## Note

# GRAPH EXPONENTIATION AND NEIGHBORHOOD RECONSTRUCTION 

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#### Abstract

Any graph $G$ admits a neighborhood multiset $\mathscr{N}(G)=\left\{N_{G}(x) \mid x \in\right.$ $V(G)\}$ whose elements are precisely the open neighborhoods of $G$. We say $G$ is neighborhood reconstructible if it can be reconstructed from $\mathscr{N}(G)$, that is, if $G \cong H$ whenever $\mathscr{N}(G)=\mathscr{N}(H)$ for some other graph $H$. This note characterizes neighborhood reconstructible graphs as those graphs $G$ that obey the exponential cancellation $G^{K_{2}} \cong H^{K_{2}} \Longrightarrow G \cong H$.


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Our graphs are finite and may have loops, but not parallel edges. The open neighborhood of a vertex $x$ of a graph $G$ is $N_{G}(x):=\{y \in V(G) \mid x y \in E(G)\}$. Notice that $x \in N_{G}(x)$ if and only if $x x \in E(G)$, that is, there is a loop at $x$.

To any graph $G$ there is an associated neighborhood multiset $\mathscr{N}(G)=\left\{N_{G}(x)\right.$ $\mid x \in V(G)\}$ whose elements are the open neighborhoods of $G$. It is possible that $\mathscr{N}(G)=\mathscr{N}(H)$ but $G \neq H$. Figure 1 shows the simplest instance of this. Here $G \not \approx H$ but $\mathscr{N}(G)=\{\{0\},\{1\}\}=\mathscr{N}(H)$. Figure 2 shows a more complex and interesting example.


Figure 1. Two non-isomorphic graphs with the same neighborhood multiset.

[^0]

Figure 2. The Petersen graph is not neighborhood reconstructible. It is paired here with a different graph that has the same neighborhood multiset. Example from Mizzi [5, § 3.9].

A graph $G$ is called neighborhood reconstructible if $\mathscr{N}(G)=\mathscr{N}(H)$ implies $G \cong H$ for any graph $H$ with $V(H)=V(G)$. Figure 2 shows that the Petersen graph is not neighborhood reconstructible. Aigner and Triesch [1] attribute the neighborhood reconstruction problem to Sós [9]. They note that deciding if a graph is neighborhood reconstructible is NP-complete.

Given graphs $G$ and $K$, the graph exponential $G^{K}$ is the graph whose vertex set is the set of all functions $V(K) \rightarrow V(G)$, where two functions $f, g$ are adjacent precisely if $f(x) g(y) \in E(G)$ for all $x y \in E(K)$. (See [6, 8].) If $V(K)=\left\{v_{1}, \ldots, v_{n}\right\}$, then a function $f: V(K) \rightarrow V(G)$ can be identified with an $n$-tuple $f=\left(x_{1}, \ldots, x_{n}\right) \in V(G)^{n}$ signifying $f\left(v_{i}\right)=x_{i}$.

We are interested exclusively in $G^{K_{2}}$. Note $V\left(G^{K_{2}}\right)=V(G) \times V(G)$, and two functions $\left(x_{1}, x_{2}\right)$ and ( $y_{1}, y_{2}$ ) are adjacent if and only if $x_{1} y_{2} \in E(G)$ and $x_{2} y_{1} \in E(G)$. That is,

$$
E\left(G^{K_{2}}\right)=\left\{\left(x_{1}, x_{2}\right)\left(y_{1}, y_{2}\right) \mid x_{1} y_{2} \in E(G) \text { and } x_{2} y_{1} \in E(G)\right\} .
$$

See Figure 3, which shows that $G^{K} \cong H^{K}$ does not necessarily imply $G \cong H$.

Figure 3. Two exponentials $G^{K_{2}}$ and $H^{K_{2}}$. This shows $G^{K} \cong H^{K}$ may not imply $G \cong H$.
Actually, the conditions under which $G^{K} \cong H^{K}$ implies $G \cong K$ are not fully understood today. (The issue is further complicated by the fact that there are at least two definitions of graph exponentiation; compare [4].) This note links one instance of this exponential cancellation to neighborhood reconstruction. Our
main result is that $G$ is neighborhood reconstructible if and only if $G^{K_{2}} \cong H^{K_{2}}$ implies $G \cong H$ for all graphs $H$. To understand why we might expect this, consider Proposition 1 below, whose proof is almost automatic. (Figures 1 and 3 illustrate Proposition 1.)

Proposition 1. If $G$ and $H$ are two graphs on the same vertex set and $\mathscr{N}(G)=$ $\mathscr{N}(H)$, then $G^{K_{2}} \cong H^{K_{2}}$.

Proof. Say $\mathscr{N}(G)=\mathscr{N}(H)$. As $G$ and $H$ have the same neighborhood multiset, there is a bijection $\varphi: V(G) \rightarrow V(H)$ for which $N_{G}(x)=N_{H}(\varphi(x))$ for each $x \in V(G)$. (Such map $\varphi$ is unique if no two vertices of $G$ have the neighborhood; otherwise there is more than one $\varphi$.) The bijection $\lambda: V\left(G^{K_{2}}\right) \rightarrow V\left(H^{K_{2}}\right)$ where $\lambda(x, y)=(\varphi(x), y)$ is an isomorphism. Indeed,

$$
\begin{aligned}
(x, y)(u, v) \in E\left(G^{K_{2}}\right) & \Longleftrightarrow v \in N_{G}(x) \text { and } y \in N_{G}(u) \\
& \Longleftrightarrow v \in N_{H}(\varphi(x)) \text { and } y \in N_{H}(\varphi(u)) \\
& \Longleftrightarrow(\varphi(x), y)(\varphi(u), v) \in E\left(H^{K_{2}}\right) \\
& \Longleftrightarrow \lambda(x, y) \lambda(u, v) \in E\left(H^{K_{2}}\right) .
\end{aligned}
$$

We will use this proposition in the proof of our main result. We will also need the direct product of graphs: $G \times H$ is the graph whose vertex set is the set Cartesian product $V(G \times H)=V(G) \times V(H)$, and whose edges are

$$
E(G \times H)=\left\{(x, y)\left(x^{\prime} y^{\prime}\right) \mid x x^{\prime} \in E(G) \text { and } y y^{\prime} \in E(H)\right\}
$$

See Chapter 8 of [2] for a survey of the direct product.
For a positive integer $k$, the direct power $G^{k}$ is $G \times \cdots \times G$ ( $k$ factors). Any square $G^{2}$ admits a mirror automorphism $\mu: G^{2} \rightarrow G^{2}$ of order 2 , where $\mu(x, y)=(y, x)$. From the definitions it is immediate that

$$
\begin{array}{lll}
(x, y)(u, v) \in E\left(G^{2}\right) & \text { if and only if } & (x, y) \mu(u, v) \in E\left(G^{K_{2}}\right) \\
(x, y)(u, v) \in E\left(G^{K_{2}}\right) & \text { if and only if } & \mu(x, y)(u, v) \in E\left(G^{2}\right) \tag{2}
\end{array}
$$

Recall the following two results (by Lovász) concerning direct powers and products. (They are Theorems 2 and 5, respectively, in [7].)

Proposition 2. If $G^{k} \cong H^{k}$ for a positive integer $k$, then $G \cong H$.
Proposition 3. If $G \times K \cong H \times K$, then there is an isomorphism $G \times K \rightarrow H \times K$ of form $(x, y) \mapsto(\lambda(x, y), y)$ for some map $\lambda: G \times K \rightarrow H$.

Actually, we will only need a weaker instance of Proposition 3, one that is easy to prove from scratch. If $G \times K_{2} \cong H \times K_{2}$, then there exists an isomorphism $G \times K_{2} \rightarrow H \times K_{2}$ of form $(x, y) \mapsto(\lambda(x, y), y)$.

We are ready for our main theorem.
Theorem 4. A graph $G$ is neighborhood reconstructible if and only if the exponential cancellation law $G^{K_{2}} \cong H^{K_{2}} \Rightarrow G \cong H$ holds for any graph $H$.

Proof. Say the exponential cancellation law $G^{K_{2}} \cong H^{K_{2}} \Rightarrow G \cong H$ holds. Let $\mathscr{N}(G)=\mathscr{N}(H)$ for a graph $H$ with $V(H)=V(G)$. Proposition 1 yields $G^{K_{2}} \cong H^{K_{2}}$, whence $G \cong H$. Thus $G$ is neighborhood reconstructible.

Conversely, suppose $G$ is neighborhood reconstructible. Say $G^{K_{2}} \cong H^{K_{2}}$ for some graph $H$. We must show $G \cong H$.

Put $V\left(K_{2}\right)=\{0,1\}$. Take an isomorphism $\varphi: G^{K_{2}} \rightarrow H^{K_{2}}$. Using (1) and (2), observe that

$$
\begin{aligned}
(x, y)(u, v) \in E\left(G^{2}\right) & \Longleftrightarrow(x, y) \mu(u, v) \in E\left(G^{K_{2}}\right) \\
& \Longleftrightarrow \varphi(x, y) \varphi \mu(u, v) \in E\left(H^{K_{2}}\right) \\
& \Longleftrightarrow \mu \varphi(x, y) \varphi \mu(u, v) \in E\left(H^{2}\right)
\end{aligned}
$$

From this we get an isomorphism $\Theta: G^{2} \times K_{2} \rightarrow H^{2} \times K_{2}$ defined as

$$
\Theta((x, y), \varepsilon)= \begin{cases}(\varphi \mu(x, y), \varepsilon) & \text { if } \varepsilon=0 \\ (\mu \varphi(x, y), \varepsilon) & \text { if } \varepsilon=1\end{cases}
$$

From $G^{2} \times K_{2} \cong H^{2} \times K_{2}$ we get $G^{2} \times K_{2} \times K_{2} \cong H^{2} \times K_{2} \times K_{3}$, yielding $\left(G \times K_{2}\right)^{2} \cong\left(H \times K_{2}\right)^{2}$. By Proposition 2 we have $G \times K_{2} \cong H \times K_{2}$. Then Proposition 3 guarantees an isomorphism $\theta: G \times K_{2} \rightarrow H \times K_{2}$ having form

$$
\theta(x, \varepsilon)= \begin{cases}\left(\lambda_{0}(x), \varepsilon\right) & \text { if } \varepsilon=0 \\ \left(\lambda_{1}(x), \varepsilon\right) & \text { if } \varepsilon=1\end{cases}
$$

for two bijections $\lambda_{0}, \lambda_{1}: V(G) \rightarrow V(H)$, which (by definition of the direct product) necessarily satisfy $x y \in E(G)$ if and only if $\lambda_{0}(x) \lambda_{1}(y) \in E(H)$.

Now form a graph $H^{\prime}$ on $V(G)$ whose edges are precisely $\lambda_{1}^{-1}(u) \lambda_{1}^{-1}(v)$ for each $u v \in E(H)$. Thus $\lambda_{1}^{-1}: H \rightarrow H^{\prime}$ is an isomorphism.

We claim that $N_{G}(x)=N_{H^{\prime}}\left(\lambda_{1}^{-1} \lambda_{0}(x)\right)$ for each $x \in V(G)=V\left(H^{\prime}\right)$. Note $y \in N_{G}(x)$ if and only if $x y \in E(G)$, if and only if $\lambda_{0}(x) \lambda_{1}(y) \in E(H)$, if and only if $\lambda_{1}^{-1} \lambda_{0}(x) \lambda_{1}^{-1} \lambda_{1}(y) \in E\left(H^{\prime}\right)$, if and only if $\lambda_{1}^{-1} \lambda_{0}(x) y \in E\left(H^{\prime}\right)$, if and only if $y \in N_{H^{\prime}}\left(\lambda_{1}^{-1} \lambda_{0}(x)\right)$. Thus indeed $N_{G}(x)=N_{H^{\prime}}\left(\lambda_{1}^{-1} \lambda_{0}(x)\right)$.

Consequently $\mathscr{N}(G)=\mathscr{N}\left(H^{\prime}\right)$, so $G \cong H^{\prime}$ because $G$ is neighborhood reconstructible. But $H^{\prime} \cong H$, so $G \cong H$.

The present note is a sequel to [3], which characterizes neighborhood reconstructible graphs as those graphs $G$ which obey the cancellation law $G \times K \cong$ $H \times K \Rightarrow G \cong K$ for all graphs $H$ and $K$.

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