Discussiones Mathematicae
Graph Theory 41 (2021) 685–690
doi:10.7151/dmgt.2187

Note

HAMILTONIAN CYCLE PROBLEM IN STRONG k-QUASI-TRANSITIVE DIGRAPHS WITH LARGE DIAMETER

Ruixia Wang

School of Mathematical Sciences
Shanxi University
Taiyuan, Shanxi, 030006, P.R. China

e-mail: wangrx@sxu.edu.cn

Abstract

Let k be an integer with $k \geq 2$. A digraph is k-quasi-transitive, if for any path $x_0 x_1 \ldots x_k$ of length k, x_0 and x_k are adjacent. Let D be a strong k-quasi-transitive digraph with even $k \geq 4$ and diameter at least $k+2$. It has been shown that D has a Hamiltonian path. However, the Hamiltonian cycle problem in D is still open. In this paper, we shall show that D may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for D to be Hamiltonian.

Keywords: quasi-transitive digraph, k-quasi-transitive digraph, Hamiltonian cycle.

2010 Mathematics Subject Classification: 05C20.

1. Terminology and Introduction

We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1] for terminology not defined here. We only consider finite digraphs without loops or multiple arcs. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $x \rightarrow y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $x \rightarrow y$ or $y \rightarrow x$. For disjoint subsets X and Y of $V(D)$, $X \rightarrow Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \rightarrow Y$ and $X \Rightarrow Y$ hold. For subsets X, Y of $V(D)$, we define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. If $X = \{x\}$, then we write (x, Y) instead of $(\{x\}, Y)$. Likewise, if $Y = \{y\}$, then we write (X, y) instead of $(X, \{y\})$. Let

D' be a subdigraph of D and $x \in V(D) \setminus V(D')$. We say that x and D' are adjacent if x and some vertex of D' are adjacent. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S.

Let x and y be two vertices of $V(D)$. The distance from x to y in D, denoted $d(x, y)$, is the minimum length of an (x, y)-path, if y is reachable from x, and otherwise $d(x, y) = \infty$. The distance from a set X to a set Y of vertices in D is $d(X, Y) = \max\{d(x, y) : x \in X, y \in Y\}$. The diameter of D is $\mathrm{diam}(D) = d(V(D), V(D))$. Clearly, D has finite diameter if and only if it is strong.

Let $P = v_1v_2 \cdots v_p$ be a path or a cycle of D. For $i \neq j$, $v_i, v_j \in V(P)$ we denote by $P[v_i, v_j]$ the subpath of P from v_i to v_j. Let $Q = u_1u_2 \cdots u_q$ be a vertex-disjoint path or cycle with P in D. If there exist $v_i \in V(P)$ and $u_j \in V(Q)$ such that $v_iu_j \in A(D)$, then we will use $P[v_1, v_i]Q[u_j, u_q]$ to denote the path $v_1v_2 \cdots v_iu_ju_{j+1} \cdots u_q$.

A digraph is quasi-transitive, if for any path $x_0x_1x_2$ of length 2, x_0 and x_2 are adjacent. The concept of k-quasi-transitive digraphs was introduced in [2] as a generalization of quasi-transitive digraphs. A digraph is k-quasi-transitive, if for any path $x_0x_1 \cdots x_k$ of length k, x_0 and x_k are adjacent. The k-quasi-transitive digraphs have been studied in [2–7].

In [7], Wang and Zhang showed that a strong k-quasi-transitive digraph D with even $k \geq 4$ and $\mathrm{diam}(D) \geq k + 2$ has a Hamiltonian path and proposed the following problem. Let k be an even integer with $k \geq 4$. Is it true that every strong k-quasi-transitive digraph with diameter at least $k + 2$ is Hamiltonian?

In this paper, we shall show that D may contain no Hamiltonian cycle with $k \geq 6$ and give the sufficient condition for it to be Hamiltonian.

2. Main Results

For the rest of this paper, let k be an even integer with $k \geq 4$ and D denote a strong k-quasi-transitive digraph with $\mathrm{diam}(D) \geq k+2$. There exist two vertices u, v such that $d(u, v) = k + 2$ in D. Let $P = x_0x_1 \cdots x_{k+2}$ denote a shortest (u, v)-path in D, where $u = x_0$ and $v = x_{k+2}$.

Theorem 1 [7]**.** *The subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \to x_i$ for $1 \leq i + 1 < j \leq k + 2$.*

Lemma 2 [5]**.** *Let k be an integer with $k \geq 2$ and D be a strong k-quasi-transitive digraph. Suppose that $C = x_0x_1 \cdots x_{n-1}x_0$ is a cycle of length n with $n \geq k$ in D. Then for any $x \in V(D) \setminus V(C)$, x and C are adjacent.*

By Theorem 1, $x_{k+2} \to x_0$. So $x_0x_1 \cdots x_{k+2}x_0$ is a cycle of length $k + 3$. By Lemma 2, every vertex of $V(D) \setminus V(P)$ is adjacent to P. Hence we can divide

$V(D) \setminus V(P)$ into three subsets:

$$O = \{x \in V(D) \setminus V(P) : V(P) \Rightarrow x\},$$

$$I = \{x \in V(D) \setminus V(P) : x \Rightarrow V(P)\},$$

and

$$B = V(D) \setminus (V(P) \cup O \cup I).$$

One of I, O and B may be empty.

Theorem 3 [7]**.** *The subdigraph induced by $V(D) \setminus V(P)$ is a semicomplete digraph.*

Lemma 4 [7]**.** *For any $x \in B$, either x is adjacent to every vertex of $V(P)$ or $\{x_{k+2}, x_{k+1}, x_k, x_{k-1}\} \mapsto x \mapsto \{x_0, x_1, x_2, x_3\}$. In particular, if $k = 4$, then x is adjacent to every vertex of $V(P)$.*

From the proof of Lemma 2.11 in [7], we have the following result.

Lemma 5 [7]**.** *$V(P) \mapsto O$ and $I \mapsto V(P)$.*

By Theorems 1 and 3 and Lemmas 4 and 5, a strong 4-quasi-transitive digraph D with $\mathrm{diam}(D) \geq 6$ is in fact a semicomplete digraph. It is well known that a strong semicomplete digraph is Hamiltonian. Hence, for the rest of this paper, we consider the case $k \geq 6$.

Lemma 6. *Let H be a digraph and $u, v \in V(H)$ such that $d(u, v) = n$ with $n \geq 4$. Let $Q = x_0x_1 \cdots x_n$ be a shortest (u, v)-path in H. If $H[V(Q)]$ is a semicomplete digraph, then, for any $x_i, x_j \in V(Q)$ with $0 \leq i < j \leq n$, there exists a path of length p from x_j to x_i with $p \in \{2, 3, \ldots, n-1\}$ in $H[V(Q)]$.*

Proof. We prove the result by induction on n. For $n = 4$, it is not difficult to check that the result is true. Suppose $n \geq 5$. Assume $j - i = n$. It must be $j = n$ and $i = 0$. Then the length of the path $x_nP[x_2, x_p]x_0$ is p with $p \in \{2, 3, \ldots, n-1\}$. Now assume $1 \leq j - i \leq n - 1$. Then $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$ or $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, assume that $x_i, x_j \in \{x_0, x_1, \ldots, x_{n-1}\}$. By induction, there exists a path of length p from x_j to x_i with $p \in \{2, 3, \ldots, n-2\}$. Now we only need to show that there exists a path of length $n-1$ from x_j to x_i. If $j - i = 1$, then $P[x_j, x_{n-1}]P[x_0, x_i]$ is the desired path. If $j - i = 2$, then $P[x_j, x_n]P[x_0, x_i]$ is the desired path. If $3 \leq j - i \leq n-1$, then $P[x_j, x_n]P[x_{i+2}, x_{j-1}]P[x_0, x_i]$ is the desired path. ■

By Lemma 6, we can obtain the following lemma.

Lemma 7. *For any $x \in V(D) \setminus V(P)$ and $x_i \in V(P)$, if $x \to x_i$, then x and every vertex of $\{x_0, x_1, \ldots, x_{i-1}\}$ are adjacent; if $x_i \to x$, then x and every vertex of $\{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\}$ are adjacent.*

Proof. If $x \to x_i$, then for any $x_j \in \{x_0, x_1, \ldots, x_{i-1}\}$, by Lemma 6, there exists a path Q of length $k-1$ from x_i to x_j. Then the path xQ implies $\overline{xx_j}$. If $x_i \to x$, then for any $x_j \in \{x_{i+1}, x_{i+2}, \ldots, x_{k+2}\}$, by Lemma 6, there exists a path R of length $k-1$ from x_j to x_i. Then the path Rx implies $\overline{xx_j}$. ■

Using Lemma 7, Lemma 4 can be improved to the following result.

Lemma 8. *For any $x \in B$, either x and every vertex of $V(P)$ are adjacent or there exist two vertices $x_t, x_s \in V(P)$ with $4 \le t+1 < s \le k-1$ such that $\{x_s, \ldots, x_{k+2}\} \mapsto x \mapsto \{x_0, \ldots, x_t\}$.*

Proof. If x and every vertex of $V(P)$ are adjacent, then we are done. Suppose not. By the definition of B, $(x, V(P)) \neq \emptyset$ and $(V(P), x) \neq \emptyset$. Take $t = \max\{i : x \to x_i\}$ and $s = \min\{j : x_j \to x\}$. By Lemma 7, x and every vertex $\{x_0, \ldots, x_t\} \cup \{x_s, \ldots, x_{k+2}\}$ are adjacent. Moreover, since x and some vertex of $V(P)$ are not adjacent, we can conclude $s > t+1$ and $\{x_s, \ldots, x_{k+2}\} \mapsto x \mapsto \{x_0, \ldots, x_t\}$. By Lemma 4, $t \ge 3$ and $s \le k-1$. ■

Lemma 9. *Let $Q = z_0 z_1 \cdots z_n$ be a path of length n with $1 \le n \le k-1$ in $D - V(P)$. For some $x_i \in V(P)$, if $z_n \to x_i$, then z_0 and $x_{i+(k-n-1)}$ are adjacent; if $x_i \to z_0$, then z_n and $x_{i-(k-n-1)}$ are adjacent, where the subscripts are taken modulo $k+3$.*

Proof. Using the definition of k-quasi-transitive digraphs, the proof is easy and so we omit it. ■

According to Lemma 8, we can divide B into two subsets. Let $B_1 = \{x \in B : x$ and some vertex of $V(P)$ are not adjacent$\}$ and $B_2 = \{x \in B : x$ and every vertex of $V(P)$ are adjacent$\}$. Now we consider the arcs among I, O, B_1 and B_2. First we show $I \mapsto O$. Let $x \in I$ and $y \in O$ be arbitrary. By Lemma 5, $x \mapsto V(P)$ and $V(P) \mapsto y$. If $y \to x$, then the path $x_0 y x x_{k+2}$ contradicts $d(x_0, x_{k+2}) = k+2 \ge 8$. Thus $I \mapsto O$. Let $z \in B_1$ be arbitrary. By the definition of B_1, z and some vertex of $V(P)$ are not adjacent, say x_{n_0}. By Lemma 8, $3 < n_0 < k-1$. It is not difficult to see that $I \mapsto B_1$ and $B_1 \mapsto O$, otherwise, by Lemma 9, $I \mapsto V(P)$ and $V(P) \mapsto O$, z and every vertex of $V(P)$ are adjacent. Since D is strong, $(B_2, I) \neq \emptyset$ and $(O, B_2) \neq \emptyset$. Let $B_2' = \{u \in B_2 : (u, I) \neq \emptyset\}$ and $B_2'' = \{v \in B_2 : (O, v) \neq \emptyset\}$. Let $u \in B_2'$ and $v \in B_2''$ be two arbitrary vertices. By the definition of B_2', there exist $x_i \in V(P)$ and $x' \in I$ such that $x_i \to u \to x'$. Then the path $x_i u x' x_{k+2}$ implies that $d(x_i, x_{k+2}) \le 3$. Hence $i \ge k-1$, which means $u \mapsto \{x_0, x_1, \ldots, x_{k-2}\}$. By the definition of B_2'', there exist $x_j \in V(P)$ and $y' \in O$ such that $y' \to v \to x_j$. Then the path $x_0y'vx_j$ implies that $d(x_0, x_j) \leq 3$. Hence $j \leq 3$, which means $\{x_4, x_5, \ldots, x_{k+2}\} \mapsto v$. Note that $k-2 \geq 4$. Thus $u \neq v$, which implies $B_2' \cap B_2'' = \emptyset$. Hence $B_2' \mapsto O$ and $I \mapsto B_2''$. If $z \to u$, then considering the path zux', by Lemma 9, z and every vertex of $V(P)$ are adjacent, a contradiction. Hence $B_2' \mapsto B_1$. If $v \to z$, then considering the path $y'vz$, by Lemma 9, z and every vertex of $V(P)$ are adjacent, a contradiction. Hence $B_1 \mapsto B_2''$. If $k = 6$, denote the path $R = x_{k+2}x_{n_0}$. If $k \geq 7$, by Lemma 6, there exists a path of length $k-5$ from x_{k+2} to x_{n_0}, denote it by R. If $v \to u$, then $zy'vux'R$ implies $\overline{zx_{n_0}}$, a contradiction. Hence $B_2' \mapsto B_2''$.

Theorem 10. *If $D - V(P)$ is strong, then D is Hamiltonian.*

Proof. By Lemma 3, $D - V(P)$ is a semicomplete digraph. Hence $D - V(P)$ contains a Hamiltonian cycle, denote it by $H = y_0y_1 \cdots y_my_0$. Clearly, if there exists a pair of arcs $x_ix_{i+1} \in A(P)$ and $y_jy_{j+1} \in A(H)$ such that $x_i \to y_{j+1}$ and $y_j \to x_{i+1}$, then D contains a Hamiltonian cycle $x_iH[y_{j+1}, y_j]P[x_{i+1}, x_i]$. Next we shall find out such a pair of arcs. Suppose $O \neq \emptyset$. Since D is strong, $B \cup I \neq \emptyset$ and there exists $y_j \in V(H)$ such that $y_j \in B \cup I$ and $y_{j+1} \in O$. There exists $x_i \in V(P)$ such that $y_j \to x_i$. Then y_jy_{j+1} and $x_{i-1}x_i$ are the desired arcs. Now assume $O = \emptyset$. Analogously, assume $I = \emptyset$ and so $V(D) \setminus V(P) = B$. If $B_1 = \emptyset$, then D is semicomplete and so D is Hamiltonian. Now assume that $B_1 \neq \emptyset$. If $|V(H)| = 1$, then $y_0 \in B_1$ and $x_{k+2}y_0x_0x_1 \cdots x_{k+2}$ is a Hamiltonian cycle of D. Assume $|V(H)| \geq 2$. If there exist two consecutive vertices $y_j, y_{j+1} \in B_1$, then y_jy_{j+1} and $x_{k+2}x_0$ are the desired arcs. Assume there is no such a pair of arcs. So there exists a pair of vertices $y_j, y_{j+1} \in V(H)$ such that $y_j \in B_2$ and $y_{j+1} \in B_1$. If $y_j \to x_0$, then y_jy_{j+1} and $x_{k+2}x_0$ are the desired arcs. Assume $x_0 \mapsto y_j$. If $|V(H)| = 2$, then $x_0y_jy_{j+1}x_1x_2 \cdots x_{k+2}x_0$ is a Hamiltonian cycle of D. Assume $|V(H)| \geq 3$. According to the above argument, $y_{j+2} \in B_2$. If $y_{j+2} \to x_{k+2}$, then $x_0y_jy_{j+1}y_{j+2}x_{k+2}$ is a path of length 4 from x_0 to x_{k+2}, a contradiction to $d(x_0, x_{k+2}) \geq 8$. Thus $x_{k+2} \to y_{i+2}$. Then $y_{j+1}y_{j+2}$ and $x_{k+2}x_0$ are the desired arcs. ■

Theorem 11. *If $B_2 = \emptyset$ or for any $x \in B_2$, $x_{k+2} \to x \to x_0$, then D is Hamiltonian.*

Proof. If $D - V(P)$ is strong, then, by Theorem 10, we are done. If $D - V(P)$ is not strong, then let $D_1, D_2, \ldots, D_t$ be strong components of $D - V(P)$, where $t \geq 2$. Since D is strong, there exist $x \in V(D_1)$ and $y \in V(D_t)$ such that $(V(P), x) \neq \emptyset$ and $(y, V(P) \neq \emptyset$. By the hypothesis of this theorem and Lemmas 4 and 5, $x_{k+2} \to x$ and $y \to x_0$. It is easy to see that there exists a Hamiltonian path R from x to y in $D - V(P)$. So $x_{k+2}Rx_0x_1 \cdots x_{k+2}$ is a Hamiltonian cycle of D. ■

Suppose $D - V(P)$ is not strong and there exists a vertex $u \in B_2$ such that $u \mapsto x_{k+2}$, we may construct some k-quasi-transitive digraphs such that they are not Hamiltonian. For example, let $V(D) \setminus V(P) = \{u, v\}$ and $u \to v$, $\{x_{k-1}, x_k, x_{k+1}, x_{k+2}\} \to v \to \{x_0, x_1, x_2, x_3\}$ and $x_{k+1} \to u \to \{x_0, x_1, \ldots, x_k, x_{k+2}\}$. It is not difficult to see that D contains no Hamiltonian cycle.

Acknowledgement

This work is supported by the National Natural Science Foundation for Young Scientists of China (11401354) (11501490) (11501341).

References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications (Springer-Verlag, London, 2000).

[2] C. Hernández-Cruz and H. Galeana-Sánchez, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530. doi:10.1016/j.disc.2012.05.005

[3] H. Galeana-Sánchez. I.A. Goldfeder and I. Urrutia, *On the structure of strong* 3-*quasi-transitive digraphs*, Discrete Math. **310** (2010) 2495–2498. doi:10.1016/j.disc.2010.06.008

[4] H. Galeana-Sánchez, C. Hernández-Cruz and M.A. Juárez-Camacho, *On the existence and number of* $(k+1)$-*kings in k-quasi-transitive digraphs*, Discrete Math. **313** (2013) 2582–2591. doi:10.1016/j.disc.2013.08.007

[5] R. Wang and W. Meng, *k-kings in k-quasitransitive digraphs*, J. Graph Theory **79** (2015) 55–62. doi:10.1002/jgt.21814

[6] R. Wang, $(k+1)$-*kernels and the number of k-kings in k-quasi-transitive digraphs*, Discrete Math. **338** (2015) 114–121. doi:10.1016/j.disc.2014.08.009

[7] R. Wang and H. Zhang, *Hamiltonian paths in k-quasi-transitive digraphs*, Discrete Math. **339** (2016) 2094–2099. doi:10.1016/j.disc.2016.02.020

Received 9 May 2018
Accepted 5 November 2018