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CHANGING AND UNCHANGING OF THE DOMINATION NUMBER OF A GRAPH: PATH ADDITION NUMBERS

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Abstract

Given a graph $G = (V, E)$ and two its distinct vertices u and v, the (u, v)-P_k-*addition graph* of G is the graph $G_{u,v,k-2}$ obtained from disjoint union of G and a path $P_k : x_0, x_1, \ldots, x_{k-1}$, $k \geq 2$, by identifying the vertices u and x_0, and identifying the vertices v and x_{k-1}. We prove that $\gamma(G) - 1 \leq \gamma(G_{u,v,k})$ for all $k \geq 1$, and $\gamma(G_{u,v,k}) > \gamma(G)$ when $k \geq 5$. We also provide necessary and sufficient conditions for the equality $\gamma(G_{u,v,k}) = \gamma(G)$ to be valid for each pair $u, v \in V(G)$. In addition, we establish sharp upper and lower bounds for the minimum, respectively maximum, k in a graph G over all pairs of vertices u and v in G such that the (u, v)-P_k-addition graph of G has a larger domination number than G, which we consider separately for adjacent and non-adjacent pairs of vertices.

Keywords: domination number, path addition.

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1. Introduction

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes *et al.* [8]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The complement $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. We write K_n for the *complete graph* of order n, $K_{m,n}$ for the *complete bipartite graph* with partite sets of order m and n, and P_n for the *path* on n vertrices. Let C_m denote the *cycle* of length m. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of x is $deg(x, G) = |N_G(x)|$. The *minimum* and *maximum* degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $A \subseteq V(G)$, let $N_G(A) = \bigcup_{x \in A} N_G(x)$ and $N_G[A] = N_G(A) \cup A$. A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. Let G be a graph and uv be an edge of G. By subdividing the edge uv we mean forming a graph H from G by adding a new vertex w and replacing the edge uv by uw and wv. Formally, $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) \backslash \{uv\}) \cup \{uw, wv\}$. For a graph G, let $x \in S \subseteq V(G)$. A vertex $y \in V(G)$ is a *S-private neighbor* of x if $N_G[y] \cap S = \{x\}$. The set of all S-private neighbors of x is denoted by $pn_G[x, S]$.

The study of domination and related subset problems is one of the fastest growing areas in graph theory. For a comprehensive introduction to the theory of domination in graphs we refer the reader to Haynes *et al.* [8]. A *dominating set* for a graph G is a subset $D \subseteq V(G)$ of vertices such that every vertex not in D is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality $\gamma(G)$ is called a *γ-set of* G. The concept of γ-bad/good vertices in graphs was introduced by Fricke *et al.* in [5]. A vertex v of a graph G is called

(i) [5] *γ-good*, if v belongs to some γ-set of G, and

(ii) [5] *γ-bad*, if v belongs to no γ-set of G.

A graph G is said to be *γ-excellent* whenever all its vertices are γ-good [5]. Brigham *et al.* [3] defined a vertex v of a graph G to be *γ-critical* if $\gamma(G - v) < \gamma(G)$, and G to be *vertex domination-critical* (from now on called *vc-graph*) if each vertex of G is γ-critical. For a graph G we define $V^-(G) = \{x \in V(G) \mid \gamma(G - x) < \gamma(G)\}$.

It is often of interest to known how the value of a graph parameter μ is affected when a change is made in a graph, for instance vertex or edge removal, edge addition, edge subdivision and edge contraction. In this connection, here we consider this question in the case $\mu = \gamma$ when a path is added to a graph.

Path-addition is an operation that takes a graph and adds an internally vertex-disjoint path between two vertices together with a set of supplementary edges. This operation can be considered as a natural generalization of the edge addition. Formally, let u and v be distinct vertices of a graph G. The *(u, v)-P_k-addition graph* of G is the graph $G_{u,v,k-2}$ obtained from disjoint union of G and a path $P_k : x_0, x_1, \dots, x_{k-1}$, $k \geq 2$, by identifying the vertices u and x_0, and identifying the vertices v and x_{k-1}. When $k \geq 3$ we call $x_1, x_2, \dots, x_{k-2}$ *path-addition vertices*. By $pa_\gamma(u, v)$ we denote the minimum number k such that $\gamma(G) < \gamma(G_{u,v,k})$. For every graph G with at least 2 vertices we define

▷ the *e-path addition* (*$\overline{e}$-path addition*) *number with respect to domination*, denoted $epa_\gamma(G)$ ($\overline{e}pa_\gamma(G)$, respectively), to be

- $epa_\gamma(G) = \min\{pa_\gamma(u,v) \mid u,v \in V(G), uv \in E(G)\}$,
- $\overline{e}pa_\gamma(G) = \min\{pa_\gamma(u,v) \mid u,v \in V(G), uv \notin E(G)\}$, and

▷ the *upper e-path addition* (*upper $\overline{e}$-path addition*) *number with respect to domination*, denoted $Epa_\gamma(G)$ ($\overline{E}pa_\gamma(G)$, respectively), to be

- $Epa_\gamma(G) = \max\{pa_\gamma(u,v) \mid u,v \in V(G), uv \in E(G)\}$,
- $\overline{E}pa_\gamma(G) = \max\{pa_\gamma(u,v) \mid u,v \in V(G), uv \notin E(G)\}$.

If G is complete, then we write $\overline{E}pa_\gamma(G) = \overline{e}pa_\gamma(G) = \infty$, and if G is edgeless then $epa_\gamma(G) = Epa_\gamma(G) = \infty$. In what follows the subscript γ will be omitted from the notation.

The remainder of this paper is organized as follows. In Section 2, we prove that $1 \le epa(G) \le 3$ and $2 \le Epa(G) \le 3$, and we present necessary and sufficient conditions for $pa(u,v) = i$, $i = 1,2,3$, where $uv \in E(G)$. In Section 3, we show that $1 \le \overline{e}pa(G) \le \overline{E}pa(G) \le 5$, and we give necessary and sufficient conditions for $\overline{e}pa(G) = \overline{E}pa(G) = j$, $1 \le j \le 5$. We conclude in Section 4 with open problems.

We end this section with some known results which will be useful in proving our main results.

Lemma 1 [2]**.** *If G is a graph and H is any graph obtained from G by subdividing some edges of G, then $\gamma(H) \ge \gamma(G)$.*

Lemma 2. *Let G be a graph and $v \in V(G)$.*

(i) [5] *If v is γ-bad, then $\gamma(G - v) = \gamma(G)$.*

(ii) [3] *v is γ-critical if and only if $\gamma(G - v) = \gamma(G) - 1$.*

(iii) [5] *If v is γ-critical, then all its neighbors are γ-bad vertices of $G - v$.*

(iv) [11] *If $e \in E(\overline{G})$, then $\gamma(G) - 1 \le \gamma(G + e) \le \gamma(G)$.*

In most cases, Lemma 2 will be used in the sequel without specific reference.

2. THE ADJACENT CASE

The aim of this section is to prove that $1 \le pa(u,v) \le 3$ and to find necessary and sufficient conditions for $pa(u,v) = i$, $i = 1,2,3$, where $uv \in E(G)$.

Observation 3. *If u and v are adjacent vertices of a graph G, then $\gamma(G) = \gamma(G_{u,v,0}) \le \gamma(G_{u,v,k}) \le \gamma(G_{u,v,k+1})$ for $k \ge 1$.*

Proof. The equality $\gamma(G) = \gamma(G_{u,v,0})$ is obvious. For any γ-set M of $G_{u,v,1}$ both $M_u = (M \backslash \{x_1\}) \cup \{u\}$ and $M_v = (M \backslash \{x_1\}) \cup \{v\}$ are dominating sets of G, and at least one of them is a γ-set of $G_{u,v,1}$. Hence $\gamma(G) \leq \min\{|M_u|, |M_v|\} = \gamma(G_{u,v,1})$. The rest follows by Lemma 1. ■

Theorem 4. *Let u and v be adjacent vertices of a graph G. Then $\gamma(G) \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1$ and the following is true.*

(i) *$\gamma(G) = \gamma(G_{u,v,1})$ if and only if at least one of u and v is a γ-good vertex of G.*

(ii) *$\gamma(G_{u,v,1}) = \gamma(G) + 1$ if and only if both u and v are γ-bad vertices of G.*

Proof. The left side inequality follows by Observation 3. If D is a γ-set of G, then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$, which implies $\gamma(G_{u,v,1}) \leq \gamma(G) + 1$.

If at least one of u and v belongs to some γ-set D_1 of G, then D_1 is a dominating set of $G_{u,v,1}$. This clearly implies $\gamma(G) = \gamma(G_{u,v,1})$.

Let now both u and v are γ-bad vertices of G, and suppose that $\gamma(G_{u,v,1}) = \gamma(G)$. In this case for any γ-set M of $G_{u,v,1}$ is fulfilled $u, v \notin M$ and $x_1 \in M$. But then $(M \backslash \{x_1\}) \cup \{u\}$ is a γ-set for both G and $G_{u,v,1}$, a contradiction. ■

Corollary 5. *Let G be a graph with edges. Then $Epa(G) \geq 2$ and $epa(G) = 1$ if and only if the set of all γ-bad vertices of G is neither empty nor independent.*

Theorem 6. *Let u and v be adjacent vertices of a graph G. Then $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Moreover,*

($\mathbb{A}$) *$\gamma(G_{u,v,2}) = \gamma(G) + 1$ if and only if at least one of the following holds:*

- (i) *both u and v are γ-bad vertices of G,*
- (ii) *at least one of u and v is γ-good, $u, v \notin V^-(G)$ and each γ-set of G contains at most one of u and v.*

($\mathbb{B}$) *$\gamma(G_{u,v,2}) = \gamma(G)$ if and only if at least one of the following is true:*

- (iii) *there exists a γ-set of G which contains both u and v,*
- (iv) *at least one of u and v is in $V^-(G)$.*

Proof. The left side inequality follows by Observation 3. If D is an arbitrary γ-set of G, then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,2}$. Hence $\gamma(G_{u,v,2}) \leq \gamma(G)+1$.

($\mathbb{A}$) $\Rightarrow$ Assume that the equality $\gamma(G_{u,v,2}) = \gamma(G) + 1$ holds. By Theorem 4 we know that $\gamma(G_{u,v,1}) \in \{\gamma(G), \gamma(G) + 1\}$. If $\gamma(G_{u,v,1}) = \gamma(G) + 1$, then again by Theorem 4, both u and v are γ-bad vertices of G. So let $\gamma(G) = \gamma(G_{u,v,1})$. Then at least one of u and v is a γ-good vertex of G (Theorem 4). Clearly there is no γ-set of G which contains both u and v. If $u \in V^-(G)$ and U is a γ-set of $G - u$, then $U \cup \{x_1\}$ is a dominating set of $G_{u,v,2}$ and $|U \cup \{x_1\}| = \gamma(G)$, a contradiction. Thus $u, v \notin V^-(G)$.

($\mathbb{A}$) $\Leftarrow$ If both u and v are γ-bad vertices of G, then $\gamma(G_{u,v,1}) = \gamma(G) + 1$(Theorem 4). But we know that $\gamma(G_{u,v,1}) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$; hence

$\gamma(G_{u,v,2}) = \gamma(G) + 1$. Finally let (ii) hold and M be a γ-set of $G_{u,v,2}$. If $x_1, x_2 \notin M$, then $u, v \in M$ which leads to $\gamma(G_{u,v,2}) > \gamma(G)$. If $x_1, x_2 \in M$, then $(M \backslash \{x_1, x_2\}) \cup \{u, v\}$ is a dominating set of G of cardinality more than $\gamma(G)$. Now let without loss of generality $x_1 \in M$ and $x_2 \notin M$. If $M \backslash \{x_1\}$ is a dominating set of G, then $\gamma(G) + 1 \leq |M| = \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. So, let $M \backslash \{x_1\}$ be no dominating set of G. Hence $M \backslash \{x_1\}$ is a dominating set of $G - u$. Since $u \notin V^-(G)$, $\gamma(G) \leq \gamma(G - u) \leq |M \backslash \{x_1\}| < \gamma(G_{u,v,2})$.

($\mathbb{B}$) $\Rightarrow$ Let $\gamma(G_{u,v,2}) = \gamma(G)$. Suppose that neither (iii) nor (iv) is valid. Hence $u, v \notin V^-(G)$ and no γ-set of G contains both u and v. But then at least one of (i) and (ii) holds, and from ($\mathbb{A}$) we conclude that $\gamma(G_{u,v,2}) = \gamma(G) + 1$, a contradiction.

($\mathbb{B}$) $\Leftarrow$ Let at least one of (iii) and (iv) be hold. Then neither (i) nor (ii) is fulfilled. Now by ($\mathbb{A}$) we have $\gamma(G_{u,v,2}) \neq \gamma(G) + 1$. Since $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$, we obtain $\gamma(G) = \gamma(G_{u,v,2})$. ■

The *independent domination number* of a graph G, denoted by $i(G)$, is the minimum size of an independent dominating set of G. It is obvious that $i(G) \geq \gamma(G)$. In a graph G, $i(G)$ is *strongly equal to* $\gamma(G)$, written $i(G) \equiv \gamma(G)$, if each γ-set of G is independent. It remains an open problem to characterize the graphs G with $i(G) \equiv \gamma(G)$ [7].

Corollary 7. *Let G be a graph with edges. Then* (a) *$epa(G) \geq 2$ if and only if the set of all γ-bad vertices is either empty or independent, and* (b) *$Epa(G) = 2$ if and only if $i(G) \equiv \gamma(G)$.*

Proof. (a) Immediately by Corollary 5.

(b) $\Rightarrow$ Let $Epa(G) = 2$. If D is a γ-set of G and $u, v \in D$ are adjacent, then D is a dominating set of $G_{u,v,2}$, a contradiction.

(b) $\Leftarrow$ Let all γ-sets of G be independent. Suppose $u \in V^-(G)$ and D is a γ-set of $G - u$. Then $D_1 = D \cup \{v\}$ is a γ-set of G, where v is any neighbor of u. But D_1 is not independent. Hence $V^-(G)$ is empty. Thus, for any 2 adjacent vertices u and v of G is fulfilled either ($\mathbb{A}$)(i) or ($\mathbb{A}$)(ii) of Theorem 6. Therefore $Epa(G) \leq 2$. The result now follows by Corollary 5. ■

Denote by $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ the additive group of order n. Let S be a subset of $\mathbb{Z}_n$ such that $0 \notin S$ and $x \in S$ implies $-x \in S$. The *circulant graph* with distance set S is the graph $C(n; S)$ with vertex set $\mathbb{Z}_n$ and vertex x adjacent to vertex y if and only if $x - y \in S$.

Let $n \geq 3$ and $k \in \mathbb{Z}_n \backslash \{0\}$. The *generalized Petersen graph* $P(n, k)$ is the graph on the vertex-set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ with adjacencies $x_i x_{i+1}, x_i y_i$, and $y_i y_{i+k}$ for all i.

Example 8. A special case of graphs G with $Epa(T) = 2$ are graphs for which each γ-set is efficient dominating (an efficient dominating set in a graph G is a set S such that $\{N[s] \mid s \in S\}$ is a partition of $V(G)$). We list several examples of such graphs [10].

(a) A *crown graph* $H_{n,n}$, $n \geq 3$, which is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

(b) Circulant graphs $G = C(n = (2k+1)t; \{1, \dots, k\} \cup \{n-1, \dots, n-k\})$, where $k, t \geq 1$.

(c) Circulant graphs $G = C(n; \{\pm 1, \pm s\})$, where $2 \leq s \leq n-2$, $s \neq n/2$, $5 \mid n$ and $s \equiv \pm 2 \pmod 5$.

(d) The *generalized Petersen graph* $P(n, k)$, where $n \equiv 0 \pmod 4$ and k is odd.

Theorem 9. *If u and v are adjacent vertices of a graph G, then $\gamma(G_{u,v,3}) = \gamma(G) + 1$.*

Proof. If D is a γ-set of G, then $D \cup \{x_2\}$ is a dominating set of G. Hence $\gamma(G_{u,v,3}) \leq \gamma(G) + 1$.

Let M be a γ-set of $G_{u,v,3}$. Then at least one of x_1, x_2 and x_3 is in M. If $x_2 \in M$, then clearly $\gamma(G_{u,v,3}) = \gamma(G) + 1$. If $x_2 \notin M$ and $x_1, x_3 \in M$, then $(M \backslash \{x_1, x_3\}) \cup \{u\}$ is a dominating set of G. If $x_2, x_3 \notin M$ and $x_1 \in M$, then $v \in M$ and $M \backslash \{x_1\}$ is a dominating set of G. All this leads to $\gamma(G_{u,v,3}) = \gamma(G) + 1$. ■

Corollary 10. *Let G be a graph with edges. Then $epa(G) \leq Epa(G) \leq 3$. Moreover, $Epa(G) = 3$ if and only if G has a γ-set that is not independent, and $epa(G) = 3$ if and only if for each pair of adjacent vertices u and v at least one of the following is valid.*

(i) *There exists a γ-set of G which contains both u and v.*

(ii) *At least one of u and v is in $V^-(G)$.*

Proof. By Corollary 5 and Theorem 9 we have $1 \leq epa(G) \leq Epa(G) \leq 3$ and $2 \leq Epa(G)$. Since $Epa(G) = 2$ if and only if $i(G) \equiv \gamma(G)$ (by Corollary 7), $Epa(G) = 3$ if and only if G has a γ-set that is not independent.

Clearly $epa(G) = 3$ if and only if $\gamma(G_{u,v,2}) = \gamma(G)$ for each pair of adjacent vertices u and v of G. Then because of Theorem 6($\mathbb{B}$), we have that $epa(G) = 3$ if and only if for each pair of adjacent vertices u and v of G at least one of (i) and (ii) holds. ■

Corollary 11. *Let G be a graph with edges. If $V^-(G)$ has a subset which is a vertex cover of G, then $epa(G) = 3$. In particular, if G is a vc-graph then $epa(G) = 3$.*

We define the following classes of graphs G with $\Delta(G) \geq 1$.

- $\mathcal{A} = \{G \mid epa(G) = 3\}$,

- $\mathcal{A}_1 = \{G \mid V^-(G)$ is a vertex cover of $G\}$,
- $\mathcal{A}_2 = \{G \mid$ each two adjacent vertices belongs to some γ-set of $G\}$,
- $\mathcal{A}_3 = \{G \mid G$ is a vc-graph$\}$.

Clearly, $\mathcal{A}_3 \subseteq \mathcal{A}_1$ and by Corolaries 10 and 11, $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{A}$. These relationships are illustrated in the Venn diagram in Figure 1(left). To continue we relabel this diagram in six regions $\mathbf{R_0}$–$\mathbf{R_5}$ as shown in Figure 1(right). In what follows in this section we show that none of $\mathbf{R_0}$–$\mathbf{R_5}$ is empty. The *corona* of a graph H is the graph $G = H \circ K_1$ obtained from H by adding a degree-one neighbor to every vertex of H. If F and H are disjoint graphs, $v_F \in V(F)$ and $v_H \in V(H)$, then the *coalescence* $(F \cdot H)(v_F, v_H : v)$ of F and H via v_F and v_H, is the graph obtained from the union of F and H by identifying v_F and v_H in a vertex labeled v.

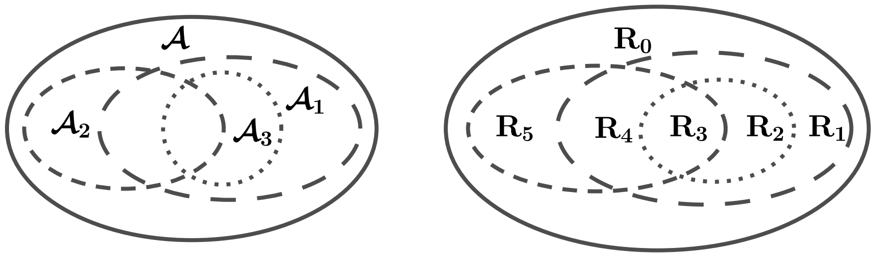

Figure 1. Left: Classes of graphs with *epa* = 3. Right: Regions of Venn diagram.

Remark 12. It is easy to see that all the following hold.

(i) If H is a connected graph of order $n \geq 2$, then $G = H \circ K_1 \in \mathbf{R_0}$.

(ii) Let G_k^1 be a graph obtained from the cycle $C_{3k+1} : x_0, x_1, x_2, \ldots, x_{3k}, x_0$, $k \geq 2$, by adding a vertex y and edges yx_0, yx_2. Then $\gamma(G_k^1) = k+1$, G_k^1 is γ-excellent, $V^-(G_k^1) = \{x_0, x_2\} \cup \bigcup_{r=1}^{k-1}\{x_{3r+1}, x_{3r+2}\}$ is a vertex cover of G, and there is no γ-set of G_k^1 that contains both x_{3r+1} and x_{3r+2}. Thus G_k^1 is in $\mathbf{R_1}$.

(iii) The graph H_{10} depicted in Figure 2 is in $\mathcal{A}_3$ and $\gamma(H_{10}) = 3$ [1]. It is obvious that no γ-set of H_{10} contains both u and v. Hence $H_{10} \in \mathbf{R_2}$. Consider now the graph $G_k^2 = (C_{3k+1} \cdot H_{10})(x_0, w : z)$, where $C_{3k+1} : x_0, x_1, x_2, \ldots, x_{3k}, x_0$, $k \geq 2$, is a cycle on $3k+1$ vertices and w is any of the two common neighbors of u and v in H_{10}. Since both C_{3k+1} and H_{10} are vc-graphs, by [4] we have that G_k^2 is vc-graph and $\gamma(G_k^2) = \gamma(C_{3k+1}) + \gamma(H_{10}) - 1$. Let D be an arbitrary γ-set of G_k^2, $D_1 = D \cap V(H_{10})$ and $D_2 = D \cap V(C_{3k+1})$. Then exactly one of the following holds.

  (a) $z \in D$, D_1 is a γ-set of H_{10} and D_2 is a γ-set of C_{3k+1}.

  (b) $z \notin D$, D_1 is a γ-set of H_{10} and $D_2 \cup \{x_0\}$ is a γ-set of C_{3k+1}.

(c) $z \notin D$, $D_1 \cup \{w\}$ is a γ-set of H_{10} and D_2 is a γ-set of C_{3k+1}.

Since no γ-set of H_{10} contains both u and v, by (a), (b) and (c) we conclude that at most one of u and v is in D. Thus $G_k^2 \in \mathbf{R_2}$.

(iv) $C_{3k+1} \in \mathbf{R_3}$ for all $k \geq 1$.

(v) $K_{2,n} \in \mathbf{R_4}$ for all $n \geq 3$.

(vi) $K_{n,n} \in \mathbf{R_5}$ for all $n \geq 3$.

Thus all regions $\mathbf{R_0}, \mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \mathbf{R_5}$ are nonempty.

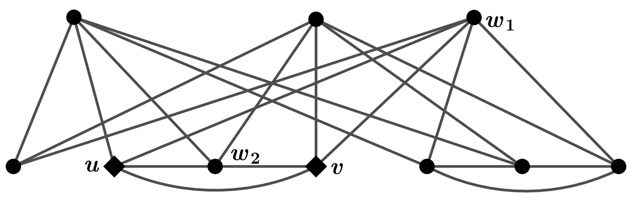

Figure 2. Graph H_{10} is in $\mathbf{R_2}$.

3. The Nonadjacent Case

In this section we show that $1 \leq \overline{e}pa(G) \leq \overline{E}pa(G) \leq 5$ and we obtain necessary and sufficient conditions for $\overline{e}pa(G) = \overline{E}pa(G) = j$, $1 \leq j \leq 5$.

We begin with an easy observation which is an immediate consequence by Lemma 2(iv) and Lemma 1.

Observation 13. *Let u and v be nonadjacent vertices of a graph G. Then $\gamma(G) - 1 \leq \gamma(G_{u,v,0}) \leq \gamma(G)$ and $\gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$ for $k \geq 0$.*

Theorem 14. *Let u and v be nonadjacent vertices of a graph G. Then $\gamma(G)-1 \leq \gamma(G_{u,v,1}) \leq \gamma(G) + 1$. Moreover,*

(i) *$\gamma(G) - 1 = \gamma(G_{u,v,1})$ if and only if $\gamma(G - \{u, v\}) = \gamma(G) - 2$.*

(ii) *$\gamma(G_{u,v,1}) = \gamma(G) + 1$ if and only if both u and v are γ-bad vertices of G, $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$. If $\gamma(G_{u,v,1}) = \gamma(G) + 1$, then $x_1 \in V^-(G_{u,v,1})$.*

Proof. The left side inequality follows by Observation 13.

(i) $\Rightarrow$ Assume the equality $\gamma(G) - 1 = \gamma(G_{u,v,1})$ holds and let M be any γ-set of $G_{u,v,1}$. Then at least one and not more than two of x_1, u and v must be in M. Hence $M_1 = (M \setminus \{x_1\}) \cup \{u, v\}$ is a dominating set of G and $\gamma(G) \leq |M_1| \leq |M| + 1 = \gamma(G_{u,v,1}) + 1 = \gamma(G)$. This immediately implies that M_1 is a γ-set of G. Hence $x_1 \in M$ and $pn[x_1, M] = \{x_1, u, v\}$. Since $M_1 \backslash \{u, v\}$ is a dominating set of $G - \{u, v\}$, we have $\gamma(G) - 2 \leq \gamma(G - \{u, v\}) \leq |M_1 \backslash \{u, v\}| = \gamma(G) - 2$.

(i) $\Leftarrow$ Suppose now $\gamma(G - \{u,v\}) = \gamma(G) - 2$. Then for any γ-set U of $G-\{u,v\}$, the set $U \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$. This leads to $\gamma(G_{u,v,1}) \leq |U \cup \{x_1\}| = \gamma(G) - 1 \leq \gamma(G_{u,v,1})$.

Now we will prove the right side inequality. Let D be any γ-set of G. If at least one of u and v is in D, then D is a dominating set $G_{u,v,1}$ and $\gamma(G_{u,v,1}) \leq \gamma(G)$. So, let neither u nor v belong to some γ-set of G. Then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(G_{u,v,1}) \leq \gamma(G) + 1$.

(ii) $\Rightarrow$ Assume that $\gamma(G_{u,v,1}) = \gamma(G)+1$. Then u and v are γ-bad vertices of G and for any γ-set D of G, $D \cup \{x_1\}$ is a γ-set of $G_{u,v,1}$. Hence $x_1 \in V^-(G_{u,v,1})$. Suppose $u \in V^-(G-v)$ and let U be a γ-set of $G-\{u,v\}$. Then $U_1 = U \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(G)+1 = \gamma(G_{u,v,1}) \leq |U_1| = 1+\gamma((G-v)-u) = \gamma(G-v) = \gamma(G)$, a contradiction. Thus $u \notin V^-(G-v)$ and by symmetry, $v \notin V^-(G-u)$.

(ii) $\Leftarrow$ Let both u and v be γ-bad vertices of G, $u \notin V^-(G-v)$ and $v \notin V^-(G-u)$. Hence $\gamma(G-\{u,v\}) \geq \gamma(G)$. Consider any γ-set M of $G_{u,v,1}$. If one of u and v belongs to M, then $\gamma(G)+1 = \gamma(G_{u,v,1})$. So, let x_1 is in each γ-set of $G_{u,v,1}$. But then $pn[x_1, M] = \{x_1, u, v\}$. Hence $\gamma(G_{u,v,1}) - 1 = \gamma(G - \{u,v\}) \geq \gamma(G) \geq \gamma(G_{u,v,1}) - 1$. ∎

Corollary 15. *Let G be a noncomplete graph. Then $1 \leq \overline{e}pa(G) \leq \overline{E}pa(G)$ and the following assertions hold.*

(i) *$\overline{e}pa(G) = 1$ if and only if there are nonadjacent γ-bad vertices u and v of G such that $u \notin V^-(G-v)$ and $v \notin V^-(G-u)$.*

(ii) *$\overline{E}pa(G) = 1$ if and only if $\gamma(G) = 1$.*

Proof. Observation 13 implies $1 \leq \overline{e}pa(G)$.

(i) Immediately by Theorem 14.

(ii) If $\gamma(G) = 1$, then clearly $\overline{E}pa(G) = 1$. If $\gamma(G) \geq 2$, then G has 2 nonadjacent vertices at least one of which is γ-good. By Theorem 14, $\overline{E}pa(G) \geq 2$. ∎

Theorem 16. *Let u and v be nonadjacent vertices of a graph G. Then $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. Moreover,*

($\mathbb{C}$) *$\gamma(G_{u,v,2}) = \gamma(G)$ if and only if one of the following holds.*

  (i) *There is a γ-set of G which contains both u and v.*

  (ii) *At least one of u and v is in $V^-(G)$.*

($\mathbb{D}$) *$\gamma(G_{u,v,2}) = \gamma(G)+1$ if and only if $u, v \notin V^-(G)$ and any γ-set of G contains at most one of u and v.*

Proof. For any γ-set D of G, $D \cup \{x_2\}$ is a dominating set of $G_{u,v,2}$. Hence $\gamma(G_{u,v,2}) \leq \gamma(G)+1$. Suppose $\gamma(G_{u,v,2}) \leq \gamma(G)-1$ and let M be a γ-set of $G_{u,v,2}$. Then at least one of x_1 and x_2 is in M. If $x_1, x_2 \in M$, then $M_1 = (M \backslash \{x_1, x_2\}) \cup$

$\{u, v\}$ is a dominating set of G and $|M_1| \leq \gamma(G_{u,v,2})$, a contradiction. So let without loss of generality, $x_1 \in M$ and $x_2 \notin M$. If $u \in M$ or $v \in M$, then again M_1 is a dominating set of G and $|M_1| \leq \gamma(G_{u,v,2})$, a contradiction. Thus $x_1 \in M$ and $u, v \notin M$. But then $(M \backslash \{x_1\}) \cup \{u\}$ is a dominating set of G, contradicting $\gamma(G_{u,v,2}) < \gamma(G)$. Thus $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$.

($\mathbb{C}$) $\Rightarrow$ Let $\gamma(G_{u,v,2}) = \gamma(G)$. Assume that neither (i) nor (ii) hold. Let M be a γ-set of $G_{u,v,2}$. If $x_1, x_2 \in M$, then $M_1 = (M \backslash \{x_1, x_2\}) \cup \{u, v\}$ is a dominating set of G of cardinality not more than $\gamma(G)$ and $u, v \in M_1$, a contradiction. Let without loss of generality $x_1 \in M$ and $x_2 \notin M$. Since $M \backslash \{x_1\}$ is no dominating set of G, $u \in pn[x_1, M]$. But then $M_3 = (M \backslash \{x_1\}) \cup \{u\}$ is a γ-set of G and $u \in V^-(G)$, a contradiction. Thus at least one of (i) and (ii) is valid.

($\mathbb{C}$) $\Leftarrow$ If both u and v belong to some γ-set D of G, then D is a dominating set of $G_{u,v,2}$. Hence $\gamma(G_{u,v,2}) = \gamma(G)$. Finally let $u \in V^-(G)$ and D a γ-set of $G - u$. Then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,2}$ of cardinality $\gamma(G)$. Thus $\gamma(G_{u,v,2}) = \gamma(G)$.

($\mathbb{D}$) Immediately by ($\mathbb{C}$) and $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$. ■

Corollary 17. *Let G be a noncomplete graph. Then the following assertions hold.*

(i) $\overline{e}pa(G) \leq 2$ *if and only if there are nonadjacent vertices $u, v \in V(G) \backslash V^-(G)$ such that any γ-set of G contains at most one of them.*

(ii) $\overline{E}pa(G) = 2$ *if and only if $\gamma(G) \geq 2$ and each γ-set of G is a clique.*

Proof. (i) Immediately by Theorem 16.

(ii) $\Rightarrow$ Let $\overline{E}pa(G) = 2$. By Corollary 15, $\gamma(G) \geq 2$. Suppose G has a γ-set, say D, which is not a clique. Then there are nonadjacent $u, v \in D$. By Theorem 16($\mathbb{C}$), $\gamma(G_{u,v,2}) = \gamma(G)$, which contradict $\overline{E}pa(G) = 2$. Thus, each γ-set of G is a clique.

(ii) $\Leftarrow$ Let $\gamma(G) \geq 2$ and let each γ-set of G be a clique. If G has a vertex $z \in V^-(G)$ and M_z is a γ-set of $G - z$, then $M = M_z \cup \{z\}$ is a γ-set of G and z is an isolated vertex of the graph induced by M, a contradiction. Thus $V^-(G)$ is empty. Now by Theorem 16($\mathbb{D}$), $\overline{E}pa(G) = 2$. ■

Example 18. The join of two graphs G_1 and G_2 with disjoint vertex sets is the graph, denoted by $G_1 + G_2$, with the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Let $\gamma(G_i) \geq 3$, $i = 1, 2$. Then $\gamma(G_1 + G_2) = 2$ and each γ-set of $G_1 + G_2$ contains exactly one vertex of G_i, $i = 1, 2$. Hence $\overline{E}pa(G_1 + G_2) = 2$. In particular, $\overline{E}pa(K_{m,n}) = 2$ when $m, n \geq 3$.

Theorem 19. *Let u and v be nonadjacent vertices of a graph G. Then $\gamma(G) \leq \gamma(G_{u,v,3}) \leq \gamma(G) + 1$. Moreover, $\gamma(G_{u,v,3}) = \gamma(G)$ if and only if at least one of the following holds.*

(i) $u \in V^-(G)$ *and* v *is a* γ*-good vertex of* $G - u$,

(ii) $v \in V^-(G)$ *and* u *is a* γ*-good vertex of* $G - v$.

Proof. If D is a dominating set of G, then $D \cup \{x_2\}$ is a dominating set of $G_{u,v,3}$. Hence $\gamma(G_{u,v,3}) \leq \gamma(G) + 1$. We already know that $\gamma(G) \leq \gamma(G_{u,v,2})$ and $\gamma(G_{u,v,2}) \leq \gamma(G_{u,v,3})$. But then $\gamma(G) \leq \gamma(G_{u,v,3})$.

$\Rightarrow$ Let $\gamma(G_{u,v,3}) = \gamma(G)$ and let M be a γ-set of $G_{u,v,3}$ such that $Q = M \cap \{x_1, x_2, x_3\}$ has minimum cardinality. Clearly $|Q| = 1$. If $\{x_2\} = Q$, then $M\backslash\{x_2\}$ is a dominating set of G, contradicting $\gamma(G_{u,v,3}) = \gamma(G)$. Let without loss of generality $\{x_1\} = Q$. This implies $v \in M$, $x_3 \in pn[v, M]$ and $pn[x_1, M] = \{u, x_1, x_2\}$. Then $M_2 = (M\backslash\{x_1\}) \cup \{u\}$ is a γ-set of G, $pn[u, M_2] = \{u\}$ and $v \in M_2$; hence (i) holds.

$\Leftarrow$ Let without loss of generality (i) is true. Then there is a γ-set D of G such that $u, v \in D$ and $D\backslash\{u\}$ is a γ-set of $G - u$. But then $(D\backslash\{u\}) \cup \{x_1\}$ is a dominating set of $G_{u,v,3}$, which implies $\gamma(G) \geq \gamma(G_{u,v,3})$. ■

Corollary 20. *Let* G *be a noncomplete graph. Then the following holds.*

($\mathbb{E}$) $\overline{e}pa(G) \leq 3$ *if and only if there is a pair of nonadjacent vertices* u *and* v *such that neither* (i) *nor* (ii) *is valid, where*

(i) $u \in V^-(G)$ *and* v *is a* γ*-good vertex of* $G - u$,

(ii) $v \in V^-(G)$ *and* u *is a* γ*-good vertex of* $G - v$.

($\mathbb{F}$) $\overline{e}pa(G) = \overline{E}pa(G) = 3$ *if and only if all vertices of* G *are* γ*-good,* $V^-(G)$ *is empty and for every* 2 *nonadjacent vertices* u *and* v *of* G *there is a* γ*-set of* G *which contains them both.*

Proof. ($\mathbb{F}$) $\Rightarrow$ Let $\overline{e}pa(G) = \overline{E}pa(G) = 3$. If $u \in V^-(G)$ and D is a γ-set of $G - u$, then for u and each $v \in D$ is fulfilled (i) of Theorem 19. But then $\overline{E}pa(G) \neq 3$, a contradiction. So, $V^-(G)$ is empty. Suppose that G has γ-bad vertices. Then there is a γ-bad vertex which is nonadjacent to some other vertex of G. But Theorem 16($\mathbb{D}$) implies $\overline{e}pa(G) < 3$, a contradiction. Thus all vertices of G are γ-good. Now let $u, v \in V(G)$ be nonadjacent. If there is no γ-set of G which contains both u and v, then by Theorem 16($\mathbb{D}$) we have $\gamma(G_{u,v,2}) = \gamma(G) + 1$, a contradiction.

($\mathbb{F}$) $\Leftarrow$ Let $V^-(G)$ be empty and for each pair u, v of nonadjacent vertices of G there is a γ-set D_{uv} of G with $u, v \in D_{uv}$. By Theorem 19, $\gamma(G_{u,v,3}) = \gamma(G) + 1$, and by Theorem 16, $\gamma(G_{u,v,2}) = \gamma(G)$. Hence $pa(u, v) = 3$. ■

Example 21. Denote by $\mathcal{U}$ the class of all graphs G with $\overline{e}pa(G) = \overline{E}pa(G) = 3$. Then all the following holds. (a) Circulant graphs $C(2k + 1; \{\pm 1, \pm 2, \ldots, \pm (k - 1)\}) \in \mathcal{U}$ for all $k \geq 1$. (b) Let G be a disconnected graph. Then $G \in \mathcal{U}$ if and only if G has no isolated vertices and each its component is either in $\mathcal{U}$ or is complete.

Theorem 22. *Let u and v be nonadjacent vertices of a graph G. Then $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. Moreover, the following assertions are valid.*

($\mathbb{G}$) $\gamma(G_{u,v,4}) = \gamma(G) + 2$ *if and only if* $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

($\mathbb{H}$) *If* $\gamma(G_{u,v,1}) = \gamma(G)$ *and* $\gamma(G_{u,v,i}) = \gamma(G) + 1$ *for some* $i \in \{2, 3\}$, *then* $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

($\mathbb{I}$) *Let* $\gamma(G_{u,v,3}) = \gamma(G)$. *Then* $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$ *and the equality holds if and only if* $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$.

($\mathbb{J}$) $\gamma(G_{u,v,4}) = \gamma(G)$ *if and only if* $\gamma(G - \{u, v\}) = \gamma(G) - 2$.

Proof. Since $\gamma(G) \leq \gamma(G_{u,v,3})$ (by Theorem 19) and $\gamma(G_{u,v,3}) \leq \gamma(G_{u,v,4})$ (by Observation 13), we have $\gamma(G) \leq \gamma(G_{u,v,4})$. Let S be a γ-set of G. Then $S \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$, which leads to $\gamma(G_{u,v,4}) \leq \gamma(G) + 2$.

Claim 1. *If* $\gamma(G_{u,v,1}) \leq \gamma(G)$, *then* $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Proof. Assume that v is a γ-bad vertex of G, $u \in V^-(G - v)$ and R a γ-set of $G - \{u, v\}$. Then $|R| = \gamma((G - v) - u) = \gamma(G - v) - 1 = \gamma(G) - 1$ and $R \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) \leq |R| + 2 = \gamma(G) + 1$.

Assume now that D is a γ-set of G with $u \in D$. Then $D \cup \{x_3\}$ is a dominating set of $G_{u,v,4}$. Hence again $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. Now by Theorem 14 we immediately obtain the required. □

($\mathbb{G}$) Let $\gamma(G_{u,v,4}) = \gamma(G) + 2$. By Claim 1, $\gamma(G_{u,v,1}) > \gamma(G)$ and by Theorem 14, $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

Let now $\gamma(G_{u,v,1}) = \gamma(G) + 1$. By Theorem 14, u and v are γ-bad vertices of G, $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$. Let M be a γ-set of $G_{u,v,4}$ such that $R = M \cap \{x_1, x_2, x_3, x_4\}$ has minimum cardinality. Clearly $|R| \in \{1, 2\}$. Assume first $|R| = 1$ and without loss of generality $\{x_2\} = M$. Then $M \backslash \{x_2\}$ is a dominating set of G with $v \in M \backslash \{x_2\}$. Since v is a γ-bad vertex of G, $|M \backslash \{x_2\}| > \gamma(G)$ and then $\gamma(G_{u,v,4}) = |M| > \gamma(G) + 1$. Let now $|R| = 2$ and without loss of generality $x_1, x_4 \in M$. Since $|M \cap \{x_1, x_2, x_3, x_4\}|$ is minimum, $u, v \notin M$ and $M \backslash \{x_1, x_4\}$ is a dominating set of $G - \{u, v\}$. But then $\gamma(G_{u,v,4}) = 2 + |M \backslash \{x_1, x_4\}| \geq 2 + \gamma((G - u) - v) \geq 2 + \gamma(G - u) = 2 + \gamma(G)$.

($\mathbb{H}$) Let $\gamma(G_{u,v,1}) = \gamma(G)$. By Claim 1, $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$. If $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{1, 2\}$, then since $\gamma(G_{u,v,4}) \geq \gamma(G_{u,v,i})$, we obtain $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

($\mathbb{I}$) Let $\gamma(G_{u,v,3}) = \gamma(G)$. Hence at least one of (i) and (ii) of Theorem 19 holds, and by ($\mathbb{E}$), $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.

Assume that the equality holds. If $\gamma(G - \{u, v\}) = \gamma(G) - 2$, then for any γ-set U of $G - \{u, v\}$, $U \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Hence $\gamma(G_{u,v,4}) = \gamma(G)$, a contradiction.

Let now $\gamma(G-\{u,v\}) \geq \gamma(G)-1$ and without loss of generality condition (i) of Theorem 19 be satisfied. Suppose $\gamma(G_{u,v,4}) = \gamma(G)$. Hence for each γ-set M of $G_{u,v,4}$ are fulfilled: $x_1, x_4 \in M$, $x_2, x_3, u, v \notin M$, $pn[x_1, M] = \{x_1, x_2, u\}$ and $pn[x_4, M] = \{x_3, x_4, , v\}$. But then $\gamma(G-\{u,v\}) = \gamma(G) - 2$, a contradiction. Thus $\gamma(G_{u,v,4}) = \gamma(G) + 1$.

($\mathbb{J}$) If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,3}) = \gamma(G)$ and by ($\mathbb{G}$), $\gamma(G-\{u,v\}) = \gamma(G) - 2$.

Now let $\gamma(G-\{u,v\}) = \gamma(G) - 2$. But then for each γ-set D of $G-\{u,v\}$, the set $D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$. Thus $\gamma(G_{u,v,4}) = \gamma(G)$. ∎

Theorem 23. *Let u and v be nonadjacent vertices of a graph G. If $\gamma(G_{u,v,k}) = \gamma(G)$, then $k \leq 4$. If $k \geq 5$, then $\gamma(G_{u,v,k}) > \gamma(G)$. If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,5}) = \gamma(G) + 1$.*

Proof. By Theorem 22, $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$. If $\gamma(G_{u,v,4}) > \gamma(G)$, then $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$ because of Observation 13. So, let $\gamma(G_{u,v,4}) = \gamma(G)$. By Theorem 22($\mathbb{H}$), $\gamma(G-\{u,v\}) = \gamma(G)-2$. But then for each γ-set D of $G-\{u,v\}$, the set $D\cup\{x_1, x_3, x_5\}$ is a dominating set of $G_{u,v,5}$. Hence $\gamma(G_{u,v,5}) \leq \gamma(G)+1$. Let now M be a γ-set of $G_{u,v,5}$. Then at least one of x_2, x_3, x_4 is in M and hence $\gamma(G_{u,v,5}) = |M| \geq \gamma(G) + 1$. Thus $\gamma(G_{u,v,5}) = \gamma(G) + 1$. Now using again Observation 13 we conclude that $\gamma(G_{u,v,k}) > \gamma(G)$ for all $k \geq 5$. ∎

Corollary 24. *Let G be a noncomplete graph. Then $\overline{e}pa(G) \leq \overline{E}pa(G) \leq 5$. Moreover, the following holds.*

(i) $\overline{E}pa(G) = 5$ *if and only if there are nonadjacent vertices u and v of G with* $\gamma(G-\{u,v\}) = \gamma(G) - 2$.

(ii) $\overline{e}pa(G) = 5$ *if and only if G is edgeless.*

(iii) $\overline{e}pa(G) = \overline{E}pa(G) = 4$ *if and only if for each pair u, v of nonadjacent vertices of G, $\gamma(G-\{u,v\}) \geq \gamma(G)-1$ and at least one of the following holds:*

  (a) $u \in V^-(G)$ *and v is a γ-good vertex of $G-u$,*

  (b) $v \in V^-(G)$ *and u is a γ-good vertex of $G-v$.*

Proof. By Theorem 23, $\overline{E}pa(G) \leq 5$.

(i) $\Rightarrow$ Let $\overline{E}pa(G) = 5$. Then there is a pair u, v of nonadjacent vertices of G such that $\gamma(G_{u,v,4}) = \gamma(G)$. Now by Theorem 22($\mathbb{H}$), $\gamma(G-\{u,v\}) = \gamma(G) - 2$.

(i) $\Leftarrow$ Let $\gamma(G-\{u,v\}) = \gamma(G) - 2$ and D be a γ-set of $G-\{u,v\}$, where u and v are nonadjacent vertices of G. Hence $D_1 = D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$ and $|D_1| = \gamma(G)$. This implies $\gamma(G_{u,v,4}) = \gamma(G)$. The result now follows by Theorem 23.

(ii) If G has no edges, then the result is obvious. So let G have edges and $\overline{e}pa(G) = 5$. Then for any 2 nonadjacent vertices u and v of G is satisfied

$\gamma(G - \{u, v\}) = \gamma(G) - 2$ (by (i)). Hence we can choose u and v so that they have a neighbor in common, say w. But then w is a γ-bad vertex of $G - u$ which implies $v \notin V^-(G - u)$. This leads to $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$, a contradiction.

(iii) $\Rightarrow$ Let $\bar{e}pa(G) = \overline{E}pa(G) = 4$. Then for each two nonadjacent $u, v \in V(G)$ we have $\gamma(G) = \gamma(G_{u,v,3}) < \gamma(G_{u,v,4})$. Now by Theorem 22($\mathbb{G}$), $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and by Theorem 19, at least one of (a) and (b) is valid.

(iii) $\Leftarrow$ Consider any two nonadjacent vertices u, v of G. Then $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$ and at least one of (a) and (b) is valid. Theorem 19 now implies $\gamma(G) = \gamma(G_{u,v,3})$, and by Theorem 22, $pa(u, v) = 4$. ∎

Example 25. Let G_n be the Cartesian product of two copies of K_n, $n \geq 2$. We consider G_n as an $n \times n$ array of vertices $\{x_{i,j} \mid 1 \leq i \leq j \leq n\}$, where the closed neighborhood of $x_{i,j}$ is the union of the sets $\{x_{1,j}, x_{2,j}, \ldots, x_{n,j}\}$ and $\{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\}$. Note that $V(G_n) = V^-(G_n)$ and $\gamma(G_n) = n$ [6]. It is easy to see that the following sets are γ-sets of $G_n - x_{1,1}$: $D_i = \{x_{2,i}, x_{3,i+1}, \ldots, x_{n,n+i-2}\}$, $i = 2, 3, \ldots, n$, where $x_{k,j} = x_{k,j-n+1}$ for $j > n$ and $2 \leq k \leq n$. Since $D = \bigcup_{i=2}^{n} D_i = V(G_n) \backslash N[x_{1,1}]$, all γ-bad vertices of $G_n - x_{1,1}$ are the neighbors of $x_{1,1}$ in G_n. Since each vertex of D is adjacent to some neighbor of $x_{1,1}$, $V^-(G_n - x_{1,1})$ is empty. Now by Theorem 19 we have $pa(x_{1,1}, y) \geq 4$, and by Theorem 22($\mathbb{H}$), $pa(x_{1,1}, y) < 5$. Thus $pa(x_{1,1}, y) = 4$. By reason of symmetry, we obtain $\bar{e}pa(G_n) = \overline{E}pa(G_n) = 4$.

4. Observations and Open Problems

A constructive characterization of the trees T with $i(T) \equiv \gamma(T)$, and therefore a constructive characterization of the trees T with $Epa(T) = 2$ (by Corollary 7), was provided in [9].

Problem 26. Characterize all unicyclic graphs G with $Epa(G) = 2$.

Problem 27. Find results on γ-excellent graphs G with $\overline{E}pa(G) = 2$.

Problem 28. Characterize all graphs G with $\bar{e}pa(G) = \overline{E}pa(G) = 4$.

Corollary 29. *Let G be a connected noncomplete graph with edges. Then*

(i) $2 \leq epa(G) + \overline{E}pa(G) \leq 8$,
(ii) $2 \leq epa(G) + \bar{e}pa(G) \leq 7$,
(iii) $3 \leq Epa(G) + \overline{E}pa(G) \leq 8$,
(iv) $3 \leq Epa(G) + \bar{e}pa(G) \leq 7$.

Proof. (i)–(iv) The left-side inequalities immediately follow by Corollary 5 and Corollary 15. The right-side inequalities hold because of Corollary 10 and Corollary 24. ∎

Note that all bounds stated in Corollary 29 are attainable. We leave finding examples demonstrating this to the reader.

Problem 30. Characterize all graphs G that attain the bounds in Corollary 29.

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