamma_{tr2}(G) \leq 2\beta(G) + r,$$

where r is the number of isolated vertices in the subgraph induced by V_c . This bound is sharp for the graphs in Figure 2.

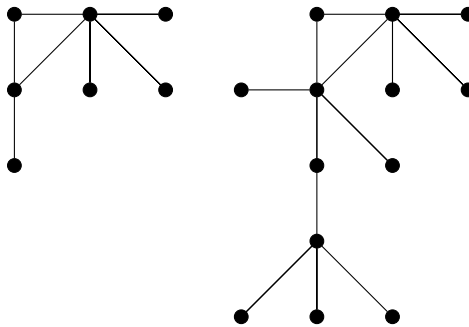


Figure 2. Two graphs G with $\gamma_{tr2}(G) = 2\beta(G) + r$.

Proof. Let V_c be a minimum vertex cover of G and I the set of isolated vertices in $G[V_c]$. Let $K = V(G) - V_c$. Since K is a maximum independent set, every vertex of V_c has a neighbor in K . Let D be a smallest subset of vertices of K that dominates all vertices of I . Obviously, $|D| \leq |I| = r$. Now define a function $f : V(G) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(x) = \{1, 2\}$ if $x \in V_c$, $f(x) = \{1\}$ if $x \in D$ and $f(x) = \emptyset$ otherwise. Clearly, f is a T2RDF of G of weight $2|V_c| + |D| \leq 2|V_c| + r$. \blacksquare

The proof of the next result is inspired by the proof of Theorem 2 in [9].

Theorem 11. *Let T be a tree of order $n \geq 3$ and let S' be the set of isolated vertices in the subgraph induced by the set of support vertices of T . Then*

$$\gamma_{tr2}(T) \leq 2\beta(T) + |S'|.$$

This bound is sharp for the graph in Figure 3.

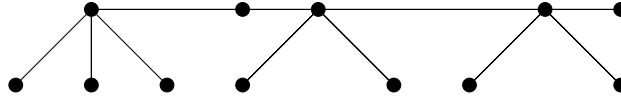


Figure 3. A tree T with $\gamma_{tr2}(T) = 2\beta(T) + |S'|$.

Proof. Let L and S denote the set of leaves and support vertices of a tree T , respectively. Let V_I be a maximum independent set that contains all leaves of T . Then $V_c = V - V_I$ is a vertex cover set of T . Note that $S \subseteq V_c$. If no support vertex of T is isolated in $T[V_c]$, then the result holds by Lemma 10. Hence, assume that u is a support vertex which is isolated in $T[V_c]$. Root T at u and let $A_1 = \{u\}$ and $A_2 = N(u)$. Clearly, $A_1 \subseteq V_c$ and $A_2 \subseteq V_I$. Assume that $A_3 = (N(A_2) - A_1) \cup B_{N(A_2)-A_1}$, where $B_{N(A_2)-A_1} = \{v \in V_c \mid v \text{ is in a component of } T[V_c] \text{ with a vertex of } N(A_2) - A_1\}$. Set $A_4 = N(A_3) - A_2$. Then we have $A_3 \subseteq V_c$ and $A_4 \subseteq V_I$.

We repeat this process so that at some odd number step $2k + 1$, we put

$$A_{2k+1} = (N(A_{2k}) - A_{2k-1}) \cup B_{N(A_{2k})-A_{2k-1}},$$

where $B_{N(A_{2k})-A_{2k-1}} = \{v \in V_c \mid v \text{ is in a component of } T[V_c] \text{ with a vertex of } N(A_{2k}) - A_{2k-1}\}$ and we set $A_{2k+2} = N(A_{2k+1}) - A_{2k}$. This process will terminate at some m^{th} step where m is even and A_m composed only of leaves. Note that $A_1 \cup \dots \cup A_m$ is a partition of $V(T)$. Obviously, $V_I = A_2 \cup A_4 \cup \dots \cup A_{m-2} \cup A_m$ and $V_c = A_1 \cup A_3 \cup \dots \cup A_{m-3} \cup A_{m-1}$. Note that if $v \in A_i$, for $i > 1$, has a neighbor in A_{i-1} , then it has only one neighbor in A_{i-1} .

Let $D_1 = V_c$. If $T[V_c]$ has isolated vertices that are support vertices in T , then let K be a smallest subset of vertices of $V_I - L$ that dominates these isolated support vertices. Clearly, $|K| \leq |S'|$. Now we consider the isolated vertices of $T[V_c]$ that are not support vertex in T . In decreasing order, we visit each A_i with odd index i , where $3 \leq i \leq m - 1$. We start with A_{m-1} and observe that if there is an isolate of $T[V_c]$ in A_{m-1} , then it is a support vertex and some vertex of K is adjacent to it. Now for each non-support isolated vertex v of $T[V_c]$ which is in A_{m-3} , if $N(v) \cap A_{m-2}$ is dominated by $A_{m-1} \cap V_c$, then remove v from D_1 and add to D_1 its unique neighbor in A_{m-4} , otherwise we leave v in D_1 . Continue this way for each odd i in decreasing order. That is, in general for A_i where i is odd,

if a non-support isolated vertex v of $T[V_c]$ is in A_i and $N(u) \cap A_{i+1}$ are dominated by $A_{i+2} \cap V_c$, then remove v from D_1 and add its unique neighbor in A_{i-1} to D_1 , otherwise we leave v in D_1 . This process terminates after $i = 3$. Now, if some vertex of A_2 is in K , then we are done. Otherwise remove u from D_1 and add to D_1 one of its neighbors. Note that $|D_1|$ has not increased. Now let $D_2 = D_1 \cup K$. Using an argument similar to that described in the proof of Theorem 2 in [9], we see that the induced subgraph $T[D_2]$ has no isolated vertex. Define the function $f : V(T) \rightarrow \mathcal{P}(\{1, 2\})$ by $f(x) = \{1, 2\}$ for $x \in D_1$, $f(x) = \{1\}$ for $x \in K$ and $f(x) = \emptyset$ otherwise. Clearly, f is a T2RDF of T and thus

$$\gamma_{tr2}(T) \leq 2|V_c| + |K| \leq 2\beta(T) + |S'|.$$

This achieves that proof. ■

4. COMPLEXITY

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as TOTAL 2-RAINBOW DOMINATION:

TOTAL 2-RAINBOW DOMINATION

Instance. Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question. Does G have a total 2-rainbow dominating function of weight at most k ?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, EXACT-3-COVER (X3C), to TOTAL 2-RAINBOW DOMINATION.

EXACT 3-COVER (X3C)

Instance. A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X .

Question. Is there a subset C' of C such that every element of X appears in exactly one element of C' ?

Theorem 12. TOTAL 2-RAINBOW DOMINATION is NP-complete for bipartite graphs.

Proof. TOTAL 2-RAINBOW DOMINATION is a member of NP, since we can check in polynomial time that a function $f : V \rightarrow \{0, 1, 2\}$ has weight at most k and is a T2RDF. Now let us show how to transform any instance of X3C into an instance of TOTAL 2-RAINBOW DOMINATION so that one of them has a

solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \dots, x_{3q}\}$ and $C = \{C_1, C_2, \dots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we build a graph H_i obtained from a path $P_2 : x_i - y_i$ and two stars $K_{1,3}$ with centers a_i and b_i , by adding edges $y_i a_i$ and $y_i b_i$. Hence, each H_i has order 10. For each $C_j \in C$, we build a double star $S_{3,3}$ with support vertices u_j and v_j . Let c_j be a leaf of the double star $S_{3,3}$. Let $Y = \{c_1, c_2, \dots, c_t\}$. Now to obtain a graph G , we add edges $c_j x_i$ if $x_i \in C_j$. Clearly, G is a bipartite graph (for example, see Figure 4). Set $k = 4t + 16q$. Observe that for every T2RDF f on G , each H_i has weight at least 5 and each double star $S_{3,3}$ has weight at least 4.

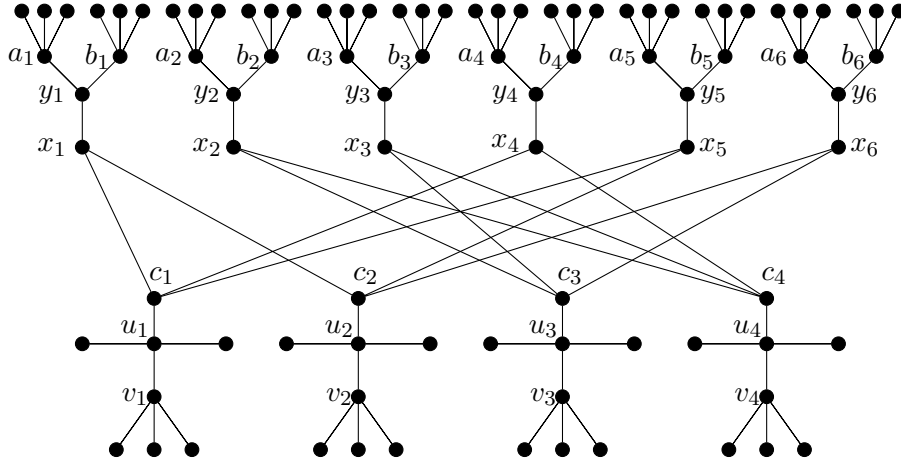


Figure 4. NP-completeness for bipartite graphs.

Suppose that the instance X, C of X3C has a solution C' . We construct a T2RDF f on G of weight k . For each i , assign the set $\{1, 2\}$ to a_i, b_i , the set $\{1\}$ to y_i and \emptyset to the remaining vertices of H_i . For every j , assign $\{1, 2\}$ to u_j and v_j , and \emptyset to each leaf. In addition, if for every C_j , assign to c_j the set $\{2\}$ if $C_j \in C'$ and \emptyset if $C_j \notin C'$. Note that since C' exists, its cardinality is precisely q , and so the number of c_j 's assigned $\{2\}$ is q , having disjoint neighborhoods in $\{x_1, x_2, \dots, x_{3q}\}$. Since C' is a solution for X3C, every vertex x_i in X satisfies $f(N[x_i]) = \{1, 2\}$. Hence, it is straightforward to see that f is a T2RDF with weight $f(V) = 4t + q + 15q = k$.

Conversely, suppose that G has a T2RDF with weight at most k . Among all such functions, let $g = (V_\emptyset, V_1, V_2, V_{12})$ be one such that the number of vertices of $\{y_1, y_2, \dots, y_{3q}\}$ assigned $\{1, 2\}$ is as small as possible. As observed above, since each H_i has weight at least 5, we may assume that $g(a_i) = g(b_i) = \{1, 2\}$ and $|g(y_i)| > 0$ so that vertices a_i, b_i are not isolated in the subgraph induced by $V_1 \cup V_2 \cup V_{12}$. Hence each leaf neighbor of a_i or b_i is assigned \emptyset under g . Assume

that $g(y_i) = \{1, 2\}$ for some i . Observe that if $|g(x_i)| > 0$, then reassigning $\{1\}$ to y_i provides a T2RDF g' with less vertices y_i assigned $\{1, 2\}$ than under g , contradicting our choice of g . Hence $g(x_i) = \emptyset$. But then reassigning $\{1\}$ to each of y_i and x_i instead of $\{1, 2\}$ and \emptyset , respectively, provides a T2RDF g' with less vertices y_i assigned $\{1, 2\}$ than under g , a contradiction too. Therefore $|g(y_i)| = 1$ for every $i \in \{1, 2, \dots, 3q\}$. On the other hand, the total weight of all double stars corresponding to elements of C is $4t$. In this case, we can assume that $g(u_j) = g(v_j) = \{1, 2\}$ and so each leaf neighbor of u_j or v_j is assigned \emptyset under g . Note that each c_j can be assigned \emptyset since $g(u_j) = \{1, 2\}$. Since $w(g) \leq 4t + 16q$ and the total weight assigned to vertices of $V(G) - (X \cup Y)$ is $4t + 15q$, we have to assign to vertices of $(X \cup Y)$ sets whose total cardinalities not exceeding q so that each vertex $x_i \in X$ has either $|g(x_i)| > 0$ or has two neighbors in $V_1 \cup V_2$ so that $f(N[x_i]) = \{1, 2\}$. Since $|X| = 3q$, it is clear that this is only possible if there are q vertices of $\{c_1, c_2, \dots, c_t\}$ belonging to $V_1 \cup V_2$. Since each c_j has a exactly three neighbors in $\{x_1, x_2, \dots, x_{3q}\}$, we deduce that $C' = \{C_j : |g(c_j)| = 1\}$ is an exact cover for C . ■

The next result is obtained by using the same proof as for Theorem 12 on the (same) graph G built for the transformation by adding all edges between the c_j 's so that the resulting graph is chordal.

Theorem 13. TOTAL 2-RAINBOW DOMINATION is NP-complete for chordal graphs.

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REFERENCES

- [1] H. Abdollahzadeh Ahangar, J. Amjadi, N. Jafari Rad and V. Samodivkin, *Total k -rainbow domination in graphs*, Comm. Combin. Optim. **3** (2018) 37–50.
doi:10.22049/CCO.2018.25719.1021
- [2] H. Abdollahzadeh Ahangar, M.A. Henning, V. Samodivkin and I.G. Yero, *Total Roman domination in graphs*, Appl. Anal. Discrete Math. **10** (2016) 501–517.
doi:10.2298/AADM160802017A
- [3] H. Abdollahzadeh Ahangar, M. Khaibari, N. Jafari Rad and S.M. Sheikholeslami, *Graphs with large total 2-rainbow domination number*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2018) 841–846.
doi:10.1007/s40995-017-0465-9

- [4] G.J. Chang, J. Wu and X. Zhu, *Rainbow domination on trees*, Discrete Appl. Math. **158** (2010) 8–12.
doi:10.1016/j.dam.2009.08.010
- [5] M. Chellali and N. Jafari Rad, *On 2-Rainbow domination and Roman domination in graphs*, Australas. J. Combin. **56** (2013) 85–93.
- [6] B. Brešar, M.A. Henning and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008) 201–213.
- [7] B. Brešar and T.K. Sumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007) 2394–2400.
doi:10.1016/j.dam.2007.07.018
- [8] N. Dehgardi, S.M. Sheikholeslami and L. Volkmann, *The rainbow domination subdivision numbers of graphs*, Mat. Vesnik **67** (2015) 102–114.
- [9] E. DeLaVina, C.E. Larson, R. Pepper and B. Waller, *On total domination and support vertices of a tree*, AKCE Int. J. Graphs Comb. **7** (2010) 85–95.
- [10] M. Falahat, S.M. Sheikholeslami and L. Volkmann, *New bounds on the rainbow domination subdivision number*, Filomat **28** (2014) 615–622.
doi:10.2298/FIL1403615F
- [11] C.-H. Liu and G.J. Chang, *Roman domination on strongly chordal graphs*, J. Comb. Optim. **26** (2013) 608–619.
doi:10.1007/s10878-012-9482-y
- [12] Y. Wu and N. Jafari Rad, *Bounds on the 2-Rainbow domination number of graphs*, Graphs Combin. **29** (2013) 1125–1133.
doi:10.1007/s00373-012-1158-y
- [13] Y. Wu and H. Xing, *Note on 2-rainbow domination and Roman domination in graphs*, Appl. Math. Lett. **23** (2010) 706–709.
doi:10.1016/j.aml.2010.02.012

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