Discussiones Mathematicae
Graph Theory 41 (2021) 335–339
doi:10.7151/dmgt.2186

Note

GRAPH EXPONENTIATION AND NEIGHBORHOOD RECONSTRUCTION

Richard H. Hammack[1]

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284-2014, USA

e-mail: rhammack@vcu.edu

Abstract

Any graph G admits a neighborhood multiset $\mathscr{N}(G) = \{N_G(x) \mid x \in V(G)\}$ whose elements are precisely the open neighborhoods of G. We say G is neighborhood reconstructible if it can be reconstructed from $\mathscr{N}(G)$, that is, if $G \cong H$ whenever $\mathscr{N}(G) = \mathscr{N}(H)$ for some other graph H. This note characterizes neighborhood reconstructible graphs as those graphs G that obey the exponential cancellation $G^{K_2} \cong H^{K_2} \Longrightarrow G \cong H$.

Keywords: neighborhood reconstructible graphs, graph exponentiation.

2010 Mathematics Subject Classification: 05C60, 05C76.

Our graphs are finite and may have loops, but not parallel edges. The open neighborhood of a vertex x of a graph G is $N_G(x) := \{y \in V(G) \mid xy \in E(G)\}$. Notice that $x \in N_G(x)$ if and only if $xx \in E(G)$, that is, there is a loop at x.

To any graph G there is an associated *neighborhood multiset* $\mathscr{N}(G) = \{N_G(x) \mid x \in V(G)\}$ whose elements are the open neighborhoods of G. It is possible that $\mathscr{N}(G) = \mathscr{N}(H)$ but $G \ncong H$. Figure 1 shows the simplest instance of this. Here $G \ncong H$ but $\mathscr{N}(G) = \big\{\{0\},\{1\}\big\} = \mathscr{N}(H)$. Figure 2 shows a more complex and interesting example.

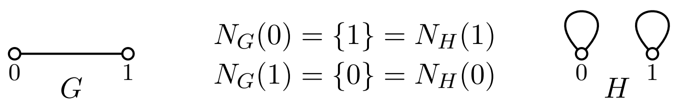
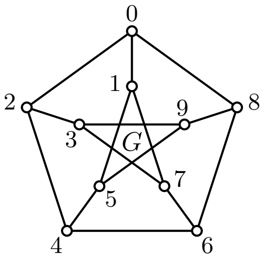


Figure 1. Two non-isomorphic graphs with the same neighborhood multiset.

[1]Supported by Simons Collaboration Grant for Mathematicians 523748.



$$\begin{array}{rcccl}
N_G(0) & = & \{1,2,8\} & = & N_H(0)\\
N_G(1) & = & \{0,5,7\} & = & N_H(1)\\
N_G(2) & = & \{0,3,4\} & = & N_H(8)\\
N_G(3) & = & \{2,7,9\} & = & N_H(6)\\
N_G(4) & = & \{2,5,6\} & = & N_H(9)\\
N_G(5) & = & \{1,4,9\} & = & N_H(5)\\
N_G(6) & = & \{4,7,8\} & = & N_H(3)\\
N_G(7) & = & \{1,3,6\} & = & N_H(7)\\
N_G(8) & = & \{0,6,9\} & = & N_H(2)\\
N_G(9) & = & \{3,5,8\} & = & N_H(4)
\end{array}$$

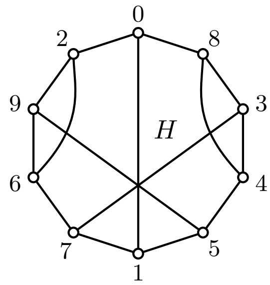


Figure 2. The Petersen graph is not neighborhood reconstructible. It is paired here with a different graph that has the same neighborhood multiset. Example from Mizzi [5, § 3.9].

A graph G is called *neighborhood reconstructible* if $\mathscr{N}(G) = \mathscr{N}(H)$ implies $G \cong H$ for any graph H with $V(H) = V(G)$. Figure 2 shows that the Petersen graph is not neighborhood reconstructible. Aigner and Triesch [1] attribute the neighborhood reconstruction problem to Sós [9]. They note that deciding if a graph is neighborhood reconstructible is NP-complete.

Given graphs G and K, the *graph exponential* G^K is the graph whose vertex set is the set of all functions $V(K) \to V(G)$, where two functions f, g are adjacent precisely if $f(x)g(y) \in E(G)$ for all $xy \in E(K)$. (See [6, 8].) If $V(K) = \{v_1, \dots, v_n\}$, then a function $f : V(K) \to V(G)$ can be identified with an n-tuple $f = (x_1, \dots, x_n) \in V(G)^n$ signifying $f(v_i) = x_i$.

We are interested exclusively in G^{K_2}. Note $V\big(G^{K_2}\big) = V(G) \times V(G)$, and two functions (x_1, x_2) and (y_1, y_2) are adjacent if and only if $x_1y_2 \in E(G)$ and $x_2y_1 \in E(G)$. That is,

$$E\big(G^{K_2}\big) = \big\{(x_1, x_2)(y_1, y_2) \mid x_1y_2 \in E(G) \text{ and } x_2y_1 \in E(G)\big\}.$$

See Figure 3, which shows that $G^K \cong H^K$ does not necessarily imply $G \cong H$.

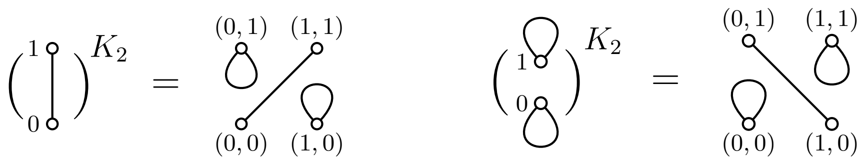


Figure 3. Two exponentials G^{K_2} and H^{K_2}. This shows $G^K \cong H^K$ may not imply $G \cong H$.

Actually, the conditions under which $G^K \cong H^K$ implies $G \cong K$ are not fully understood today. (The issue is further complicated by the fact that there are at least two definitions of graph exponentiation; compare [4].) This note links one instance of this *exponential cancellation* to neighborhood reconstruction. Our main result is that G is neighborhood reconstructible if and only if $G^{K_2} \cong H^{K_2}$ implies $G \cong H$ for all graphs H. To understand why we might expect this, consider Proposition 1 below, whose proof is almost automatic. (Figures 1 and 3 illustrate Proposition 1.)

Proposition 1. *If G and H are two graphs on the same vertex set and $\mathscr{N}(G) = \mathscr{N}(H)$, then $G^{K_2} \cong H^{K_2}$.*

Proof. Say $\mathscr{N}(G) = \mathscr{N}(H)$. As G and H have the same neighborhood multiset, there is a bijection $\varphi : V(G) \to V(H)$ for which $N_G(x) = N_H(\varphi(x))$ for each $x \in V(G)$. (Such map φ is unique if no two vertices of G have the neighborhood; otherwise there is more than one φ.) The bijection $\lambda : V\left(G^{K_2}\right) \to V\left(H^{K_2}\right)$ where $\lambda(x, y) = (\varphi(x), y)$ is an isomorphism. Indeed,

$$\begin{aligned}
(x,y)(u,v) \in E\left(G^{K_2}\right) &\iff v \in N_G(x) \text{ and } y \in N_G(u) \\
&\iff v \in N_H(\varphi(x)) \text{ and } y \in N_H(\varphi(u)) \\
&\iff \big(\varphi(x), y\big)\big(\varphi(u), v\big) \in E\big(H^{K_2}\big) \\
&\iff \lambda(x,y)\,\lambda(u,v) \in E\big(H^{K_2}\big).
\end{aligned}$$
∎

We will use this proposition in the proof of our main result. We will also need the *direct product* of graphs: $G \times H$ is the graph whose vertex set is the set Cartesian product $V(G \times H) = V(G) \times V(H)$, and whose edges are

$$E(G \times H) = \left\{(x,y)(x'y') \mid xx' \in E(G) \text{ and } yy' \in E(H)\right\}.$$

See Chapter 8 of [2] for a survey of the direct product.

For a positive integer k, the *direct power* G^k is $G \times \cdots \times G$ (k factors). Any square G^2 admits a *mirror automorphism* $\mu : G^2 \to G^2$ of order 2, where $\mu(x,y) = (y,x)$. From the definitions it is immediate that

$$(x,y)(u,v) \in E\big(G^2\big) \quad \text{if and only if} \quad (x,y)\mu(u,v) \in E\big(G^{K_2}\big), \tag{1}$$

$$(x,y)(u,v) \in E\big(G^{K_2}\big) \quad \text{if and only if} \quad \mu(x,y)(u,v) \in E\big(G^2\big). \tag{2}$$

Recall the following two results (by Lovász) concerning direct powers and products. (They are Theorems 2 and 5, respectively, in [7].)

Proposition 2. *If $G^k \cong H^k$ for a positive integer k, then $G \cong H$.*

Proposition 3. *If $G \times K \cong H \times K$, then there is an isomorphism $G \times K \to H \times K$ of form $(x,y) \mapsto (\lambda(x,y), y)$ for some map $\lambda : G \times K \to H$.*

Actually, we will only need a weaker instance of Proposition 3, one that is easy to prove from scratch. If $G \times K_2 \cong H \times K_2$, then there exists an isomorphism $G \times K_2 \to H \times K_2$ of form $(x,y) \mapsto (\lambda(x,y), y)$.

We are ready for our main theorem.

Theorem 4. *A graph G is neighborhood reconstructible if and only if the exponential cancellation law $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$ holds for any graph H.*

Proof. Say the exponential cancellation law $G^{K_2} \cong H^{K_2} \Rightarrow G \cong H$ holds. Let $\mathscr{N}(G) = \mathscr{N}(H)$ for a graph H with $V(H) = V(G)$. Proposition 1 yields $G^{K_2} \cong H^{K_2}$, whence $G \cong H$. Thus G is neighborhood reconstructible.

Conversely, suppose G is neighborhood reconstructible. Say $G^{K_2} \cong H^{K_2}$ for some graph H. We must show $G \cong H$.

Put $V(K_2) = \{0,1\}$. Take an isomorphism $\varphi : G^{K_2} \to H^{K_2}$. Using (1) and (2), observe that

$$\begin{aligned} (x,y)(u,v) \in E\big(G^2\big) &\iff (x,y)\,\mu(u,v) \in E\big(G^{K_2}\big) \\ &\iff \varphi(x,y)\,\varphi\mu(u,v) \in E\big(H^{K_2}\big) \\ &\iff \mu\varphi(x,y)\,\varphi\mu(u,v) \in E\big(H^2\big). \end{aligned}$$

From this we get an isomorphism $\Theta : G^2 \times K_2 \to H^2 \times K_2$ defined as

$$\Theta\big((x,y),\varepsilon\big) = \begin{cases} (\varphi\mu(x,y),\varepsilon) & \text{if } \varepsilon = 0, \\ (\mu\varphi(x,y),\varepsilon) & \text{if } \varepsilon = 1. \end{cases}$$

From $G^2 \times K_2 \cong H^2 \times K_2$ we get $G^2 \times K_2 \times K_2 \cong H^2 \times K_2 \times K_3$, yielding $(G \times K_2)^2 \cong (H \times K_2)^2$. By Proposition 2 we have $G \times K_2 \cong H \times K_2$. Then Proposition 3 guarantees an isomorphism $\theta : G \times K_2 \to H \times K_2$ having form

$$\theta(x,\varepsilon) = \begin{cases} (\lambda_0(x),\varepsilon) & \text{if } \varepsilon = 0, \\ (\lambda_1(x),\varepsilon) & \text{if } \varepsilon = 1 \end{cases}$$

for two bijections $\lambda_0, \lambda_1 : V(G) \to V(H)$, which (by definition of the direct product) necessarily satisfy $xy \in E(G)$ if and only if $\lambda_0(x)\lambda_1(y) \in E(H)$.

Now form a graph H' on $V(G)$ whose edges are precisely $\lambda_1^{-1}(u)\lambda_1^{-1}(v)$ for each $uv \in E(H)$. Thus $\lambda_1^{-1} : H \to H'$ is an isomorphism.

We claim that $N_G(x) = N_{H'}\big(\lambda_1^{-1}\lambda_0(x)\big)$ for each $x \in V(G) = V(H')$. Note $y \in N_G(x)$ if and only if $xy \in E(G)$, if and only if $\lambda_0(x)\lambda_1(y) \in E(H)$, if and only if $\lambda_1^{-1}\lambda_0(x)\,\lambda_1^{-1}\lambda_1(y) \in E(H')$, if and only if $\lambda_1^{-1}\lambda_0(x)\,y \in E(H')$, if and only if $y \in N_{H'}\big(\lambda_1^{-1}\lambda_0(x)\big)$. Thus indeed $N_G(x) = N_{H'}\big(\lambda_1^{-1}\lambda_0(x)\big)$.

Consequently $\mathscr{N}(G) = \mathscr{N}(H')$, so $G \cong H'$ because G is neighborhood reconstructible. But $H' \cong H$, so $G \cong H$. ■

The present note is a sequel to [3], which characterizes neighborhood reconstructible graphs as those graphs G which obey the cancellation law $G \times K \cong H \times K \Rightarrow G \cong K$ for all graphs H and K.

Acknowledgment

The author thanks the referee for a prompt response.

References

[1] M. Aigner and E. Triesch, *Realizability and uniqueness in graphs*, Discrete Math. **136** (1994) 3–20.
doi:10.1016/0012-365X(94)00104-Q

[2] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, Second Edition, Series: Discrete Mathematics and its Applications (CRC Press, 2011).

[3] R. Hammack and C. Mullican, *Neighborhood reconstruction and cancellation of graphs*, Electron. J. Combin. **24** (2017) #P2.8

[4] B. Jónsson, *Arithmetic of ordered sets*, in: I. Rival (Ed.), Ordered Sets (Banff Alta, 1981), Volume 83 of NATO Advanced Study Inst. Ser. C. Math. Phys. Sci. 3–43.

[5] R. Mizzi, Two-Fold Isomorphisms, Ph.D. Dissertation (University of Malta, 2016).

[6] L. Lovász, *Operations with structures*, Acta Math. Acad. Sci. Hungar. **18** (1967) 321–328.
doi:10.1007/BF02280291

[7] L. Lovász, *On the cancellation law among finite relational structures*, Period. Math. Hungar. **1** (1971) 145–156.
doi:10.1007/BF02029172

[8] N. Sauer, *Hedetniemi's conjecture—a survey*, Discrete Math. **229** (2001) 261–292.
doi:10.1016/S0012-365X(00)00213-2

[9] V.T. Sós, *Problem*, in: A. Hajnal and V.T. Sós, (Eds.), Combinatorics Coll. Math. Soc. J. Bolyai 18 (North-Holland, Amsterdam, 1978) 1214.

Received 3 September 2018
Accepted 27 October 2018