# ON THE RESTRICTED SIZE RAMSEY NUMBER INVOLVING A PATH $P_{3}$ 

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#### Abstract

For any pair of graphs $G$ and $H$, both the size Ramsey number $\hat{r}(G, H)$ and the restricted size Ramsey number $r^{*}(G, H)$ are bounded above by the size of the complete graph with order equals to the Ramsey number $r(G, H)$, and bounded below by $e(G)+e(H)-1$. Moreover, trivially, $\hat{r}(G, H) \leq$ $r^{*}(G, H)$. When introducing the size Ramsey number for graph, Erdős et al. (1978) asked two questions; (1) Do there exist graphs $G$ and $H$ such that $\hat{r}(G, H)$ attains the upper bound? and (2) Do there exist graphs $G$ and $H$ such that $\hat{r}(G, H)$ is significantly less than the upper bound?

In this paper we consider the restricted size Ramsey number $r^{*}(G, H)$. We answer both questions above for $r^{*}(G, H)$ when $G=P_{3}$ and $H$ is a connected graph.


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## 1. Introduction

Let $G$ be a graph, where the vertex set, edge set, order, size, minimum degree, maximum degree, and its complement are $V(G), E(G), v(G), e(G), \delta(G), \Delta(G)$, and $\bar{G}$, respectively. The degree of a vertex $v$ in $G$ is denoted by $d(v)$. If $H$ is a subgraph of $G$, then $G-H$ is a graph obtained from $G$ by deleting the edges of $H$ [13]. Thus, $V(G-H)=V(G)$. For further terminologies in graphs, please see [6].

A graph $F$ is called a graph arrowing a pair of graphs $G$ and $H$, denoted by $F \rightarrow(G, H)$, if any 2 -coloring (say red and blue) of the edges of $F$ contains a red $G$ or a blue $H$. In 1978, Erdős et al. in [9] introduced the question of how few edges in such an arrowing graph can be. The size Ramsey number $\hat{r}(G, H)$ is the smallest number of edges that an arrowing graph can have. If the order of the arrowing graph is equal to the Ramsey number $r(G, H)$, the smallest number of edges in the arrowing graph is called the restricted size Ramsey number $r^{*}(G, H)$. The Ramsey number $r(G, H)$ itself is the smallest number $r$ such that $K_{r}$ is an arrowing graph for a pair of graphs $G$ and $H$. For a diagonal case, when $G=H$, we write $\hat{r}(G)$ and $r^{*}(G)$ instead of $\hat{r}(G, G)$ and $r^{*}(G, G)$, respectively.

To find the exact values for the (restricted) size Ramsey number for a pair of graphs is a challenging but difficult problem even for small order graphs. The complete list of the (restricted) size Ramsey numbers for all pairs of graphs of order at most four with no isolates can be found in [13], and for the size and the restricted size Ramsey numbers for all pairs of forest graphs of order at most five with no isolates can be found in [18]. For further results about the (restricted) size Ramsey number for graphs can be found in $[2,3,12]$. The current results on the exact values of the (restricted) size Ramsey number for graphs can be found in $[16,19,21-23]$, and on the bounds of the (restricted) size Ramsey number involving paths in $[7,8,16,17]$.

The (restricted) size Ramsey number for any pair of graphs $G$ and $H$ meet the following inequalities.

$$
\begin{equation*}
e(G)+e(H)-1 \leq \hat{r}(G, H) \leq r^{*}(G, H) \leq\binom{ r(G, H)}{2} \tag{1}
\end{equation*}
$$

The first inequality was given by Erdős and Faudree in [10]. The diagonal version of these bounds for the size Ramsey number was given by Harary and Miller in [15].

When introducing the size Ramsey number for graphs, Erdős et al. in [9] proposed two preliminary questions.
(i) Do there exist graphs $G$ and $H$ such that $\hat{r}(G, H)$ attains the upper bound?
(ii) Do there exist $o$-sequences?

An $o$-sequence is defined to give the precise meaning to the idea that $\hat{r}(G, H)$ is 'significantly' less than $\binom{r(G, H)}{2}$. For diagonal case, a sequence of graphs $\left\{G_{n}\right\}$ is called $o$-sequence if $\hat{r}\left(G_{n}\right)=o\binom{r\left(G_{n}\right)}{2}$.

Erdős et al. [9] provided a positive answer to the first question by showing that $\hat{r}\left(K_{m}, K_{n}\right)=r^{*}\left(K_{m}, K_{n}\right)=\binom{r\left(K_{m}, K_{n}\right)}{2}$ for all values of $m$ and $n$. This result is due to Chvátal (by personal communication). It is also true for some pairs of small graphs, namely, $\left(P_{3}, C_{4}\right),\left(C_{4}, C_{4}\right)$, and $\left(C_{4}, K_{4}-e\right)$ (see $\left.[11,13]\right)$.

Erdős et al. [9] also give a positive answer to the second question by showing that it is true for the size Ramsey number of stars and some graphs obtained by star operation. By considering the value of $\hat{r}\left(n K_{2}\right)$ (see [15]) and the value of $r\left(n K_{2}\right)$ (see [20]), we conclude that $\left\{n K_{2}\right\}$ is also an $o$-sequence. Note that the size Ramsey number of stars, $\hat{r}\left(K_{1, m}, K_{1, n}\right)=m+n-1$, is attaining the lower bound. However, it is still open whether every sequence of graph $\{(G, H)\}$ for which $\hat{r}(G, H)$ is attaining the lower bound always belong to $o$-sequence. For a special case, when $G=H=P_{3}$, the lower and upper bounds are the same, which is $\hat{r}\left(P_{3}\right)=r^{*}\left(P_{3}\right)=3$. They are equal to the size of $K_{3}[4,10]$.

In this paper we are concerned with the restricted size Ramsey number $r^{*}\left(P_{3}, H\right)$ for any connected graph $H$. We characterize all connected graphs $H$ such that $r^{*}\left(P_{3}, H\right)$ attains the upper and lower bounds. We also show that $\left\{\left(P_{3}, H\right)\right\}$ with $r^{*}\left(P_{3}, H\right)$ attaining the lower bound belongs to restricted $o$-sequence. The main results of this paper are the following.

Theorem 1. Let $H$ be a connected graph and $v(H)=n$.

$$
r^{*}\left(P_{3}, H\right)=\binom{r\left(P_{3}, H\right)}{2}
$$

if and only if one of the following holds:
(a) $n$ is even, $n \geq 4$, and $H=K_{n}-\frac{n}{2} K_{2}$,
(b) $n$ is odd, $n \geq 5$, and $H$ is one of the following
(b1) $H$ with $\beta(\bar{H})=\frac{n-1}{2}$ and $\Delta(H)=n-1$,
(b2) $H=K_{n}-\left(P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$, or
(b3) $H=K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$.
Theorem 2. Let $H$ be a connected graph and $v(H)=n$.

$$
r^{*}\left(P_{3}, H\right)=e\left(P_{3}\right)+e(H)-1
$$

if and only if $H=K_{1, n-1}$ and $n$ is even.

## 2. Preliminaries

In 1972, Chvátal and Harary [5] gave the Ramsey number for $P_{3}$ and any graph with no isolates, as stated in Theorem A. In finding $r^{*}\left(P_{3}, H\right)$, the order of graph $F$ satisfying $F \rightarrow\left(P_{3}, H\right)$ will be determined by this result.

Theorem A [5]. For any graph $H$ with no isolates,

$$
r\left(P_{3}, H\right)= \begin{cases}v(H), & \bar{H} \text { has } 1 \text {-factor }, \\ 2 v(H)-2 \beta(\bar{H})-1, & \text { otherwise }\end{cases}
$$

where $\beta(\bar{H})$ is the maximum number of independent edges in the complement of $H$.

From the definition of the (restricted) size Ramsey number, the following monotonicity property is obvious. If $G^{\prime} \subseteq G$ and $H^{\prime} \subseteq H$, then

$$
\begin{equation*}
\hat{r}\left(G^{\prime}, H^{\prime}\right) \leq \hat{r}(G, H) \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
r^{*}\left(G^{\prime}, H^{\prime}\right) \leq r^{*}(G, H) \tag{3}
\end{equation*}
$$

Let $F$ be a graph with all the edges colored by red and blue. Following the idea of Faudree and Sheehan in [14], we define a graph that represents the subgraph induced by the blue edges in such a coloring. A graph $G_{F}$ has $V\left(G_{F}\right)=$ $V(F)$ and $E\left(G_{F}\right)$ that consists of red edges in the coloring of $F$ and edges in $\bar{F}$. It is important to notice that $\bar{G}_{F}$ is precisely the subgraph of $F$ induced by blue edges. The notation of the graph $G_{F}$ will be extensively used to prove the main results.

## 3. Proof of Theorem 1

The following lemmas are used to prove this theorem.
Lemma 3 [14]. For a positive integer $n \geq 2$,

$$
\hat{r}\left(P_{3}, K_{n}\right)=r^{*}\left(P_{3}, K_{n}\right)=2(n-1)^{2} .
$$

Lemma 4. Let $H$ be a connected graph of order $n$ such that $\beta=\beta(\bar{H})$. If $n \geq 4$ and $\beta \leq\left\lfloor\frac{n}{2}\right\rfloor-1$, then

$$
r^{*}\left(P_{3}, H\right) \leq\binom{ 2 n-2 \beta-1}{2}-1
$$

Proof. Note that $\beta(\bar{H})$ is the maximum number of independent edges in the complement of $H$. For $\beta=0$, by Theorem A we obtain $r\left(P_{3}, H\right)=2 n-1$ and from Lemma 3 we have $r^{*}\left(P_{3}, H\right)=2(n-1)^{2}<\binom{2 n-1}{2}-1$.

For $1 \leq \beta \leq\left\lfloor\frac{n}{2}\right\rfloor-1$, by Theorem A we obtain $r\left(P_{3}, H\right)=2 n-2 \beta-1$. Note that for any $\beta$, there will be a plenty of non-isomorphic graphs $H$ such that $\beta(\bar{H})=\beta$. However, such a graph $H$ must satisfy $H \subseteq K_{n}-\beta K_{2}$. By (3) we have $r^{*}\left(P_{3}, H\right) \leq r^{*}\left(P_{3}, K_{n}-\beta K_{2}\right)$. Therefore, we only need to show that $r^{*}\left(P_{3}, K_{n}-\beta K_{2}\right) \leq\left({ }_{2}^{2 n-2 \beta-1}\right)-1$.

Now, let $F=K_{2 n-2 \beta-1}-K_{2}$. We will show that $F \rightarrow\left(P_{3}, K_{n}-\beta K_{2}\right)$ for any $\beta$, where $1 \leq \beta \leq\left\lfloor\frac{n}{2}\right\rfloor-1$. To do so, consider any 2-coloring of the edges of $F$ having no red $P_{3}$. The graph $G_{F}$ will be isomorphic to a subgraph of either $P_{4} \cup(n-\beta-3) K_{2} \cup K_{1}$ or $P_{3} \cup(n-\beta-2) K_{2}$. Since $K_{n-\beta} \subseteq K_{n}-\beta K_{2}$, we will show that we can construct $K_{n}-\beta K_{2}$ in $\bar{G}_{F}$ by constructing $K_{n-\beta}$ first.

If $G_{F} \subseteq P_{4} \cup(n-\beta-3) K_{2} \cup K_{1}$, then in $\bar{G}_{F}$ there is a $K_{n-\beta}$ induced by two nonadjacent vertices from $P_{4}$ (one of degree 1 and one of degree 2), vertex of $K_{1}$, and $n-\beta-3$ non-adjacent vertices in $(n-\beta-3) K_{2}$. Furthermore, in $G_{F}$ there are at least $n-\beta-2$ (one is from $P_{4}$ ) nonadjacent vertices of degree 1 (or of degree $2 n-2 \beta-3$ in $\bar{G}_{F}$ ) which are not in $K_{n-\beta}$. Since for $n \geq 4$ it is true that $n-\beta-2 \geq \beta$ and $2 n-2 \beta-3 \geq n-\beta-1$, we can extend this $K_{n-\beta}$ to $K_{n}-\beta K_{2}$ in $\bar{G}_{F}$.

If $G_{F} \subseteq P_{3} \cup(n-\beta-2) K_{2}$, then in $\bar{G}_{F}$ there is a $K_{n-\beta}$ induced by two nonadjacent vertices from $P_{3}$ and $n-\beta-2$ vertices each from $K_{2}$ in $(n-\beta-2) K_{2}$. Furthermore, in $G_{F}$ there are at least $n-\beta-2$ nonadjacent vertices of degree 1 (or of degree $2 n-2 \beta-3$ in $\bar{G}_{F}$ ) which are not in $K_{n-\beta}$. Since for $n \geq 4$ it is true that $n-\beta-2 \geq \beta$ and $2 n-2 \beta-3 \geq n-\beta-1$, we can extend this $K_{n-\beta}$ to $K_{n}-\beta K_{2}$ in $\bar{G}_{F}$.

Since $\bar{G}_{F}$ is exactly the subgraph of $F$ induced by blue edges, $F \rightarrow\left(P_{3}, K_{n}-\right.$ $\left.\beta K_{2}\right)$. Thus $r^{*}\left(P_{3}, K_{n}-\beta K_{2}\right) \leq\left({ }_{2}^{2 n-2 \beta-1}\right)-1$ for any $1 \leq \beta \leq\left\lfloor\frac{n}{2}\right\rfloor-1$.

Next, we examine $r^{*}\left(P_{3}, H\right)$ when $r\left(P_{3}, H\right)=n$. In Lemma 5 we give the first conditions for a connected graph $H$ with $r\left(P_{3}, H\right)=n$, so that $r^{*}\left(P_{3}, H\right)$ attains the upper bound of (1). Odd stars, wheels, and fans meet the conditions, so we also obtain $r^{*}\left(P_{3}, H\right)$ for $H$ are those graphs.

Lemma 5. Let $H$ be a connected graph of order $n$ such that $\beta(\bar{H})=\frac{n-1}{2}$, and $\Delta(H)=n-1$ where $n$ is odd and $n \geq 3$.

$$
r^{*}\left(P_{3}, H\right)=\binom{n}{2}
$$

Proof. Since $\beta(\bar{H})=\frac{n-1}{2}$, by Theorem A we obtain $r\left(P_{3}, H\right)=n$. As a consequence, $r^{*}\left(P_{3}, H\right) \leq\binom{ n}{2}$.

Now, let $F$ be a graph with $v(F)=n$ and $e(F)=\binom{n}{2}-1$. The only possible $F$ is isomorphic to $K_{n}-K_{2}$. Since $n$ is odd, we can give a 2-coloring of the edges of $F$ such that $\frac{n-1}{2}$ independent edges which are involving only one vertex from non-edge are red and the remaining edges are blue. In this 2-coloring, there is no red $P_{3}$ and the maximum degree of the subgraph of $F$ induced by blue edges is $n-2$. Since $\Delta(H)=n-1$, we cannot have a blue $H$. Thus $F \nrightarrow\left(P_{3}, H\right)$ and $r^{*}\left(P_{3}, H\right) \geq\binom{ n}{2}$.

The maximal graph $H$ with $v(H)=n$ and $\beta(\bar{H})=\left\lfloor\frac{n}{2}\right\rfloor$ is $K_{n}-\left\lfloor\frac{n}{2}\right\rfloor K_{2}$, which is the graph obtained by deleting a maximal matching from $K_{n}$. In Lemma 6 we consider this graph. Note that if $n$ is even, then the graph $H$ is a cocktail party graph.

Lemma 6. For $n \geq 3$,

$$
r^{*}\left(P_{3}, K_{n}-\left\lfloor\frac{n}{2}\right\rfloor K_{2}\right)=\binom{n}{2}
$$

Proof. If $n$ is odd, then $\beta(\bar{H})=\frac{n-1}{2}$ and $\Delta(H)=n-1$. From Lemma 5 we have $r^{*}\left(P_{3}, H\right)=\binom{n}{2}$.

If $n$ is even, then $H$ is an $(n-2)$-regular graph. Since $\beta(\bar{H})=\frac{n}{2}(\bar{H}$ has 1-factor), by Theorem A we obtain $r\left(P_{3}, H\right)=n$. As a consequence, $r^{*}\left(P_{3}, H\right) \leq$ $\binom{n}{2}$.

Now, let $F$ be a graph with $v(F)=n$ and $e(F)=\binom{n}{2}-1$. The only possible $F$ is isomorphic to $K_{n}-K_{2}$. Since $n$ is even, we can give a 2 -coloring of the edges of $F$ such that $\frac{n}{2}$ independent edges are red and the remaining edges are blue. In this 2-coloring, there is no red $P_{3}$ and in the subgraph of $F$ induced by blue edges there are exactly two vertices of degree less than $n-2$. Since $H$ is an $(n-2)$-regular graph, we cannot have a blue $H$. Thus $F \nrightarrow\left(P_{3}, H\right)$ and $r^{*}\left(P_{3}, H\right) \geq\binom{ n}{2}$.

Since $r^{*}\left(P_{3}, K_{n}-\left\lfloor\frac{n}{2}\right\rfloor K_{2}\right)$ is still attaining the upper bound of (1), we will go further by considering the graphs obtained by removing one more edge from $K_{n}-\left\lfloor\frac{n}{2}\right\rfloor K_{2}$.

Lemma 7. Let $H_{1}=K_{n}-\left\lfloor\frac{n}{2}\right\rfloor K_{2}$ and $H=H_{1}-e$ for $n \geq 4$. Then,

$$
r^{*}\left(P_{3}, H\right)= \begin{cases}\binom{n}{2}-1, & n \text { is even } \\ \binom{n}{2}, & n \text { is odd }\end{cases}
$$

Proof. Since $v(H)=n$ and $\beta(\bar{H})=\left\lfloor\frac{n}{2}\right\rfloor$, by Theorem A we obtain $r\left(P_{3}, H\right)=n$. We divide the proof into two cases based on the parity of $n$.

Case 1. $n$ is even. The graph $H$ will be isomorphic to $K_{n}-\left(P_{4} \cup\left(\frac{n}{2}-2\right) K_{2}\right)$. Let $F=K_{n}-K_{2}$. We will show that $F \rightarrow\left(P_{3}, H\right)$. To do so, consider any 2 coloring of $F$ having no red $P_{3}$. The graph $G_{F}$ will be a subgraph of $P_{4} \cup$ $\left(\frac{n}{2}-2\right) K_{2}$. As a consequence $\bar{G}_{F}$ must contain $H$. Thus $r^{*}\left(P_{3}, H\right) \leq\binom{ n}{2}-1$.

Now, let $F$ be a graph with $v(F)=n$ and $e(F)=\binom{n}{2}-2$. Thus, $F$ must be isomorphic to either $K_{n}-P_{3}$ or $K_{n}-2 K_{2}$. We will show that $F \nrightarrow\left(P_{3}, H\right)$. First, consider $F=K_{n}-P_{3}$. There is exactly one vertex in $F$ of degree $n-3$. We can color an edge incident to this vertex by red and the remaining edges by blue. In this 2-coloring, there is no red $P_{3}$ in $F$ and the induced subgraph of $F$ by blue edges will contain a vertex of degree $n-4$. Since $\delta(H)=n-3$, we cannot have a blue $H$ in $F$.

Consider $F=K_{n}-2 K_{2}$. There are exactly four vertices in $F$ of degree $n-2$. We can color independent edges incident to these four vertices by red and the remaining edges by blue. In this 2 -coloring, there is no red $P_{3}$ in $F$, and the induced subgraph of $F$ by blue edges will contain at least four vertices of degree $n-3$. Since in $H$ there are exactly two vertices of minimum degree, which is $n-3$, we cannot have a blue $H$. Thus, in any cases we have that $F \nrightarrow\left(P_{3}, H\right)$ and so $r^{*}\left(P_{3}, H\right) \geq\binom{ n}{2}-1$.

Case 2. $n$ is odd. In this case, the graph $H$ must be isomorphic to either $K_{n}-\left(P_{4} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$ or $K_{n}-\left(P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$. If $H=K_{n}-$ $\left(P_{4} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$, then $\beta(\bar{H})=\frac{n-1}{2}$ and $\Delta(H)=n-1$. From Lemma 5 we have $r^{*}\left(P_{3}, H\right)=\binom{n}{2}$.

Let $H=K_{n}-\left(P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$. Since $r\left(P_{3}, H\right)=n, r^{*}\left(P_{3}, H\right) \leq\binom{ n}{2}$. Now, let $F$ be a graph with $v(F)=n$ and $e(F)=\binom{n}{2}-1$. Thus, the graph $F$ must be isomorphic to $K_{n}-K_{2}$. There are exactly two vertices in $F$ of degree $n-2$. Now, color the independent edges incident to these two vertices by red and the remaining edges by blue. In this 2-coloring, there is no red $P_{3}$ in $F$ and the induced subgraph of $F$ by blue edges will contain two vertices of degree $n-3$. Since $H$ contains exactly one vertex of degree $n-3$ (minimum), we cannot have a blue $H$ in $F$. Thus $F \nrightarrow\left(P_{3}, H\right)$ and $r^{*}\left(P_{3}, H\right) \geq\binom{ n}{2}$.

We see in Lemma 7 that $r^{*}\left(P_{3}, K_{n}-\left\lfloor\frac{n}{2}\right\rfloor K_{2}-e\right)$ still attains the upper bound of (1) if $n$ is odd. So, we can consider its subgraph $H$ by deleting one edge and $\Delta(H)<n-1$. (If $n$ is odd and $\Delta(H)=n-1$, then $r^{*}\left(P_{3}, H\right)$ has been given in Lemma 5). We get four different graphs, namely, $H_{1}=K_{n}-$ $\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)(n \geq 5), H_{2}=K_{n}-\left(P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)(n \geq 5), H_{3}=$ $K_{n}-\left(P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)(n \geq 7)$, and $H_{4}=K_{n}-\left(S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$ ( $n \geq 5$ ), where $S_{k, n}$ is a graph obtained from $K_{1, n}$ by subdividing one edge $k$ times [1].

The amalgamation of graphs $H_{1}$ and $H_{2}$, denoted by $\operatorname{Amal}\left(H_{1}, H_{2}\right)$, is formed by identifying a vertex from $H_{1}$ to a vertex from $H_{2}$.

Lemma 8. Let $H_{1}=K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)(n \geq 5), H_{2}=K_{n}-\left(P_{5} \cup\left(\frac{n-1}{2}\right.\right.$ $\left.-2) K_{2}\right)(n \geq 5), H_{3}=K_{n}-\left(P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)(n \geq 7)$, and $H_{4}=$ $K_{n}-\left(S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)(n \geq 5)$. For odd $n$

$$
r^{*}\left(P_{3}, H_{1}\right)=\binom{n}{2}, r^{*}\left(P_{3}, H_{2}\right)=r^{*}\left(P_{3}, H_{3}\right)=r^{*}\left(P_{3}, H_{4}\right)=\binom{n}{2}-1
$$

Proof. Since $v\left(H_{i}\right)=n, n$ is odd, and $\beta\left(\overline{H_{i}}\right)=\frac{n-1}{2}$, by Theorem A we obtain $r\left(P_{3}, H_{i}\right)=n$ for $1 \leq i \leq 4$.

Case 1. $H_{1}=K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$. Since $r\left(P_{3}, H_{1}\right)=n, r^{*}\left(P_{3}, H_{1}\right) \leq$ $\binom{n}{2}$. Now, let $F$ be a graph with $v(F)=n$ and $e(F)=\binom{n}{2}-1$. Then, $F$ must be isomorphic to $K_{n}-K_{2}$. Consider a 2-coloring of the edges of $F$ having no red $P_{3}$ such that the graph $G_{F}$ is $P_{4} \cup\left(\frac{n-1}{2}-2\right) K_{2} \cup K_{1}$. In this 2-coloring, $G_{F}$ is not a subgraph of $C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}$ and so $\bar{G}_{F}$ contains no $H_{1}$. Thus $F \nrightarrow\left(P_{3}, H_{1}\right)$ and $r^{*}\left(P_{3}, H_{1}\right) \geq\binom{ n}{2}$.

Case 2. $H_{2}=K_{n}-\left(P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right), H_{3}=K_{n}-\left(P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)$, or $H_{4}=K_{n}-\left(S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$. If $n=5$, then Silaban et al. have shown that $r^{*}\left(P_{3}, H_{2}\right)=r^{*}\left(P_{3}, H_{4}\right)=\binom{n}{2}-1=9$ [23].

For $n \geq 7$, let $F=K_{n}-K_{2}$. We will show that $F \rightarrow\left(P_{3}, H_{i}\right)$ for $2 \leq i \leq 4$. To do so, consider any 2 -coloring of $F$ that have no red $P_{3}$. Then $G_{F}$ will be isomorphic to a subgraph of either $P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}$ or $P_{4} \cup K_{1} \cup\left(\frac{n-1}{2}-2\right) K_{2}$. Since each of $P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}, P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}$, and $S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}$ is a subgraph of both $P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}$ and $P_{4} \cup K_{1} \cup\left(\frac{n-1}{2}-1\right) K_{2}, \bar{G}_{F}$ must contain $H_{i}$ for $2 \leq i \leq 4$. Thus $F \rightarrow\left(P_{3}, H_{i}\right)$ and $r^{*}\left(P_{3}, H_{i}\right) \leq\binom{ n}{2}-1$ for $2 \leq i \leq 4$.

Now, let $F$ be a graph with $v(F)=n$ and $e(F)=\binom{n}{2}-2$. Then, $F$ must be isomorphic to either $K_{n}-P_{3}$ or $K_{n}-2 K_{2}$. We will show that $F \nrightarrow\left(P_{3}, H_{i}\right)$ for $2 \leq i \leq 4$ for such a graph $F$.

If $F=K_{n}-P_{3}$, then $F$ must contain exactly one vertex of degree $n-3$ and two vertices of degree $n-2$. Now, color three independent edges incident to these three vertices by red so that $G_{F}$ contains an $\operatorname{Amal}\left(P_{4}, P_{3}\right)$ in a vertex of $P_{4}$ of degree 2 and leaf of $P_{3}$. All the remaining edges are colored by blue. In this 2-coloring, there is no red $P_{3}$. Since $\operatorname{Amal}\left(P_{4}, P_{3}\right) \nsubseteq P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}$, $\operatorname{Amal}\left(P_{4}, P_{3}\right) \nsubseteq P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}$, and $\operatorname{Amal}\left(P_{4}, P_{3}\right) \nsubseteq S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}$, we cannot have a blue $H_{i}$ for $2 \leq i \leq 4$. Thus $F \nrightarrow\left(P_{3}, H_{i}\right)$ for $2 \leq i \leq 4$.

If $F=K_{n}-2 K_{2}$, then $F$ contains exactly four vertices of degree $n-2$. Now, color three independent edges incident to these four vertices of degree $n-2$ by red such that $G_{F}$ contains a $P_{6}$ and the remaining edges by blue. In this 2-coloring, there is no red $P_{3}$. Since $P_{6} \nsubseteq P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}, P_{6} \nsubseteq P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}$, and $P_{6} \nsubseteq S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}$, we cannot have a blue $H_{i}$ for $2 \leq i \leq 4$. Thus $F \nrightarrow\left(P_{3}, H_{i}\right)$.

Hence $r^{*}\left(P_{3}, H_{i}\right) \geq\binom{ n}{2}-1$ for $2 \leq i \leq 4$.

Since $r^{*}\left(P_{3}, K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)\right)=\binom{n}{2}$, then we can consider the subgraph of $K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$ by deleting one edge for $n$ being odd. This subgraph will be isomorphic to either $\left.H_{1}=K_{n}-\left(\operatorname{Amal}\left(P_{3}, C_{3}\right) \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)\right)$ for $n \geq 5$ or $\left.H_{2}=K_{n}-\left(C_{3} \cup P_{4} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)\right)$ for $n \geq 7$ where $\operatorname{Amal}\left(P_{3}, C_{3}\right)$ is the amalgamation of a leaf in $P_{3}$ with a vertex in $C_{3}$. Since $K_{n}-\left(\operatorname{Amal}\left(P_{3}, C_{3}\right) \cup\right.$ $\left.\left.\left(\frac{n-1}{2}-2\right) K_{2}\right)\right) \subseteq K_{n}-\left(S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$ and $\left.K_{n}-\left(C_{3} \cup P_{4} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)\right) \subseteq$ $P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}$ ), we have Corollary 9 .

Corollary 9. Let $H_{1}=K_{n}-\left(\operatorname{Amal}\left(P_{3}, C_{3}\right) \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)(n \geq 5)$ and $H_{2}=$ $K_{n}-\left(C_{3} \cup P_{4} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)(n \geq 7)$ where $n$ is odd. Then,

$$
r^{*}\left(P_{3}, H_{1}\right)=\left\{\begin{array}{ll}
8, & n=5, \\
\binom{n}{2}-1, & n \geq 7,
\end{array} \quad \text { and } r^{*}\left(P_{3}, H_{2}\right)=\binom{n}{2}-1 .\right.
$$

Now, we are ready to prove Theorem 1.
Proof. $(\Leftarrow)$ If $n$ is even and $H=K_{n}-\frac{n}{2} K_{2}$, then by Theorem A we have $r\left(P_{3}, H\right)=n$ and the result follows by Lemma 6 . Now, let us consider the case if $n$ is odd. If $\beta(\bar{H})=\frac{n-1}{2}, \Delta(H)=n-1$, then by Theorem A we have $r\left(P_{3}, H\right)=n$ and the result follows by Lemma 5 .

If $K_{n}-\left(P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$, then by Theorem A we have $r\left(P_{3}, H\right)=n$ and the result follows by Lemma 7 . If $K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$, then by Theorem A we have $r\left(P_{3}, H\right)=n$ and the result follows by Lemma 8 .
$(\Rightarrow)$ Suppose to the contrary that $H$ is not one of (a)-(b) and $r^{*}\left(P_{3}, H\right)=$ $\binom{r\left(P_{3}, H\right)}{2}$. If $\beta(\bar{H}) \leq\left\lfloor\frac{n}{2}\right\rfloor-1$, then by Theorem A we have $r\left(P_{3}, H\right)=2 n-2 \beta-1$ and by Lemma 4 we have $r^{*}\left(P_{3}, H\right)<\binom{r\left(P_{3}, H\right)}{2}$.

So our next consideration is for the case if $\beta(\bar{H})=\left\lfloor\frac{n}{2}\right\rfloor$. By Theorem A $r\left(P_{3}, H\right)=n$. If $n$ is even and $H \neq K_{n}-\frac{n}{2} K_{2}$, then $H \subseteq K_{n}-\left(P_{4} \cup\left(\frac{n}{2}-2\right) K_{2}\right)$. By Lemma 7 and (3) we have $r^{*}\left(P_{3}, H\right)<\left({ }_{\left({ }^{r}\left(P_{3}, H\right)\right.}^{2}\right)$.

If $n$ is odd, $\Delta(H)<n-1, H \neq K_{n}-\left(P_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$, and $H \neq$ $K_{n}-\left(C_{3} \cup\left(\frac{n-1}{2}-1\right) K_{2}\right)$, then $H \subseteq K_{n}-\left(P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$ or $H \subseteq K_{n}-$ $\left(P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)$ or $H \subseteq K_{n}-\left(S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$.

If $H \subseteq K_{n}-\left(P_{5} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$ or $H \subseteq K_{n}-\left(P_{4} \cup P_{3} \cup\left(\frac{n-1}{2}-3\right) K_{2}\right)$ or $H \subseteq K_{n}-\left(S_{1,3} \cup\left(\frac{n-1}{2}-2\right) K_{2}\right)$, then by Lemma 8 and (3) we have $r^{*}\left(P_{3}, H\right)<$ $\binom{r\left(\overline{P_{3}}, H\right)}{2}$.

## 4. Proof of Theorem 2

The following lemmas are used to prove Theorem 2.

Lemma 10 [9]. For $m, n \geq 1$,

$$
\hat{r}\left(K_{1, m}, K_{1, n}\right)=m+n-1 .
$$

Clearly, $F=K_{1, m+n-1}$ satisfies that $F \rightarrow\left(K_{1, m}, K_{1, n}\right)$.
Lemma 11. Let $H$ be a connected graph. If $r^{*}\left(P_{3}, H\right)=e\left(P_{3}\right)+e(H)-1$, then $H$ is a star.

Proof. Let $H$ be a connected graph. Suppose $e(H)=m$. Since $r^{*}\left(P_{3}, H\right)=$ $e\left(P_{3}\right)+e(H)-1=m+1$, there is a graph $F$ with $e(F)=m+1$ such that $F \rightarrow\left(P_{3}, H\right)$. It means that any 2-coloring of the edges of $F$ having no red $P_{3}$ must contain a blue $H$. As a consequence, there is at most one independent edge in $F$. Thus, the only possible $F$ is either $K_{3}$ or $K_{1, m+1}$. If $F=K_{3}$, then $H$ must be $K_{1,2}$. If $F=K_{1, m+1}$, then $H$ must be a subgraph of $K_{1, m}$. In both cases, $H$ is a star.

Lemma 12. For integer $n \geq 2$,

$$
r^{*}\left(P_{3}, K_{1, n-1}\right)=\left\{\begin{array}{cl}
n, & n \text { is even } \\
\binom{n}{2}, & n \text { is odd. }
\end{array}\right.
$$

Proof. We know $v\left(K_{1, n-1}\right)=n$. If $n$ is odd, then the assertion holds from Lemma 5 since $\beta\left(\bar{K}_{1, n-1}\right)=\frac{n-1}{2}$ and $\Delta\left(K_{1, n-1}\right)=n-1$. If $n$ is even, then $\beta\left(\bar{K}_{1, n-1}\right)=\frac{n}{2}-1$. By Theorem A we obtain $r\left(P_{3}, K_{1, n-1}\right)=n+1$. From Lemma 10 we have $\hat{r}\left(P_{3}, K_{1, n-1}\right)=n$ and the graph $F=K_{1, n}$ satisfies $F \rightarrow\left(P_{3}, K_{1, n-1}\right)$. Since $v\left(K_{1, n}\right)=n+1=r\left(P_{3}, K_{1, n-1}\right)$, we have $r^{*}\left(P_{3}, K_{1, n-1}\right)=\hat{r}\left(P_{3}, K_{1, n-1}\right)=$ $n$.

As given in Lemma 12, if $n$ is odd, then $r^{*}\left(P_{3}, K_{1, n-1}\right)=\binom{n}{2}$. In fact, this value is the upper bound of (1). On the other hand, Lemma 10 gives that $\hat{r}\left(P_{3}, K_{1, n-1}\right)=n$, the lower bound of (1). This is an example of a pair of graphs for which the size Ramsey number attains the lower bound of (1) while the restricted size Ramsey number attains the upper bound of (1). If $n$ is even, then we have a different phenomenon. Both the size and the restricted size Ramsey number, $\hat{r}\left(P_{3}, K_{1, n-1}\right)=r^{*}\left(P_{3}, K_{1, n-1}\right)=n$ are attaining the lower bound.

Now, we show the proof of Theorem 2.
Proof. $(\Leftarrow)$ Let $H=K_{1, n-1}$ and $n$ be even. By Lemma 12 we have $r^{*}\left(P_{3}, H\right)=$ $n=e\left(P_{3}\right)+e(H)-1$.
$(\Rightarrow)$ Let $H$ be a connected graph, $v(H)=n$ and $r^{*}\left(P_{3}, H\right)=e\left(P_{3}\right)+e(H)-1$. According to Lemma 11, $H$ must be a star. By Lemma 12 we have $H=K_{1, n-1}$ and $n$ is even.

## 5. Restricted o-Sequence

Erdős et al. [9] defined o-sequence for diagonal case of the size Ramsey number. We can adopt the concept for non-diagonal restricted size Ramsey number and we call it as a restricted o-sequence. Then, we have Corollary 13.

Corollary 13. For even $n$ the sequence $\left\{\left(P_{3}, K_{1, n-1}\right)\right\}$ is a restricted o-sequence.
Proof. Since $v\left(K_{1, n-1}\right)=n$, and by Theorem A we have $r\left(P_{3}, K_{1, n-1}\right)=n+1$ if $n$ is even, by Lemma 12 we obtain (if $n$ is even),

$$
\lim _{n \rightarrow \infty} \frac{r^{*}\left(P_{3}, K_{1, n-1}\right)}{\binom{r\left(P_{3}, K_{1, n-1}\right)}{2}}=\lim _{n \rightarrow \infty} \frac{n}{\binom{n+1}{2}}=0
$$

In the introduction we have discussed that $\left\{\left(K_{1, m}, K_{1, n}\right)\right\}$, some graphs obtained by star operation, and $\left\{n K_{2}\right\}$ belong to o-sequence. However, in Corollary 13 we showed that if $m=2$, then $\left\{\left(K_{1, m}, K_{1, n}\right)\right\}$ belongs to restricted $o$-sequence only if $n+1$ is even.

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